Strategic Resource Allocation for Competitive Influence in Social Networks

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Abstract

One of the main objectives of data mining is to help companies determine to which potential customers to market and how many resources to allocate to these potential customers. Most previous works on competitive influence in social networks focus on the first issue. In this work, our focus is on the second issue, i.e., we are interested on the competitive influence of marketing campaigns who need to simultaneously decide how many resources to allocate to their potential customers to advertise their products. Using results from game theory, we are able to completely characterize the optimal strategic resource allocation for the voter model of social networks and prove that the price of competition of this game is unbounded. This work is a step towards providing a solid foundation for marketing advertising in more general scenarios.

1 Introduction

In contrast to mass marketing, where a product is promoted indiscriminately to all potential customers, direct marketing promotes a product only to customers likely to be profitable. The groundbreaking works of Domingos and Richardson [7, 27] incorporated the influence of peers on the decision making process of potential customers deciding between different products or services promoted by competing marketing campaigns through direct marketing. This aggregated value of a customer has been called the network value of a customer. If each customer was making a buying decision independently of all other customers, then we should only consider the intrinsic value of a customer (i.e., the expected profit from sales to him). However, an individual's decision to buy a product or service is often strongly influenced by his friends, acquaintances, etc. A customer whose intrinsic value is lower than the cost of marketing may be worth marketing to when his network value is considered. Conversely, marketing to a profitable consumer may be redundant if network effects already make him very likely to buy [7].

Most of the existing literature assumes there is an incumbent that holds the market and a challenger who needs to allocate advertisement through direct marketing for certain individuals in order to promote the challenger product or service. Notable exceptions to that trend are the works on competitive influence in social networks of Bharathi et al. [1], Sanjeev and Kearns [14], He and Kempe [17], Borodin et al. [4] and Chasparis and Shamma [5]. Bharathi et al. [1] proposed a generalization of the independent cascade model [12] and gave a $(1 - 1/e)$ approximation algorithm for computing the best response to an already known opponent's strategy and proved that the price of competition of the game (resulting from the lack of coordination among the agents) is at most 2. Sanjeev and Kearns [14] considered two independent functions denoted switching function and selection function. The switching function takes into account the probability of a consumer switching from non-adoption

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to adoption and the selection function specifies, conditional on switching, the probability that the consumer adopts one of them. Both players simultaneously choose some number of nodes to initially seed. The authors make the simplifying assumption that once a node is infected, it never switches again, and proceed to study some specific families of switching and adoption functions. He and Kempe [17] motivated by [14] studied the price of anarchy of that framework and found an upper bound of 2 on that price. Borodin et al. [4] showed that for a broad family of competitive influence models is NP-hard to achieve an approximation that is better that the square root of the optimal solution. Chasparis and Shamma [5] found optimal advertising policies using dynamic programming based on the models of [8] and [10].

In the present work, our focus is different from previously described works where the focus was to which potential customers to market, assuming that we have knowledge about the cost of adoption of potential customers, while the focus in our work is on how many resources to allocate to potential customers for them to prefer one product or service versus another. We are interested on the scenario when two competing marketing campaigns need to simultaneously decide how many resources to allocate to potential customers to advertise their products. The process and dynamics by which influence is spread is given by the voter model.

Under that model, our main results are the following:

Theorem 1. *Given a graph $G = (V, E)$ representing a social network of n potential customers. The symmetric strategic resource allocation problem by only taking into account the intrinsic value of the customers, for any target time τ, is given by a probability distribution function F^* of $x \in \Delta^{n-1}$, such that each vector coordinate x_i is uniformly distributed on $[0, 2B/n]$ for $i \in \{1, \ldots, n\}$, where B is the available budget and Δ^{n-1} is the set of available allocations.*

Theorem 2. *Given a graph $G = (V, E)$ representing a social network of n potential customers. The symmetric strategic resource allocation problem for target time τ has a solution given by a probability distribution function F^*_τ of $x \in \Delta^{n-1}$, such that each vector coordinate x_i is uniformly distributed on $[0, 2B\sum_{j=1}^{n} M^\tau(i,j)]$ for $i \in \{1, \ldots, n\}$, where B is the available budget, M is the normalized transition matrix and Δ^{n-1} is the set of available allocations. Moreover, the long term case has a solution given by a probability distribution function F^*_∞ of $x \in \Delta^{n-1}$, such that each vector coordinate x_i is uniformly distributed on $[0, Bd_i/|E|]$ for $i \in \{1, \ldots, n\}$, where B is the available budget, d_i is the degree of potential customer i, $|E|$ is the total number of edges of the graph and Δ^{n-1} is the set of available allocations.*

Theorem 3. *The price of competition of the game (resulting from lack of coordination among the agents) is unbounded.*

Theorem 1 gives an optimal policy for the allocation of resources in the case when we consider potential customers in isolation or their intrinsic value, while Theorem 2 gives an optimal policy for the allocation of resources in the case when we also include the network value of potential customers. Theorem 3 gives the price of competition (resulting from the lack of coordination among the agents) for the strategic allocation problem. We notice that this is a very different result than the one obtained by Bharathi et al. [1] where they found that the price of competition is at most a factor of 2 when the focus is on which potential customers to recruit.

1.1 Related Work

The (meta) problem of influence maximization was first defined by Domingos and Richardson [7, 27]. They studied this problem in a probabilistic setting and provided heuristics to compute an influence maximizing set. Following this work, Kempe et al. [19, 20] and Mossel and Roch [23], based on the results of Nemhauser et al. [26] and Vetta [31], proved that for very natural activation functions (monotone and submodular or in economic terms, with decreasing marginal utility), the function of the expected number of active nodes at termination is a submodular function and thus can be approximated through a greedy approach with a $(1-1/e-\varepsilon)$-approximation algorithm for the spread maximization set problem. A slightly different model but with similar flavor, the voter model, was introduced by Clifford and Sudbury [6] and Holley and Liggett [18]. In that model of social network, Even-Dar and Shapira [9] found an exact solution to the spread maximization set problem when all the nodes have the same cost and provided an FPTAS (Fully Polynomial Time Approximation Scheme is an algorithm that for any ε approximates the optimal solution up to an error $(1+\varepsilon)$ in time $\text{poly}(n/\varepsilon)$) for the case in which different nodes may have different costs.

In this work, we study the case when two marketing campaigns competing to promote a product or service need to decide how many resources to allocate to potential customers to advertise their product or service. Other works related to competitive influence where the already described [1, 4, 5, 14, 17].

To study this case, we use recent advances of game theory, and in particular of Colonel Blotto games. In the simplest version of the Colonel Blotto game, two generals want to capture three equally valued battlefields. Each general disposes of one divisible unit of military resources. The generals have to simultaneously allocate these resources across the three battlefields. A battlefield is captured by a general if he allocates more resources there than his opponent. The goal of each general is to maximize the number of captured battlefields.

The relationship between Colonel Blotto games and our work is the following. We establish a parallel between the marketing campaigns and the generals; and between the potential customers and the battlefields. Each marketing campaign needs to strategically allocate advertising resources to outperform the competing marketing campaign. This needs to be done while knowing that the competing marketing campaign is trying to do the same. It is thus a typical situation in which game theory comes into play. In our case, of course, we will not be dealing with three potential customers so we need to extend this case to include any number of potential customers. By including the network value of customers, each potential customer will not be equally valued so we will need also to consider different payoffs for different potential customers.

The Colonel Blotto game, was first solved for the case of three battlefields by Borel [2, 3]. For the case of equally valued battlefields, also known as homogeneous battlefields case, this result was generalized for any number of battlefields by Gross and Wagner [16]. Roberson [28] focused on the case of homogeneous battlefields and different budgets (also known as asymmetric budgets case). Gross [15] proved the existence and a method to construct the joint probability distribution. Laslier and Picard [22] and Thomas [30] provided alternative methods to construct the joint distribution by extending the disc method proposed by Gross and Wagner [16]. We will extensively use the work of Gross [15] to derive our results.

The plan of this work is as follows. In Section 2, we provide some preliminaries about the tools we use to find the optimal strategy. The reader acquainted with game theory and Colonel Blotto games can skip this section. In Section 3, we explain the voter model of social networks. In Section 4, we derive the main results of this work, and in Section 5, we conclude this work and provide future perspectives for continuing our work.

2 Preliminaries on game theory

Colonel Blotto games (or Divide a Dollar games) in their classic version are a class of two-person zero-sum games, in which both players need to simultaneously allocate limited resources over several objects. Colonel Blotto games are usually described in a military context where the limited resources are troops and the objects are battlefields. In that context, the player devoting the most troops (or resources) to a battlefield (or object) captures that battlefield (and the payoff associated with that battlefield) and its total payoff is the sum of the individual payoffs across captured battlefields. To analyze these games, we need to introduce some basic concepts in game theory.

In this subsection, we follow the notation of [31]. Consider we have two agents (or players) and disjoint groundsets V_1 and V_2. Each element in V_i represents an act that agent i may make, $i \in \{1, 2\}$, in our case, the allocation of troops to a battlefield. Let $a_i \subseteq V_i$ be an action (set of acts) available to agent i, for example, the set of troops allocations across battlefields. We want to restrict the set of actions an agent may make; thus we may not allow every subset of V_i to be a feasible action. We let $\mathcal{A}_i = \{a_i \subseteq V_i : a_i \text{ is a feasible action}\}$ be the set of all available actions to agent i. In our case, the set of possible allocations is limited by the budget of the agents. We call $\mathcal{A}_i$ the action space for agent i. A *pure strategy* is one in which the agent decides to carry out a specific action. A *mixed strategy* is one in which the agent decides upon an action according to some probability distribution. The strategy space $\mathcal{S}_i$ of agent i is the set of mixed strategies. We let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, and $V = V_1 \cup V_2$. Given an action set $A \in \mathcal{A}$, let $A \oplus a'_i$ denote the action set obtained if agent i changes its action from a_i to a'_i. Similarly, given a strategy set $S \in \mathcal{S}$, let $S \oplus s'_i$ denote the strategy set obtained if agent i changes its strategy from s_i to s'_i. For every agent i, there is a private utility function $\alpha_i : 2^V \to \mathbb{R}$. The expected value of $\alpha_i(S)$ on the strategy set S, denoted $\bar{\alpha}_i(S)$, is given by

$$\bar{\alpha}_i(S) = \sum_{A \in \mathcal{A}} \alpha_i(A)\mathbb{P}(A|S),$$

where $\mathbb{P}(A|S)$ is the probability that action set A is implemented given that the agents are using the strategy set S. The goal of each agent is to maximize its expected private utility. We say that a set of strategies $S \in \mathcal{S}$ is a *Nash equilibrium* if no agent has an incentive to change strategy. That is, for any agent i,

$$\bar{\alpha}_i(S) \geq \bar{\alpha}_i(S \oplus s'_i) \quad \forall s'_i \in S_i.$$

We say that a Nash equilibrium is a pure strategy Nash equilibrium if, for each agent i, s_i is a pure strategy. Otherwise, we say that the Nash equilibrium is a mixed strategy Nash equilibrium. Throughout this work the term equilibrium refers to Nash equilibrium, although, since the game is a zero sum game, these equilibrium strategies are also optimal strategies.

The following result, due to Nash [25], shows that there exists at least one Nash equilibrium for any finite game.

Theorem 4 (Nash [25])**.** *Any finite, k-person, non-co- operative game has at least one Nash equilibrium.*

2.1 Colonel Blotto games

In this subsection, we include the notation of [30]. As previously said in Section 1, in the simplest version of the Colonel Blotto game, two generals want to capture three equally valued battlefields. Each general disposes of one divisible unit of military resources and the generals have to simultaneously allocate these resources across the three battlefields. A battlefield is captured by a general if he allocates more resources there than his opponent and the goal of each general is to maximize the number of captured battlefields.

In that game, a pure strategy for a player, denoted X, is a 3-dimensional allocation vector $\mathbf{x} = (x_1, x_2, x_3)$ where x_i is the amount of resources allocated to the ith battlefield for $i \in \{1, 2, 3\}$. The set of pure strategies is the 2-dimensional unit simplex

$$\Delta^2 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0 \quad \text{and} \quad x_1+x_2+x_3 = 1\}.$$

A mixed strategy is a tri-variate distribution function $F : \Delta^2 \to [0, 1]$. Similarly for his enemy, denoted Y, we define $\mathbf{y}$ the allocation vector, y_i the proportion of resources allocated to the ith battlefield for $i \in \{1, 2, 3\}$, Δ^2 the set of pure strategies and $G : \Delta^2 \to [0, 1]$ a mixed strategy.

The natural extension of the classic version of the Colonel Blotto game is to study the case with n battlefields where each captured battlefield gives a different payoff. Consider that we have two players, denoted X and Y, and n objects. Player X has budget B_X and it can allocate for an object i a proportion of his budget x_i for $1 \leq i \leq n$. Similarly, player Y has budget B_Y and it can allocate for an object i a proportion of his budget y_i for $1 \leq i \leq n$. In this work, we limit ourselves to the case when both players have the same total divisible budget, i.e., $B_X = B_Y$, and without loss of generality we consider this budget to be equal to 1. We consider this to simplify the derivations, however it is easy to derive the results by considering the budgets to be equal to B instead of 1.

For the general case, a pure strategy for player X can be written as an n-dimensional allocation vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with

$$\sum_{i=1}^{n} x_i = 1, x_i \in [0, 1],$$

where x_i represents the fraction of budget allocated to front i. Thus, the set of pure strategies is the $(n-1)$-dimensional simplex

$$\Delta^{(n-1)} = \{(x_1, \ldots, x_n) : x_i \geq 0,\, 1 \leq i \leq n \text{ and } \sum_{i=1}^{n} x_i = 1\}.$$

A mixed strategy is an n-variate distribution function

$F : \Delta^{(n-1)} \to [0,1]$. Let F_i denote the ith one-dimensional marginal of F, i.e., the unconditional distribution of x_i.

The object i has an associated non-negative payoff A_i for $1 \leq i \leq n$. We denote the sum of the payoffs of all objects by A, i.e.,

$$A = \sum_{i=1}^{n} A_i.$$

For all i, let us define

$$a_i = \frac{A_i}{A},$$

which represents the relative value of object i and note that

$$\sum_{k=1}^{n} a_i = 1.$$

We assume that the player devoting the most resources to a battlefield captures that battlefield. Ties are resolved by flipping a coin.

For any pair $(\mathbf{x}, \mathbf{y})$ of pure strategies, the excess aggregate value for player X, denoted by $g(\mathbf{x}, \mathbf{y})$, of objects captured by player X if he plays the pure strategy $\mathbf{x}$ while player Y plays the pure strategy $\mathbf{y}$ is given by

$$g(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} a_i \mathrm{sgn}(x_i - y_i), \tag{1}$$

where $\mathrm{sgn}(\cdot)$ is the sign function defined as

$$\mathrm{sgn}(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -1 & \text{if } u < 0. \end{cases} \tag{2}$$

The excess aggregate value $g(\mathbf{x}, \mathbf{y})$ is the gain for pure strategy $\mathbf{x}$ against pure strategy $\mathbf{y}$.

Definition 1. *We define the Colonel Blotto game as the two-player, zero-sum game defined by the payoff function g.*

If F and G are two mixed strategies the payoff to mixed strategy F against mixed strategy G is:

$$K(F, G) = \int_{\mathbf{x} \in \Delta^{(n-1)}} \int_{\mathbf{y} \in \Delta^{(n-1)}} g(\mathbf{x}, \mathbf{y}) \, dF(\mathbf{x}) \, dG(\mathbf{y}). \tag{3}$$

The expected payoff for mixed strategy F against pure strategy $\mathbf{y}$ is given by

$$K(\mathbf{y}) = \int_{\mathbf{x} \in \Delta^{(n-1)}} g(\mathbf{x}, \mathbf{y}) \, dF(\mathbf{x}). \tag{4}$$

The game is symmetric so to prove that a strategy is optimal we only need to show that $K(\mathbf{y}) \geq 0$ for every $\mathbf{y}$.

From eq. (1) and eq. (4), we have that

$$K(\mathbf{y}) = \sum_{i=1}^{n} a_i \left(\mathbb{P}(x_i > y_i) - \mathbb{P}(y_i > x_i) \right). \tag{5}$$

We assume $n \geq 2$ otherwise the game always ends in a tie.

We observe the following:

(a) For the case $n = 2$ (originally solved in [16]), it is optimal to put all the budget in the object of maximum value, i.e., in $i^* \in \{1, 2\}$ such that $a_{i^*} = \max\{a_1, a_2\}$. In case $a_1 = a_2$, choose any of them. Indeed, without loss of generality assume the second object gives higher payoff than the first one, i.e. $a_2 \geq a_1$, then the expected payoff against a strategy $\mathbf{y} = (y, 1 - y)$ will be

$$a_1 \mathrm{sgn}(-y) + a_2 \mathrm{sgn}(y) \geq 0,$$

which always gives a non-negative payoff for all $y \in [0, 1]$.

(b) For the case when there exists an object i with relative value $a_i \geq 1/2$, then the optimal strategy is to put all the budget in object i. Indeed, without loss of generality assume the last object has relative value $a_n \geq 1/2$, then the expected payoff against a strategy $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ will be

$$\sum_{i=1}^{n-1} a_i \mathrm{sgn}(-y_i) + a_n \mathrm{sgn}\left(\sum_{i=1}^{n-1} y_i \right) \geq 0,$$

which always gives a non-negative payoff.

(c) For the case when there exists an object i with relative value $a_i = 0$, then from the payoff function, the optimal strategy is not to put any budget in object i.

From (b) and (c), in the following we assume that

$$0 < a_i < 1/2, \quad \forall i \in \{1, \ldots, n\}, \tag{6}$$

or equivalently that

$$0 < A_i < \sum_{j \neq i} A_j, \quad \forall i \in \{1, \ldots, n\}.$$

It can be shown that under the previous assumption for $n > 2$ there is no pure strategy Nash equilibrium in the general case. Indeed, consider a pure strategy $\mathbf{x}$, select the object of minimum value where the pure strategy is not zero. Then $\mathbf{x}$ will lose with respect to the strategy $\mathbf{y}$ that allocates no resources to the i-th object and more resources to all the other objects, i.e.,

$$y_i = 0, \quad y_j = x_j + \frac{x_i}{n-1} \quad \forall j \neq i.$$

Therefore, we need to search for optimal mixed strategies. For the case of three battlefields, Gross and Wagner [16] proved the existence of a mixed strategy solution given as follows.

Theorem 5 (Gross and Wagner [16]). *For $n = 3$, the Colonel Blotto game with heterogeneous battlefield values has a mixed strategy equilibrium in which the marginal distribution over front i is uniform on $[0, 2a_i]$ for $i \in \{1, 2, 3\}$.*

We need to construct a joint distribution such that each marginal distribution is uniform on $[0, 2a_i]$ for $i \in \{1, 2, 3\}$ and such that the sum of the values given by the marginal distributions is equal to 1. The difficulty comes from this last condition, otherwise we could always define a joint distribution from its marginal distributions. In the following, we show how this joint distribution can be constructed. In the case of three battlefields, there is a geometric construction of the joint n-variate distribution function with uniform marginal distribution functions.

Proof. We construct a non-degenerate triangle having sides of length a_1, a_2 and a_3 (see Fig. 1), and then inscribe a circle within it and erect a hemisphere upon this circle. We notice that we can construct a non-degenerate triangle since a degenerate triangle would violate the assumption given by eq. (6). Then, we choose a point from a density uniformly distributed over the surface of the hemisphere and project this point straight down into the plane of the triangle (we denote by P the projected point over the plane). We then divide the forces in respective proportion to the triangular areas subtended by P and the sides, i.e., $x_1 : x_2 : x_3 = \mathrm{A}_1 : \mathrm{A}_2 : \mathrm{A}_3$ (see Fig. 1).

Let $F_i(x_i)$ denote the respective marginal distribution function of Blotto's continuous mixed strategy F. Blotto's expectation from battlefield i is given by

$$a_i[1 - F_i(y_i)] - a_i F_i(y_i) = a_i[1 - 2F_i(\mathbf{y})],$$

and hence his total expectation is given by

$$K(\mathbf{y}) = \sum_{i=1}^{3} a_i[1 - 2F_i(y_i)]. \tag{7}$$

Let h_i denote the altitude of the triangle of area A_i subtended by P. From a well-known property of the surface area of a sphere, we see that h_i is uniformly distributed over $(0, 2r)$, r being the radius of the sphere. Now, $\mathrm{A}_i = \frac{1}{2} a_i h_i$, and hence A_i is uniformly distributed over $(0, a_i r)$. Also, since $x_1 + x_2 + x_3 = 1$ and $x_1 : x_2 : x_3 = \mathrm{A}_1 : \mathrm{A}_2 : \mathrm{A}_3$, it follows that $x_i = \frac{\mathrm{A}_i}{\Delta}$ where Δ is the area of the originally constructed triangle. Thus x_i is uniformly distributed over $(0, \frac{a_i r}{\Delta})$. But $\Delta = \frac{1}{2}(a_1 r + a_2 r + a_3 r)$, hence x_i is uniformly distributed over $\left(0, \frac{2a_i}{a_1 + a_2 + a_3}\right)$, i.e.,

$$F_i(x_i) = \min\left[1, \frac{a_1 + a_2 + a_3}{2a_i} x_i\right].$$

Consequently, from eq. (7),

$$K(y) = \sum_{i=1}^{3} a_i \left\{1 - 2\min\left[1, \frac{a_1 + a_2 + a_3}{2a_i} y_i\right]\right\}.$$

Note that if $\alpha > 0$, $y \geq 0$, then $\min(1, \alpha y) \geq \alpha y$, and hence

$$\begin{aligned} K(y) &\geq \sum_{i=1}^{3} a_i \left\{1 - \frac{a_1 + a_2 + a_3}{a_i} y_i\right\}, \\ &= a_1 + a_2 + a_3 - (a_1 + a_2 + a_3) \sum_{i=1}^{3} y_i \\ &= 0 \end{aligned}$$

since $\sum_{i=1}^{3} y_i = 1$. $\square$

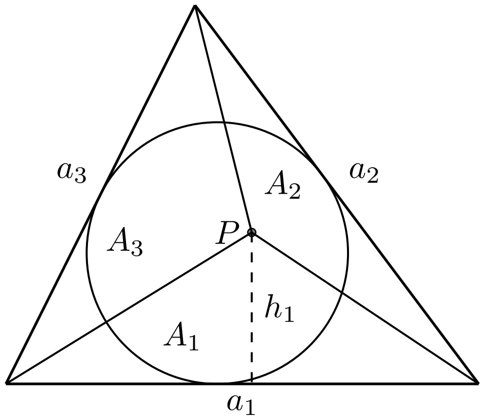

Figure 1: Triangle of lengths a_1, a_2 and a_3.

For the case of any number of battlefields n, Gross [15] proved the existence and a method to construct such joint distributions.

Theorem 6 (Gross [15]). *Consider the Colonel Blotto Game with heterogeneous battlefield values. Let F^* be a probability distribution of $\mathbf{x} \in \Delta^{n-1}$ such that each vector coordinate x_i is uniformly distributed on $[0, 2a_i]$ for $i \in \{1, \ldots, n\}$. Then (F^*, F^*) constitutes a symmetric Nash equilibrium.*

We notice that Friedman [11] used this solution for advertisement expenditures without considering the network value of customers. Robertson [28] showed that for homogeneous battlefield values ($a_i = 1/n, \forall i \in \{1, \ldots, n\}$) uniform univariate marginals are also a necessary condition for equilibrium. Laslier and Picard [22] and Thomas [30] provided alternative methods to construct the joint distribution. In the following we will use Theorem 6 to the strategic resource allocation on the voter model of social networks.

3 Voter model

The voter model is one of the most natural probabilistic models to represent the diffusion of opinions in social networks. In each step, each potential customer changes its opinion by choosing one of its neighbors at random and adopting its neighbor's opinion.

The voter model is quite different from the threshold model [19, 20], however it still has the same key property that a person is more likely to change its opinion to the one held by most of its neighbors. It has another characteristic to its advantage, the fact that in the threshold models once a node adopts a product or service it stays with that product or service forever. These models are called monotone models of diffusion. However, the voter model allows to change preferences, which may be more suitable for non-monotone processes.

Let $G = (V, E)$ be an undirected graph with self-loops where V is the set of nodes in the graph which represent the potential customers of the competing marketing campaigns and E is the set of edges which represent the influence between individuals. We consider that the graph G has n nodes, i.e. $|V| = n$. As we will see in the following, the use of the same notation n, as for the number of battlefields, is not a coincidence since each potential customer will represent a battlefield for competing marketing campaigns.

For a node $v \in V$, we denote by $N(v)$ the set of neighbors of v in G, i.e. $N(v) = \{u \in V : \{u, v\} \in E\}$ and by d_v the degree of node v, i.e. $d_v = |N(v)|$.

We recall that the players of the game are the competing marketing campaigns and the nodes of the graph correspond to the potential customers. We label a node $v \in V$ by its initial preference between different players, X or Y, denoted by function f_0. We denote by $f_0(v) = 1$ when node $v \in V$ prefers the product promoted by marketing campaign X, $f_0(v) = -1$ when node v prefers the product promoted by marketing campaign Y, and $f_0(v) = 0$ when node v is indifferent between both products.

We assume that the quantity of marketing budget allocated to node i determines its initial preference, i.e.,

$$f_0(i) = \begin{cases} 1 & \text{if } x_i > y_i, \\ 0 & \text{if } x_i = y_i, \\ -1 & \text{if } x_i < y_i. \end{cases}$$

We notice that $f_0(i)$ corresponds to $\text{sgn}(x_i - y_i)$ where $\text{sgn}(\cdot)$ is given by eq. (2). We also observe that since the bids are real numbers, ties have measure zero. However, in the unlikely event that bids coincide in the maximum, we choose uniformly at random the winner between both players. This defines a new function and for simplicity we use the same notation,

$$f_0(i) = \begin{cases} 1 & \text{if } x_i > y_i, \\ 1 & \text{if } x_i = y_i \quad \text{and} \quad R = 1, \\ -1 & \text{if } x_i = y_i \quad \text{and} \quad R = 0, \\ -1 & \text{if } x_i < y_i, \end{cases}$$

where R is a Bernoulli random variable with success probability equal to 1/2.

The evolution of the system will be described by the voter model. Starting from any arbitrary initial preference assignment to the vertices of G, at each time $t \geq 1$, each node picks uniformly at random one of its neighbors and adopts its opinion. More formally, starting from any assignment $f_0 : V \to \{-1, 1\}$, we inductively define

$$f_{t+1}(v) = \begin{cases} 1 & \text{with prob. } \frac{|\{u \in N(v): f_t(u)=1\}|}{|N(v)|}, \\ -1 & \text{with prob. } \frac{|\{u \in N(v): f_t(u)=-1\}|}{|N(v)|}. \end{cases}$$

The objective function for player X is to maximize at a certain target time τ the expected number of nodes:

$$\mathbb{E}\left[\sum_{v \in V} f_\tau(v)\right].$$

We thus define the strategic resource allocation problem as follows.

Definition 2 (Competitive Influence). *Let $G = (V, E)$ be a graph with n nodes representing a social network with n potential customers. Consider two players, X and Y, which represent two competing marketing campaigns with equal budget B which need to simultaneously allocate resources across potential customers. A node $i \in V$ with respective allocations x_i and y_i such that $x_i > y_i$ (or $y_i > x_i$) will choose the product or service proposed by player X (or Y). Otherwise, if $x_i = y_i$, it will flip a coin to decide between both marketing campaigns. Then the strategic resource allocation problem is the problem of finding an initial assignment of resources $x = (x_1, x_2, \ldots, x_n)$ for player X that will maximize at a target time τ the expectation*

$$\mathbb{E}\left[\sum_{v \in V} f_\tau(v)\right]$$

subject to the budget constraint

$$\sum_{v \in V} x_v \leq B.$$

4 Results

In this section, we establish the main results of our work, we find the optimal marginal probability density function when we consider the intrinsic value of potential customers and when we consider the total value of potential customers by incorporating their network value. We also give a distance to compare both probability density functions.

We notice that in the voter model, the probability that node v adopts the opinion of one its neighbors u is precisely $1/|N(v)|$. Equivalently, this is the probability that a random walk of length 1 that starts at v ends up in u. Generalizing this observation by induction on t, we obtain the following proposition.

Proposition 1 (Even-Dar and Shapira [9]). *Let $p^t_{u,v}$ denote the probability that a random walk of length t starting at node u stops at node v. Then the probability that after t iterations of the voter model, node u will adopt the opinion that node v had at time $t = 0$ is precisely $p^t_{u,v}$.*

Let M be the normalized transition matrix of G, i.e.

$$M(v, u) = 1/|N(v)| \quad \text{if } u \in N(v).$$

By linearity of expectation, we have that for player X

$$\mathbb{E}\left[\sum_{v \in V} f_\tau(v)\right] = \sum_{v \in V} \left(\mathbb{P}[f_\tau(v) = 1] - \mathbb{P}[f_\tau(v) = -1]\right).$$

For a subset $S \subseteq \{1, \ldots, n\}$, we denote by 1_S the 0/1 column vector whose ith entry is 1 if and only if $i \in S$. Then, the probability that a random walk of length t starting at u ends in v, is given by the (u, v) entry of the matrix M^t, or equivalently, by $1^T_{\{u\}} M^t 1_{\{v\}}$. Then

$$\begin{aligned} \mathbb{P}[f_t(v) = 1] &= \sum_{u \in V} p^t_{u,v} \mathbb{P}[f_0(u) = 1] \\ &= \sum_{u \in V} 1^T_{\{u\}} M^t 1_{\{v\}} \mathbb{P}[x_u > y_u]. \end{aligned}$$

Similarly,

$$\mathbb{P}[f_t(v) = -1] = \sum_{u \in V} 1^T_{\{u\}} M^t 1_{\{v\}} \mathbb{P}[x_u < y_u].$$

Then

$$\mathbb{E}\left[\sum_{v\in V} f_t(v)\right] = \sum_{v\in V}\sum_{u\in V} 1_{\{u\}}^T M^t 1_{\{v\}} \left(\mathbb{P}[x_u > y_u] - \mathbb{P}[x_u < y_u]\right). \quad (8)$$

If we are interested only on the intrinsic value of potential customers, this case is equivalent to consider that each node is influenced only by itself, i.e.,

$$M(u,v) = \begin{cases} 1 & u = v, \\ 0 & u \neq v. \end{cases}$$

Equivalently, that $M = I_{n\times n}$, where $I_{n\times n}$ is the $n \times n$ identity matrix. Then, for every target time τ we have that

$$\mathbb{E}\left[\sum_{v\in V} f_\tau(v)\right] = \sum_{u\in V} \left(\mathbb{P}[x_u > y_u] - \mathbb{P}[x_u < y_u]\right).$$

From equation (5), we notice that this case is equivalent to the homogeneous Colonel Blotto, i.e., with relative values $a_i = 1/n$ for $1 \le i \le n$, and from Theorem 6, we are able to conclude Theorem 1.

If we consider a complete graph, then the normalized transition matrix is given by

$$M = \frac{1}{n}\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & 1 \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

We notice that $M^t = M$ for all $t \in \mathbb{N}$. Thus independently of the target time τ, the objective function for player X is to maximize:

$$\mathbb{E}\left[\sum_{v\in V} f_\tau(v)\right] = \sum_{u\in V} \left(\mathbb{P}[x_u > y_u] - \mathbb{P}[x_u < y_u]\right).$$

Similarly to the previous case, from eq. (5), we see that this particular case is equivalent to the homogeneous Colonel Blotto game and the solution is the same as in the intrinsic value case.

If we could only solve this challenge in the intrinsic case and the complete graph it would not be a compelling model. However, as we will see we can compute this when we are interested on much more general cases.

Indeed, from eq. (8), redefining

$$a_u = \sum_{v\in V} M^t(u,v),$$

we have the optimal strategy from Theorem 6 and we conclude the first part of Theorem 2.

Recall the well known fact that for any graph G with self-loops, a random walk starting from any node v, converges to the steady state distribution after $O(n^3)$ steps (see [24]). Then the (unique) steady state distribution is that the probability of being at node u is $d_u/2|E|$. In other words, if $t \gg n^3$ then $M^t_{u,v} = (1+o(1))d_u/2|E|$.

If we have $t \gg n^5$, the error in each entry is within a factor of $1 + o(1/n^2)$ of the exact value. Then

$$K(\mathbf{y}) = o\left(\frac{1}{n}\right) + \sum_{i=1}^{n} \frac{d_i}{2|E|}\left(\mathbb{P}(x_i > y_i) - \mathbb{P}(x_i < y_i)\right). \quad (9)$$

Then, in the long term, i.e., when $t \to +\infty$, from Theorem 6, the optimal strategy is to play a mixed strategy such that the vector coordinate x_i is uniformly distributed between 0 and $d_i/|E|$, which concludes Theorem 2.

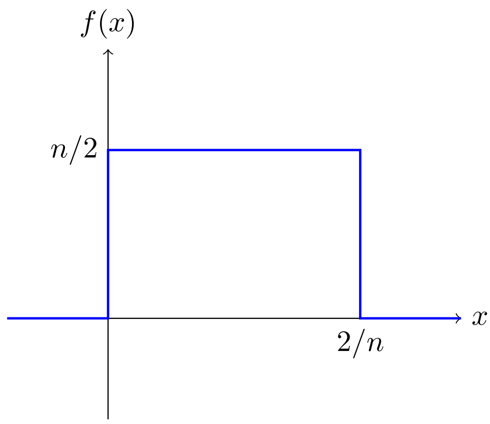

Figure 2: Probability density function for the optimal marginal allocation for the intrinsic value case.

4.1 Distance measure between the intrinsic value and the total value of potential customers

In this subsection, we are interested to find the distance measure between the intrinsic value of po-

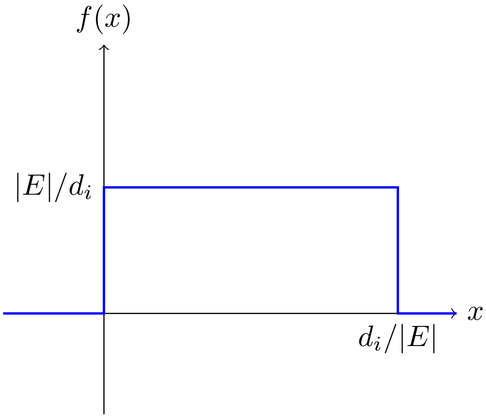

Figure 3: Probability density function for the optimal marginal allocation for the total value case.

tential customers and the total value of potential customers. From Theorem 1, we know that if we only take into account the intrinsic value of customers, the optimal strategy for the long term is to choose from marginal distributions uniform $x_i^{(1)} \sim \mathcal{U}(0, 2/n)$ (see Fig. 2). From Theorem 2, we know that, by also taking into account the network value of the customers, the optimal strategy is to choose from marginal distributions uniform $x_i^{(2)} \sim \mathcal{U}(0, d_i/|E|)$ (see Fig. 3).

The marginal distribution of the difference, $Z = x_i^{(1)} - x_i^{(2)}$, is given by:

- For the case $\frac{d_i}{|E|} \leq \frac{2}{n}$:

$$f_Z(z) = \begin{cases} \frac{|E|}{d_i} + \frac{n}{2}\frac{|E|}{d_i} z & -\frac{2}{n} \leq z \leq \frac{d_i}{|E|} - \frac{2}{n}, \\ \frac{n}{2} & \frac{d_i}{|E|} - \frac{2}{n} \leq z \leq 0, \\ -\frac{n}{2}\frac{|E|}{d_i} z + \frac{n}{2} & 0 \leq z \leq \frac{d_i}{|E|}. \end{cases}$$

- For the case $\frac{d_i}{|E|} \geq \frac{2}{n}$:

$$f_Z(z) = \begin{cases} \frac{|E|}{d_i} + \frac{n}{2}\frac{|E|}{d_i} z & -\frac{2}{n} \leq z \leq 0, \\ \frac{|E|}{d_i} & 0 \leq z \leq \frac{d_i}{|E|} - \frac{2}{n}, \\ -\frac{n}{2}\frac{|E|}{d_i} z + \frac{n}{2} & \frac{d_i}{|E|} - \frac{2}{n} \leq z \leq \frac{d_i}{|E|}. \end{cases}$$

We notice that in both cases f_Z is a trapezoidal distribution of mean

$$\mu_Z = \frac{1}{2}\left(\frac{d_i}{|E|} - \frac{2}{n}\right),$$

and variance

$$\sigma_Z^2 = \frac{1}{12}\left(\left(\frac{2}{n}\right)^2 + \left(\frac{d_i}{|E|}\right)^2\right).$$

However, the marginal distribution of the difference is not a good distance measure since one of the properties that one would like to have is the identity of indiscernibles, i.e., that the distance between two equal probability distributions is zero. Because of that, we consider the total variation distance of probability measures.

Definition 3. *For μ and ν probability measures on $\mathbb{R}$. The total variation distance between μ and ν is defined as*

$$\delta(\mu, \nu) = \frac{1}{2}\sup_f \left| \int f(t)\, d\mu(t) - \int f(t)\, d\nu(t) \right|,$$

where the supremum is taken over continuous functions which are bounded by 1 and vanish at infinity.

Informally, the total variation distance is the largest possible difference between the probabilities that the two probability distributions can assign to the same event (see Fig. 4).

Thus, we have the following two cases:

- For the case $\frac{d_i}{|E|} \leq \frac{2}{n}$:

$$\delta(x_i^{(1)}, x_i^{(2)}) = 1 - \frac{n}{2}\frac{d_i}{|E|} \geq 0.$$

- For the case $\frac{d_i}{|E|} \geq \frac{2}{n}$:

$$\delta(x_i^{(1)}, x_i^{(2)}) = 1 - \frac{|E|}{d_i}\frac{2}{n} \geq 0.$$

We define the average total variation between both strategies across all the nodes as

$$\delta = \frac{1}{n}\sum_{i=1}^{n} \delta(x_i^{(1)}, x_i^{(2)}).$$

This distance has some good properties (non-negativity, identity of indiscernibles, symmetry, triangle inequality). For the long term, it can be seen that for any k-regular graph, $\delta = 0$. Give the particular probability densities considered, the effect of δ will be given by how much different are $\frac{d_i}{|E|}$

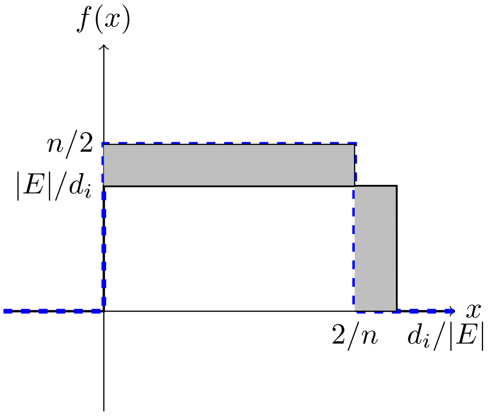

Figure 4: Total variation between both probability density functions (shaded region).

from $\frac{2}{n}$. For example, for $n \geq 2$, a graph with self-loops where all other edges are given by one node connected to every other node, there is one node of degree $(n+1)$ and $(n-1)$ nodes of degree 3. The total number of edges in the graph is $2n-1$. Therefore, the average variation is given by

$$\frac{1}{n}\left\{(n-1)\left(1-\frac{n}{2}\frac{3}{2n-1}\right)+\left(1-\frac{2}{n}\frac{2n-1}{n+1}\right)\right\}$$
$$=\frac{(n-1)(n-2)n(n+1)+2(2n-1)(n-1)(n-2)}{2(2n-1)n^2(n+1)},$$

which converges to 1/4 when $n \to \infty$. As we have seen, this is very different for other graphs, for example for k-regular graphs.

4.2 Price of competition

In this subsection, we study the price of competition in the strategic resource allocation problem for competitive influence. We define the price of competition as the ratio of the total cost to a competitive duopoly to that of a monopoly. Informally, it is the price resulting from the lack of coordination among the agents [1].

Until now, we have considered that for any $\epsilon > 0$ such that $x_i - y_i \geq \epsilon$ user i will choose the product or service of player X versus the product or service of player Y. Under that scenario the price of competition will be unbounded. Indeed, the cost for two marketing campaigns cooperating (as in a monopoly) could put together for each user ε/n (thus ε for the whole population), while for the non-cooperating scenario each marketing campaign needs to invest a budget of $B > 0$ for the whole population, thus the price of competition is B/ε which is unbounded for $\varepsilon \to 0$.

We notice that this is a very different result than the one obtained by Bharathi et al. [1] where they found that the price of competition is at most a factor of 2 when the focus is on which potential customers to recruit. We have thus obtained Theorem 3.

5 Conclusions and Future Work

5.1 Conclusions

In this work, we were interested on the strategic resource allocation of competing marketing campaigns who need to simultaneously decide how many resources to allocate to potential customers to advertise their products. Using game theory, and in particular Colonel Blotto games, we were able to completely characterize the optimal strategic allocation of resources for the voter model of social networks. We were able to prove that for this case the price of competition is unbounded.

5.2 Future Work

One generalization to several marketing campaigns consists on the simple case of analyzing pairwise competitions as previously described. This case has the advantage that we already know the solution, given by the results of the previous section. However, this is not a realistic case for competitions in which each customer chooses only one product from the competing marketing campaigns. To see this, consider the example of three competing marketing campaigns X, Y, and Z and four customers (for the sake of simplification). Consider the pure strategies $\mathbf{x} = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\mathbf{y} = (0, 0, 0, 0.5, 0.5)$ and $\mathbf{z} = (0.5, 0.5, 0, 0, 0)$. In that case, the pairwise competition gives that X has captured 3 out of 5 potential customers to Y, and that X has captured 3 out of 5 potential customers to Z, thus winning in a pairwise competition against both competitors. However, since each customer will only choose one product, the final outcome will be 2 customers for Y, 2 customers for Z, while only 1 customer for X.

For exploring this generalization, we need to search for another framework. We notice that there is a tight relationship between Colonel Blotto games and auctions. A Colonel Blotto game can be seen as a simultaneous all-pay auction of multiple items of complete information. An all-pay auction is an auction in which every bidder must forfeit its bid regardless of whether it wins the object which is awarded to the highest bidder. It is an auction of complete information since the value of the object is known to every bidder. In another context, this was already noted by Szentes and Rosenthal [29], Roberson [28] and Kvasov [21].

Through this new perspective, we can generalize Colonel Blotto games to more than two players, where the winner of an object will be the highest bidder for that object. We notice that this is a different generalization than the work of Goldman and Page [13] who consider another payment function. This general framework does not have yet a known solution and we consider it an interesting extension to study.

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References

[1] Bharathi, S., Kempe, D., and Salek, M. Competitive influence maximization in social networks. In *Internet and Network Economics*, X. Deng and F. Graham, Eds., vol. 4858 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2007, pp. 306–311.

[2] Borel, É. La théorie du jeu et les équations intégrales à noyau symétrique. *Comptes Rendus de l'Académie des Sciences 173* (1921), 1304–1308.

[3] Borel, É., and Ville, J. *Application de la théorie des probabilités aux jeux de hasard.* 1938.

[4] Borodin, A., Filmus, Y., and Oren, J. Threshold models for competitive influence in social networks. In *Proceedings of the 6th International Conference on Internet and Network Economics* (Berlin, Heidelberg, 2010), WINE'10, Springer-Verlag, pp. 539–550.

[5] Chasparis, G. C., and Shamma, J. Control of preferences in social networks. In *Proceedings of the 49th IEEE Conference on Decision and Control (CDC)* (Dec 2010), pp. 6651–6656.

[6] Clifford, P., and Sudbury, A. A model for spatial conflict. *Biometrika 60*, 3 (1973), 581–588.

[7] Domingos, P., and Richardson, M. Mining the network value of customers. In *Proceedings of the 7th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining* (New York, Aug. 26–29 2001), ACM Press, pp. 57–66.

[8] Dubey, P., Garg, R., and Meyer, B. D. Competing for customers in a social network. Cowles Foundation Discussion Papers 1591, Cowles Foundation for Research in Economics, Yale University, Nov. 2006.

[9] Even-Dar, E., and Shapira, A. A note on maximizing the spread of influence in social networks. In *Proceedings of the 3rd International Conference on Internet and Network Economics (WINE'07)* (Berlin, Heidelberg, 2007), Springer-Verlag, pp. 281–286.

[10] Friedkin, N. E. Norm formation in social influence networks. *Social Networks 23*, 3 (July 2001), 167–189.

[11] Friedman, L. Game-theory models in the allocation of advertising expenditures. *Operations Research 6*, 5 (1958), 699–709, http://pubsonline.informs.org/doi/pdf/10.1287/opre.6.5.699.

[12] Goldenberg, J., Libai, B., and Muller, E. Talk of the network: A complex systems look at the underlying process of word-of-mouth. *Marketing Letters 12*, 3 (2001), 211–223.

[13] Golman, R., and Page, S. E. General blotto: games of allocative strategic mismatch. *Public Choice 138*, 3-4 (2009), 279–299.

[14] Goyal, S., and Kearns, M. Competitive contagion in networks. In *Proceedings of the 44th Symposium on Theory of Computing* (New York, NY, USA, 2012), STOC '12, ACM, pp. 759–774.

[15] Gross, O. The symmetric blotto game. In *RAND Corporation RM-424* (1950).

[16] Gross, O., and Wagner, R. A continuous colonel blotto game. In *RAND Corporation RM-408* (1950).

[17] He, X., and Kempe, D. Price of anarchy for the n-player competitive cascade game with submodular activation functions. In *Web and Internet Economics*, Y. Chen and N. Immorlica, Eds., vol. 8289 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2013, pp. 232–248.

[18] Holley, R. A., and Liggett, T. M. Ergodic theorems for weakly interacting infinite systems and the voter model. *The Annals of Probability 3*, 4 (1975), 643–663.

[19] Kempe, D., Kleinberg, J., and Tardos, É. Maximizing the spread of influence through a social network. In *Proceedings of the 9th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining* (New York, Aug. 24–27 2003), ACM Press, pp. 137–146.

[20] Kempe, D., Kleinberg, J., and Tardos, É. Influential nodes in a diffusion model for social networks. In *ICALP: Annual International Colloquium on Automata, Languages and Programming* (2005).

[21] Kvasov, D. Contests with limited resources. *Journal of Economic Theory 136*, 1 (September 2007), 738–748.

[22] Laslier, J.-F., and Picard, N. Distributive politics and electoral competition. *Journal of Economic Theory 103*, 1 (March 2002), 106–130.

[23] Mossel, and Roch. On the submodularity of influence in social networks. In *STOC: ACM Symposium on Theory of Computing (STOC)* (2007).

[24] Motwani, R., and Raghavan, P. *Randomized Algorithms*. Cambridge University Press, New York, NY, USA, 1995.

[25] Nash, J. Non-cooperative games. *Annals of Mathematics 54* (September 1951), 268–295.

[26] Nemhauser, G., Wolsey, L., and Fisher, M. An analysis of approximations for maximizing submodular set functions i. *Mathematical Programming 14*, 1 (1978), 265–294.

[27] Richardson, M., and Domingos, P. Mining knowledge-sharing sites for viral marketing. In *Proceedings of the 8th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining* (New York, July 23-26 2002), ACM Press, pp. 61–70.

[28] Roberson, B. The colonel blotto game. *Economic Theory 29*, 1 (2006), 1 – 24.

[29] Szentes, B., and Rosenthal, R. W. Three-object two-bidder simultaneous auctions: chopsticks and tetrahedra. *Games and Economic Behavior 44*, 1 (2003), 114–133.

[30] Thomas, C. N-dimensional blotto game with asymmetric battlefield values. 2013.

[31] Vetta, A. Nash equilibria in competitive societies, with applications to facility location, traffic routing and auctions. In *Proceedings of the 43rd Symposium on Foundations of Computer Science (FOCS)* (Los Alamitos, Nov. 16–19 2002), IEEE Computer Society Press, pp. 416–428.