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Abstract: We study network coding for multi-hop wireless networks. In particular, we consider the case of broadcasting: a source transmits information (packets) to all nodes in the network. Wireless communication is modeled as a hyper-graph where the same transmission from one node achieves many of its neighbors and we analyze the case where the nodes are arranged on a torus grid. We provide the broadcast capacity of wireless network coding when all nodes have the same transmission rate, with the exception of the source. In order to do this we translate the min-cut problem on a hypergraph in an equivalent problem of additive combinatorics and we use tools from group theory. In addition, in this case the network coding is "near optimal" in terms of energy efficiency.

Key-words: wireless multi-hop networks, network coding, broadcasting, min-cut/max-flow, energy efficiency performances

Efficacité de diffusion avec codage réseau dans les réseaux sans fil

Résumé : Dans ce document, nous étudions les performances du codage réseau appliqué à une forme spécifique de multidiffusion, la diffusion, où l'information (paquets) est envoyée d'une source à tous les noeuds d'un réseau multi-sauts sans fil. La communication sans fil est modélisée comme un hyper-graphe, i.e. la même transmission d'un noeud atteint simultanément plusieurs de ses voisins. Nous analysons le cas particulier où les noeuds sont organisés dans une grille torique. Nous étudions la capacité de diffusion du codage réseau sans-fil sur cette topologie, en utilisant des outils de la géométrie discrète et de la théorie des groupes. Une implication est que le codage réseau est ici "quasi optimal" en termes d'efficacité énergétique, dans le sens où une transmission apporte de nouvelles informations à "quasiment" chaque récepteur.

Mots-clés : réseaux sans fil multi-sauts, codage réseau, diffusion, flot-max/coupe-min, efficacité énergétique

Contents

1   Introduction

The idea of *network coding* has been introduced by Ahlswede, Cai, Li and Yeung in [2]: interme-
diate nodes are mixing information from different flows before forwarding it. This approach is
different from the classic routing strategy, where intermediate nodes simply replicate and forward
the received information. This technique offers different benefits, with respect to classic routing
strategy, such as improvements in capacity, security, complexity, etc.

In multi-hop wireless networks, a natural application of network coding is to reduce the num-
ber of transmissions required to transmit some amount of information to the destinations. This
kind of application allows to achieve energy efficiency for networks where the cost of wireless
communication is a critical design factor. We focus on one specific form of communication:
broadcasting information from some sources to all the nodes in a wireless multi-hop network.
Such communication is commonly used in wireless networks, for instance, for management, in-
formation dissemination, multimedia content distribution, or as a simplified form of multicast.
The idea of energy efficient broadcast communication can be expressed as follows:

- With some given broadcast sources, minimize the total number of (re)transmissions
  used to allow all nodes in the network to get the information.

The efficiency provided by network coding in multiscat and broadcast networks has been
studied for instance, by Lun et al. [11], and Wu et al. [13]. In particular, they provide methods
for determining optimal network coding parameters for a given network with specific model
assumptions. The work of Fragouli et al. [5] gives insights for all-to-all broadcast and illustrate
how gains could be obtained compared to routing.

Problem Statement
This article seeks and provides some answers for the following question:

- How efficient is broadcast with network coding?

In [1] for homogeneous planar random networks, where the density and the area of the
network would increase towards infinity, the authors have shown that, wireless network coding
was asymptotically "optimal" for a strong definition of optimality, *transmission-level optimality*,
where nearly every received packet would be innovative. This relied on an intermediate step,
the computation of the broadcast rate when nodes organized on a grid, and on a specific *rate
selection*, where most nodes would have the same rate. However, because the grid was not a
torus, nodes near the edge of the network would have a smaller neighborhood, and would be in
the bottleneck for the computation of the maximum broadcast rate. For this reason, the rate
selection was modified to handle them specially.

Our Results
In this paper, we analyze the case where the nodes are arranged on a torus grid and the trans-
mission rate (coded packets per second) is identical for all nodes except for the source. We
investigate the maximum broadcast capacity which is the maximum rate (packets per second)
at which the source can inject packets, while ensuring that the receivers nodes can decode (with
probability tending to 1). We find this capacity in the considered topologies to be equal to the
number of neighbors of a node. In order to do this we translate the min-cut problem on a hy-
pergraph in an equivalent problem of additive combinatorics. Moreover, network coding in such
networks is " near optimal " in terms of energy efficiency, in the sense that each transmission will
provide innovative information (outside the vicinity of the source).

2 Background

In this article, we study the problem of broadcasting information from one source to all nodes in the network.

2.1 Model

We will consider multi-hop wireless networks with a certain number of nodes, without mobility. We also assume an ideal wireless model. More precisely, wireless transmissions are without losses, collisions or interferences. We assume that each node of the network is operating well below its maximum transmission capacity. Additionally, the network is a packet network with fixed packet size. We consider two network topologies: torus grid and integer lattice.

A fundamental concept is represented by the idea of "neighbors". We give here a definition.

Definition 1 *We say that two nodes in the network are neighbors if their distance is less than a fixed radius that we denote by r (integer).*

2.2 Notations

We will use the following general notation in the rest of the article:

- $\mathcal{V}$: set of nodes in the network;

- C_v: the retransmission rate of packets of a node v;

Some of the notation is more specifically targeted to a network of nodes organized on a lattice. Assume that $\mathcal{V}$ is included in a larger set $\hat{\mathcal{V}}$ (for a lattice, $\mathcal{V} \subset \hat{\mathcal{V}} = \mathbb{Z}^n$). We use the following notations for concepts related to neighborhood:

- $\mathcal{N}(X)$: open set of neighbors of $X \in \mathcal{V}$, $\mathcal{N}(X) \subset \hat{\mathcal{V}}$;

- $\mathcal{N}[X]$: closed set of neighbors of $X \in V$, that is nodes and their neighbors $\mathcal{N}[X] \triangleq \mathcal{N}(X) \cup X$;

- R: the set of neighbors of the origin node;

- M: the number of nodes in R, $M \triangleq |R|$;

- $\mathcal{L}$: the integer lattice, $\mathcal{L} \triangleq \mathbb{Z}^n$ for n integer > 2.

3 Network Coding Fundamentals

The basic idea behind network coding is performing coding operations of packets at intermediate nodes instead of simply replying and forwarding them as in routing protocols. It has been shown in [2] that coding in the network reaches its maximum broadcast capacity, while in the general case, without network coding, this capacity can not be achieved.

In linear coding [8], packets are seen as vectors of a fixed Galois field $\mathcal{F}_p$. Linear combinations of them are computed in order to code them. In this case, decoding means inverting the coding matrix (matrix of coding vectors needed for generating coded packets) to recover the original packets from the received linear combinations. The authors of [8] prove that linear coding is sufficient to achieve the min-cut max-flow capacity in multicast setting.

In random linear coding [6], instead, when a node transmits a packet it chooses randomly the coefficients used to perform linear combination of packets. This means that coding is no

longer a predeterminated operation and does not require coordination at intermediate nodes. The authors of [10] show that this coding technique performs asymptotically as efficiently as any other network coding method in terms of capacity achieving, for the case of single source multicast [10], and its performance is determined entirely by the average rates of nodes [9]. The source may transmit at a rate arbitrarily close to some fixed rate, the maximum broadcast rate, which is the min-cut of an hypergraph, and at the end of the broadcast process all destinations can decode with an error probability p_e. The error probability p_e can be made arbitrarily small by increasing the generation size.

3.1   Network Coding: Maximum Broadcast Rate

The maximum broadcast rate for the source represents the rate limit for the source which ensures that every destination in the network may decode. It is given by the minimum cut from the source to each particular destination in the network, where connectivity is described as an hypergraph [4].

An *hypergraph* is a graph where edges are replaced by hyper-arcs which are generalizations of arcs that may have more than one end node.

Let us consider the source s, and one of the multicast destinations $t \in \mathcal{V}$.

Definition 2 *An s-t* cut *is a partition of the set of nodes V in two sets S, T such that $s \in S$ and $t \in T$.*

Let $Q(s,t)$ be the set of such *s-t cuts*: $(S,T) \in Q(s,t)$.

We denote ΔS the set of nodes of S that are neighbors of at least one node of T:

$$\Delta_S \triangleq \{v \in S | \mathcal{N}(v) \cap T \neq \emptyset\}. \tag{1}$$

The *capacity of the cut* $C(S)$ is defined as the maximum rate between the nodes in S and the nodes in T:

$$C(S) \triangleq \sum_{v \in \Delta_S} C_v. \tag{2}$$

In other terms, the idea is to cut the network into two parts, and check the total rate transmitted from nodes in the part including the source, to nodes of the other part. The *min-cut* between s and t, that we denote by $C_{\min}(s,t)$, is the cut of $Q(s,t)$ with the minimum capacity. When we consider the multicast case, there are several destinations t for the same source s, the min-cut is the minimum of the $s - t$ min-cut for all destinations t:

$$C_{\min}(s,t) \triangleq \min_{(S,T) \in Q(s,t)} C(S). \tag{3}$$

In the case of broadcast to all nodes, the min-cut is the minimum for all nodes different from s; we denote the broadcast capacity $C_{\min}(s)$,

$$C_{\min}(s) \triangleq \min_{t \in \mathcal{V} \setminus \{s\}} C_{\min}(s,t). \tag{4}$$

4   Broadcast Capacity in Torus Grid

In order to compute the global min-cut $C_{\min}(s)$ in the considered topologies, we consider a destination node t in the network. We link the capacity of the cut between the nodes of S and the nodes of T with the number of nodes of S who are neighbors of nodes of T, that can be written by a Minkowski sum. Moreover, we use tools from group theory in order to verify that there is no problem of neighborhood for the nodes that are at the border.

4.1 Minowski Sum and Neighborhood

The Minkowski addition is a classical way to express the neighborhood of an area, see [7] and the figure within.

Definition 3 *Given two subsets A and B of a group, the Minkowski sum of the two sets $A \oplus B$ is defined as the set of all vector sums generated by all pairs of points in A and B, respectively:*

$$A \oplus B \triangleq \{a + b \ : \ a \in A, b \in B\}. \tag{5}$$

In the torus grid or the integer lattice, the closed set of neighbors of one node t, that we denote $\mathcal{N}[t]$, can then be redefined in terms of Minkowski sum as

$$\mathcal{N}[t] = \{t\} \oplus R. \tag{6}$$

This is indeed a translation of R, the set of neighbors of the node at $(0,0)$ to the node t.

This extends to the neighborhood of subsets:

$$\mathcal{N}(A) = A \oplus R. \tag{7}$$

To see why, the expression (5) can be rewritten as:

$$A \oplus R = \bigcup_{a \in A} \{a\} \oplus R,$$

which corresponds to the union of the closed neighborhood of each node in A.

We consider the torus grid G, and we write the 2-dimensional torus grid as:

$$G = \frac{\mathbb{Z}}{n_X \mathbb{Z}} \times \frac{\mathbb{Z}}{n_Y \mathbb{Z}}, \tag{8}$$

where n_X and n_Y are the width and height of the grid and $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is the set of integers modulo n. Since each $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is a group, the minkowski sum of subsets of G is well-defined. The set R, neighborhood of the node $(0,0)$, can also be more precisely defined. Let $r > 0$ be the radio range, we define:

$$\begin{aligned}
R \quad &\triangleq \quad \{(x \bmod n_X, y \bmod n_Y) : \\
&\qquad (x,y) \in \mathbb{Z}^2 \text{ and } x^2 + y^2 \le r^2\}. \tag{9}
\end{aligned}$$

We observe that R is symmetric with respect to the origin, that is $(x,y) \in R \Rightarrow (-x,-y) \in R$. In paricular, the number of elements in R less 1 represents the number of neighbors of a node : $M = |R| - 1$. In Figure 1,we show an example of R with range $r = 3$ and the Minkowski sum with a set of nodes A.

We now introduce the essential notion of "large neighborhood" related to the fact that we have no problem of neighborhood for the nodes that are on the border of the grid.

Definition 4 *If for all subsets $A \subset G$ at least one of this conditions is verified:*

$$A \oplus R \quad = \quad G \tag{10}$$

$$\text{or} \ |A \oplus R| \quad \ge \quad |A| + |R| - 1 \tag{11}$$

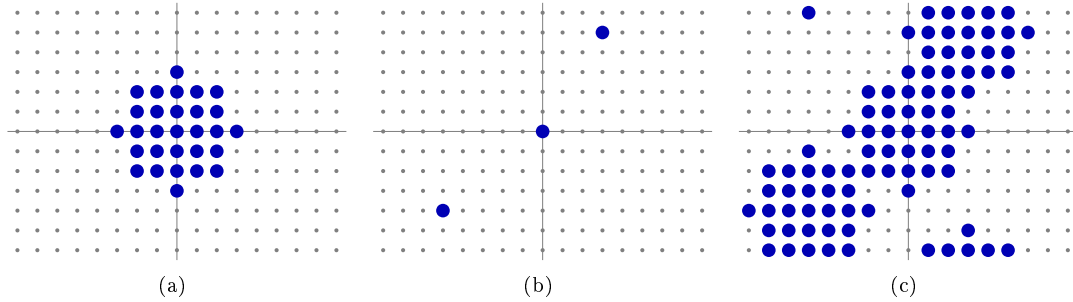*we say that G verifies the large neighborhood condition.*

Figure 1: (a) R for $r = 3$; (b) Set of nodes A; (c) Example of neighborhood with the Minkowski addition.

4.2 Maximum Broadcast Capacity

We focus on our main problem, computing the maximum broadcast capacity of the source s.

Theorem 1 *We consider a network G which is represented by a torus grid with a neighborhood defined by the set R, and with the following rate selection:*

- *rate $C_v = 1$ for all nodes $v \neq s$,*

- *rate $C_s = M = |R| - 1$ for the source s.*

if G verifies the large neighborhood condition then the maximum broadcast capacity of the source is $|R| - 1$.

Proof: Consider a fixed source s. In the previous section, we said that the maximum broadcast rate of the source is the min-cut $C_{\min}(s)$. We will assume that the source transmits at the maximum broadcast rate, that is $C_s = C_{\min}(s)$. Let us now consider any cut $(S, T) \in Q(s, t)$. The capacity of this cut is

$$C(S) \triangleq \sum_{v \in \Delta_S} C_v \text{ with } \Delta_S \triangleq \{v \in S : \mathcal{N}(v) \cap T \neq \emptyset\}. \tag{12}$$

• Case (i): If $s \in \Delta_S$, then T includes at least one node which is neighbor of the source. Thus $C(S) \geq C_s$, and this cut never constraints the maximum broadcast rate since $C_s = |R| - 1$ and therefore $C(S) \geq |R| - 1$.

• Case (ii): Otherwise, Δ_S includes only nodes different from the source, hence with transsmission rate 1. Therefore,

$$C(S) = \sum_{v \in \Delta_S} C_v = |\Delta_S|. \tag{13}$$

Since Δ_S represents the set of nodes of S which are neighbor of at least a node of T, Δ_S can be rewritten as: $\Delta_S = \mathcal{N}[T] \setminus T$, where $\mathcal{N}[T] \triangleq \mathcal{N}(T) \cup T$ is the "closed neighborhood" of nodes of T. Then,

$$\begin{aligned}
|C(S)| &= |\Delta_S| \\
&= |\mathcal{N}[T] \setminus T| \\
&\stackrel{(a)}{=} |\mathcal{N}[T]| - |T| \\
&\stackrel{(b)}{=} |T \oplus R| - |T|
\end{aligned}$$

where (*a*) is coming from $T \subset \mathcal{N}[T]$ and (*b*) from $\mathcal{N}[T] = T \oplus R$. Now the hypothesis is that one of the conditions (10) or (11) is true and we use it for $A = T$. The condition (10), implies that $N[T] = G$. This means that the source is neighbor of T, and since s in never in T we have that $s \in \Delta_S$. But we know that Δ_S includes only nodes different from the source, then T can never verify (10). As a consequence (11) must be true: $|T \oplus R| \geq |T| + |R| - 1$. By combining it with the previous expression of $C(s)$, we obtain $C(S) \geq |R| - 1$. ∎

4.3 The logic for Energy-Efficiency

In this section, we see why the previous results imply energy-efficiency in the network.

We have proved in the considered networks that the maximum broadcast capacity of the source is equal to the number of its neighbors. If we consider a node which is not neighbor of the source, it will receive on average M coded packets per unit time. Our result implies that, on average, it receives in particular M "innovative" coded packets per unit time, where innovative are the packets that provide new informations. This means that on average each transmission will be innovative for each receiver. In other words, the transmission in these networks is efficace in terms of energy since we could not do better.

We underline that this is not true in general (see experiments in [3] for instance) but it is strictly linked to the network topology and its homogeneity. We have extended here the results presented in [1], where a modification of rate selection is needed since the network is not a torus and nodes near the border of the network would have a smaller neighborhood. Without this modification the network would be in the bottleneck for the computation of the maximum broadcast rate.

4.4 Inequalities for Sumsets

Our goal is to prove in our case sufficient conditions that appear in Theorem 1, and thus in Definition 4.

In the case of a torus, these relations are a difficult problem and closely linked to the number theory and additive combinatorics. To prove the conditions (10) and (11) we use the following result due to Kneser [12].

Proposition 1 (Kneser's Theorem) *Let G be a finite abelian group, A and B nonempty finite subsets:*

$$|A \oplus B| \quad \geq \quad |A \oplus H| + |B \oplus H| - |H| \tag{14}$$

where $H \triangleq \{h \in G : x + h \in A \oplus B, \ \forall x \in (A \oplus B)\}$ is a subgroup of G and it is called stabilizer.

In our case, if n_x and n_y are prime (equal or not), we prove the desired properties.

Theorem 2 *Let n_x and n_y be prime, A a nonempty finite subset of G, and R defined in (9). Then*

$$|A \oplus R| \geq |A| + |R| - 1$$
$$\text{or} \qquad A \oplus R = G.$$

Proof: We consider Kneser's relation (14) with $B = R$:

$$|A \oplus R| \geq |A \oplus H| + |R \oplus H| - |H|. \tag{15}$$

H is a subgroup of G and we know that the subgroups of G are: $\{0\}$, G, $\left\{(x,0) : x \in \frac{\mathbb{Z}}{n_Y \mathbb{Z}}\right\}$ and $\left\{(0,y) : x \in \frac{\mathbb{Z}}{n_X \mathbb{Z}}\right\}$.

• Case $H = \{\mathbf{0}\}$: We have

$$
\begin{aligned}
|A \oplus R| &\geq |A \oplus \{\mathbf{0}\}| + |R \oplus \{\mathbf{0}\}| - |\{\mathbf{0}\}| \\
&\geq |A| + |R| - 1.
\end{aligned}
\tag{16}
$$

• Case $H = \left\{(x,0) : x \in \frac{\mathbb{Z}}{n_Y \mathbb{Z}}\right\}$: We observe that

$$
|A \oplus H| \geq |A|.
\tag{17}
$$

This is true since $\mathbf{0} \in H$ implies that $A \oplus \{\mathbf{0}\} \subset A \oplus H$ which gives $A \subset A \oplus H$.

We focus, now, our attention on $|R \oplus H|$. Since we are on a torus, the Minkowski sum of an horizontal line H and R is equal to a rectangle (see Figure 2), where the height is the diameter of R equal to $2r + 1$ and the width is the lenght of the line H. We consider the upper edge of
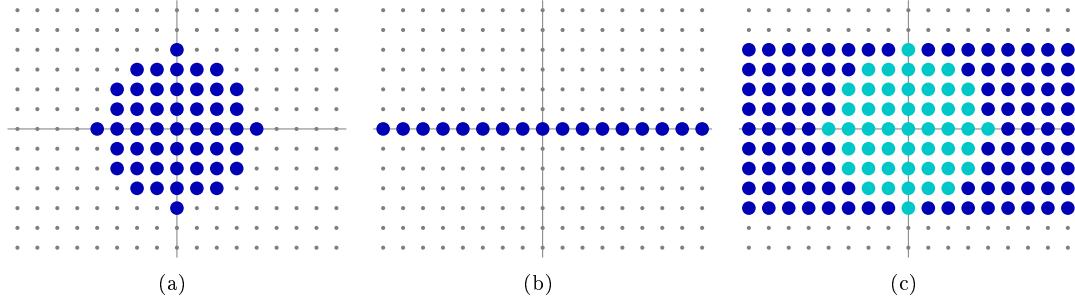

$$
\begin{array}{ccc}
\text{(a)} & \text{(b)} & \text{(c)}
\end{array}
$$

Figure 2: (a) R for $r = 4$; (b) $H = \left\{(x,0) : x \in \frac{\mathbb{Z}}{n_Y \mathbb{Z}}\right\}$; (c) Minkowski sum of R and H.

the rectangle which is given by the horizontal line denoted by

$$
L' = \left\{(x,y) \in \mathbb{Z}^2 : y = r\right\}
$$

which has the same lenght of the line H : $|L'| = |H|$. By definition of Minkowski sum and of R, this line passes by the unique point with coordinates $(0, r)$. We observe that

$$
\begin{aligned}
(L' \setminus \{(0,r)\}) &\cap R = \emptyset \quad \text{and} \\
(L' \setminus \{(0,r)\}) &\cup R \subseteq R \oplus H.
\end{aligned}
\tag{18}
$$

This means that

$$
\begin{aligned}
|R \oplus H| &\geq |L' \setminus \{(0,r)\} \cup R| = |L'| - 1 + |R| + \\
&\quad - |L' \setminus \{(0,r)\} \cap R| = |H| + |R| - 1.
\end{aligned}
\tag{19}
$$

Then, if we consider equations (24), (17) and (19), we obtain

$$
\begin{aligned}
|A \oplus R| &\geq |A \oplus H| + |R \oplus H| - |H| \geq \\
&\geq |A| + |R| + |H| - 1 - |H| = |A| + |R| - 1.
\end{aligned}
$$

- Case $H = \left\{ (0, y) : x \in \frac{\mathbb{Z}}{n_x \mathbb{Z}} \right\}$: It is similar to the case $H = \left\{ (x, 0) : x \in \frac{\mathbb{Z}}{n_Y \mathbb{Z}} \right\}$.
- Case $H = G$: If the stabilizer is G by definition we have that

$$(A \oplus R) \oplus G = A \oplus R.$$

This implies that

$$|A \oplus R| = |(A \oplus R) \oplus G| \geq |G|$$

since $G \subset (A \oplus R) \oplus G$. Therefore, $A \oplus B = G$. ∎

We observe that Theorem 2 allows us to prove Theorem 1 in the case the neighboroohd R is a discretized circle given in (9). What happens if R is a general subset of integer points? In the following Theorem we give some sufficient conditions such that the inequalities (10) and (11) hold for a genaral neighboroohd.

Definition 5 *A set $B \subset G$ is connected if for all u and v in B there exixts a path from u to v in B such that any two consecutive points in the path differ by at most one in each coordinate.*

Theorem 3 *Let n_x and n_y be prime, A a nonempty finite subset of G, and B a subset of G. Let n_1 and n_2 be two positive integers such that $n_x = n_1 + n_2 + d_{max}$ where $d_{max} = max\left\{ |x_u - x_v|, \forall u, v \in B \right\}$. Then*

$$|A \oplus B| \geq |A| + |B| - 1 \tag{20}$$

$$\text{or} \quad A \oplus B = G \tag{21}$$

is true in the following cases:

1. *In the case B is connected, (20) or (21) is true if*

$$d_{max} \leq (n_1 + n_2)(h - 1) + 1 \tag{22}$$

where $h = max\left\{ |y_u - y_v|, \forall u, v \in B \right\}$.

2. *In the case B is disconnected, (20) or (21) is true if*

$$d_{max} \leq (n_1 + n_2)(\hat{h} - 1) + 1 \tag{23}$$

where $\hat{h}$ is the number of rows of B with at least one element.

Proof: We consider Kneser's relation (14)

$$|A \oplus B| \geq |A \oplus H| + |B \oplus H| - |H|. \tag{24}$$

H is a subgroup of G and we know that the subgroups of G are: $\{0\}$, G, $\left\{ (x, 0) : x \in \frac{\mathbb{Z}}{n_Y \mathbb{Z}} \right\}$ and $\left\{ (0, y) : x \in \frac{\mathbb{Z}}{n_X \mathbb{Z}} \right\}$.

The cases $H = \{0\}$ and $H = G$ are similar to the ones discussed in the previous Theorem. In particular,

- Case $H = \{\mathbf{0}\}$: We have
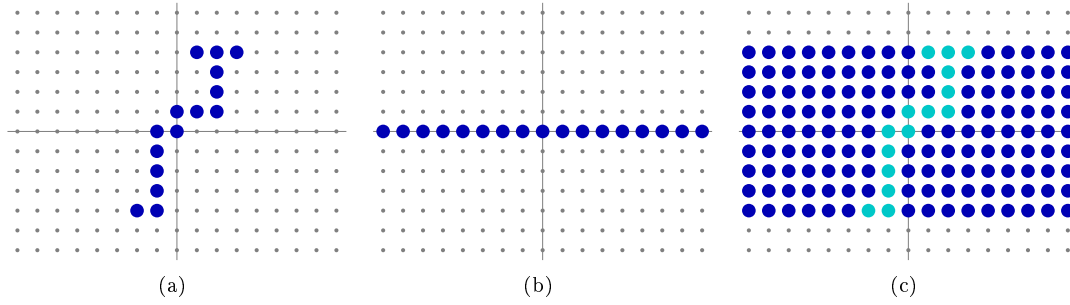
$$|A \oplus B| \geq |A| + |B| - 1.$$

Figure 3: (a) B connected; (b) $H = \left\{(x,0) : x \in \frac{\mathbb{Z}}{n_Y \mathbb{Z}}\right\}$; (c) Minkowski sum of B and H.

- Case $H = G$: We have

$$|A \oplus B| = |(A \oplus B) \oplus G| \geq |G|.$$

- Case $H = \left\{(x,0) : x \in \frac{\mathbb{Z}}{n_Y \mathbb{Z}}\right\}$: We suppose B connected. The Minkowski sum of B and H is equal to a rectangle where the height is h and the width is the lenght of the line H which is n_x, see Figure 3.

Since $n_x = n_1 + n_2 + d_{max}$, the rectangle can be written as disjoint union of three smaller rectangles S_1, S_2 ans S_3, where S_1 has $n_1 h$ elements, S_2 has $d_{max} h$ elements and it includes B and S_3 with $n_2 h$ elements. So,

$$\text{Rectangle} = S_1 \cup S_2 \cup S_3. \tag{25}$$

Therefore, we have

$$
\begin{aligned}
|\text{Rectangle}| &= |S_1 \cup S_2 \cup S_3| \\
&= |S_1| + |S_2| + |S_3| \\
&\geq n_1 h + |B| + n_2 h \\
&= |B| + n_1 + n_2 + n_1(h-1) + n_2(h-1) \\
&\overset{(a)}{\geq} |B| + n_1 + n_2 + d_{max} - 1 \\
&= |B| + |H| - 1, \tag{26}
\end{aligned}
$$

where (a) comes from applying hypothesis $d_{max} \leq (n_1 + n_2)(\hat{h} - 1) + 1$.

We consider now B a disconnected subset of G. We have by hypothesis that $\hat{h}$ is the number of rows of B with at least one element. In this case, the Minkowski sum of B and H is equivalent to a rectagle with height $\hat{h}$ and width n_x, see Figure 4.

Then, as in the previous case, we can write the rectangle as disjoint union of three rectangles
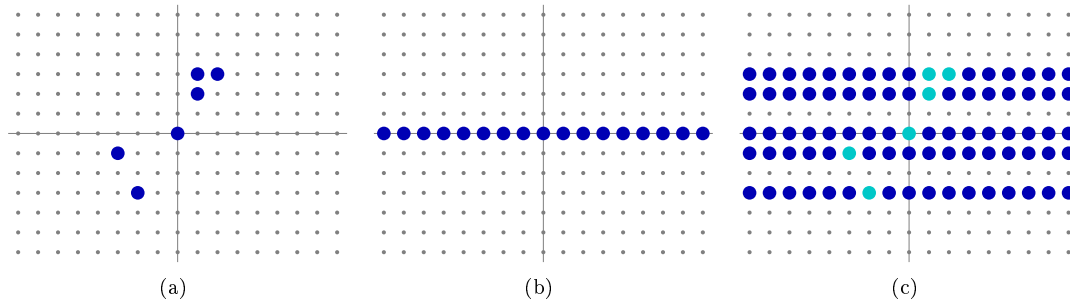
Figure 4: (a) B disconnected; (b) $H = \left\{ (x,0) : x \in \frac{\mathbb{Z}}{n_Y \mathbb{Z}} \right\}$; (c) Minkowski sum of B and H.

and, in partcular, we have

$$
\begin{aligned}
|\text{Rectangle}| &= |S_1 \cup S_2 \cup S_3| \\
&= |S_1| + |S_2| + |S_3| \\
&\geq n_1 \hat{h} + |B| + n_2 \hat{h} \\
&= |B| + n_1 + n_2 + n_1(\hat{h} - 1) + n_2(\hat{h} - 1) \\
&\overset{(b)}{\geq} |B| + n_1 + n_2 + d_{max} - 1 \\
&= |B| + |H| - 1,
\end{aligned}
\tag{27}
$$

where (b) comes from applying hypothesis (23).

- Case $H = \left\{ (0,y) : x \in \frac{\mathbb{Z}}{n_x \mathbb{Z}} \right\}$: It is similar to the case $H = \left\{ (x,0) : x \in \frac{\mathbb{Z}}{n_Y \mathbb{Z}} \right\}$. ∎

We observe that hypothesis (22) and (23) allows us to exclude the case where the neighborhood is to large, which represents a situation not interesting from the point of view of network coding. Moreover, these conditions imply that h and $\hat{h}$ are > 1, which means that there is at least one node besides the source in the network.

5   Conclusions

In this work, we studied network coding applied to the case of information broadcast in wireless networks. We have provided the maximum broadcast capacity of the source, for networks modeled by hyper-arcs such as torus grid, which is equal to the number of neighbors. In particular, each node receives, in average, M innovative packets where M is the number of neighbors. Network coding in such networks is efficace in terms of energy efficiency, in the sense that each transmission will provide innovative information. In order to prove this we have translated the min-cut problem on a hypergraph in an equivalent problem of additive combinatorics and we use tools from group theory.

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