Belief revision in the propositional closure of a qualitative algebra (extended version)

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Abstract

Belief revision is an operation that aims at modifying old beliefs so that they become consistent with new ones. The issue of belief revision has been studied in various formalisms, in particular, in qualitative algebras (QAs) in which the result is a disjunction of belief bases that is not necessarily representable in a QA. This motivates the study of belief revision in formalisms extending QAs, namely, their propositional closures: in such a closure, the result of belief revision belongs to the formalism. Moreover, this makes it possible to define a contraction operator thanks to the Harper identity. Belief revision in the propositional closure of QAs is studied, an algorithm for a family of revision operators is designed, and an open-source implementation is made freely available on the web.

Introduction

Belief revision is an operation of belief change that consists in modifying minimally old beliefs so that they become consistent with new beliefs (Alchourrón et al., 1985). One way to study this issue following a knowledge representation angle is to consider a formalism and to study some belief revision operators defined on it: how they are defined and how they can be implemented.

In particular, it is rather simple to define a revision operator on a qualitative algebra (such as the Allen algebra) by reusing the work of Condotta et al. (2010) about the related issue of belief merging. The result of such a belief revision is a set of belief bases to be interpreted disjunctively, and which is not necessarily representable as a sole belief base: qualitative algebras are not closed under disjunction.

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[*]This technical report constitutes an extended version of Dufour-Lussier et al. (2014).

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This gives a first motivation for the study of belief revision in the propositional closure of a qualitative algebra: the revision operator in such a closure gives a result necessarily representable in the formalism.

The first section of the paper contains some preliminaries about various notions used throughout the paper; this section is rather long since it contains notions from which a big part of the rest of the paper ensues, e.g. propositional closure of a formalism, qualitative algebras, and belief revision based on distances. Then, the paper describes some motivations about the study of belief revision in the propositional closure of a qualitative algebra. The next section briefly describes some properties of such a formalism. Finally, an algorithm and an implementation of this algorithm for a revision operator in the propositional closure of a qualitative algebra are presented with some detailed examples.

Preliminaries

Main terminology and assumptions about knowledge representation formalisms

A (knowledge representation) *formalism* is a pair $(\mathcal{L}, \models)$ where $\mathcal{L}$ is a language and $\models$ is a binary relation on $\mathcal{L}$. A *formula* φ is an element of $\mathcal{L}$. $\models$ is called the entailment relation. For $\varphi_1, \varphi_2 \in \mathcal{L}$, $\varphi_1 \equiv \varphi_2$ means that $\varphi_1 \models \varphi_2$ and $\varphi_2 \models \varphi_1$ and is read "φ_1 and φ_2 are equivalent".

The entailment relation of the formalisms used in this paper can always be characterized as follows—according to a model-theoretic semantics with a class of interpretations that is a set: It is assumed that there is a set Ω whose elements are called the *interpretations*. There is a relation also denoted by $\models$ on $\Omega \times \mathcal{L}$. If $\omega \models \varphi$, for $\omega \in \Omega$ and $\varphi \in \mathcal{L}$, ω is said to be a *model* of φ. The set of models of φ is called by $\mathcal{M}(\varphi)$. Therefore, the entailment relation is defined as follows: for $\varphi_1, \varphi_2 \in \mathcal{L}$, $\varphi_1 \models \varphi_2$ if $\mathcal{M}(\varphi_1) \subseteq \mathcal{M}(\varphi_2)$. From that, it can be implied that $\varphi_1 \equiv \varphi_2$ is equivalent to $\mathcal{M}(\varphi_1) = \mathcal{M}(\varphi_2)$.

A formula φ is *consistent* (or satisfiable) if $\mathcal{M}(\varphi) \neq \emptyset$. φ is a *tautology* if $\mathcal{M}(\varphi) = \Omega$.

$\mathcal{L}$ is assumed to be *closed under conjunction*, which means that for any $\varphi_1, \varphi_2 \in \mathcal{L}$ there exists $\varphi \in \mathcal{L}$ such that $\mathcal{M}(\varphi) = \mathcal{M}(\varphi_1) \cap \mathcal{M}(\varphi_2)$; φ is unique up to equivalence and is written $\varphi_1 \wedge \varphi_2$. $\wedge$ is associative wrt equivalence, so one can write $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$: no matter where the parentheses are placed, the formula will have the same set of models. Thus, the formalism is simplified, without loss of expressiveness, by removing such useless parentheses. It is also commutative wrt equivalence.

A *knowledge base* KB is a finite subset of $\mathcal{L}$. It is assimilated as the conjunction of its elements.

A formalism $(\mathcal{L}, \models)$ is *closed under disjunction* if for any $\varphi_1, \varphi_2 \in \mathcal{L}$ there exists $\varphi \in \mathcal{L}$ such that $\mathcal{M}(\varphi) = \mathcal{M}(\varphi_1) \cup \mathcal{M}(\varphi_2)$; then φ is unique up to equivalence and is written $\varphi_1 \vee \varphi_2$. $\vee$ is commutative and associative wrt equivalence.

A formalism $(\mathcal{L}, \models)$ is *closed under negation* if for any $\varphi_1 \in \mathcal{L}$ there exists $\varphi \in \mathcal{L}$ such that $\mathcal{M}(\varphi) = \Omega \setminus \mathcal{M}(\varphi_1)$; then φ is unique up to equivalence and is denoted by $\neg\varphi_1$.

A formalism $(\mathcal{L}, \models)$ is *propositionally closed* if it is closed under conjunction and negation. In this situation, it is also closed under disjunction (consider $\varphi_1 \vee \varphi_2$ as an abbreviation for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$).

The *propositional closure* of a formalism $(\mathcal{L}, \models)$ is the formalism $(\widehat{\mathcal{L}}, \widehat{\models})$ such that $\widehat{\mathcal{L}}$ is the smallest superset of $\mathcal{L}$ verifying:

- If $\varphi_1, \varphi_2 \in \widehat{\mathcal{L}}$ then $\varphi_1 \wedge \varphi_2 \in \widehat{\mathcal{L}}$;
- If $\varphi_1, \varphi_2 \in \widehat{\mathcal{L}}$ then $\varphi_1 \vee \varphi_2 \in \widehat{\mathcal{L}}$;
- If $\varphi \in \widehat{\mathcal{L}}$ then $\neg\varphi \in \widehat{\mathcal{L}}$;

and $\widehat{\models}$ is the entailment relation defined by the $\widehat{\mathcal{M}}$ function which extends $\mathcal{M}$ on $\widehat{\mathcal{L}}$ and is such that $\widehat{\mathcal{M}}(\varphi_1 \wedge \varphi_2) = \widehat{\mathcal{M}}(\varphi_1) \cap \widehat{\mathcal{M}}(\varphi_2)$, $\widehat{\mathcal{M}}(\varphi_1 \vee \varphi_2) = \widehat{\mathcal{M}}(\varphi_1) \cup \widehat{\mathcal{M}}(\varphi_2)$, and $\widehat{\mathcal{M}}(\neg\varphi_1) = \Omega \setminus \widehat{\mathcal{M}}(\varphi_1)$ (for any $\varphi_1, \varphi_2 \in \widehat{\mathcal{L}}$). The meta-language expression $\varphi_1 \mathrel{\widehat{\equiv}} \varphi_2$ means that $\varphi_1 \models \varphi_2$ and $\varphi_2 \models \varphi_1$. In the following, when the context is explicit, hats will be omitted ($\models$ and $\equiv$ instead of $\widehat{\models}$ and $\widehat{\equiv}$).

Let us consider a propositionally closed formalism $(\mathcal{L}, \models)$. An *atom* is a formula without any occurrence of the symbols $\neg$, $\vee$ and $\wedge$ (e.g. in propositional logic, atoms are propositional variables). A *literal* either is an atom (positive literal) or is of the form $\neg a$ where a is an atom (negative literal). A formula is under disjunctive normal form (*DNF*) if it is a disjunction of conjunctions of literals. Every formula φ is equivalent to a formula under DNF. To prove this, first, it can be proven that the following equivalences hold:

$$
\begin{aligned}
\varphi \wedge (\varphi_1 \vee \varphi_2 \vee \ldots \varphi_n) &\equiv (\varphi \wedge \varphi_1) \vee \ldots \vee (\varphi \wedge \varphi_n) \\
\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) &\equiv \neg\varphi_1 \vee \ldots \vee \neg\varphi_n \\
\neg(\varphi_1 \vee \ldots \vee \varphi_n) &\equiv \neg\varphi_1 \wedge \ldots \wedge \neg\varphi_n \\
\neg\neg\varphi &\equiv \varphi
\end{aligned}
\tag{1}
$$

for any $\varphi, \varphi_1, \ldots, \varphi_n \in \mathcal{L}$. Then, applying these equivalences from left to right until it is not possible to do this, starting with φ, results in a formula under DNF equivalent to φ.

Distance functions

A distance function on a set X is a function $d : X^2 \to \mathbb{R}_+$ (where $\mathbb{R}_+$ is the set of non negative real numbers) verifying the separation axiom ($d(x,y) = 0$ iff $x = y$), the symmetry ($d(x,y) = d(y,x)$) and the triangular inequality ($d(x,z) \leq d(x,y) + d(y,z)$).

Given $A, B \in 2^X$ and $y \in X$, $d(A, y)$ is an abbreviation for $\inf_{x \in A} d(x,y)$ and $d(A, B)$ is an abbreviation for $\inf_{x \in A, y \in B} d(x,y)$.

Qualitative algebras

Qualitative algebras (QAs) are formalisms that are widely used for representation depending on time and/or on space (Stock, 1997). Formulas built upon QAs are closed under conjunction, though the symbol $\wedge$ is not systematically used. Some of the usual notations and conventions of QAs are changed to better fit the scope of this paper. In particular, the representation of knowledge by graphs (namely, qualitative constraint networks) is not well-suited here, because of the propositional closure introduced afterwards.

First, the Allen algebra is introduced: it is one of the most famous QAs and it will be used in our examples throughout the paper. Then, a general definition of QAs is given.
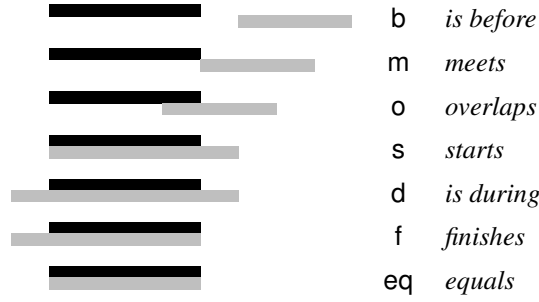
(a) Intuitive meaning.

$\mathcal{D}$ is the set of closed and bounded intervals $[a, b]$ of $\mathbb{Q}$ (the set of rational numbers) such that $a < b$. The base relations are defined as follows, with $d_1, d_2 \in \mathcal{D}$, $d_1 = [a_1, b_1]$, $d_2 = [a_2, b_2]$:

$$
\begin{aligned}
(d_1, d_2) \in \widetilde{\mathsf{eq}} \quad & \text{if } a_1 = a_2 \text{ and } b_1 = b_2 \\
(d_1, d_2) \in \widetilde{\mathsf{b}} \quad & \text{if } b_1 < a_2 \\
(d_1, d_2) \in \widetilde{\mathsf{m}} \quad & \text{if } a_2 = b_1 \\
(d_1, d_2) \in \widetilde{\mathsf{o}} \quad & \text{if } a_1 < a_2,\ a_2 < b_1 \text{ and } b_1 < b_2 \\
(d_1, d_2) \in \widetilde{\mathsf{s}} \quad & \text{if } a_1 = a_2 \text{ and } b_1 < b_2 \\
(d_1, d_2) \in \widetilde{\mathsf{f}} \quad & \text{if } a_1 > a_2 \text{ and } b_1 = b_2 \\
(d_1, d_2) \in \widetilde{\mathsf{d}} \quad & \text{if } a_1 > a_2 \text{ and } b_1 < b_2 \\
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{bi} = \mathsf{b}^- \quad & \mathsf{mi} = \mathsf{m}^- \quad & \mathsf{oi} = \mathsf{o}^- \\
\mathsf{si} = \mathsf{s}^- \quad & \mathsf{fi} = \mathsf{f}^- \quad & \mathsf{di} = \mathsf{d}^- \\
\end{aligned}
$$

(b) Semantics based on a domain.

Figure 1: The base relations of $\mathcal{L}_{\text{Allen}}$.

The Allen algebra

is used for representing relations between time intervals (Allen, 1983). A formula of the Allen Algebra can be seen as a conjunction of constraints, where a constraint is an expression of the form $x \ r \ y$ stating that the interval x is related to the interval y by the relation r. 13 base relations are introduced (cf. figure 1(a)); a relation r is either one of these base relations or the union of base relations $r_1, \ldots, r_m$ denoted by $r_1 \ \big| \ \ldots \ \big| \ r_m$.

For example, if one wants to express that the maths course is immediately before the physics course which is before the English course (either with a time lapse, or immediately before it), one can write the formula:

$$\texttt{maths}\ \mathsf{m}\ \texttt{physics} \quad \wedge \quad \texttt{physics}\ \mathsf{b}\ \big|\ \mathsf{m}\ \texttt{english}$$

$\mathcal{L}_{\text{Allen}}$ is the set of the formulas of the Allen algebra.

Qualitative algebras

in general are defined below, first by their syntax and then by their semantics. Finally, some inference mechanisms are described.

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Syntax. A finite set of symbols $\mathfrak{B}$ is given (with $|\mathfrak{B}| \geq 2$). A *base relation* is an element of $\mathfrak{B}$. A *relation* is an expression of the form $r_1 \mid \ldots \mid r_m$ ($m \geq 0$), such that a base relation occurs at most once in a relation and the order is irrelevant (e.g. $r_1 \mid r_2$ and $r_2 \mid r_1$ are equivalent expressions). The set of relations is denoted by $\mathfrak{R}$, which is of cardinality $|\mathfrak{R}| = 2^{|\mathfrak{B}|}$. The relation in which all the base relations occur is named ?. The relation $r_1 \mid \ldots \mid r_m$ with $m = 0$ is named !.

A finite set of symbols $\mathcal{V}$, disjoint from $\mathfrak{B}$, is given. A *(qualitative) variable* is an element of $\mathcal{V}$.

A *constraint* is an expression of the form $x \ r \ y$ where $x, y \in \mathcal{V}$ and $r \in \mathfrak{R}$.

A *formula* φ is a conjunction of n constraints ($n \geq 1$): $x_1 \ r_1 \ y_1 \wedge \ldots \wedge x_n \ r_n \ y_n$. A constraint of φ is one of the constraints of this conjunction. Let $\mathcal{L}_{\mathrm{QA}}$ be the set of the formulas of the considered QA. The atoms of $\mathcal{L}_{\mathrm{QA}}$ are the constraints.

A formula $\varphi \in \mathcal{L}_{\mathrm{QA}}$ is under normal form if for every $x, y \in \mathcal{V}$ with $x \neq y$, there is exactly one $r \in \mathfrak{R}$ such that $x \ r \ y$ is a constraint of φ. Then, this relation r is denoted by $r_\varphi(x, y)$.

A *scenario* σ is a formula under normal form such that, for every variables x and y, $x \neq y$, $r_\sigma(x, y) \in \mathfrak{B}$. Therefore, there are $|\mathfrak{B}|^{|\mathcal{V}| \times (|\mathcal{V}|-1)}$ scenarios. Given a formula φ under normal form, $\mathrm{Scen}(\varphi)$ is the set of scenarios obtained by substituting each constraint $x \ r_1 \mid \ldots \mid r_m \ y$ ($m \geq 2$) of φ with a constraint $x \ r_k \ y$ ($1 \leq k \leq m$).

Semantics. The semantics will be described twice. The two descriptions correspond to the same entailment relation, but serve different purposes. The first one gives a semantics based on a domain $\mathcal{D}$ on which the relations are interpreted, but the class of interpretations for this semantics is difficult to use for the purpose of the paper. This motivates a second semantics, defining a finite set Ω of interpretations, where an interpretation is a consistent scenario and on which a distance function can be easily defined.

Semantics based on a domain $\mathcal{D}$. The semantics of the Allen algebra given in figure 1(b) exemplifies this section.

Let $\mathcal{D}$ be a nonempty set, and let $\tilde{\ }$ be a mapping that associates to each $r \in \mathfrak{B}$ a relation $\tilde{r}$ on $\mathcal{D}$ ($\tilde{r} \subseteq \mathcal{D}^2$) such that:

- $\widetilde{\mathfrak{B}} = \{\tilde{r} \mid r \in \mathfrak{B}\}$ is a partition of $\mathcal{D}^2$: for each $(d, e) \in \mathcal{D}^2$ there is exactly one $r \in \mathfrak{B}$ such that $(d, e) \in \tilde{r}$. Furthermore, each $\tilde{r} \in \widetilde{\mathfrak{B}}$ is nonempty.

- For each $r \in \mathfrak{B}$ there exists exactly one $s \in \mathfrak{B}$ such that $\tilde{s}$ is the inverse of the relation $\tilde{r}$. In the following, s is denoted by r^-.

- There is a base relation, denoted by eq, that is interpreted as the equality on $\mathcal{D}$: $\widetilde{\mathsf{eq}} = \{(d, d) \mid d \in \mathcal{D}\}$. eq is its own inverse: $\mathsf{eq}^- = \mathsf{eq}$.

This mapping is extended on $\mathfrak{R}$ as follows:

$$\text{if } r \in \mathfrak{R} \text{ and } r = r_1 \mid \ldots \mid r_m \text{ then } \tilde{r} = \tilde{r_1} \cup \ldots \cup \widetilde{r_m}$$

In other words: $(d, e) \in \tilde{r}$ iff there exists $i \in \{1, \ldots, m\}$ such that $(d, e) \in \tilde{r_i}$.

An interpretation $\mathcal{I}$ is a mapping from $\mathcal{V}$ to $\mathcal{D}$. $\mathcal{I}$ is a model of $x \ r \ y$ if $(\mathcal{I}(x), \mathcal{I}(y)) \in \tilde{r}$. $\mathcal{I}$ satisfies a conjunction of constraints if it satisfies every constraint in the conjunction. A formula φ is consistent if there exists an interpretation satisfying it. Finally, $\varphi_1 \models \varphi_2$ if every interpretation that satisfies φ_1 also satisfies φ_2.

According to this semantics, any constraint of the form x ? y is a tautology and any constraint of the form x ! y is inconsistent. Moreover, any formula φ is equivalent to a formula φ' under normal form.[1] Thus, in the following of the paper, all the formulas of φ are assumed to be under normal form, without lost of expressiveness.

Semantics defined by consistent scenarios. The semantics can be characterized a posteriori thanks to consistent scenarios.

Let Ω be the set of consistent scenarios on the variables of $\mathcal{V}$. It can be easily proven that $|\Omega| \leq |\mathfrak{B}|^{|\mathcal{V}| \times (|\mathcal{V}|-1)/2}$: if $x\ r\ y$ is a constraint of a consistent scenario σ then $y\ r^-\ x$ is also a constraint of σ.

Let $\mathcal{M} : \mathcal{L} \to 2^\Omega$ be defined by

$$\mathcal{M}(\varphi) = \{\sigma \in \Omega \mid \sigma \models \varphi\}$$

for $\varphi \in \mathcal{L}$, where $\models$ is the entailment relation defined below, thanks to the semantics based on a domain.

Ω and $\mathcal{M}$ make it possible to define a semantics on $\mathcal{L}$ which coincides with the semantics based on a domain (hence the same entailment relation $\models$). However, this second semantics is more practical to use for the definition of revision on QAs.

Inferences. The main inference about QAs used in this paper is the test of consistency.

It is usually implemented in the following way. Properties on formulas named *arc consistency* and *path consistency* are defined. Having those properties are a necessary condition and, in most algebras, a sufficient condition for scenarios to be consistent (a scenario σ is consistent iff it is arc-consistent and path-consistent). A way to test if $\varphi \in \mathcal{L}_{\text{QA}}$ is consistent is to test whether there exists $\sigma \in \text{Scen}(\varphi)$ that is consistent.

A formula $\varphi \in \mathcal{L}_{\text{QA}}$ is arc-consistent if:

- For all 2 variables $x, y \in \mathcal{V}$, $r_\varphi(x, y) \neq$!.

- For all 2 variables $x, y \in \mathcal{V}$, $r_\varphi(x, y) = r_\varphi(y, x)^-$.

The definition of path consistency is based on a binary operation on $\mathfrak{R}$, written ;. It is defined on $\mathfrak{B}$ (for example by a $|\mathfrak{B}| \times |\mathfrak{B}|$ table) and extended on $\mathfrak{R}$ thanks to the equalities

$$(r_1 \mid \ldots \mid r_m)\ ;\ s = (r_1\ ;\ s) \mid \ldots \mid (r_m\ ;\ s)$$
$$s\ ;\ (r_1 \mid \ldots \mid r_m) = (s\ ;\ r_1) \mid \ldots \mid (s\ ;\ r_m)$$

In $\mathcal{L}_{\text{Allen}}$, ; corresponds to the classical composition of relations: $\widetilde{s\ ;\ r} = \widetilde{s} \circ \widetilde{r}$ (i.e. $\mathcal{I} \models x\ s\ ;\ r\ y$ if there exists $d \in \mathcal{D}$ such that $(\mathcal{I}(x), d) \in \widetilde{r}$ and $(d, \mathcal{I}(y)) \in \widetilde{s}$). In some other QAs, ; corresponds to a different operation, called the weak composition (Renz and Ligozat, 2005; Ligozat and Renz, 2004).

A formula $\varphi \in \mathcal{L}_{\text{QA}}$ is path-consistent if, for all 3 variables $x, y, z \in \mathcal{V}$, the constraint deduced by composition between x and z ($x\ r_\varphi(y, z)\ ;\ r_\varphi(x, y)\ z$) is weaker than the constraint stated in φ (i.e. $x\ r_\varphi(x, z)\ z$).

---

[1]This can be proven by considering, for any $x, y \in \mathcal{V}$, $x \neq y$, the set R_{xy} of relations r such that $x\ r\ y$ is a constraint of φ. If $R = \emptyset$, let C_{xy} be the constraint x ? y. Else, let r_{xy} be the relation constituted of the base relations that occur in all relations of R_{xy} (for example, if $R_{xy} = \{$b $\mid$ m $\mid$ o, m $\mid$ o $\mid$ s$\}$ then $r_{xy} =$ m $\mid$ o). Then, C_{xy} is the constraint $x\ r_{xy}\ y$. Finally, the formula $\bigwedge\limits_{x, y \in \mathcal{V}, x \neq y} C_{xy}$ is a formula under normal form equivalent to φ.

Belief change

Belief revision

is an operation of belief change. Intuitively, given the set of beliefs ψ an agent has about a static world, it consists in considering the change of their beliefs when faced with a new set of beliefs μ, assuming that μ is considered to be unquestionable by the agent. The resulting set of beliefs is noted $\psi \dotplus \mu$, and depends on the choice of a belief revision operator $\dotplus$. In Alchourrón et al. (1985), the principle of minimal change has been stated and could be formulated as follows: ψ is minimally changed into ψ' such that the conjunction of ψ' and μ is consistent, and the result of the revision is this conjunction. Hence, there is more than one possible $\dotplus$ operator, since the definition of $\dotplus$ depends on how belief change is "measured". More precisely, the minimal change principle has been formalized by a set of postulates, known as the AGM postulates—after the names of Alchourrón, Gärdenfors, and Makinson (1985). Peppas (2008) presents a detailed survey of belief revision at a general level (for any formalisms satisfying some general properties, such as closure under conjunction) including some representation theorems and the discussion of certain related issues (other belief change operators, etc.).

In Katsuno and Mendelzon (1991b), revision has been studied in the framework of propositional logic (with a finite set of variables). The AGM postulates are translated into this formalism as follows (ψ, ψ_1, ψ_2, μ, μ_1, μ_2 and ϕ are propositional formulas):

($\dotplus$1) $\psi \dotplus \mu \models \mu$.

($\dotplus$2) If $\psi \wedge \mu$ is consistent then $\psi \dotplus \mu \equiv \psi \wedge \mu$.

($\dotplus$3) If μ is consistent then $\psi \dotplus \mu$ is consistent.

($\dotplus$4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$ then $\psi_1 \dotplus \mu_1 \equiv \psi_2 \dotplus \mu_2$.

($\dotplus$5) $(\psi \dotplus \mu) \wedge \phi \models \psi \dotplus (\mu \wedge \phi)$.

($\dotplus$6) If $(\psi \dotplus \mu) \wedge \phi$ is consistent then
$\psi \dotplus (\mu \wedge \phi) \models (\psi \dotplus \mu) \wedge \phi$.

Moreover, a family of revision operators is defined based on distance functions d on Ω, where Ω is the set of interpretations: the revision of ψ by μ according to $\dotplus^d$ ($\psi \dotplus^d \mu$) is such that

$$\mathcal{M}(\psi \dotplus^d \mu) = \{\omega \in \mathcal{M}(\mu) \mid d(\mathcal{M}(\psi), \omega) = d^*\}$$
$$\text{with } d^* = d(\mathcal{M}(\psi), \mathcal{M}(\mu)) \tag{2}$$

Intuitively, d^* measures, using d, the minimal modification of ψ into ψ' needed to make $\psi' \wedge \mu$ consistent.

It appears that it is not required for d to be a true distance function, i.e. symmetry and triangular inequality are not required: if d verifies the separation postulate, then $\dotplus^d$ verifies postulates ($\dotplus$1–6).

This approach can be extended to other formalisms for which a model-theoretic semantics can be defined such that a distance function can be defined on the set of interpretations Ω. However, in some of these formalisms, a representability issue can be raised: it may occur that a subset Σ of Ω is not representable, i.e. there is no formula φ such that $\mathcal{M}(\varphi) = \Sigma$. This representability issue is addressed below, for the case of qualitative algebras.

Belief revision has been applied to the issue of the adaptation process of a case-based reasoning system (Cojan and Lieber, 2012; Dufour-Lussier et al., 2013).
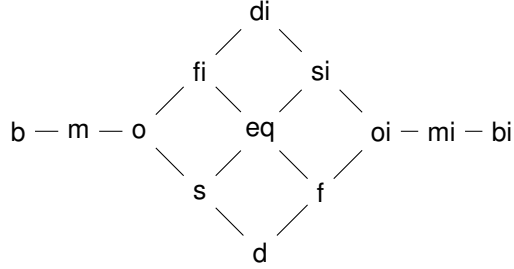
Figure 2: One possible neighborhood graph for the Allen algebra (Ligozat, 1991).

Belief contraction

is the operation of belief change that associates to a set of beliefs ψ and a set of beliefs μ, a set of beliefs $\psi \dotminus \mu$ such that $\psi \dotminus \mu \not\models \mu$. In propositionally closed formalisms, the Harper identity makes it possible to define a contraction operator $\dotminus$ thanks to a revision operator $\dotplus$ with

$$\psi \dotminus \mu = \psi \vee (\psi \dotplus \neg\mu) \tag{3}$$

Conversely, the Levi identity makes it possible to define a revision operator $\dotplus$ with

$$\psi \dotplus \mu = (\psi \dotminus \neg\mu) \wedge \mu$$

Belief merging

is another operation of belief change. Given some sets of beliefs ψ_1, ..., ψ_n, their merging is a set of beliefs Ψ that contains "as much as possible" of the beliefs in the ψ_i's. Intuitively, Ψ is the conjunction of ψ_1', ..., ψ_n' such that each ψ_i has been minimally modified into ψ_i' in order to make this conjunction consistent. Some postulates of belief merging have been proposed and discussed (Konieczny and Pérez, 2002), in a similar way as the AGM postulates.

In practice, studies on belief merging are often easy to reuse for belief revision: the revision of ψ by μ can be seen as a kind of merging of ψ and μ such that no modification is allowed on μ.

For instance, belief merging has been studied for qualitative algebras by Condotta et al. (2010) and Wallgrün and Dylla (2010). Wallgrün and Dylla have proposed *syntax-based* revision operators for qualitative algebras. Those operators do not obey the AGM postulates—most importantly, the syntax-independance postulate. Therefore, their work cannot serve as a base for developing a model distance-based, AGM revision operator. Condotta et al., on the other hand, proposed both syntax and semantic-based operators. The latter can be used as a base to create corresponding revision operators.

Belief revision in qualitative algebras

In Condotta et al. (2010) a belief merging operator is defined which is based on a distance function d on scenarios, defined as follows. Let δ be a distance function on $\mathfrak{B}$. Let $\sigma, \tau \in \Omega$, be two scenarios based on the same set of variables $\mathcal{V}$. Then, d is defined

by

$$d(\sigma, \tau) = \sum_{x,y \in \mathcal{V}, x \neq y} \delta(r_\sigma(x,y), r_\tau(x,y))$$

One of the possibilities for δ is the use of a neighborhood graph, i.e. a connected, undirected graph whose vertices are the base relations and such that $\delta(r, s)$ is the length of the shortest path between r and s. Figure 2 presents such a graph for the Allen algebra. Then, the models of the merging of $\psi_1, \ldots, \psi_n$ is the set of scenarios σ that minimizes $\sum_{i=1}^{n} d(\mathcal{M}(\psi_i), \sigma)$ (other aggregation functions than the sum can also be used). The representability issue can be raised since the set of the optimal scenarios is not necessarily representable in $(\mathcal{L}_{QA}, \models)$. One solution to address this issue is to find a formula $\varrho \in \mathcal{L}_{QA}$ whose set of models includes closely the set of optimal models. Another solution is to consider that the result of merging is a set of scenarios.

This representability issue is also raised for revision in $\mathcal{L}_{QA}$, and the second type of solution is used: for $\psi, \mu \in \mathcal{L}_{QA}$, $\psi \dotplus^d \mu$ *is the set of the scenarios that are the closest to $\mathcal{M}(\psi)$.*

In Dufour-Lussier et al. (2012) and Dufour-Lussier et al. (2013), an algorithm for $\dotplus^d$ in a qualitative algebra $(\mathcal{L}_{QA}, \models)$ is defined and its implementation in the system REVISOR/QA—for three QAs—is described. Its inputs are ψ and μ, which are in $\mathcal{L}_{QA}$. Its output is the set of the scenarios $\sigma \in \mathcal{M}(\mu)$ such that $d(\mathcal{M}(\psi), \sigma)$ is minimal. Its principle is based on an A* search (Pearl, 1984) with an admissible heuristics. For this search:

- A state is a $\varphi \in \mathcal{L}_{QA}$.

- The initial state is μ.

- A successor of a state φ is a state φ' obtained by substituting in φ a constraint $x \; r_1 \mid \ldots \mid r_m \; y \; (m \geq 2)$ with a (more specific) constraint $x \; r_k \; y \; (1 \leq k \leq m)$.

- A final state is a consistent scenario.

- The heuristic cost function is an estimation of the distance from ψ to the state φ (estimation that is exact on final states).

A slight modification wrt the classical A* algorithm is that the search stops after *all* the states at minimal cost have been generated—not as soon as a first final state is found. The result is the set of final states φ which are the models of μ that are the closest to models of ψ according to d. It can be noticed that the cost of a final state generated by an A* search is d^* (as defined in (2)).

The worst-case complexity of this algorithm depends on the amount of scenarios in μ, which is of the order of $O\left(|\mathfrak{B}|^{\frac{|V| \cdot (|V|-1)}{2}}\right)$.

Hué and Westphal (2012) have also implemented a family of revision operators on QAs. Their search algorithm is based on the GQR reasoner (Gantner et al., 2008), which does not use a heuristic search but, on the other hand, takes advantage of the existence of pre-convex relations—which under certain circumstances make it possible to guarantee consistency without having to compute scenarios.

Motivations

Let us consider the following formulas of $\mathcal{L}_{\text{Allen}}$:

$$\psi = x \text{ eq } y \ \wedge \ y \text{ eq } z$$
$$\mu = x \text{ d } z \ \wedge \ z \text{ di } x$$

The set of models of μ that are the closest to models of ψ according to d is $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ with

$$\sigma_1 = \mu \ \wedge \ x \text{ d } y \ \wedge \ y \text{ di } x \ \wedge \ y \text{ eq } z \ \wedge \ z \text{ eq } y$$
$$\sigma_2 = \mu \ \wedge \ x \text{ s } y \ \wedge \ y \text{ si } x \ \wedge \ y \text{ f } z \ \wedge \ z \text{ fi } y$$
$$\sigma_3 = \mu \ \wedge \ x \text{ f } y \ \wedge \ y \text{ fi } x \ \wedge \ y \text{ s } z \ \wedge \ z \text{ si } y$$
$$\sigma_2 = \mu \ \wedge \ x \text{ eq } y \ \wedge \ y \text{ eq } x \ \wedge \ y \text{ d } z \ \wedge \ z \text{ di } y$$

and it can be proven that no formula ϱ of the Allen algebra is such that $\mathcal{M}(\varrho) = \Sigma$.[2] So, the representability issue is raised: $\dot{+}^d$ in $\mathcal{L}_{\text{Allen}}$ does not match exactly equation (2). Thus, either $\psi \dot{+}^d \mu$ gives a result outside of $\mathcal{L}_{\text{Allen}}$ or $\psi \dot{+}^d \mu$ gives a formula ϱ that approximates the equality (2). By contrast, $\dot{+}^d$ defined by this equality can be defined in the propositional closure of the Allen algebra (which is a consequence of proposition 1, given in the next section), and this gives a first motivation for this work.

The second motivation is linked to the expressiveness of the formalisms: some knowledge are more easily represented in the propositional closure of a QA. An example will be presented that is formalized using both $\mathcal{L}_{\text{Allen}}$ and its propositional closure $\widehat{\mathcal{L}}_{\text{Allen}}$. It appears to be much simpler (or "more natural") to formalize it in $\widehat{\mathcal{L}}_{\text{Allen}}$. Moreover, still on this particular example, the computing time of the revision is shorter in the more expressive formalism $\widehat{\mathcal{L}}_{\text{Allen}}$, with the systems we have implemented.

The third motivation of this work is that a revision operator on the propositional closure of a QA can be used in the definition of a contraction operator, thanks to (3), which requires disjunction and negation connectors.

Propositional closure of a qualitative algebra

Let $(\mathcal{L}_{\text{QA}}, \models)$ be a qualitative algebra. The propositional closure of this formalism, as defined in the preliminaries, is $(\widehat{\mathcal{L}}_{\text{QA}}, \widehat{\models})$.

Proposition 1 (representability). *Every set of scenarios $\Sigma \subseteq \Omega$ is representable in $\widehat{\mathcal{L}}_{QA}$. More precisely, with $\varphi = \bigvee\limits_{\sigma \in \Sigma} \sigma$, $\mathcal{M}(\varphi) = \Sigma$.*

---

[2]To prove this, first, let us consider the formula

$$\varphi = \mu \wedge x \text{ d}|\text{s}|\text{f}|\text{eq } y \ \wedge \ y \text{ di}|\text{si}|\text{fi}|\text{eq } x \ \wedge$$
$$y \text{ d}|\text{s}|\text{f}|\text{eq } z \ \wedge \ z \text{ di}|\text{si}|\text{fi}|\text{eq } y$$

φ is such that $\Sigma \subseteq \mathcal{M}(\varphi)$ and for each $\chi \in \mathcal{L}_{\text{Allen}}$, if $\Sigma \subseteq \mathcal{M}(\chi)$ then $\varphi \models \chi$ (φ is the most specific formula whose set of models contains Σ). Now, $\Sigma \neq \mathcal{M}(\varphi)$ since, for instance, the following consistent scenario belongs to $\mathcal{M}(\varphi)$ and not to Σ:

$$\sigma = \mu \ \wedge \ x \text{ d } y \ \wedge \ y \text{ di } x \ \wedge \ y \text{ d } z \ \wedge \ z \text{ d } y$$

Therefore, there is no $\varrho \in \mathcal{L}_{\text{Allen}}$ such that $\mathcal{M}(\varrho) = \Sigma$.

Proof. First, it is proven that

$$\text{for any } \sigma \in \Omega, \quad \mathcal{M}(\sigma) = \{\sigma\} \tag{4}$$

$\sigma \in \mathcal{M}(\sigma)$ is a direct consequence of $\sigma \models \sigma$, thus it is sufficient to prove that each $\tau \in \Omega$ such that $\tau \neq \sigma$ is not a model of σ. $\tau \neq \sigma$ implies that there exists $x, y \in \mathcal{V}$ and $r, s \in \mathfrak{B}$ with $r \neq s$ such that $x \; r \; y$ and $x \; s \; y$ are respectively a constraint of σ and of τ. Since $\widetilde{r} \cap \widetilde{s} = \emptyset$ ($\mathfrak{B}$ being a partition of Ω) and $\tau \models x \; s \; y$, $\tau \not\models x \; r \; y$, and therefore, $\tau \not\models \sigma$, which proves (4).

From (4) and the semantics of $\vee$, it comes that $\mathcal{M}(\varphi) = \bigcup_{\sigma \in \Sigma} \mathcal{M}(\sigma) = \bigcup_{\sigma \in \Sigma} \{\sigma\}$ $= \Sigma$, which proves the proposition. $\qquad\square$

Every formula of $\widehat{\mathcal{L}}_{\text{QA}}$ can be written in DNF, since it is a propositionally closed formalism, but the following proposition goes beyond that.

Proposition 2 (normal forms). *Let $\varphi \in \widehat{\mathcal{L}}_{QA}$. φ can be put under the following forms:*

DNF-w/oN form *φ is equivalent to a formula in DNF using no negation symbol.*

DNF-w/oN-$\mathfrak{B}$ form *φ is equivalent to a formula in DNF using no negation symbol and such that its constraints contain only base relations.*

Proof. DNF-w/oN form. Let φ_1 be a formula under DNF equivalent to φ (it exists: cf. the section on preliminaries). Therefore φ_1 has the form $\varphi_1 = \bigvee_i \bigwedge_j \ell_{ij}$ where ℓ_{ij} is either a constraint (positive literal) or the negation of a constraint (negative literal).

Let $\neg(x \; r \; y)$ be a negative literal. Let R be the set of base relations occurring in r (if $r = r_1 \mid \ldots \mid r_m$ then $R = \{r_1, \ldots, r_m\}$) and $\overline{R} = \mathfrak{B} \setminus R$. Let s be the relation based on the relations of $\overline{R}$. Then, it comes that:

$$\neg(x \; r \; y) \quad \equiv \quad x \; s \; y$$

(for example, $\neg(x \; ? \; y) \equiv x \; ! \; y$). Therefore every negative literal can be substituted by an equivalent positive literal and, by doing such substitutions on φ_1, the result is a formula φ_2, equivalent to φ, which proves that φ can be put under DNF-w/oN form. *DNF-w/oN-$\mathfrak{B}$ form.* First, it is proven that any constraint $x \; r \; y$ is equivalent to a formula containing constraints based only on base relations (i.e. no occurrence of the symbol $\mid$). If $r = !$, then $x \; r \; y$ is an inconsistent formula and therefore is equivalent to any inconsistent formula, for example $x \; r \; y \; \wedge \; x \; s \; y$ ($r, s \in \mathfrak{B}$, $r \neq s$), which is only based on base relations. If $r \neq !$ then $r = r_1 \mid \ldots \mid r_m$ with $m \geq 1$ and then

$$x \; r \; y \quad \equiv \quad x \; r_1 \; y \; \vee \ldots \vee \; x \; r_m \; y$$

Second, let φ_2 be a formula equivalent to φ that is under DNF-w/oN form. By substituting in φ_2 all the constraints by equivalent formulas based only on base relations, the resulting formula, φ_3, is equivalent to φ, and contains only base relations and no negation. Finally, φ_3 can be put under DNF as explained in the preliminaries of the paper (i.e. according to the set of equivalences (1)) resulting in a formula φ_4 that is under DNF-w/oN-$\mathfrak{B}$ and which is equivalent to φ. $\qquad\square$

Other authors as well stressed the interest of being able to handle temporal constraints disjunctions, such as "the trip takes either 5 minutes (by car) or 15 minutes (by bus)." These disjunctions are generally not taken into account in the existing representations of qualitative relational algebras. Some work proposed to handle disjunctions

in the point algebra (Vilain and Kautz, 1986). In Gerevini and Schubert (1995), for instance, qualitative relations between intervals are represented by disjunctions of relations between the ends of the intervals—e.g. "the beginning of interval y is before the beginning of interval x or the end of x is before the beginning of y." Formalisms representing temporal metric constraints are more frequent, following the proposition of Dechter et al. (1991). In Barber (2000), disjunctions of constraints are handled using a notion of temporal context. As far as we know, none of these works has addressed the issue of propositional closure, though.

Belief revision in $(\widehat{\mathcal{L}}_{\mathbf{QA}}, \widehat{\models})$

Given a distance function d on Ω, a revision operator on $(\widehat{\mathcal{L}}_{\mathrm{QA}}, \widehat{\models})$ can be defined according to equation (2). Indeed, proposition 1 implies that $\{\omega \in \mathcal{M}(\mu) \mid d(\mathcal{M}(\psi), \omega) = d^*\}$ is representable.

An algorithm for computing $\dotplus^d$ in $\widehat{\mathcal{L}}_{\mathbf{QA}}$

The principle of the algorithm is based on the following proposition.

Proposition 3 (revision of disjunctions). *Let ψ and μ be two formulas of $\widehat{\mathcal{L}}_{QA}$ and $\{\psi_i\}_i$ and $\{\mu_j\}_j$ be two finite families of $\widehat{\mathcal{L}}_{QA}$ such that $\psi = \bigvee_i \psi_i$ and $\mu = \bigvee_j \mu_j$.*

Let $d_{ij}^* = d(\mathcal{M}(\psi_i), \mathcal{M}(\mu_j))$ for any i and j. Then:

$$\psi \dotplus^d \mu \equiv \bigvee_{i,j,d_{ij}^* = d^*} \psi_i \dotplus^d \mu_j$$
$$\textit{with } d^* = d(\mathcal{M}(\psi), \mathcal{M}(\mu))$$
$$\textit{Moreover,} \quad d^* = \min_{ij} d_{ij}^* \tag{5}$$

Proof. First, (5) is proven:

$$d^* = d(\mathcal{M}(\psi), \mathcal{M}(\mu)) = d\left(\bigcup_i \mathcal{M}(\psi_i), \bigcup_j \mathcal{M}(\mu_j)\right)$$
$$= \min_{ij} d(\mathcal{M}(\psi_i), \mathcal{M}(\mu_j)) = \min_{ij} d_{ij}^*$$

Second, let $\omega \in \mathcal{M}(\psi \dotplus^d \mu)$. Thus, there exists $\nu \in \mathcal{M}(\psi)$ such that $d(\nu, \omega) = d^*$. Let i and j be such that $\nu \in \mathcal{M}(\psi_i)$ and $\omega \in \mathcal{M}(\mu_j)$. So, the following chain of relations holds:
$$d^* = d(\nu, \omega) \geq d(\mathcal{M}(\psi_i), \omega) \geq d_{ij}^* \geq d^*$$

Therefore, all the numbers in this chain are equal and $d(\mathcal{M}(\psi_i), \omega) = d_{ij}^* = d^*$, so $\omega \in \mathcal{M}(\psi_i \dotplus^d \mu_j)$ for i and j, such that $d_{ij}^* = d^*$. To summarize, if $\omega \in \mathcal{M}(\psi \dotplus^d \mu)$ then $\omega \in \mathcal{M}\left(\bigvee_{i,j,d_{ij}^* = d^*} \psi_i \dotplus^d \mu_j\right)$.

Conversely, let $\omega \in \mathcal{M}(\psi_i \dotplus^d \mu_j)$ for i and j such that $d_{ij}^* = d^*$. This entails that $d(\mathcal{M}(\psi_i), \omega) = d^*$, hence the following chain of relations:

$$d^* \leq d(\mathcal{M}(\psi), \omega) \leq d(\mathcal{M}(\psi_i), \omega) = d^*$$

so $d(\mathcal{M}(\psi), \omega) = d^*$ with $\omega \in \mathcal{M}(\mu)$, consequently $\omega \in \mathcal{M}(\psi \dot{+}^d \mu)$.

To conclude, $\omega \in \mathcal{M}(\psi \dot{+}^d \mu)$ iff $\omega \in \mathcal{M}\left(\bigvee_{i,j,d_{ij}^*=d^*} \psi_i \dot{+}^d \mu_j\right)$, which proves the proposition. $\qquad\square$

The algorithm for $\dot{+}^d$ in $\widehat{\mathcal{L}}_{\text{QA}}$ consists roughly in putting ψ and μ in DNF-w/oN form then applying proposition 3 on them, using the $\dot{+}^d$ algorithm on $\mathcal{L}_{\text{QA}}$ for computing the $\psi_i \dot{+}^d \mu_j$'s.

This requires some small modifications in the algorithm for $\dot{+}^d$ in $(\mathcal{L}_{\text{QA}}, \models)$:

- The revision algorithm inputs a triple $(\psi, \mu, d_{\max})$ where $\psi, \mu \in \mathcal{L}_{\text{QA}}$ and $d_{\max}$ is a non negative number which gives a maximal admissible value for $d^* = d(\mathcal{M}(\psi), \mathcal{M}(\mu))$.

- The search in the state space is stopped (and returns a "failure symbol") when the cost associated to a state is greater than $d_{\max}$.

- The output of the algorithm is either the failure symbol or a pair (ϱ, d^*) where $\varrho \in \widehat{\mathcal{L}}_{\text{QA}}$ is the disjunction of scenarios of $\psi \dot{+}^d \mu$.

The algorithm is shown in figure 3. It is based on the proposition 3 and on the modified algorithm for $\dot{+}^d$ in $(\mathcal{L}_{\text{QA}}, \models)$—line 6 makes use of this modified algorithm.

REVISOR/PCQA: an implementation of $\dot{+}^d$ in $(\widehat{\mathcal{L}}_{\textbf{QA}}, \widehat{\models})$

REVISOR is a collection of several revision engines that are open-source and freely available.[3]

In particular, REVISOR/QA implements $\dot{+}^d$ in three QAs: the Allen algebra, INDU —an extension of the Allen algebra taking into account relations between intervals according to their lengths (Pujari et al., 1999)—and RCC8—a QA for representing topological relations between regions of space (Randell et al., 1992). Moreover, it is easy to use a different qualitative algebra, by specifying in the code the value of $s \; ; \; r$ for each $r, s \in \mathfrak{B}$, the value of r^- for each $r \in \mathfrak{B}$, and the neighborhood graph. The engine is written in Perl, but can be used through a Java library. The worst-case complexity of this implementation is of the order of $O\left(|\mathfrak{B}|^{\frac{|\mathcal{V}| \cdot (|\mathcal{V}| - 1)}{2}}\right)$.

REVISOR/PCQA implements $\dot{+}^d$ on the propositional closures of the QAs $\mathcal{L}_{\text{Allen}}$, INDU and RCC8: it actually uses REVISOR/QA and is one of the engines of REVISOR. The worst-case complexity of this implementation is of the order of $O\left(|\mathcal{V}|^4 |\mathfrak{B}|^{\frac{|\mathcal{V}| \cdot (|\mathcal{V}| - 1)}{2}}\right)$, according to a coarse analysis.

Examples

The following examples have been executed using REVISOR/PCQA, and are included with the source code. The README file associated with REVISOR/QA on the REVISOR website explains how they can be executed.

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[3] http://revisor.loria.fr

$\mathtt{Revision}_{\widehat{\mathcal{L}}_{\mathrm{QA}}}(\psi, \mu)$

input $\psi, \mu \in \widehat{\mathcal{L}}_{\mathrm{QA}}$

output $\varrho \in \widehat{\mathcal{L}}_{\mathrm{QA}}$ such that $\varrho \equiv \psi \dotplus^d \mu$

1 $\psi \leftarrow \mathrm{DNF\text{-}w/oN}(\psi)$ $\psi = \bigvee_i \psi_i$ where $\psi_i \in \mathcal{L}_{\mathrm{QA}}$

2 $\mu \leftarrow \mathrm{DNF\text{-}w/oN}(\mu)$ $\mu = \bigvee_j \mu_j$ where $\mu_j \in \mathcal{L}_{\mathrm{QA}}$

3 $\mathtt{result} \leftarrow \emptyset$

4 $d_{\max} \leftarrow +\infty$

5 **for each** i **and each** j **do**

6 $\quad$ $\mathtt{rev} \leftarrow \mathtt{Revision}_{\mathcal{L}_{\mathrm{QA}}}(\psi_i, \mu_j, d_{\max})$

7 $\quad$ **if** $\mathtt{rev} \neq \mathtt{failure}$ **then**

8 $\quad\quad$ $(\varrho_{ij}, d^*_{ij}) \leftarrow \mathtt{rev}$ // $\varrho_{ij} = \psi_i \dotplus^d \mu_j \in \widehat{\mathcal{L}}_{\mathrm{QA}}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ // $d^*_{ij} = d(\mathcal{M}(\psi_i), \mathcal{M}(\mu_j))$

9 $\quad\quad$ **if** $d^*_{ij} < d_{\max}$ **then**

10 $\quad\quad\quad$ $d_{\max} \leftarrow d^*_{ij}$

11 $\quad\quad\quad$ $\mathtt{result} \leftarrow \{\varrho_{ij}\}$

12 $\quad\quad$ **else if** $d^*_{ij} = d_{\max}$ **then**

13 $\quad\quad\quad$ $\mathtt{result} \leftarrow \mathtt{result} \cup \{\varrho_{ij}\}$

14 $\quad\quad$ **end if**

15 $\quad$ **end if**

16 **end for**

17 $\varrho \leftarrow \bigvee_{\sigma \in \mathtt{result}} \sigma$

18 **return** ϱ

Figure 3: Algorithm for $\dotplus^d$ in $(\widehat{\mathcal{L}}_{\mathrm{QA}}, \widehat{\models})$.

The first example aims at showing that some revision problems are more easily expressed in $\widehat{\mathcal{L}}_{\text{QA}}$ than in $\mathcal{L}_{\text{QA}}$. Let us consider Zoé, a school principal that has to schedule a morning with 4 courses in biology, English, history and maths for a group of students. For this purpose, she plans to reuse the previous year schedule:

$$\pi = \texttt{English eq 8-9} \wedge \texttt{biology eq 9-10}$$
$$\wedge \texttt{history eq 10-11} \wedge \texttt{maths eq 11-12}$$

stating, e.g., that the English course takes place from 8 to 9 a.m.

She also has some background knowledge that she expresses first in $\mathcal{L}_{\text{Allen}}$. She knows the relation between the 4 time periods:

$$\beta_1 = \texttt{8-9 m 9-10} \wedge \texttt{9-10 m 10-11} \wedge \texttt{10-11 m 11-12}$$

Then, she states that every course c_1 has no intersection (except, possibly, on one of the boundaries) with another course c_2:

$$\beta_2 = \bigwedge_{c_1,c_2 \in \texttt{Courses}, c_1 \neq c_2} c_1 \; \mathsf{b} \mid \mathsf{bi} \mid \mathsf{m} \mid \mathsf{mi} \; c_2$$
$$\text{with } \texttt{Courses} = \{\texttt{biology}, \texttt{English}, \texttt{history}, \texttt{maths}\}$$

Then, she aims at representing the fact that each course corresponds to one of the 4 time periods. Since there is no disjunction in $\mathcal{L}_{\text{Allen}}$, she uses the following trick: asserting that each course is either equal or has no intersection (except on the boundaries) with any period:

$$\beta_3 = \bigwedge_{c \in \texttt{Courses}, p \in \texttt{Periods}} c \; \mathsf{eq} \mid \mathsf{b} \mid \mathsf{bi} \mid \mathsf{m} \mid \mathsf{mi} \; p$$
$$\text{with } \texttt{Periods} = \{\texttt{8-9}, \texttt{9-10}, \texttt{10-11}, \texttt{11-12}\}$$

In order to prevent the courses and the periods to exceed the boundaries of the morning, the variable $\texttt{8-12}$ is introduced and the following knowledge about it is asserted:

$$\beta_4 = \texttt{8-9 s 8-12} \wedge \texttt{9-10 d 8-12} \wedge \texttt{10-11 d 8-12}$$
$$\wedge \texttt{11-12 f 8-12} \wedge \bigwedge_{c \in \texttt{Courses}} c \; \mathsf{s} \mid \mathsf{d} \mid \mathsf{f} \; \texttt{8-12}$$

Let $\beta = \beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4$. Then, the knowledge about the previous year is $\psi = \beta \wedge \pi$. For the current year, a new constraint is that the biology and history teachers should not meet (for some reason):

$$\gamma = \texttt{biology} \; \mathsf{b} \mid \mathsf{bi} \; \texttt{history}$$

Since the background knowledge has not changed, the knowledge about this year is $\mu = \beta \wedge \gamma$. Thus, to propose a new schedule, Zoé will revise ψ by μ. If she uses the $\dot{+}^d$ revision operator defined above, there are two models that consist in switching English with biology or history with maths.

Now, Zoé wants to formalize its knowledge in $\widehat{\mathcal{L}}_{\text{Allen}}$. The previous year schedule π and the new constraint γ for the current years are kept. What changes is the representation of background knowledge: $\widehat{\beta} = \beta_1 \wedge \widehat{\beta_2} \wedge \widehat{\beta_3}$, with $\widehat{\beta_2}$ expressing the fact that two courses cannot occur in the same period of time

$$\widehat{\beta_2} = \bigwedge_{c_1,c_2 \in \texttt{Courses}, c_1 \neq c_2} \neg(c_1 \; \mathsf{eq} \; c_2)$$

n	p	REVISOR/QA			REVISOR/PCQA		
		#Variables	Avg distance	Avg time (s)	#Variables	Avg distance	Avg time (s)
3	0	7	24.0	1.387	6	22.0	3.809
3	1	8	21.0	5.407	6	20.0	7.744
4	0	9	25.3	444.927	8	24.7	119.136
4	1	10	29.3	765.125	8	29.3	183.945
4	2	11	14.0	2040.551	8	14.0	266.667
5	0	11	—	*> 1 hour*	10	26.0	3052.398
5	1	12	—	*> 1 hour*	10	—	*> 1 hour*

Table 1: Average distance and average time according to the problem, parametrized by n and p, where n is the number of courses and of time periods and p is the number of breaks during the global time period. "Avg distance" is the average of the d^* values on the set of revision problems generated for a given pair (n, p).

and $\widehat{\beta_3}$ expressing the fact that each course is in one of the 4 periods:

$$\widehat{\beta_3} = \bigwedge_{c \in \texttt{Courses}} \bigvee_{p \in \texttt{Periods}} c \ \mathsf{eq} \ p$$

The revision of $\widehat{\psi} = \widehat{\beta} \wedge \pi$ by $\widehat{\mu} = \widehat{\beta} \wedge \gamma$ also gives two models, corresponding to the two same course exchanges. (Formally, they are not the same models, since the sets of variables are different—there is an additional variable in the first formalization: 8-12.)

Our claim is that the second formalization is simpler than the first one, which has required a "trick". Furthermore, REVISOR/QA requires about 6 minutes to solve this problem (in the first formalization) whereas REVISOR/PCQA only requires about 2 minutes.

The second example generalizes the first one. It consists in a family of examples parametrized by n and p, where n is the number of courses and of time periods (the first example corresponds to $n = 4$) and p is the number of breaks during the global time period (the first example corresponds to $p = 0$). Moreover, the breaks in the examples are uniformly spread throughout the whole period. It has been experimented with $n \in \{3, 4, 5\}$ and $p \in \{0, 1, 2\}$. Following similar formalizations in $\mathcal{L}_{\text{Allen}}$ (with $2n + p + 1$ variables) and $\widehat{\mathcal{L}}_{\text{Allen}}$ (with $2n$ variables), the result were the same (except for the additional variables) and the computing times are presented in table 1. The average time, for each line, is computed with series of $n - 1$ tests with $d > 0$, on a computer with a 2.53 GHz processor and 8 GB of available memory. For example, for $n = 4$ and $p = 1$, the average distance is 14.0 for REVISOR/QA and REVISOR/PCQA and the average time is 765.125 s for REVISOR/QA and 183.945 s for REVISOR/PCQA. The average time increases with the number of variables for REVISOR/QA and for REVISOR/PCQA. For the same number of variables, REVISOR/QA is faster than REVISOR/PCQA. However, as fewer additional variables are introduced under REVISOR/PCQA, more complex problems can be solved with REVISOR/PCQA than with REVISOR/QA.

The third example uses a belief contraction operator. As stated by equation (3), a contraction operator can be defined based on the revision operator $\dotplus^d$. Let $\dot{-}^d$ be this

operator. Now let us consider the set of beliefs ψ of an agent called Maurice about the dates of birth and death of famous mathematicians. Maurice thought that Boole was born after de Morgan and died before him and that de Morgan and Weierstraß were born the same year (say, at the same time) but the former died before the latter:

$$\psi = \texttt{Boole d De Morgan} \wedge \texttt{De Morgan s Weierstraß}$$

where, `Boole` is the interval of time between the birth and the death of Boole, and so on. Now, Germaine, a friend of Maurice, tells him that she is not sure whether Boole was born strictly after Weierstraß. Since Maurice trusts Germaine (and her doubts), he wants to make the contraction of its original beliefs ψ by μ with

$$\mu = \texttt{Boole bi} \mid \texttt{mi} \mid \texttt{oi} \mid \texttt{f} \mid \texttt{d Weierstraß}$$

The result, computed by REVISOR/PCQA in less than one second, is $\psi \mathbin{\dot{-}}^d \mu$, equivalent to the following formula:

$$
\begin{aligned}
&(\texttt{Boole d De Morgan} \wedge \texttt{De Morgan s Weierstraß}) \\
\vee\ &(\texttt{Boole s Weierstraß} \wedge \texttt{De Morgan di Weierstraß}) \\
\vee\ &(\texttt{Boole s De Morgan} \wedge \texttt{De Morgan s Weierstraß}) \\
\vee\ &\begin{pmatrix} \texttt{Boole d De Morgan} \wedge \texttt{Boole s Weierstraß} \\ \wedge\ \texttt{De Morgan o Weierstraß} \end{pmatrix}
\end{aligned}
$$

Actually, the last term of this disjunction corresponds to the reality, provided that the intervals of time correspond to a year granularity.[4]

Conclusion

This paper has presented an algorithm for distance-based belief revision in the propositional closure $\widehat{\mathcal{L}}_{\mathrm{QA}}$ of a qualitative algebra $\mathcal{L}_{\mathrm{QA}}$, using the revision operation on $\mathcal{L}_{\mathrm{QA}}$. This work is motivated by the fact that it gives a revision operation whose result is representable in the formalism, by the fact that some practical examples are easily represented in $\widehat{\mathcal{L}}_{\mathrm{QA}}$ whereas they are quite difficult to represent in $\mathcal{L}_{\mathrm{QA}}$, and by the fact that it makes it possible to define a contraction operator thanks to the Harper identity (which requires disjunction and negation). The preprocessing of the algorithm consists in putting the formulas into a disjunctive normal form without negation. Then, proposition 3, which reduces a revision of disjunctions to a disjunction of the least costly revisions, is applied. REVISOR/PCQA is an implementation of this revision operator for the Allen algebra, INDU and RCC8.

A first direction of research following this work is the improvement of the computation time of the REVISOR/PCQA system. One way to do it is to parallelize it, which should not be very difficult (parallelizing the main loop). A sequential optimization would consist in finding a heuristic for ranking the pairs (i, j), with the aim of starting from the best candidates, in order to obtain a low upper bound $d_{\max}$ sooner.

The approach depicted in this paper for an algorithm of $\mathbin{\dot{+}}^d$ in $\widehat{\mathcal{L}}_{\mathrm{QA}}$ built using an algorithm of $\mathbin{\dot{+}}^d$ in $\mathcal{L}_{\mathrm{QA}}$ has actually little dependence on the peculiarities of QAs (except for the fact that negations can be removed in $\widehat{\mathcal{L}}_{\mathrm{QA}}$ according to proposition 2).

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[4]George Boole (1815-1864), Augustus De Morgan (1806-1871), Karl Weierstraß (1815-1897).

Indeed, it could be reused as such for designing an algorithm of a revision on the disjunctive closure of a formalism $\mathcal{L}$, provided that an algorithm of $\dot{+}^d$ has been designed in $\mathcal{L}$. For example, the REVISOR/CLC system has been implemented in the formalism $\mathcal{L}_{\text{CLC}}$ of conjunction of linear constraints (on integers and real numbers), with a city block distance (Cojan and Lieber, 2008). However, reusing this approach for having an algorithm of $\dot{+}^d$ in a *propositional* closure $\widehat{\mathcal{L}}$ raises additional issues. In particular, the minimal distance between sets of models (i.e. n-tuples of numbers) is not necessarily reached, thus violating the postulate ($\dot{+}3$). Working on this issue is a second direction of research.

This paper has described an algorithm for belief revision in $\widehat{\mathcal{L}}_{\text{QA}}$, which can be straightforwardly used for belief contraction. The third direction of research is to study how other belief change operations can be implemented in this formalism, in particular belief merging (Konieczny and Pérez, 2002) and knowledge update (Katsuno and Mendelzon, 1991a).

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[5]`http://kolflow.univ-nantes.fr`

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