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# An Overview of Transience Bounds in Max-Plus Algebra

Thomas Nowak and Bernadette Charron-Bost

ABSTRACT. We survey and discuss upper bounds on the length of the transient phase of max-plus linear systems and sequences of max-plus matrix powers. In particular, we explain how to extend a result by Nachtigall to yield a new approach for proving such bounds and we state an asymptotic tightness result by using an example given by Hartmann and Arguelles.

## 1. Introduction

Max-plus linear algebra is used to describe production and transportation systems, and several distributed algorithms, due to the occurrence of a temporal maximum operation when events are synchronized. These systems are described by the repeated application of a fixed matrix to an initial vector. A body of research spawned examining the behavior and parameters of interest of max-plus linear systems. If the system matrix is irreducible, i.e., if the digraph described by it is strongly connected, one observes a periodic behavior after an initial transient phase whose length we refer to as the *transient*: If  $x(k)$  denotes the system vector after  $k$  applications, then  $x_i(k+p) = x_i(k) + c$  with a constant  $c$  independent of index  $i$  for all  $k$  large enough, i.e., greater or equal to the transient. This was first shown by Cohen et al. [9]. In fact, they showed that the entries in the sequence of max-plus powers  $A^{\otimes k}$  of every irreducible matrix  $A$  are eventually periodic in the same sense, i.e.,  $A_{i,j}^{\otimes(k+p)} = A_{i,j}^{\otimes k} + c$  with some  $c$  independent of the index  $(i, j)$  for all  $k$  large enough. This obviously implies the result for systems. In the same vein, the transient of a system is always upper bounded by the transient of the sequence of powers of its system matrix.

It is the purpose of this paper to survey the existing upper bounds on the transient of max-plus linear systems and matrices. Preceding the first general transience bound by Hartmann and Arguelles [13], a number of upper bounds on the transient of certain max-plus linear systems in computer science were established (e.g., [12, 17, 6, 8]). All these results are, as far as we are aware of them, covered by the bounds we chose to present here.

Hartmann and Arguelles [13] proved, as a corollary of their upper bound on the transient, that the computation of the exact value of the transient of a system or a matrix can be done in time polynomial in the size of a list representation. Their

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algorithm first calculates an upper bound on the transient and then identifies the transient by doing a binary search. Consequently, better upper bounds improve the running time of this algorithm. Also, bounds involving certain parameters of the systems or the matrix allow to design for a small transient.

The paper is structured as follows: Section 2 defines the basic notions used in the paper and presents some preliminary results used later. Section 3 discusses selected upper bounds on the transient for the special case of Boolean matrices. In Section 4, we present the decomposition of the sequence of matrix powers as a maximum of matrices with bounded transients established by Nachtigall [19]. We also explain how his arguments can be extended and completed to show a bound on the transient of the sequence of powers. This extension was presented by the authors at the workshop TROPICAL-12 [7]. Section 5 gives the first general transience bound proved by Hartmann and Arguelles [13]. It also includes the generalization of one of their examples showing a form of asymptotic tightness of their bound, and also of other bounds. In Section 6, we present a bound for a special class of max-plus matrices that was proved with a technique used by Akian et al. [1] and Bouillard and Gaujal [2]. Section 7 gives the bound by Soto y Koelemeijer [24] for the special case of max-plus matrices whose all entries are finite. Section 8 presents the bounds by Charron-Bost et al. [5] that use two graph parameters (girth and cyclicity). In Section 9, we present the method by Charron-Bost et al. to transform transience bounds for systems into transience bounds for matrices. With Section 10, we conclude the paper by comparing the bounds from both a quantitative and a qualitative viewpoint.

## 2. Definitions and Preliminaries

In max-plus algebra, one endows the set  $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$  with the addition  $a \oplus b = \max\{a, b\}$  and the multiplication  $a \otimes b = a + b$ . The identity with respect to  $\oplus$  is  $-\infty$  and 0 with respect to  $\otimes$ .

A max-plus matrix is a matrix with entries in  $\mathbb{R}_{\max}$ . The max-plus product of two matrices  $A$  and  $B$  of compatible size is defined in the usual way by setting  $(A \otimes B)_{i,j} = \bigoplus_h A_{i,h} \otimes B_{h,j}$ . We write  $A^{\otimes k}$  for the  $k$ th max-plus power of a square matrix  $A$ .

A digraph is a pair  $G = (V, E)$  of a nonempty set  $V$  of nodes and a set  $E \subseteq V \times V$  of edges. A walk in the digraph is a *path* if every node occurs only once. A closed walk is a *cycle* if only the start and end node occurs twice. We write  $\ell(W)$  for the length of walk  $W$ .

The length of the shortest cycle in a digraph  $G$  is called the *girth* of  $G$ . If a digraph is strongly connected, the greatest common divisor of its cycle lengths is called its *cyclicity*. The cyclicity of a (possibly not strongly connected) digraph is the least common multiple of the cyclicities of its strongly connected components.

To every  $n \times n$  max-plus matrix  $A$  corresponds a digraph  $G(A)$  with node set  $V = \{1, 2, \dots, n\}$  containing an edge  $(i, j)$  if and only if  $A_{i,j} \neq -\infty$ . We refer to  $A_{i,j}$  as the weight of edge  $(i, j)$ . Matrix  $A$  is *irreducible* if  $G(A)$  is strongly connected. If  $W$  is a walk in  $G(A)$ , we define its weight  $A(W)$  as the sum of the weights of its edges. The entry  $A_{i,j}^{\otimes k}$  is the maximum weight of walks from  $i$  to  $j$  of length  $k$ .<sup>1</sup> If  $v$  is a max-plus column vector of size  $n$ , then the entry  $(A^{\otimes k} \otimes v)_i$

<sup>1</sup>We follow the convention that  $\max \emptyset = -\infty$ .

is the maximum of the values  $A(W) + v_j$  where the maximum is formed over all nodes  $j$  and all walks  $W$  from  $i$  to  $j$  of length  $k$ .

Denote by  $\lambda(A)$  the maximum mean weight  $A(\Gamma)/\ell(\Gamma)$  of cycles in  $G(A)$ . We call *critical* every cycle with maximum mean weight. The subgraph of  $G(A)$  induced by edges on critical cycles is called the *critical subgraph*.

Let  $X$  be a fixed index set. A sequence  $a : \mathbb{N} \rightarrow \mathbb{R}_{\max}^X$  is called *eventually periodic* with period  $p \geq 1$  and ratio  $\varrho \in \mathbb{R}$  if there exists a  $K \in \mathbb{N}$  such that  $a(k+p) = a(k) + p \cdot \varrho$  for all  $k \geq K$ , where the addition is to be understood component-wise. The smallest such  $K$  is the *transient* of the sequence. The ratio is unique if not all components of  $a(k)$  are eventually constant  $-\infty$ . For every eventual periodicity of sequence  $a(k)$  with ratio  $\varrho$ , the sequence  $a(k) - k \cdot \varrho$  is eventually periodic with ratio 0.

The following lemma shows that the transient is independent of the considered period. For a proof see, for instance, [13, Lemma 11].

LEMMA 2.1. *Let  $a(k)$  be a sequence,  $p, q$  positive integers,  $K_p, K_q$  nonnegative integers, and  $\varrho$  a real number. If  $a(k+p) = a(k) + p \cdot \varrho$  for all  $k \geq K_p$  and  $a(k+q) = a(k) + q \cdot \varrho$  for all  $k \geq K_q$ , then  $a(k + \gcd(p, q)) = a(k) + \gcd(p, q) \cdot \varrho$  for all  $k \geq \max\{K_p, K_q\}$ .*

Cohen et al. [9] showed that the sequence of powers of an irreducible max-plus matrix, and hence of all systems with irreducible matrix, are eventually periodic. Denote by  $\gamma_c(A)$  the cyclicity of the critical subgraph of  $G(A)$ .

THEOREM 2.2 (Cohen et al., 1983). *The sequence of powers  $A^{\otimes k}$  of an irreducible square max-plus matrix  $A$  is eventually periodic with ratio  $\lambda(A)$  and period  $\gamma_c(A)$ .*

Theorem 2.2 is based on the fact that maximum weight walk eventually include in the majority critical cycles. To give an explicit upper bound on when they visit at least one critical cycle, several authors defined what they considered to be the “second most significant” cycle mean. This can be done in a number of ways, depending on the specific proof technique used: One possibility, used by the authors’ extension of Nachtigall’s decomposition and by Soto y Koelemeijer, is to consider the second largest cycle mean  $\lambda_2(A)$ . Bouillard and Gaujal and Charron-Bost et al. both considered the largest cycle mean disjoint to all critical cycles, which we denote by  $\lambda_{nc}(A)$ . Hartmann and Arguelles used a third parameter,  $\lambda_0(A)$ , which is defined in terms of the max-balancing [20] of  $G(A)$ . We do not formally define the three parameters, but give their relative ordering, also with respect to  $\lambda(A)$ :

$$(2.1) \quad \lambda(A) > \lambda_2(A) \geq \lambda_{nc}(A) \geq \lambda_0(A)$$

We denote by  $\|A\|$  the difference between the greatest and smallest finite entry in matrix  $A$ .

### 3. The Boolean Case: Index of Convergence

A Boolean matrix is a max-plus matrix whose entries are either  $-\infty$  or 0 and it corresponds to a digraph. The behavior of the sequence of powers of Boolean matrices, or equivalently the set of possible walk lengths between nodes in a digraph, has been extensively studied (see, e.g., [4] or [15] for an overview). If the digraph is strongly connected, Theorem 2.2 shows that every such sequence is eventually

periodic; for the critical subgraph is equal to the whole digraph. Its transient is commonly referred to as the *index of convergence* of the matrix resp. the digraph (also sometimes the *exponent* if the cyclicity is equal to 1). Clearly, the case of Boolean matrices is an important special case for the study of transients in max-plus algebra.

The first bound on the index of convergence was given by Wielandt [25] for the case of *primitive* digraphs, i.e., digraphs whose cyclicity is equal to 1. He also gave a class of examples showing his bound is tight.

**THEOREM 3.1** (Wielandt, 1950). *The index of convergence of a strongly connected primitive digraph with  $n$  nodes is at most  $(n - 1)^2 + 1$ . Furthermore, for every  $n \geq 2$  there exists a strongly connected primitive digraph with  $n$  nodes whose index of convergence is equal to  $(n - 1)^2 + 1$ .*

The bound of  $(n - 1)^2 + 1$  was refined independently by Dulmage and Mendelsohn [11] and by Denardo [10] in terms of the digraph's girth  $g$ . They arrived at the same bound, which is in the order of  $O(g \cdot n)$ . This suggests that the lower the girth, the lower the index of convergence.

Later, Schwarz [21] extended Theorem 3.1 to non-primitive digraphs.<sup>2</sup> Interestingly, he showed that the bound of  $(n - 1)^2 + 1$  remains true and that even a lower upper bound holds, which is in the order of  $O(n^2/\gamma)$  where  $\gamma$  denotes the cyclicity. This suggests that the higher the cyclicity, the lower the index of convergence.

Because the girth of a strongly connected digraph is always greater or equal to the cyclicity, the two results suggest a necessary trade-off between the two parameters for attaining a small index of convergence. For instance, the two parameters need to be equal for attaining the minimal index of convergence of 0.

Kim [14] showed a new upper bound, which generalizes both the bounds of Dulmage, Mendelsohn, and Denardo, and Schwarz:

**THEOREM 3.2** (Kim, 1979). *The index of convergence of a strongly connected digraph with  $n$  nodes, girth  $g$ , and cyclicity  $\gamma$  is at most*

$$n + g \cdot \left( \left\lfloor \frac{n}{\gamma} \right\rfloor - 2 \right) .$$

#### 4. Nachtigall Decomposition

A significant step in the direction of a transience bound for non-Boolean matrices was done by Nachtigall [19]. While he did not prove a bound on the transient, he showed that the sequence of matrix powers can be written as a maximum of eventually periodic sequences with bounded transients. Such a decomposition in the form of a maximum, by itself, does not yield a bound on the transient of the original sequence; it does not even imply that it is eventually periodic. As a matter of fact, Nachtigall shows the existence of such a decomposition not only for irreducible matrices, but for general square max-plus matrices, for which the sequence of powers is not necessarily eventually periodic.

The authors [7] have observed that the specific structure of the Nachtigall decomposition in the case of irreducible matrices allows to deduce a transience bound. We discuss this after the description of the decomposition. Nachtigall's decomposition was also studied by Molnárová [18] and Sergeev and Schneider [22].

<sup>2</sup>Shao and Li [23] gave an alternative proof.

**THEOREM 4.1** (Nachtigall, 1997). *Let  $A$  be an  $n \times n$  max-plus matrix. Then there exist eventually periodic matrix sequences  $A_1(k), A_2(k), \dots, A_n(k)$  with transients at most  $3n^2$  such that for all  $k \geq 0$ :*

$$A^{\otimes k} = A_1(k) \oplus A_2(k) \oplus \dots \oplus A_n(k)$$

Nachtigall proved Theorem 4.1 by recursively picking a cycle  $\Gamma$  with maximal ratio  $A(\Gamma)/\ell(\Gamma)$  and by partitioning the sets of walks in  $G(A)$  into the sets of walks that do and do not visit cycle  $\Gamma$ . Walks that do not visit  $\Gamma$  are walks in the subgraph of  $G(A)$  that has all edges incident to  $\Gamma$  removed. This subgraph is the digraph of the matrix obtained from  $A$  by setting to  $-\infty$  all rows and columns corresponding to nodes in  $\Gamma$ ; its effective size is strictly smaller than the size of  $A$ , which enables a recursive descent. If no cycle exists in  $G(A)$  at all, then the transient of  $A$  is at most  $n$  since in this case,  $A_{i,j}^{\otimes k} = -\infty$  for all  $i, j$  and all  $k \geq n$ .

One can see that the sequence of the maximum weights of walks of length  $k$  from a node  $i$  to a node  $j$  that do visit cycle  $\Gamma$  has a transient of at most  $3n^2$  in the following way: Take a node  $h$  of  $\Gamma$  and set  $B = A^{\otimes \ell(\Gamma)}$ . As  $G(B)$  contains a self-loop with maximal ratio at node  $h$ , the sequences  $B_{i,h}^{\otimes k}$  and  $B_{h,j}^{\otimes k}$  have transients at most  $n - 1$ , which implies that the sequences  $A_{i,h}^{\otimes k}$  and  $A_{h,j}^{\otimes k}$  have transients at most  $(n - 1) \cdot \ell(\Gamma)$ . Both have period  $\ell(\Gamma)$  and ratio  $A(\Gamma)/\ell(\Gamma)$ . It is not hard to show that, in this case, their *max-plus convolution*

$$\bigoplus_{k_1+k_2=k} A_{i,h}^{\otimes k_1} \otimes A_{h,j}^{\otimes k_2}$$

has the same period and ratio, and a transient of at most  $2 \cdot (n - 1) \cdot \ell(\Gamma) + \ell(\Gamma) \leq 2n^2 - n$ . This convolution is equal to the sequence of maximum weights of walks of length  $k$  from  $i$  to  $j$  that visit node  $h$ . Because the ratios of these sequences, for all  $h$  in  $\Gamma$ , are equal to  $A(\Gamma)/\ell(\Gamma)$  and their transients are at most  $2n^2 - n$ , the sequence of maxima, formed over all  $h$  in  $\Gamma$ , has the same ratio and a transient of at most  $2n^2 - n$ . This argument, which is essentially identical to the one given by Nachtigall, yields a bound of  $2n^2 - n$ , improving the bound of  $3n^2$  in Theorem 4.1.

We would like to point out that, if the ratios differ, the transient of a maximum of eventually periodic sequences need not be bounded by the maximum of the sequences' transients (see Figure 1). It is possible that the maximum is not even eventually periodic: If  $a(k)$  and  $b(k)$  are two eventually periodic scalar sequences such that  $a(k)$ 's ratio is strictly larger than that of  $b(k)$ , then the maximum  $c(k) = a(k) \oplus b(k)$  is eventually periodic if and only if, for all  $k$  large enough,  $a(k) = -\infty$  implies  $b(k) = -\infty$ . This condition is not necessary for eventual periodicity if the two ratios are equal.

Thus, Theorem 4.1 by itself, even if  $A$  is irreducible, i.e.,  $A^{\otimes k}$  is eventually periodic, does not give an upper bound on the transient of  $A^{\otimes k}$ . However, inspection of its proof does: By construction, if the  $(i, j)$ th entry of matrix  $A_r(k)$  is finite, then there exists a walk of length  $k$  in  $G(A)$  from  $i$  to  $j$ . Because  $G(A)$  is strongly connected, by the finiteness of its index of convergence (see Section 3), if  $k$  is large enough, there exists a walk of length  $k$  in  $G(A)$  from  $i$  to  $j$  that visits an arbitrary prescribed node  $h$ . This shows that, if  $k$  is large enough, then also the  $(i, j)$ th entry

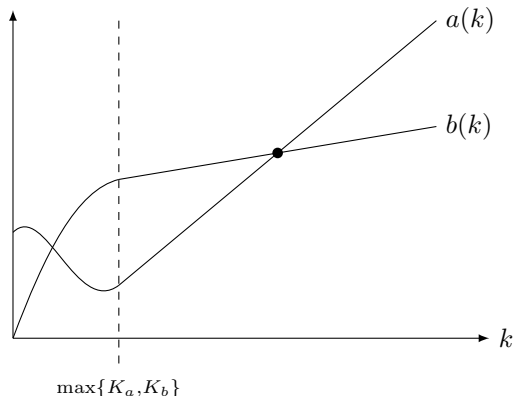


FIGURE 1. Two eventually periodic sequences with differing ratios and respective transients  $K_a$  and  $K_b$

of  $A_1(k)$  is finite.<sup>3</sup> Hence all sequences of the form  $A_1(k) \oplus A_r(k)$  are eventually periodic. Their ratios are all equal to  $\lambda(A)$  since the first cycle to be removed is a critical one. Hence the transient of the sequence  $A^{\otimes k} = A_1(k) \oplus A_2(k) \oplus \dots \oplus A_n(k)$  is bounded by the maximum transient of the  $A_1(k) \oplus A_r(k)$ 's.

By elementary calculations, one can show the following lemma which provides a bound on the transient of the maximum of two eventually periodic sequences if the maximum itself is eventually periodic. It enables us to bound the transient of  $A_1(k) \oplus A_r(k)$ .

**LEMMA 4.2.** *Let  $a(k)$  and  $b(k)$  be two eventually periodic scalar sequences with respective ratios  $\varrho_a = 0 > \varrho_b$ , transients at most  $K$ , and periods at most  $p$  such that  $a(k) = -\infty$  implies  $b(k) = -\infty$  for all  $k$  large enough. Then the maximum  $a(k) \oplus b(k)$  is eventually periodic with transient at most  $K + p - 1 + \Delta/(\varrho_a - \varrho_b)$  where  $\Delta$  is the maximum value of the form  $b(k) - a(l)$  with  $k, l \in \{K, \dots, K + p - 1\}$  and  $a(l) \neq -\infty$ .*

This lemma enables us to deduce the following transience bound from the proof of the Nachtigall decomposition. Without loss of generality, we can assume that  $\lambda(A) = 0$ . When applying the lemma to  $A_1(k)$  and  $A_r(k)$ , the number  $K$  is at most  $2n^2 - n$  and  $p$  is at most  $n$ , which implies that  $\Delta$  is at most  $2n^2\|A\|$ .

**THEOREM 4.3.** *Let  $A$  be an irreducible  $n \times n$  max-plus matrix. Then the transient of the sequence of powers  $A^{\otimes k}$  is at most*

$$2n^2 + \frac{2n^2\|A\|}{\lambda(A) - \lambda_2(A)}.$$

## 5. Bound by Hartmann and Arguelles

Hartmann and Arguelles [13] gave the first general transience bound for arbitrary irreducible max-plus matrices. Their proof is purely graph-theoretic.

<sup>3</sup>We do not need to quantify the threshold for  $k$  because eventual periodicity of  $A_1(k)$  and  $A_r(k)$  shows that the implication is true from  $2n^2 - n$  on, i.e., for all  $k \geq 2n^2 - n$ , as soon as we know that it is true from some (unknown) threshold on. This suffices for our purposes.

When analyzing their proof, one can extract a global proof strategy, variants of which are also found in later proofs of transience bounds [24, 5]. It has been described explicitly by Charron-Bost et al. [5, Section 3]. In order to prove that some number  $B$  is an upper bound on the transient of the sequence  $A^{\otimes k}$  for an irreducible matrix  $A$ , do the following:

- (1) Show that one can assume  $\lambda(A) = 0$ , i.e., the sequence  $A^{\otimes k}$  is eventually periodic with ratio 0.
- (2) Fix two nodes  $i$  and  $j$ , and a congruence class  $[k_0]$  modulo some period  $p$  of the sequence  $A^{\otimes k}$ .
- (3) The assumption  $\lambda = 0$  guarantees that the maximum  $\max_{k \in M} A_{i,j}^{\otimes k}$  formed over an arbitrary nonempty set  $M$  of nonnegative integers exists. We choose the set  $M$  to consist of those elements of class  $[k_0]$  that are greater or equal to  $B$ . Since the maximum exists, there exists a walk  $W$  from  $i$  to  $j$  with length in  $M$  that attains it. If  $B$  is indeed an upper bound on the transient, the values  $A_{i,j}^{\otimes k}$  with  $k \in M$  will all be equal.
- (4) Show that, whenever the length of  $W$  is greater or equal to some “critical bound”  $B_c \leq B$ , then it necessarily shares a node with a critical cycle  $\Gamma$ .
- (5) Show that one can reduce walk  $W$  by removing subcycles such that it is possible to attain all lengths in  $M$  greater or equal to some “pumping bound”  $B_p \leq B$  by adding critical cycles. The assumption  $\lambda = 0$  implies that all subcycles have weight at most 0 and critical cycles have weight equal to 0. Thus the weights of walks obtained in this way cannot be lower than that of  $W$ ; hence they are equal to that of  $W$ .
- (6) We then have shown, because the choice of  $[k_0]$  was arbitrary, that the transient of  $A_{i,j}^{\otimes k}$  is at most  $B \geq \max\{B_c, B_p\}$ .

Hartmann and Arguelles used  $p = \gamma_c(A)$  in step (2). For step (5), they described a walk reduction based on the following basic application of the pigeonhole principle:

LEMMA 5.1. *Let  $d$  be a positive integer. Every collection of at least  $d$  integers has a nonempty subcollection whose sum is divisible by  $d$ .*

They used this lemma to reduce walk  $W$  in step (5). After their reduction their walk could be disconnected, but they showed that adding a copy of (critical) cycle  $\Gamma$  reestablishes connection [13, Theorem 4]:

LEMMA 5.2 (Hartmann and Arguelles, 1999). *Let  $W$  be a walk that shares a node with some cycle  $\Gamma$  and let  $k$  be an integer such that  $k \equiv \ell(W) \pmod{\ell(\Gamma)}$  and  $k \geq n^2$  where  $n$  denotes the number of nodes in the graph. Then there exists a walk  $\tilde{W}$  obtained from  $W$  by removing cycles and possibly adding copies of  $\Gamma$  such that  $\ell(\tilde{W}) = k$ .*

To pump the walk length after the walk reduction, they used a result by Brauer [3] on the Frobenius problem to combine critical cycles to attain a multiple of  $\gamma_c(A)$ . The use of Brauer’s theorem introduces a term that is necessarily quadratic in  $n$  to the transience bound. We want to note at this point that this use of Brauer’s theorem can be avoided by considering a period in step (2) different from the critical subgraph’s cyclicity because of Lemma 2.1.<sup>4</sup>

<sup>4</sup>Hartmann and Arguelles actually prove Lemma 2.1 later in the paper [13, Lemma 11], but do not use it in the proof of their transience bound.



The same strategy as described above can be adapted to show transience bounds for systems  $A^{\otimes k} \otimes v$ . In the case that all entries of  $v$  are finite, it is possible to show a sharper bound because the walks under consideration do not have both the start and the end node fixed, but only the start node. This allows to circumvent the necessity of showing the existence of walks of prescribed length between two fixed nodes (see Section 3).

**THEOREM 5.3** (Hartmann and Arguelles, 1999). *Let  $A$  be an irreducible  $n \times n$  max-plus matrix. Then the transient of the sequence of powers  $A^{\otimes k}$  is at most*

$$\max \left\{ 2n^2, \frac{2n^2 \|A\|}{\lambda(A) - \lambda_0(A)} \right\}.$$

*If, additionally,  $v$  is a column vector of size  $n$  with only finite entries, then the transient of the system  $A^{\otimes k} \otimes v$  is at most*

$$\max \left\{ 2n^2, \frac{\|v\| + n\|A\|}{\lambda(A) - \lambda_0(A)} \right\}.$$

Hartmann and Arguelles also proved a form of asymptotic tightness of their transience bound for matrices. They gave, for every  $n$  of the form  $n = 3m - 1$  and all positive reals  $\lambda$  and  $\lambda_0$  with  $\lambda > \lambda_0$ , an irreducible  $n \times n$  max-plus matrix  $A$  with  $\lambda(A) = \lambda$  and  $\lambda_0(A) = \lambda_0$  (see [13, Figure 1]). Their example has the property that  $\lambda_0(A) = \lambda_{nc}(A) = \lambda_2(A)$  and  $\|A\| = \lambda$ . They showed by explicit calculation that  $A$ 's transient is at least  $3 + m(m - 2)\lambda/(\lambda - \lambda_0)$ .

We can generalize their example to arbitrary  $n$  by inserting additional nodes that do not change the transient. This then shows that, even if one can prescribe all the other parameters in the matrix bound of Theorem 5.3, it is asymptotically tight when  $n$  tends to infinity:

**THEOREM 5.4.** *Let  $D_n$  and  $M_n$  be two sequences of positive real numbers such that  $D_n \leq M_n$ . Then there exists a sequence of irreducible  $n \times n$  max-plus matrices  $A_n$  such that  $\lambda(A_n) - \lambda_2(A_n) = D_n$ ,  $\|A_n\| = M_n$ , and the transient of the sequence of matrix powers  $A_n^{\otimes k}$  is*

$$\Omega \left( \frac{n^2 \|A_n\|}{\lambda(A_n) - \lambda_2(A_n)} \right).$$

Because  $\lambda_2 = \lambda_{nc} = \lambda_0$  in Hartmann and Arguelles' example, Theorem 5.4 also holds with either  $\lambda_{nc}$  or  $\lambda_0$  replacing  $\lambda_2$ .

## 6. A Bound for Primitive Matrices

A certain class of graph-theoretic arguments has been developed for the case that the matrix is *primitive*, i.e., if its critical subgraph has a cyclicity equal to 1.<sup>5</sup> This class of arguments was used by both Akian et al. [1, Remark 7.14] and Bouillard and Gaujal [2]. To explicitly state a bound emerging from these arguments, we present the bound of Bouillard and Gaujal in this section. For ease of notation, we give it for the case  $\lambda(A) = 0$ .

<sup>5</sup>This definition is consistent with the definition of primitivity for Boolean matrices (Section 3) because all cycles are critical in the Boolean case.

**THEOREM 6.1** (Bouillard and Gaujal, 2001). *Let  $A$  be a primitive irreducible  $n \times n$  max-plus matrix with  $\lambda(A) = 0$ . Then the transient of the sequence of powers  $A^{\otimes k}$  is at most*

$$\max \left\{ 2n - 2 + H + (n_c - 2H) \cdot \hat{g} , \frac{\max_{i,j} |W_{i,j}^{(nc)} - W_{i,j}^{(c)}|}{-\lambda_{nc}(A)} + (n - n_c) \right\}$$

where  $n_c$  is the number of critical nodes,  $H$  is the number of critical components,  $\hat{g}$  is the maximum girth of components of the critical subgraph,  $W_{i,j}^{(nc)}$  is the maximum weight of walks from  $i$  to  $j$  not visiting a critical node, and  $W_{i,j}^{(c)}$  is the maximum weight of walks from  $i$  to  $j$  that do visit a critical node.

Bouillard and Gaujal explained how to extend their result to the case of non-primitive matrices: If  $A$ 's critical subgraph has cyclicity  $\gamma_c$ , then  $A^{\otimes \gamma_c}$  is primitive. It is not necessarily irreducible, but it is guaranteed to be *completely reducible*, i.e., permutation similar to a blockwise diagonal matrix whose diagonal blocks are irreducible. Also, every irreducible block contains at least one critical cycle, i.e., their eigenvalues are equal, which implies that the sequence of powers is eventually periodic. If  $K$  is the transient of the sequence  $A^{\otimes k \gamma_c}$ , then the transient of  $A^{\otimes k}$  is at most  $K \gamma_c$ .

Unfortunately, the cyclicity  $\gamma_c$  can be exponential in the size  $n$  of the matrix. This was shown by Malka et al. [16, Theorem 4] who constructed matrices whose critical subgraphs are disjoint unions of cycles of prime lengths. Using the Prime Number Theorem, one sees that it is possible to construct a critical subgraph with cyclicity  $\gamma_c = e^{\Omega(\sqrt{n})}$ . Malka et al. improved this observation by showing that even the *minimal* period can be in the same order:

**THEOREM 6.2** (Malka et al., 1993). *There exists a sequence of irreducible  $n \times n$  max-plus matrices  $A_n$  such that the minimal period of the sequence of matrix powers  $A_n^{\otimes k}$  is  $\exp(\Omega(\sqrt{n}))$ .*

## 7. When All Entries Are Finite

Soto y Koelemeijer [24, Theorem 3.5.12] established a transience bound in the case that all matrix entries are finite, i.e., the corresponding digraph is the complete graph. His approach is similar to that of Hartmann and Arguelles, but the assumption of existence of all edges in the corresponding digraph allows to construct shorter walks. Utilizing this fact, he arrived at a bound that can be lower than that of Hartmann and Arguelles (first part of Theorem 5.3).

**THEOREM 7.1** (Soto y Koelemeijer, 2003). *Let  $A$  be an  $n \times n$  max-plus matrix with only finite entries. Then the transient of the sequence of powers  $A^{\otimes k}$  is at most*

$$\max \left\{ 2n^2 , \left\lceil \frac{2\|A\|}{\lambda(A) - \lambda_2(A)} \right\rceil + n - 1 \right\} .$$

## 8. Inclusion of Cyclicity and Girth

Charron-Bost et al. [5] gave two transience bounds for systems; one that involves the maximum girth of connected components of the critical subgraph, and one that involves the maximum cyclicity.

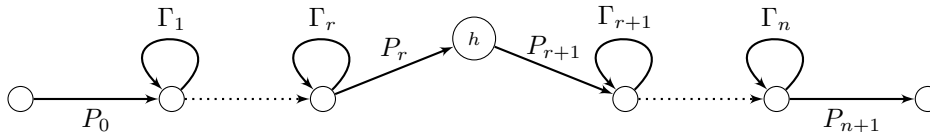


FIGURE 2. Structure of the reduced walk  $\hat{W} = \text{Red}_{d,h}(W)$

For both, they used the general proof strategy that we detailed in Section 5. With respect to this strategy, they introduced two different methods of deleting and adding cycles in step (5). Their use is enabled by considering a period  $p$  equal to the least common multiple of all critical cycle lengths in step (2).

The two methods are called the *repetitive* and the *explorative* method. Denote by  $h$  the a critical node of walk  $W$  in step (5). Depending on the strongly connected component of the critical subgraph that  $h$  lies in, they choose a positive integer  $d$  as a parameter for the walk reduction  $\text{Red}_{d,h}$ , whose definition we present later. It has the following properties:

LEMMA 8.1 (Charron-Bost et al., 2012). *Let  $W$  be a walk containing node  $h$  and let  $d$  be a positive integer. Then there exists a walk  $\hat{W} = \text{Red}_{d,h}(W)$  obtained from  $W$  by removing subcycles such that (i)  $\hat{W}$  still contains node  $h$ , (ii)  $\ell(\hat{W}) = \ell(W) \pmod{d}$ , and (iii)  $\ell(\hat{W}) \leq 2 \cdot d \cdot (n - 1) + d - 1$ , where  $n$  denotes the number of nodes in the digraph.*

In the repetitive method, they choose some critical cycle  $\Gamma$  which node  $h$  is part of and choose  $d = \ell(C)$ . For pumping the reduced walk, they add copies of  $\Gamma$ . Since  $d$  divides  $p$ , all lengths in the congruence class  $[\ell(W)]$  modulo  $p$  can be reached that are greater or equal to  $\ell(\hat{W})$ .

In the explorative method, they choose  $d$  equal to the cyclicity of  $h$ 's strongly connected component in the critical subgraph. For pumping in the explorative method, they add a closed walks in the component starting at  $h$  using the notion of index of convergence (see Section 3). Again,  $d$  divides  $p$ . Hence all lengths in the congruence class  $[\ell(W)]$  modulo  $p$  can be reached that are greater or equal to  $\ell(\hat{W}) + \text{ind}$ , where  $\text{ind}$  denotes the component's index of convergence.

The walk reduction  $\text{Red}_{d,h}(W)$  is defined as follows: Starting at  $W$ , one repeatedly removes nonempty collections of subcycles such that (a) their combined length is a multiple of  $d$  and (b) after their removal,  $h$  is still a node of the resulting walk. If there are more than one such collections, choose any. Eventually, all collections of subcycles that satisfy (a) and (b) will be empty. At this point, the walk reduction stops and returns the last walk. The resulting walk then has a structure as depicted in Figure 2: It is a sequence of paths  $P_t$  connecting the start node to the end node via some cycles  $\Gamma_t$  and the node  $h$ . By using Lemma 5.1, one sees that there can be at most  $d - 1$  cycles  $\Gamma_t$  because otherwise they could be removed as they would satisfy (a) and (b). It follows that there are at most  $d + 1$  paths  $P_t$ . Since the paths have length at most  $n - 1$  and the cycles at most  $n$ , the upper bound of Lemma 8.1 on the reduced walk length follows.

THEOREM 8.2 (Charron-Bost et al., 2012). *Let  $A$  be an irreducible  $n \times n$  matrix and let  $v$  be a column vector of size  $n$  with only finite entries. Then the*

transient of the system  $A^{\otimes k} \otimes v$  is less or equal to both

$$\max \left\{ 2\hat{g} \cdot (n-1) + \hat{g} - 1, \frac{\|v\| + (n-1)\|A\|}{\lambda(A) - \lambda_{nc}(A)} \right\}$$

and

$$\max \left\{ \hat{\text{ind}} + 2\hat{\gamma} \cdot (n-1) + \hat{\gamma} - 1, \frac{\|v\| + (n-1)\|A\|}{\lambda(A) - \lambda_{nc}(A)} \right\},$$

where  $\hat{g}$ ,  $\hat{\gamma}$ , and  $\hat{\text{ind}}$  denote the greatest girth, cyclicity, and index of critical components of  $G(A)$ , respectively.

## 9. From Systems to Matrices

Charron-Bost et al. [5] also showed how to transform bounds for system transients into bounds for matrix transients. They used the following idea: The transient of the sequence of powers of a matrix  $A$  is equal to the maximum transient of systems  $A^{\otimes k} \otimes v$  where  $v$  is one of the max-plus unit vectors. However all transience bounds for systems assume the vector  $v$  to have only finite entries, which is not the case for the max-plus unit vectors. So they considered “truncated” unit vectors that have their infinite entries replaced by  $-\mu$  where  $\mu$  is an appropriately chosen real number. They used this approach, together with graph-theoretical arguments, to show the following theorem.

**THEOREM 9.1** (Charron-Bost et al., 2012). *Let  $A$  be an irreducible  $n \times n$  max-plus matrix and let  $B$  be a nonnegative integer. Set*

$$\tilde{B} = 2n - 3 + \hat{\text{ind}} + \text{ind}(G(A)) + \hat{\gamma}$$

where  $\text{ind}(G(A))$  denotes the index of convergence of  $G(A)$ , and  $\hat{\text{ind}}$  and  $\hat{\gamma}$  denote the greatest index and cyclicity of strongly connected components of the critical subgraph.

*If  $B$  is an upper bound on all transients of systems  $A^{\otimes k} \otimes v$  with  $\|v\| \leq \tilde{B} \cdot \|A\|$ , then the transient of the sequence of matrix powers  $A^{\otimes k}$  is at most  $\max\{B, \tilde{B}\}$ .*

## 10. Conclusion

We have presented various transience bounds and some of their proofs for both max-plus systems and matrices. Most of the proofs were heavily graph-theoretic, with the exception of Theorem 4.3, which is more algebraic and founded on the concept of convolution of sequences. Except for the bound of Theorem 4.3, which can be seen to be strictly greater than the others, there is no general ordering between pairs of bounds. This is due to the fact that all of them consider either a different set of parameters or a special case. A “good” choice of parameters is not an obvious to make. At one extreme, one could declare the transient itself as a parameter, which would lead to a trivial bound. At the other extreme, restricting oneself to only consider the matrix size  $n$  as a parameter is not tractable either. In fact, the tightness result of Theorem 5.4 shows that no upper bound only in terms of  $n$  exists. We think that the choice of parameters can only depend on the envisioned application of the respective transience bound. Because the problem of computing the exact transient is computationally feasible, parameters should be ones that can be controlled during the system design phase.

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LABORATOIRE D'INFORMATIQUE, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU, FRANCE  
*E-mail address:* `nowak@lix.polytechnique.fr`

CNRS, LABORATOIRE D'INFORMATIQUE, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU, FRANCE  
*E-mail address:* `charron@lix.polytechnique.fr`