Global solvability of a networked integrate-and-fire model of McKean-Vlasov type

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GLOBAL SOLVABILITY OF A NETWORKED INTEGRATE-AND-FIRE MODEL OF MCKEAN-VLASOV TYPE

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Abstract. We here investigate the well-posedness of a networked integrate-and-fire model describing an infinite population of neurons which interact with one another through their common statistical distribution. The interaction is of the self-excitatory type as, at any time, the potential of a neuron increases when some of the others fire: precisely, the kick it receives is proportional to the instantaneous proportion of firing neurons at the same time. From a mathematical point of view, the coefficient of proportionality, denoted by α, is of great importance as the resulting system is known to blow-up for large values of α. In the current paper, we focus on the complementary regime and prove that existence and uniqueness hold for all time when α is small enough.

Keywords: McKean nonlinear diffusion process; renewal process; first hitting time density estimates; integrate-and-fire network; nonhomogeneous diffusion process; neuroscience.

1. Introduction

The stochastic integrate-and-fire model for the membrane potential V across a neuron in the brain has received a huge amount of attention since its introduction (see [18] for a comprehensive review). The central idea is to model V by threshold dynamics, in which the potential is described by a simple linear (stochastic) differential equation up until it reaches a fixed threshold value V_F, at which point the neuron emits a 'spike'. Experimentally, at this point an action potential is observed, whereby the potential increases very rapidly to a peak (hyperpolarization phase) before decreasing quickly to a reset value (depolarization phase). It then relatively slowly increases once more to the resting potential (refractory period).

Since spikes are stereotyped events, they are fully characterized by the times at which they occur. The integrate-and-fire model is part of a family of spiking neuron models which take advantage of this by modeling only the spiking times and disregarding the nature of the spike itself. Specifically, in the integrate-and-fire model we observe jumps in the action potential as the voltage is immediately reset to a value V_R whenever it reaches the threshold V_F, which is motivated by the fact that the time period during which the action potential is observed is very small. Despite its simplicity, versions of the integrate-and-fire model have been able to predict the spiking times of a neuron with a reasonable degree of accuracy [9, 10].

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Many extensions of the basic integrate-and-fire model have been studied in the neuroscientific literature, including ones in which attempts are made to include noise and to describe the situation when many integrate-and-fire neurons are placed in a network and interact with each other. In [13, 15] the following equation describing how the potential V_i of the i-th neuron in a network of N behaves in time is proposed:

$$\frac{d}{dt}V_i(t) = -\lambda V_i(t) + \frac{\alpha}{N}\sum_j \sum_k \delta_0(t-\tau_k^j) + \frac{\beta}{N}\sum_{j\neq i} V_j(t) + I_i^{ext}(t) + \sigma\eta_i(t) \tag{1.1}$$

for $V_i(t) < V_F$ and where $V_i(t)$ is immediately reset to V_R when it reaches V_F. Here $I_i^{ext}(t)$ represents the external input current to the neuron, $\eta_i(t)$ is the noise (a white noise) which is importantly supposed to be independent from neuron to neuron, and the constants λ, β, α and σ are chosen according to experimental data. Moreover, the interaction term is described in terms of τ_k^j, which is the time of the k-th spike of neuron j, and the Dirac function δ_0. Precisely, it says that whenever one of the other neurons in the network spikes, the potential across neuron i receives a 'kick' of size α/N. The Dirac mass interactions give rise to the same kind of instantaneous behavior as the integrate-and-fire model. Although it is a simplification of reality, it produces some interesting phenomena from a biological perspective, see [15].

In the case of a large network, i.e. when N is large, many authors approximate the interaction term by an instantaneous rate $\nu(t)$, the so-called mean-firing rate (see for example [1, 2, 15, 17]). However, in the neuroscience literature little attention is paid to how this convergence is achieved. Mathematically the mean-field limit as $N \to \infty$ must be taken, but, as a first step, this requires a careful analysis of the asymptotic well-posedness. This is precisely the purpose of the paper: to focus on the unique solvability of the resulting nonlinear limit equation (the analysis of the convergence being left to further investigations). At first glance such a question may seem classical, given the volume of results available that guarantee the existence of a solution to distribution dependent SDEs. However, as quickly became apparent in our analysis, in the excitatory case ($\alpha > 0$) the problem is in fact a delicate one, for which, to our knowledge, there are no existing results available. This difficulty is due to the nature of the interactions, which introduce the strong possibility of a solution that 'blows up' in finite time. The validity of the study of this question, and its non-trivial nature, is further justified by the fact that several authors have recently been interested in exactly the same problem from a PDE perspective ([3, 4]). Despite some serious effort and very interesting related results on their part, we understand that they were not able to prove the existence and uniqueness of global solutions to the limit equation, which is the main result of the present paper.

1.1. **Precisions:** We now make precise the nonlinear equation of interest. Firstly, since the mathematical difficulties lie within the jump interaction term, we suppose that there is no external input current ($I_i^{ext}(t) \equiv 0$), and that the interaction term is composed solely of the jump or reset part ($\beta = 0$). Although this is a non-trivial simplification from a neuroscience perspective, it still captures all the mathematical complexity of the resulting mean-field equation.

Without loss of generality, we also take the firing threshold $V_F = 1$ and the reset value $V_R = 0$ for notational simplicity. The nonlinear stochastic mean-field equation under study here is then

$$X_t = X_0 + \int_0^t b(X_s)ds + \alpha\mathbb{E}(M_t) + \sigma W_t - M_t, \qquad t \geqslant 0, \tag{1.2}$$

where $X_0 < 1$ almost surely, $\alpha \in \mathbb{R}$, $\sigma > 0$, $(W_t)_{t\,\geqslant\,0}$ is a standard Brownian motion in $\mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. In comparison with (1.1), b must be thought of as $b(x) = -\lambda x$. Equation (1.2) is then intended to describe the potential of one *typical* neuron in the infinite network, its jumps (or resets) being given by

$$M_t = \sum_{k\,\geqslant\,1} \mathbf{1}_{[0,t]}(\tau_k)$$

where $(\tau_k)_{k\,\geqslant\,1}$ stands for the sequence of hitting times of 1 by the process $(X_t)_{t\,\geqslant\,0}$. That is $(M_t)_{t\,\geqslant\,0}$ counts the number of times X_t hits the threshold before time t, so that $\mathbb{E}(M_t)$ denotes the *theoretical* expected number of times the threshold is reached before t. Such a theoretical expectation corresponds to what we would envisage as the limit of the integral form of the interaction term

$$\frac{1}{N}\int_0^t \sum_j \sum_k \delta(s - \tau_k^j)ds = \frac{1}{N}\sum_j \sum_k \mathbf{1}_{\{\tau_k^j\,\leqslant\,t\}}$$

in (1.1) when $N \to \infty$, assuming that neurons become asymptotically independent (as is observed in more classical particle systems – see [20]).

1.2. **PDE viewpoint and 'blow-up' phenomenon:** As mentioned above, equation (1.2) has been rigorously studied from the PDE viewpoint before. When $\sigma \equiv 1$, the Fokker-Planck equation for the density $p(t, y)dy = \mathbb{P}(X_t \in dy)$ is given by

$$\partial_t p(t, y) + \partial_y \left[(b(y) + \alpha e'(t))\, p(t, y)\right] - \frac{1}{2}\partial^2_{yy} p(t, y) = \delta_0(y)e'(t), \qquad y < 1,$$

where $e(t) = \mathbb{E}(M_t)$, subject to $p(t, 1) = 0$, $p(t, -\infty) = 0$, $p(0, y)dy = \mathbb{P}(X_0 \in dy)$. Moreover, the condition that $p(t, y)$ must remain a probability density translates into the fact that

$$e'(t) = \frac{d}{dt}\mathbb{E}(M_t) = -\frac{1}{2}\partial_y p(t, 1), \qquad \forall t > 0,$$

which describes the nonlinearity of the problem. In the case when $b(x) = -x$, this nonlinear Fokker-Planck equation is exactly the one studied in [3] and [4]. Therein, the authors conclude that for some choices of parameters, no global-in-time solutions exist. The term 'blow-up' is then used to describe the situation where the solution (defined in a weak sense) ceases to exist after some finite time. With our formulation, since $e'(t)$ corresponds to the mean firing rate of the infinite network, it is very natural to define a 'blow-up' time as a time when $e'(t)$ becomes infinite. Intuitively, this can be understood as a point in time at which a large proportion of the neurons in the network all spike at exactly the same time i.e. the network *synchronizes.*

In [3] and [4] it is shown that, in the cases $\alpha = 0$ and $\alpha < 0$ (the latter one being referred to as 'self-inhibitory' in neuroscience), the nonlinear Fokker-Planck equation has a unique solution that does not blow-up in finite time. However, in the so-called 'self-excitatory' framework, i.e. for $\alpha > 0$, existence of a solution for all time is left open. Instead, a negative result is established [3, Theorem 2.2], stating that, for any $\alpha > 0$, it is possible to find an initial probability distribution $\mathbb{P}(X_0 \in dy)$ such that any solution must blow-up in finite time, i.e. such that $e'(t) = \infty$ for some $t > 0$. solvability in the long run may fail for small values of α.

1.3. **Present contribution.** In this paper we thus investigate the case $\alpha \in (0, 1)$. Our main contribution is to show that, given a starting point $X_0 = x_0$, we can find an explicit α small enough so that there does indeed exist a unique global-in-time solution to (1.2) (and hence to the associated Fokker-Planck equation) which does not blow-up (see Theorem 2.4). In view of the above discussions, our result complements and goes further than those found in [3] and [4], and the surprising difficulty of the problem is reflected in the rather involved nature of our proofs.

As already said, Equation (1.2) can be thought of as of McKean-Vlasov-type, since the process $(X_t)_{t \geqslant 0}$ depends on the distribution of the solution itself. However, it is highly non-standard, since it actually depends on the distribution of the *first hitting times* of the threshold by the solution. This renders the traditional approaches to McKean-Vlasov equations and propagation of chaos, such as those presented by Sznitman in [20], inapplicable, because we have no *a priori* smoothness on the law of the first hitting times. Thus our results are also new in this context.

The general structure of the proof is at the intersection between probability and PDEs, the deep core of the strategy being probabilistic. The main ideas are inspired from the methods used to investigate the well-posedness of Markovian stochastic differential equations involving some non-trivial nonlinearity. Precisely, the first point is to tackle unique solvability in small time: when the parameter α is (strictly) less than 1 and the density of the initial condition decays linearly at the threshold, it is proven that the system induces a natural contraction in a well-chosen space provided the time duration is small enough. In this framework, the specific notion of a solution plays a crucial role as it defines the right space for the contraction. Below, solutions are sought in such a way that the mapping $e : t \mapsto \mathbb{E}(M_t)$ is continuously differentiable: this is a crucial point as it permits to handle the process $(X_t)_{t \geqslant 0}$ as a drifted Brownian motion. The second stage is then to extend existence and uniqueness from short to long times. The point is to prove that some key quantity is preserved as time goes by. Here, we prove that the system cannot accumulate too much mass in the vicinity of 1. Equivalently, this amounts to showing that the Lipschitz constant of the mapping $e : t \mapsto \mathbb{E}(M_t)$ cannot blow-up in a finite time. This is where the condition α small enough comes in: when α is small enough, we manage to give some estimates for the density of X_t in the neighbourhood of 1, the critical value of α explicitly depending upon the available bound of the density. Generally speaking, we make use of standard Gaussian estimates of Aronson type for the density. Unfortunately, the estimates we use are rather poor as they mostly neglect the right behavior of the density of X_t at the boundary, thus yielding a non-optimal value. Anyhow, they serve as a starting point for proving a refined estimate of the gradient of the density at the boundary: this is the required ingredient for proving that, at any time t, the mass of X_t decays linearly in the neighbourhood of 1, uniformly in t in compact sets, and thus to apply iteratively the existence and uniqueness argument in small time. In this way, we prove by induction that existence and uniqueness hold on any finite interval and thus on the whole of $[0, \infty)$.

It is worth mentioning that the main lines for proving the *a priori* estimate on the Lipschitz constant of $e : t \mapsto \mathbb{E}(M_t)$ are probabilistic, thus justifying the use of a stochastic approach to handle the model. Indeed, the key step in the control of the Lipschitz constant of e is an *intermediate* estimate of Hölder type, the proof of which is inspired from the probabilistic arguments used by Krylov and Safonov [12] for establishing the Hölder regularity of solutions to non-smooth PDEs.

1.4. **Prospects.** Our result is for a general Lipschitz function b, but there are two important specific cases that we keep in mind: the Brownian case when $b \equiv 0$ and the Ornstein-Uhlenbeck case when $b(x) = -\lambda x$, $\lambda \geqslant 0$. The Ornstein-Uhlenbeck case is most relevant to neuroscience, but surprising difficulties remain in the purely Brownian case. In both these cases we are able to give an explicit α_0 depending on the deterministic starting point x_0 such that (1.2) has a global solution for all $\alpha < \alpha_0$. However, our explicit values do not appear to be optimal: simulations suggest that for a given x_0 there exist solutions that do not blow up for α bigger than our explicit α_0, while there exist solutions that blow up that do not satisfy the conditions of [3]. Thus an interesting question is to determine for a given initial condition the critical value α_c such that for $\alpha < \alpha_c$ (1.2) does not exhibit blow-up.

Another point is to relax the notion of solution in order to allow the mapping $e : t \mapsto \mathbb{E}(M_t)$ to be non-differentiable (and thus to blow up). From the modeling point of view, this would permit to describe *synchronization* in the network. Actually, based on our understanding of the problem and numerical simulations, our guess is that, in full generality, the mapping e may be decomposed into a sequence of continuously differentiable pieces separated by isolated discontinuities. In that perspective, we feel that our work could serve as a basis for investigating the unique solvability of solutions that blow up: in order to design a proper uniqueness theory, it seems indeed quite mandatory to understand how general solutions behave in continuously differentiable regime (which is the precise purpose of the present paper) and then how discontinuities can emerge (which is left to further works).

The layout of the paper is as follows. We present the main results in Section 2. Solutions are defined in Section 3 while Section 4 is devoted to proving the existence and uniqueness in small time. The proof of Theorem 2.3 is given in Section 5.

2. Main results

2.1. **Set-up.** As stated in the introduction, we are interested in solutions to the nonlinear McKean-Vlasov-type SDE

$$X_t = X_0 + \int_0^t b(X_s)ds + \alpha\mathbb{E}(M_t) + W_t - M_t, \qquad t \geqslant 0, \tag{2.1}$$

where $X_0 < 1$ almost surely, $\alpha \in (0, 1)$ and $(W_t)_{t \geqslant 0}$ is a standard Brownian motion with respect to a filtration $(\mathcal{F}_t)_{t \geqslant 0}$ satisfying the usual conditions. The jumps, or resets, of the system are described by

$$M_t = \sum_{k \geqslant 1} \mathbf{1}_{[0,t]}(\tau_k), \quad \text{with} \quad \tau_k = \inf\{t > \tau_{k-1} : X_{t-} \geqslant 1\}, \quad k \geqslant 1 \ (\tau_0 = 0). \quad (2.2)$$

We assume that $b : (-\infty, 1] \to \mathbb{R}$ is Lipschitz continuous such that

$$|b(x)| \leqslant \Lambda(|x| + 1), \qquad |b(x) - b(y)| \leqslant K|x - y|, \qquad \forall x, y \in (-\infty, 1].$$

Remark 2.1. *By the time change $u = t/\sigma^2$, we could handle more general cases when the intensity of the noise in* (2.1) *is $\sigma > 0$ instead of* 1.

As discussed in the introduction, the key point is to look for a solution for which $t \mapsto \mathbb{E}(M_t)$ is continuously differentiable, which would correspond to a solution that does not exhibit a finite time blow-up. This leads to the following definition of a solution to (2.1), where as usual $\mathcal{C}^1[0,T]$ denotes the space of continuously differentiable functions on $[0,T]$.

Definition 2.2 (Solution to (2.1))**.** *The process $(X_t, M_t)_{0 \leqslant t \leqslant T}$ will be said to be a solution to* (2.1) *up until time T if $(M_t)_{0 \leqslant t \leqslant T}$ satisfies* (2.2)*, the map $([0,T] \ni t \mapsto \mathbb{E}(M_t)) \in \mathcal{C}^1[0,T]$ and $(X_t)_{0 \leqslant t \leqslant T}$ is a strong solution of* (2.1) *up until time T.*

2.2. **Statements.** Our main result is given by the following two theorems. The first guarantees that, when α is small enough, if there exists a solution to (2.1) on some finite time interval, then the solution does not blow-up on this interval.

Theorem 2.3. *For a given $\epsilon \in (0,1)$, there exists a positive constant $\alpha_0 \in (0,1]$, depending only upon ϵ, K and Λ, such that, for any $\alpha \in (0, \alpha_0)$ and any positive time $T > 0$, there exists a constant $\mathcal{M}_T$, only depending on T, ϵ, K and Λ, such that, for any initial condition $X_0 = x_0 \leqslant 1 - \epsilon$, any solution to* (2.1) *according to Definition 2.2 satisfies $(d/dt)\mathbb{E}(M_t) \leqslant \mathcal{M}_T$, for all $t \in [0,T]$.*

The second theorem is the main global existence and uniqueness result.

Theorem 2.4. *For any initial condition $X_0 = x_0 < 1$ and $\alpha \in (0, \alpha_0)$, where $\alpha_0 = \alpha_0(x_0)$ is as in Theorem 2.3 (taking $\epsilon = 1 - x_0$), there exists a unique solution to the nonlinear equation* (2.1) *on any $[0,T]$, $T > 0$, according to Definition 2.2.*

The size of the parameter α_0 in Theorem 2.3 is found explicitly in terms of ϵ, K and Λ (Proposition 5.3), but more precisely it derives from the fact that in the course of our proof we must first show that, *a priori*, any solution on $[0,T]$ to the nonlinear equation (2.1) with $X_0 = x_0 \leqslant 1 - \epsilon$ satisfies[1]

$$\frac{1}{dx}\mathbb{P}(X_t \in dx) < \frac{1}{\alpha}, \qquad t \in [0,T], \quad (2.3)$$

in a neighborhood of the threshold 1 (see Lemma 5.2). It is this restriction that determines the α_0 in Theorem 2.3, so that it depends only on the best *a priori* estimates available for the density on the left-hand side of (2.3). The stated explicit choice for α_0 in Proposition 5.3 merely ensures that (2.3) holds for all $\alpha < \alpha_0$ for any potential solution.

[1] In the whole paper, we use the very convenient notation $\frac{1}{dx}\mathbb{P}(X \in dx)$ to denote the density at point x of the random variable X (whenever it exists).

2.3. **Illustration: The Brownian case.** To further highlight the criticality of the system, we here illustrate the blow-up phenomenon in the Brownian case. Consider equation (2.1) with $b \equiv 0$, set $e(t) = \mathbb{E}(M_t)$ and fix $X_0 = x_0 < 1$. Then the conditions of Theorem 2.4 are trivially satisfied, and so we know that there exists a global-in-time solution for all $\alpha \in (0, \alpha_0(x_0))$.

One may then ask if we ever observe a blow-up phenomenon in this case. The affirmative answer can be seen by adapting the strategy in [3] (note that the result in [3] is written for a non-zero λ but a similar argument applies when $\lambda = 0$). For instance, choosing $x_0 = 0.8$, computations show that global in time solvability must fail for $\alpha \geqslant 0.539$. Moreover, tracking all the constants in the proof of Theorem 2.3 below, we can find that $\alpha_0(0.8) \approx .104$, which suggests that the system's behavior changes radically between these two values. Such a radical change can be observed numerically by investigating the graphs of $e(t) = \mathbb{E}(M_t)$ for different values of α in order to detect the emergence of some discontinuity. Using a particle method to solve the nonlinear equation with $b \equiv 0$, we numerically observe in Figure 1 that the graph of e is regular for $\alpha = 0.38$ but has a jump for $\alpha = 0.39$. From the observations we have for other values of α, it seems that global solvability fails for $\alpha \geqslant 0.39$ and holds for $\alpha \leqslant 0.38$.

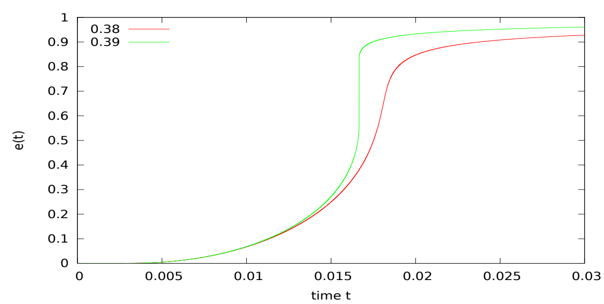

FIGURE 1. Plot of $t \mapsto e(t)$ for $x_0 = 0.8$, $b(x) \equiv 0$, $\alpha = 0.38$ (red) and $\alpha = 0.39$ (green).

As a summary, we present in Figure 2 the various regions of the α-parameter space $(0, 1)$ for $x_0 = 0.8$. The region **D** stands for the set of α's for which global solvability fails. By the numerical experiments it seems that global solvability also fails in region **C**, while by the same experiments it seems that global solutions do exist for α in region **B**. In this article we prove that global solutions exist for $\alpha \in$ **A**.

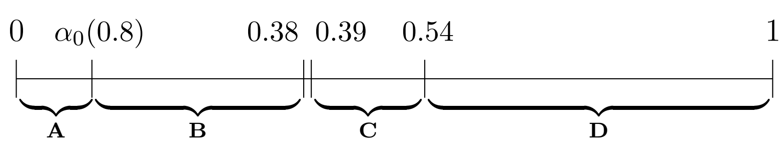

FIGURE 2. Critical regions of $\alpha \in (0, 1)$, for $x_0 = 0.8$ and $b(x) \equiv 0$.

3. Solution as a fixed point

In this section we identify a solution to the nonlinear equation (2.1) as a fixed point of an appropriate map on an appropriate space. This will reduce the problem of finding a solution to identifying a fixed point of this map.

Let $T > 0$. For a general function $e \in \mathcal{C}^1[0,T]$, consider the linear SDE

$$X_t^e = X_0 + \int_0^t b(X_s^e)ds + \alpha e(t) + W_t - M_t^e, \qquad t \in [0,T], \qquad X_0 < 1 \text{ a.s.} \tag{3.1}$$

where $(W_t)_{t \geqslant 0}$ is a standard Brownian motion, $\alpha \in (0,1)$,

$$M_t^e = \sum_{k \geqslant 1} \mathbf{1}_{[0,t]}(\tau_k^e) \tag{3.2}$$

and $\tau_k^e = \inf\{t > \tau_{k-1}^e : X_{t-}^e \geqslant 1\}$ for $k \geqslant 1$, $\tau_0^e = 0$. The drift function b is assumed to be Lipschitz as above. Note that the solution to this SDE is well defined (by solving (3.1) iteratively from any τ_k^e to the next τ_{k+1}^e and by noticing that the jumping times $(\tau_k^e)_{k \geqslant 0}$ cannot accumulate in finite time as the variations of $(X_t^e)_{t \geqslant 0}$ on any $[\tau_k^e, \tau_{k+1}^e)$, $k \geqslant 0$, are controlled in probability). We then define the map Γ by setting

$$\Gamma(e)(t) := \mathbb{E}(M_t^e). \tag{3.3}$$

We note that any fixed point of Γ that is continuously differentiable provides a solution to the nonlinear equation according to Definition 2.2 and vice versa. Thus, it is natural to look for a fixed point of Γ in a subspace of $\mathcal{C}^1[0,T]$ where we are careful to uniformly control the size of the derivative. Moreover, since it is clear from the definition that $\Gamma(e)(0) = 0$ and $t \mapsto \Gamma(e)(t)$ is non-decreasing for any $e \in \mathcal{C}^1[0,T]$, we in fact restrict the domain of Γ to the closed subspace $\mathcal{L}(T,A)$ of $\mathcal{C}^1[0,T]$ defined by

$$\mathcal{L}(T,A) := \left\{ e \in \mathcal{C}^1[0,T] : e(0) = 0,\ e(s) \leqslant e(t)\ \forall s \leqslant t, \sup_{0 \leqslant t \leqslant T} e'(t) \leqslant A \right\}$$

for some $A \geqslant 0$. The map Γ is thus defined as a map from $\mathcal{L}(T,A)$ into the set of non-decreasing functions on $[0,T]$. It in fact depends on A as its domain of definition depends on A; for this reason, it should be denoted by Γ^A. Anyhow, since the family $(\Gamma^A)_{A \geqslant 0}$ is consistent in the sense that, for any $A' \leqslant A$, the restriction of Γ^A to $\mathcal{L}(T,A')$ coincides with $\Gamma^{A'}$, we can use the simpler notation Γ.

The following *a priori* stability result provides further information about where to look for fixed points, the proof of which we leave until the end of the section.

Proposition 3.1. *Given $T > 0$, $a > 0$ and $e \in \mathcal{L}(T,A)$ it holds that*

$$\Big(\big(\forall t \in [0,T],\ e(t) \leqslant g_a(t) \big) \text{ and } \big(\mathbb{E}\big[(X_0)_+\big] \leqslant a \big) \Big) \Rightarrow \big(\forall t \in [0,T],\ \Gamma(e)(t) \leqslant g_a(t) \big),$$

where $(x)_+$ denotes the positive part of $x \in \mathbb{R}$, with

$$g_a(t) := \frac{a + (4 + \Lambda T^{1/2})t^{1/2}}{1-\alpha} \exp\left(\frac{2\Lambda t}{1-\alpha} \right). \tag{3.4}$$

Letting $g(t) := g_1(t)$, $t \geqslant 0$, since $X_0 < 1$ a.s., it thus makes sense to look for fixed points of Γ in the space

$$\mathcal{H}(T, A) := \{e \in \mathcal{L}(T, A) : e(t) \leqslant g(t)\}. \tag{3.5}$$

We equip $\mathcal{H}(T, A)$ with the norm $\|e\|_{\mathcal{H}(T,A)} = \|e\|_{\infty,T} + \|e'\|_{\infty,T}$ inherited from $\mathcal{C}^1[0, T]$. Here and throughout the paper, $\| \cdot \|_{\infty,T}$ denotes the supremum norm on $[0, T]$. $\mathcal{H}(T, A)$ is then a complete metric space, since it is a closed subspace of $\mathcal{C}^1[0, T]$.

For $e \in \mathcal{H}(T, A)$ Proposition 3.1 implies that $\Gamma(e)$ is finite and cannot grow faster that g, though it remains to show that $\Gamma(e)$ is differentiable and that its derivative is bounded by A in order to check that Γ indeed maps $\mathcal{H}(T, A)$ into itself, for a suitable value of A and T. The stability of $\mathcal{H}(T, A)$ by Γ is discussed in Section 4.3.

3.1. **Proof of Proposition 3.1:**

Proof of Proposition 3.1: Fix $T > 0$. We first note that we may write

$$\begin{aligned} M_t^e &= \sup_{s \leqslant t} \lfloor (Z_s^e)_+ \rfloor, \\ Z_t^e &= X_t^e + M_t^e = X_0 + \int_0^t b(X_s^e) ds + \alpha e(t) + W_t, \quad t \in [0, T], \end{aligned} \tag{3.6}$$

where $\lfloor x \rfloor$ denotes the floor part of $x \in \mathbb{R}$. Indeed, one can see that for $t \in [\tau_k^e, \tau_{k+1}^e)$, $k \geqslant 0$,

$$\begin{aligned} \sup_{s \leqslant t} \lfloor (Z_s^e)_+ \rfloor &= \max\Big(\max_{0 \leqslant j \leqslant k-1} \Big(\sup_{s \in [\tau_j^e, \tau_{j+1}^e)} \lfloor (X_s^e + j)_+ \rfloor \Big), \sup_{s \in [\tau_k^e, t)} \lfloor (X_s^e + k)_+ \rfloor \Big) \\ &= \max\big(\max_{0 \leqslant j \leqslant k-1} (j+1), k \big) = M_t^e, \end{aligned}$$

using the fact that $X_t^e < 1$ for all $t \geqslant 0$.

Then, given $t \in [0, T]$ such that $Z_t^e \geqslant 0$, let $\rho^e := \sup\{s \in [0, t] : Z_s^e < 0\}$ ($\sup \emptyset = 0$). Pay attention that ρ^e is not a stopping time and that it depends on t. Then, for $s \in [\rho^e, t]$,

$$|b(X_s^e)| \leqslant \Lambda(1 + |X_s^e|) \leqslant \Lambda\big(1 + |Z_s^e| + M_s^e\big) = \Lambda\big(1 + (Z_s^e)_+ + M_s^e\big). \tag{3.7}$$

By (3.6), we know that $M_s^e \leqslant \sup_{0 \leqslant r \leqslant s} (Z_r^e)_+$. Therefore,

$$|b(X_s^e)| \leqslant \Lambda \left(1 + 2 \sup_{0 \leqslant r \leqslant s} (Z_r^e)_+ \right).$$

By (3.6), we obtain:

$$Z_t^e \leqslant Z_{\rho^e}^e + \Lambda \int_{\rho^e}^t \left(1 + 2 \sup_{0 \leqslant r \leqslant s} (Z_r^e)_+ \right) ds + \alpha e(t) + W_t - W_{\rho^e}. \tag{3.8}$$

If $\rho^e > 0$, then $Z_{\rho^e}^e = 0$ as, obviously, $(Z_s^e)_{0 \leqslant s \leqslant T}$ is a continuous process. If $\rho^e = 0$, then $X_0 = Z_{\rho^e}^e \geqslant 0$ since $Z_{\rho^e}^e$ is non-negative. Therefore,

$$(Z_t^e)_+ \leqslant (X_0)_+ + \Lambda \int_0^t \left(1 + 2 \sup_{0 \leqslant r \leqslant s} (Z_r^e)_+ \right) ds + \alpha e(t) + 2 \sup_{0 \leqslant s \leqslant t} |W_s|. \tag{3.9}$$

Obviously, the above inequality still holds if $Z_t^e \leqslant 0$. We then notice that the process $(\sup_{0 \leqslant r \leqslant t}(Z_r^e)_+)_{0 \leqslant t \leqslant T}$ has finite values as $(Z_t^e)_{0 \leqslant t \leqslant T}$ is continuous. Therefore, taking the supremum in the left-hand side, applying Gronwall's lemma and taking the expectation, we deduce that $\mathbb{E}[\sup_{0 \leqslant t \leqslant T}(Z_t^e)_+]$ is finite. Taking directly the expectation in (3.9), we see that

$$\mathbb{E}\left[\sup_{0 \leqslant s \leqslant t}(Z_s^e)_+\right] \leqslant \mathbb{E}[(X_0)_+] + \Lambda \int_0^t \left(1 + 2\mathbb{E}\left[\sup_{0 \leqslant r \leqslant s}(Z_r^e)_+\right]\right) ds + \alpha e(t) + 4t^{1/2}, \tag{3.10}$$

for all $t \in [0,T]$. In particular, if $\mathbb{E}[(X_0)_+] \leqslant a$, $e(t) \leqslant g_a(t)$ for all $t \in [0,T]$ (where g_a is given by (3.4)), and R^e is the deterministic hitting time

$$R^e := \inf\left\{t \in [0,T] : \mathbb{E}\left[\sup_{0 \leqslant s \leqslant t}(Z_s^e)_+\right] > g_a(t)\right\} \quad (\inf \emptyset = +\infty),$$

then, for any $t \in (0, R^e \wedge T]$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0 \leqslant s \leqslant t}(Z_s^e)_+\right] &\leqslant a + \Lambda \int_0^t \big(1 + 2g_a(s)\big) ds + \alpha g_a(t) + 4t^{1/2} \\
&< \big(a + (4 + \Lambda T^{1/2})t^{1/2}\big)\left[1 + \int_0^t \frac{2\Lambda}{1-\alpha} \exp\left(\frac{2\Lambda s}{1-\alpha}\right) ds\right] + \alpha g_a(t) \\
&= (1-\alpha) g_a(t) + \alpha g_a(t) = g_a(t).
\end{aligned}$$

The strict inequality remains true when $t = 0$ since $\mathbb{E}[(X_0)_+] \leqslant a < g_a(0)$. Now, by the continuity of the paths of Z^e and by the finiteness of $\mathbb{E}[\sup_{0 \leqslant t \leqslant T}(Z_t^e)_+]$, we deduce that $\mathbb{E}[\sup_{0 \leqslant s \leqslant t}(Z_s^e)_+]$ is continuous in t. Therefore, if $R^e < T$, then $\mathbb{E}[\sup_{0 \leqslant s \leqslant R^e}(Z_s^e)_+]$ must be equal to $g(R^e)$, but, by the above inequalities, this sounds as a contradiction. By (3.6), this proves the announced bound. □

4. Existence and uniqueness in small time

The main result of this section is the following:

Theorem 4.1. *Suppose there exist $\beta, \epsilon > 0$ such that $\mathbb{P}(X_0 \in dx) \leqslant \beta(1-x)dx$ for any $x \in (1-\epsilon, 1]$ and that the density of X_0 on the interval $(1-\epsilon, 1]$ is differentiable at point 1. Then there exist constants $A_1 \geqslant 0$ and $T_1 \in (0,1]$, depending upon $\beta, \epsilon, \alpha, \Lambda$ and K only, such that $\Gamma\big(\mathcal{H}(A_1,T_1)\big) \subset \mathcal{H}(A_1,T_1)$. Moreover, for all $e_1, e_2 \in \mathcal{H}(A_1,T_1)$,*

$$\|\Gamma(e_1) - \Gamma(e_2)\|_{\mathcal{H}(A_1,T_1)} \leqslant \frac{1}{2}\|e_1 - e_2\|_{\mathcal{H}(A_1,T_1)}.$$

Hence there exists a unique fixed point of the restriction of Γ to $\mathcal{H}(A_1,T_1)$, which provides a solution to (2.1) according to Definition 2.2 up until time T_1 (such that $[0,T_1] \ni t \mapsto \mathbb{E}(M_t)$ is in the space $\mathcal{H}(A_1,T_1)$).

4.1. **Representation of Γ.** Let $T > 0$. As a first step towards understanding the map Γ defined above, we note that, given $e \in \mathcal{L}(T, A)$, using the definitions we can write

$$\Gamma(e)(t) = \mathbb{E}(M_t^e) = \mathbb{E}\left(\sum_{k \geqslant 1} \mathbf{1}_{[0,t]}(\tau_k^e)\right)$$
$$= \sum_{k \geqslant 1} \int_0^t \mathbb{P}\big(\tau_{k+1}^e \in (s,t] \,|\tau_k^e = s\big)\mathbb{P}(\tau_k^e \in ds) + \mathbb{P}(\tau_1^e \leqslant t),$$

where $\mathbb{P}(\tau_k^e \in ds)$ is a convenient abuse of notation for denoting the law of τ_k^e and $\mathcal{B}(\mathbb{R}) \ni A \mapsto \mathbb{P}(\tau_{k+1}^e \in A|\tau_k^e = s)$ stands for the conditional law of τ_{k+1}^e given $\tau_k^e = s$. Here $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. Moreover, observing that the solution X^e to (3.1) is a Markov process (which restarts from 0 at time τ_k^e when $k \geqslant 1$), we may write

$$\Gamma(e)(t) = \mathbb{E}(M_t^e) = \sum_{k \geqslant 1} \int_0^t \mathbb{P}\big(\tau_1^{e^{\sharp s}} \leqslant t - s|X_0^{e^{\sharp s}} = 0\big)\mathbb{P}(\tau_k^e \in ds) + \mathbb{P}(\tau_1^e \leqslant t), \quad (4.1)$$

where $e^{\sharp_s}$ stands for the mapping $([0, T - s] \ni t \mapsto e(t + s) - e(s)) \in \mathcal{L}(T - s, A)$.

With this decomposition it is clear that in order to analyse $\Gamma(e)$, and more importantly the derivative of $\Gamma(e)$ (recall we are looking for a fixed point in $\mathcal{H}(T, A)$), we must analyse the densities of the first hitting times of a barrier by a *non-homogeneous* diffusion processes with a general Lipschitz drift term. Indeed, formally taking the derivative with respect to t in (4.1) introduces terms involving the density of τ_1^e, where we recall that

$$\tau_1^e = \inf\{t > 0 : X_t^e \geqslant 1\} = \inf\left\{t > 0 : X_0 + \int_0^t b(X_s^e)ds + W_t \geqslant 1 - \alpha e(t)\right\}.$$

The analysis of such densities is well-known to be a difficult problem. These problems remain even in the case where $b \equiv 0$. However, the fact that e is continuously differentiable at least guarantees that the densities exist. In the case $b \equiv 0$ we refer to [16, Theorem 14.4]. In the general case existence of these densities will be guaranteed in the next section by Lemma 4.2.

4.2. **General bounds for the density of the first hitting time for the non-homogeneous diffusion process.** Fix $T > 0$, and for $e \in \mathcal{C}^1[0, T]$ consider the stochastic processes $(\chi_t^e)_{0 \leqslant t \leqslant T}$ which satisfies

$$d\chi_t^e = b(\chi_t^e)dt + \alpha e'(t)dt + dW_t, \quad t \in [0, T], \quad \chi_0^e < 1 \text{ a.s}, \quad (4.2)$$

together with the stopping time

$$\tau^e := \inf\{t \in [0, T] : \chi_t^e \geqslant 1\}, \qquad (\inf \emptyset = \infty).$$

Here $\alpha \in (0, 1)$ and the drift b is globally Lipschitz, exactly as above.

Lemma 4.2. *Let $e \in \mathcal{C}^1[0, T]$. Suppose there exist $\beta, \epsilon > 0$ such that $\mathbb{P}(\chi_0 \in dx) \leqslant \beta(1 - x)dx$ for any $x \in (1 - \epsilon, 1]$ and that the density of χ_0 on the interval $(1 - \epsilon, 1]$ is differentiable at point 1. Then:*

(i) *For any $t \in (0,T]$, the law of the diffusion χ_t^e killed at the threshold is absolutely continuous with respect to the Lebesgue measure.*

(ii) *Denoting the density of χ_t^e killed at the threshold by*

$$p_e(t,y) := \frac{1}{dy}\mathbb{P}\big(\chi_t^e \in dy, t < \tau^e\big), \quad t \in [0,T],\ y \leqslant 1, \tag{4.3}$$

$p_e(t,y)$ is continuous in (t,y) and continuously differentiable in y on $(0,T]\times(-\infty,1]$ and admits Sobolev derivatives of order 1 in t and of order 2 in y in any L^ς, $\varsigma \geqslant 1$, on any compact subset of $(0,T]\times(-\infty,1)$. When $\chi_0 \leqslant 1-\epsilon$ a.s. it is actually continuous and continuously differentiable in y on any compact subset of $([0,T]\times(-\infty,1])\setminus(\{0\}\times(-\infty,1-\epsilon])$.

(iii) *Almost everywhere on $(0,T]\times(-\infty,1)$, p_e satisfies the Fokker-Planck equation:*

$$\partial_t p_e(t,y) + \partial_y\big[\big(b(y)+\alpha e'(t)\big)p_e(t,y)\big] - \frac{1}{2}\partial^2_{yy}p_e(t,y) = 0, \tag{4.4}$$

with the Dirichlet boundary condition $p_e(t,1)=0$ and the measure-valued initial condition $p_e(0,y)dy = \mathbb{P}(\chi_0 \in dy)$, $p_e(t,y)$ and $\partial_y p_e(t,y)$ decaying to 0 as $y \to -\infty$.

(iv) *The first hitting time, τ^e has a density on $[0,T]$, given by*

$$\frac{d}{dt}\mathbb{P}(\tau^e \leqslant t) = -\frac{1}{2}\partial_y p_e(t,1), \quad t \in [0,T], \tag{4.5}$$

the mapping $[0,T] \ni t \mapsto \partial_y p_e(t,1)$ being continuous and its supremum norm being bounded in terms of T, α, $\|e'\|_{\infty,T}$, β and b only.

Lemma 4.2 is quite standard. The analysis of the Green function of killed processes with smooth coefficients may be found in [8, Chap. VI]. The need for considering Sobolev derivatives follows from the fact that b is Lipschitz only. The argument to pass from the case b smooth to the case b Lipschitz only is quite standard: it follows from Calderon and Zygmund estimates, see [19, Eq. (0.4), App. A], that permit the control of the L^ς norm of the second order derivatives on any compact subset of $(0,T]\times(-\infty,1)$. A complete proof may be also found in the unpublished notes [5].

When $\chi_0 = x_0$ for some deterministic $x_0 < 1$, the conditions of the above Lemma are certainly satisfied. Therefore, for $e \in \mathcal{C}^1[0,T]$ it makes sense to consider the density $p_e(t,y)$, $t \in (0,T]$, $y \leqslant 1$ of the process killed at 1 started at x_0. We will write $p_e(t,y) = p_e^{x_0}(t,y)$ in this case. The following two key results on $\partial_y p_e(t,1)$ are standard adaptations of heat kernel estimates (see for instance [7, Chapter 1]) for killed processes. The first one may be found in [8, Chap. VI, Theorem 1.10] when b is smooth and bounded. As explained in the beginning of [8, Chap. VI, Subsec. 1.5], it remains true when b is Lipschitz continuous and bounded. The argument for removing the boundedness assumption on b is explained in [6] in the case of a non-killed process. As shown in the unpublished notes [5, Cor 4.3], it can be adapted to the current case. The second result then follows from the so-called *parametric* perturbation argument following [7, Chapter 1]. Again, the complete proof can be found in the unpublished notes [5, Cor 5.3].

Proposition 4.3. *Let $e \in \mathcal{C}^1[0,T]$. Then there exists a constant $\kappa(T)$ (depending only on T and the drift function b) which increases with T such that for all $x_0 < 1$,*

$$|\partial_y p_e^{x_0}(t,1)| \leqslant \kappa(T)(\|e'\|_{\infty,T}+1)\frac{1}{t}\exp\left(-\frac{(1-x_0)^2}{\kappa(T)t}\right)$$

for all $t \leqslant \min\{[(\|e'\|_{\infty,T}+1)\kappa(T)]^{-2}, T\}$. In particular $\kappa(T)$ is independent of e.

Proposition 4.4. *Let $e_1, e_2 \in \mathcal{C}^1[0,T]$ and let $A = \max\{\|e_1'\|_{\infty,T}, \|e_2'\|_{\infty,T}\}$. Then there exists a constant $\kappa(T)$ (depending only on T and the drift function b) which increases with T such that for all $x_0 < 1$,*

$$\left|\partial_y p_{e_1}^{x_0}(t,1) - \partial_y p_{e_2}^{x_0}(t,1)\right| \leqslant \kappa(T)(A+1)\frac{1}{\sqrt{t}}\exp\left(-\frac{(1-x_0)^2}{\kappa(T)t}\right)\|e_1'-e_2'\|_{\infty,t},$$

for all $t \leqslant \min\{[(A+1)\kappa(T)]^{-2}, T\}$. In particular $\kappa(T)$ is independent of e_1 and e_2.

4.3. **Application to Γ:** In this section we return to the setting of Section 3, and apply the results of the previous subsection to complete the proof of Theorem 4.1.

The first result ensures the differentiability of $\Gamma(e)$ whenever $e \in \mathcal{L}(T,A)$, which is the first step in showing that Γ is stable on the space $\mathcal{H}(T,A)$ for some A (recall that $\mathcal{H}$ is simply a growth controlled subspace of $\mathcal{L}$).

Proposition 4.5. *Let $e \in \mathcal{L}(T,A)$ and X_0 be such that there exist $\beta, \epsilon > 0$ with $\mathbb{P}(X_0 \in dx) \leqslant \beta(1-x)dx$ for any $x \in (1-\epsilon, 1]$,and suppose that the density of X_0 on the interval $(1-\epsilon,1]$ is differentiable at point 1. Then the mapping $[0,T] \ni t \mapsto \Gamma(e)(t)$ is continuously differentiable. Moreover,*

$$\frac{d}{dt}\big[\Gamma(e)\big](t) = -\int_0^t \frac{1}{2}\partial_y p_e^{(0,s)}(t-s,1)\frac{d}{ds}\big[\Gamma(e)\big](s)ds - \frac{1}{2}\partial_y p_e(t,1), \quad t \in [0,T], \tag{4.6}$$

where p_e represents the density of the process X^e killed at 1 and $p_e^{(0,s)}$ represents the density of the process $X^{e\sharp s}$ killed at 1 with $X_0^{e\sharp s} = 0$.

Proof. We first check that $\Gamma(e)$ is Lipschitz continuous on $[0,T]$. Considering a finite difference in (4.1) and using (4.5), we get, for $t, t+h \in [0,T]$,

$$\begin{aligned}\Gamma(e)(t+h) - \Gamma(e)(t) &= \sum_{k \geqslant 1}\int_t^{t+h} \mathbb{P}\big(\tau_1^{e\sharp s} \leqslant t+h-s | X_0^{e\sharp s} = 0\big)\mathbb{P}(\tau_k^e \in ds) \\ &\quad - \frac{1}{2}\sum_{k\geqslant 1}\int_0^t\int_{t-s}^{t+h-s} \partial_y p_e^{(0,s)}(r,1)dr\mathbb{P}(\tau_k^e \in ds) - \frac{1}{2}\int_t^{t+h}\partial_y p_e(s,1)ds.\end{aligned} \tag{4.7}$$

By Lemma 4.2 (ii), we can handle the two last terms in the above to find a constant $C > 0$ (which depends on e) such that

$$\begin{aligned}\Gamma(e)(t+h) - \Gamma(e)(t) \leqslant &\sum_{k\geqslant 1}\int_t^{t+h}\mathbb{P}\big(\tau_1^{e\sharp s} \leqslant t+h-s | X_0^{e\sharp s} = 0\big)\mathbb{P}(\tau_k^e \in ds) \\ &+ Ch\big(1+\Gamma(e)(T)\big),\end{aligned}$$

the last term in the right-hand side being finite thanks to (3.9) and the argument following it. Moreover, by (3.9) and Gronwall's lemma, we deduce that

$$\begin{aligned} &\lim_{h\searrow 0} \sup_{0\leqslant s\leqslant T-h} \mathbb{P}\big(\tau_1^{e^{\sharp_s}} \leqslant h | X_0^{e^{\sharp_s}} = 0\big) \\ &= \lim_{h\searrow 0} \sup_{0\leqslant s\leqslant T-h} \mathbb{P}\left(\sup_{0\leqslant r\leqslant h} Z_r^{e^{\sharp_s}} \geqslant 1 | X_0^{e^{\sharp_s}} = 0 \right) = 0, \end{aligned} \tag{4.8}$$

where $Z^{e^{\sharp_s}}$ is given by (3.6). Therefore, there exists a mapping $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ matching 0 at 0 and continuous at 0 such that

$$\Gamma(e)(t+h) - \Gamma(e)(t) \leqslant \eta(h)\big[\Gamma(e)(t+h) - \Gamma(e)(t)\big] + Ch\big(1 + \Gamma(e)(T)\big).$$

Choosing h small enough, Lipschitz continuity easily follows.

As a consequence, we can divide both sides of (4.7) by h and then let h tend to 0. By (4.8), we have for a given $t \in [0, T)$,

$$\begin{aligned} &\lim_{h\searrow 0} h^{-1} \sum_{k\geqslant 1} \int_t^{t+h} \mathbb{P}\big(\tau_1^{e^{\sharp_s}} \leqslant t+h-s | X_0^{e^{\sharp_s}} = 0\big) \mathbb{P}(\tau_k^e \in ds) \\ &\leqslant \lim_{h\searrow 0} \left[\sup_{0\leqslant s\leqslant T-h} \mathbb{P}\big(\tau_1^{e^{\sharp_s}} \leqslant h | X_0^{e^{\sharp_s}} = 0\big) \frac{\Gamma(e)(t+h) - \Gamma(e)(t)}{h} \right] = 0. \end{aligned}$$

Handling the second term in (4.7) by Lemma 4.2 and using the Lebesgue Dominated Convergence Theorem, we deduce that

$$\frac{d}{dt}\Gamma(e)(t) = -\sum_{k\geqslant 1} \int_0^t \frac{1}{2} \partial_y p_e^{(0,s)}(t-s, 1) \mathbb{P}(\tau_k^e \in ds) - \frac{1}{2}\partial_y p_e(t, 1).$$

By Lemma 4.2, we know that $\partial_y p_e^{(0,s)}(\cdot, 1)$ and $\partial_y p_e(\cdot, 1)$ are continuous (in t). This proves that $(d/dt)\Gamma(e)$ is continuous as well.

Formula (4.6) then follows from the relationship

$$\Gamma(e)(t) = \sum_{k\geqslant 1} \int_0^t \mathbb{P}(\tau_k^e \in ds), \quad t \in [0, T]. \tag{4.9}$$

□

The second idea is to show that the difference between the derivatives of $\Gamma(e_1)$ and $\Gamma(e_2)$ is uniformly small in terms of the distance between two functions e_1 and e_2 in the space $\mathcal{H}(T, A)$ in small time.

Proposition 4.6. *Let $T > 0$ and X_0 be such that there exist $\beta, \epsilon > 0$ with $\mathbb{P}(X_0 \in dx) \leqslant \beta(1-x)dx$ for any $x \in (1-\epsilon, 1]$, and suppose that the density of X_0 on the interval $(1-\epsilon, 1]$ is differentiable at point 1.*

Suppose $e_1, e_2 \in \mathcal{H}(T, A)$ for some $A \geqslant 0$. Then there exists a constant $\kappa(T)$, independent of A, β and ϵ, and increasing in T, and a constant $\widetilde{\kappa}(T, \beta, \epsilon)$, independent of A and increasing in T, such that for any $e_1, e_2 \in \mathcal{H}(T, A)$,

$$\sup_{0\leqslant s\leqslant t} \left| \frac{d}{ds}\Big[\Gamma(e_1) - \Gamma(e_2)\Big](s) \right| \leqslant (A+1)\widetilde{\kappa}(T, \beta, \epsilon)\sqrt{t}\|e_1' - e_2'\|_{\infty,t},$$

for $t \leqslant \min\{[(A+1)\kappa(T)]^{-2}, T\}$.

Proof. We have by (4.6)

$$\begin{aligned}\left|\frac{d}{dt}\Big[\Gamma(e_1)-\Gamma(e_2)\Big](t)\right| &\leqslant \frac{1}{2}\int_{-\infty}^{1}\left|\Big[\partial_y p^x_{e_1}-\partial_y p^x_{e_2}\Big](t,1)\right|\mathbb{P}(X_0\in dx)\\ &\quad+\frac{1}{2}\int_0^t\left|\Big[\partial_y p^{(0,s)}_{e_1}-\partial_y p^{(0,s)}_{e_2}\Big](t-s,1)\right|\frac{d}{ds}\Gamma(e_1)(s)\\ &\quad+\frac{1}{2}\int_0^t\left|\partial_y p^{(0,s)}_{e_2}(t-s,1)\right|\left|\frac{d}{ds}\Big[\Gamma(e_1)-\Gamma(e_2)\Big](s)\right|ds\\ &:=\frac{1}{2}\big(L_1+L_2+L_3\big).\end{aligned} \tag{4.10}$$

Suppose $t \leqslant T$ and $\sqrt{t} \leqslant [(A+1)\kappa(T)]^{-1}$, where $\kappa(T)$ is as in Proposition 4.4. The value of $\kappa(T)$ will be allowed to increase when necessary below. Considering the first term only, we can use Proposition 4.4 to see that

$$\begin{aligned}L_1 &\leqslant (A+1)\beta\kappa(T)\left(\int_{1-\epsilon}^{1}\frac{1}{\sqrt{t}}\exp\left(-\frac{(1-x)^2}{\kappa(T)t}\right)(1-x)dx\right)\|e_1'-e_2'\|_{\infty,t}\\ &\quad+(A+1)\kappa(T)\left(\int_{-\infty}^{1-\epsilon}\frac{1}{\sqrt{t}}\exp\left(-\frac{(1-x)^2}{\kappa(T)t}\right)\mathbb{P}(X_0\in dx)\right)\|e_1'-e_2'\|_{\infty,t}.\end{aligned}$$

We deduce that there exists a constant $\widetilde{\kappa}(T,\beta,\epsilon)>0$, which is independent of A and which is allowed to increase as necessary from line to line below, such that

$$\begin{aligned}L_1 &\leqslant (A+1)\beta\kappa(T)\sqrt{t}\left(\int_0^\infty z\exp\left(-\frac{z^2}{\kappa(T)}\right)dz\right)\|e_1'-e_2'\|_{\infty,t}\\ &\quad+(A+1)\kappa(T)\frac{1}{\sqrt{t}}\exp\left(-\frac{\epsilon^2}{\kappa(T)t}\right)\|e_1'-e_2'\|_{\infty,t}\\ &\leqslant (A+1)\widetilde{\kappa}(T,\beta,\epsilon)\sqrt{t}\|e_1'-e_2'\|_{\infty,t}.\end{aligned} \tag{4.11}$$

We can then use Proposition 4.4 again to see that

$$L_2 \leqslant (A+1)\kappa(T)\sup_{0<s\leqslant t}\left[s^{-1/2}\exp\left(-\frac{1}{\kappa(T)s}\right)\right]\Gamma(e_1)(t)\|e_1'-e_2'\|_{\infty,t}.$$

By Proposition 3.1 (since $e_1\in\mathcal{H}(T,A)$), we deduce that

$$L_2 \leqslant (A+1)\kappa(T)\sqrt{t}\|e_1'-e_2'\|_{\infty,t}, \tag{4.12}$$

where $\kappa(T)$ has been increased as necessary, and we have used the elementary inequality $\exp(-1/v)\leqslant v$ for all $v\geqslant 0$. We finally turn to L_3 in (4.10). By Proposition 4.3, we have that

$$\begin{aligned}\left|\partial_y p^{(0,s)}_{e_2}(t-s,1)\right| &\leqslant \kappa(T)(A+1)\frac{1}{(t-s)}\exp\left(-\frac{1}{\kappa(T)(t-s)}\right)\\ &\leqslant \kappa(T)(A+1),\end{aligned} \tag{4.13}$$

again by increasing $\kappa(T)$. Thus, from (4.10), (4.11), (4.12) and (4.13), we deduce

$$\left|\frac{d}{dt}\Big[\Gamma(e_1)-\Gamma(e_2)\Big](t)\right| \leqslant (A+1)\widetilde{\kappa}(T,\beta,\epsilon)\sqrt{t}\|e_1'-e_2'\|_{\infty,t} + (A+1)\kappa(T)\int_0^t \left|\frac{d}{ds}\Big[\Gamma(e_1)-\Gamma(e_2)\Big](s)\right| ds.$$

By taking the supremum over all $s \leqslant t$ in the above, we have, for $t \leqslant (2\kappa(T)(A+1))^{-1}$, (which actually follows from the aforementioned condition $t \leqslant (\kappa(T)(A+1))^{-2}$ by assuming w.l.o.g. $\kappa(T) \geqslant 2$),

$$\sup_{0\,\leqslant\, s\,\leqslant\, t}\left|\frac{d}{ds}\Big[\Gamma(e_1)-\Gamma(e_2)\Big](s)\right| \leqslant 2(A+1)\widetilde{\kappa}(T,\beta,\epsilon)\sqrt{t}\|e_1'-e_2'\|_{\infty,t}.$$

□

We can then finally complete this section with the proof of Theorem 4.1.

Proof. [**Proof of Theorem 4.1**] Choose $A_1 = 2\sup_{0\,\leqslant\, t\,\leqslant\, 1}|(d/dt)\Gamma(0)(t)|+1$. Note that A_1 depends on β. Then choose $T_1 \leqslant \min\{[(A_1+1)\kappa(1)]^{-2},1\}$ such that

$$\sqrt{T_1}\widetilde{\kappa}(1,\beta,\epsilon)(A_1+1) \leqslant \frac{1}{4}, \tag{4.14}$$

where $\kappa(1)$ and $\widetilde{\kappa}(1,\beta,\epsilon)$ are as in Proposition 4.6. By that result, if $e \in \mathcal{H}(A_1,T_1)$ then

$$\left|\frac{d}{dt}\Gamma(e)(t)\right| = \frac{d}{dt}\Gamma(e)(t) \leqslant \sqrt{t}\widetilde{\kappa}(T_1,\beta,\epsilon)(A_1+1)A_1 + \frac{d}{dt}\Gamma(0)(t)$$

for all $t \leqslant \min\{[(A_1+1)\kappa(T_1)]^{-2}, T_1\} = T_1$. By definition, we have $T_1 \leqslant 1$ so that $\kappa(T_1) \leqslant \kappa(1)$ and $\widetilde{\kappa}(T_1,\beta,\epsilon) \leqslant \widetilde{\kappa}(1,\beta,\epsilon)$. Therefore

$$\frac{d}{dt}\Gamma(e)(t) \leqslant \sqrt{t}\widetilde{\kappa}(1,\beta,\epsilon)(A_1+1)A_1 + \frac{d}{dt}\Gamma(0)(t)$$

for all $t \leqslant T_1$. Hence for all $t \leqslant T_1$

$$\frac{d}{dt}\Gamma(e)(t) \leqslant \frac{A_1}{2} + \sup_{0\,\leqslant\, t\,\leqslant\, 1}\left(\frac{d}{dt}\Gamma(0)(t)\right) \leqslant A_1$$

by (4.14), so that $\Gamma(e) \in \mathcal{H}(A_1,T_1)$.

To prove that Γ is a contraction on $\mathcal{H}(A_1,T_1)$, first note that for $e \in \mathcal{H}(A_1,T_1)$

$$\|e'\|_{\infty,T_1} \leqslant \|e\|_{\mathcal{H}(A,T_1)} \leqslant 2\|e'\|_{\infty,T_1}$$

by the mean-value theorem, since $e(0) = 0$ and $T_1 \leqslant 1$. Thus for any $e_1, e_2 \in \mathcal{H}(A_1,T_1)$

$$\begin{aligned}\|\Gamma(e_1)-\Gamma(e_2)\|_{\mathcal{H}(A_1,T_1)} &\leqslant 2\,\|\Gamma(e_1)'-\Gamma(e_2)'\|_{\infty,T_1}\\ &\leqslant 2\sqrt{T_1}\widetilde{\kappa}(T_1,\beta,\epsilon)(A_1+1)\|e_1'-e_2'\|_{\infty,T_1} \leqslant \frac{1}{2}\|e_1-e_2\|_{\mathcal{H}(A_1,T_1)},\end{aligned}$$

by our choice of T_1 and using Proposition 4.6 once more. Since $\mathcal{H}(A_1,T_1)$ is a closed subspace of $\mathcal{C}^1[0,T]$ (a complete metric space), the existence of a fixed point for Γ follows from the Banach Fixed Point Theorem. □

5. Long-time estimates

In order to extend the existence and uniqueness from small time to any arbitrarily prescribed interval, we need an *a priori* bound for the Lipschitz constant of $e : t \mapsto \mathbb{E}(M_t)$ on any finite interval $[0,T]$, which is given by Theorem 2.3. The purpose of this section is to prove this result.

As already mentioned, the key point is inequality (2.3). Loosely, it says that, in (1.1), the particles that are below $1-dx$ at time t receive a kick of order $\alpha\mathbb{P}(X_t \in dx) < dx$. In other words, only the particles close to 1 can jump, which guarantees some control on the continuity of e. Precisely, Proposition 5.3 gives a bound for the 1/2-Hölder constant of e. Inquality (2.3) is proved by using *a priori* heat kernel bounds when α is small enough, this restriction determining the value of α_0 in Theorem 2.3. Once the 1/2-Hölder constant of e has been controlled, we provide in Lemma 5.5 a Hölder estimate of the oscillation (in space) of p in the neighbourhood of 1. The proof is an adaptation of [12]. Finally, in Proposition 5.6, a barrier technique yields a bound for the Lipschitz constant of p in the neighbourhood of 1.

In the whole section, for a given initial condition $X_0 = x_0 < 1$, we thus assume that there exists a solution to (2.1) according to Definition 2.2 i.e. such that $e : [0,T] \ni t \mapsto \mathbb{E}(M_t)$ is continuously differentiable.

5.1. **Reformulation of the equation and *a priori* bounds for the solution.** In the whole proof, we shall use a reformulated version of (2.1), in a similar way to Proposition 3.1 (see (3.6)). Indeed, given a solution $(X_t, M_t)_{0 \leqslant t \leqslant T}$ to (2.1) on some interval $[0,T]$ according to Definition 2.2, we set $Z_t = X_t + M_t$, $t \in [0,T]$. Then $(Z_t)_{0 \leqslant t \leqslant T}$ has continuous paths and satisfies

$$Z_t = X_0 + \int_0^t b(X_s)ds + \alpha\mathbb{E}(M_t) + W_t, \quad t \in [0,T]. \tag{5.1}$$

where

$$M_t = \lfloor \big(\sup_{0 \leqslant s \leqslant t} Z_s \big)_+ \rfloor = \sup_{0 \leqslant s \leqslant t} \lfloor (Z_s)_+ \rfloor. \tag{5.2}$$

The following is easily proved by adapting the proof of Proposition 3.1:

Lemma 5.1. *There exists a constant $B(T,\alpha,b)$, only depending upon T, α, b and non-decreasing in α, such that*

$$\sup_{0 \leqslant t \leqslant T} e(t) = e(T) \leqslant \mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} (Z_t)_+ \right] \leqslant B(T,\alpha,b). \tag{5.3}$$

A possible choice for B is

$$B(T,\alpha,b) = \frac{\mathbb{E}[(X_0)_+] + 4T^{1/2} + \Lambda T}{1-\alpha} \exp\left(\frac{2\Lambda T}{1-\alpha} \right).$$

5.2. **Local Hölder bound of the solution.** We now turn to the critical point of the proof. Indeed, in the next subsection, we shall prove that, for α small enough, the function $t \mapsto e(t) = \mathbb{E}(M_t)$ generated by some solution to (2.1) according to Definition 2.2 (so that e is continuously differentiable) satisfies an *a priori* 1/2-Hölder bound, with an explicit Hölder constant. This acts as the keystone of the argument to extend the local existence and uniqueness result into a global one. As a first step, the proof consists of establishing a local Hölder bound for e in the case when the probability that the process X lies in the neighbourhood of 1 is not too large.

Lemma 5.2. *Consider a solution $(X_t)_{0 \leqslant t \leqslant T}$ to (2.1) on some interval $[0,T]$, with $T > 0$ and initial condition $X_0 = x_0 < 1$. Assume in addition that there exists some time $t_0 \in [0,T]$ and two constants $\epsilon \in (0,1)$ and $c \in (0, 1/\alpha)$ such that for any Borel subset $A \subset [1-\epsilon, 1]$,*

$$\mathbb{P}(X_{t_0} \in A) \leqslant c|A|, \tag{5.4}$$

where $|A|$ stands for the Lebesgue measure of A. Then, with

$$\mathcal{B}_0 = \frac{\exp(2\Lambda)[(8 + 5c + 8\epsilon^{-1})\Lambda + 4(2 + c + \epsilon^{-1})]}{1 - c\alpha},$$

it holds that, for any $h \in (0,1)$,

$$\left.\begin{array}{l} \mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leqslant \epsilon/2 \\ t_0 + h \leqslant T \end{array}\right\} \Rightarrow e(t_0 + h) - e(t_0) \leqslant \mathcal{B}_0 h^{1/2}.$$

Proof. By the Markov property, we can assume $t_0 = 0$, with T being understood as $T - t_0$. Indeed, setting

$$X_t^{\sharp t_0} := X_{t_0+t}, \quad t \in [0, T - t_0], \tag{5.5}$$

we observe that, for $t \in [0, T - t_0]$,

$$X_t^{\sharp t_0} = X_{t_0} + \int_0^t b(X_r^{\sharp t_0})dr + \alpha\mathbb{E}(M_{t+t_0} - M_{t_0}) + W_{t+t_0} - W_{t_0} - \big(M_{t+t_0} - M_{t_0}\big). \tag{5.6}$$

Here $M_{t+t_0} - M_{t_0}$ represents the number of times the process X reaches 1 within the interval $(t_0, t+t_0]$. Therefore, this also matches the number of times the process $X^{\sharp t_0}$ hits 1 within the interval $(0,t]$, so that $X^{\sharp t_0}$ indeed satisfies the nonlinear equation (2.1) on $[0, T - t_0]$, with $X_0^{\sharp t_0} = X_{t_0}$ as initial condition and with respect to the shifted Brownian motion $(W_t^{\sharp t_0} := W_{t_0+t} - W_{t_0})_{0 \leqslant t \leqslant T - t_0}$. In what follows, t_0 is thus assumed to be zero, the new T standing for the previous $T - t_0$ and the new X_0 matching the previous X_{t_0} and thus satisfying (5.4).

For a given $h \in (0,1)$, such that $h \leqslant T$, and a given $\mathcal{B}_0 > 0$ (the value of which will be fixed later), we then define the deterministic hitting time:

$$R = \inf\big\{t \in [0,h] : \mathbb{E}(M_t) = e(t) \geqslant \mathcal{B}_0 h^{1/2}\big\}.$$

Following the proof of (3.9) (see more specifically (3.7)), we have, for any $t \in [0, h \wedge R]$,

$$\begin{aligned} M_t \leqslant \sup_{0 \leqslant s \leqslant t} (Z_s)_+ &\leqslant (X_0)_+ + \Lambda \int_0^t \big(1 + (Z_s)_+ + M_s\big)ds + \alpha e(t) + 2 \sup_{0 \leqslant s \leqslant t} |W_s| \\ &\leqslant (X_0)_+ + 2\Lambda \int_0^t \big(1 + M_s\big)ds + \alpha\mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant t} |W_s| \\ &\leqslant (X_0)_+ + 2\Lambda h + 2\Lambda \int_0^t M_s ds + \alpha\mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant t} |W_s|, \end{aligned}$$

where we have used (5.2) to pass from the first to the second line. By Gronwall's Lemma, we obtain

$$\begin{aligned} M_t &\leqslant \exp(2\Lambda h)\left[(X_0)_+ + 2\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2\sup_{0\leqslant s\leqslant h}|W_s|\right] \\ &\leqslant (X_0)_+ + \exp(2\Lambda h)\left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2\sup_{0\leqslant s\leqslant h}|W_s|\right], \end{aligned} \tag{5.7}$$

as $\exp(2\Lambda h) \leqslant 1 + 2\Lambda h\exp(2\Lambda h)$ and $(X_0)_+ \leqslant 1$.

Assume that $\mathcal{B}_0\exp(2\Lambda h)h^{1/2} \leqslant \epsilon/2 \leqslant 1/2$. Then, by Doob's L^2 inequality for martingales,

$$\begin{aligned} \sum_{k\geqslant 2}\mathbb{P}(M_t \geqslant k) &\leqslant \sum_{k\geqslant 2}\mathbb{P}\left(\exp(2\Lambda h)\left[4\Lambda h + 2\sup_{0\leqslant s\leqslant h}|W_s|\right] \geqslant k - 3/2\right) \\ &\leqslant 2\exp(2\Lambda h)\mathbb{E}\big[4\Lambda h + 2\sup_{0\leqslant s\leqslant h}|W_s|\big] \\ &\leqslant \exp(2\Lambda h)\big[8\Lambda h + 8h^{1/2}\big]. \end{aligned} \tag{5.8}$$

Moreover,

$$\begin{aligned} &\mathbb{P}(M_t \geqslant 1) \\ &\leqslant \mathbb{P}\left((X_0)_+ + \exp(2\Lambda h)\left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2\sup_{0\leqslant s\leqslant h}|W_s|\right] \geqslant 1\right) \\ &\leqslant \mathbb{P}\left(X_0 \in [1-\epsilon, 1], X_0 + \exp(2\Lambda h)\left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2\sup_{0\leqslant s\leqslant h}|W_s|\right] \geqslant 1\right) \\ &\quad + \mathbb{P}\left(\exp(2\Lambda h)\left[4\Lambda h + 2\sup_{0\leqslant s\leqslant h}|W_s|\right] \geqslant \epsilon/2\right) \\ &:= I_1 + I_2, \end{aligned}$$

where we have used $\mathcal{B}_0\exp(2\Lambda h)h^{1/2} \leqslant \epsilon/2$ in the third line.

By Doob's L^1 maximal inequality, we deduce that

$$I_2 \leqslant 2\exp(2\Lambda h)\epsilon^{-1}\mathbb{E}\big[4\Lambda h + 2|W_h|\big] \leqslant \exp(2\Lambda h)\epsilon^{-1}\big[8\Lambda h + 4h^{1/2}\big]. \tag{5.9}$$

We now switch to I_1. By independence of X_0 and $(W_s)_{0\leqslant s\leqslant T}$ and by (5.4),

$$\begin{aligned} I_1 &\leqslant c\int_0^{\epsilon}\mathbb{P}\left(\exp(2\Lambda h)\left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2\sup_{0\leqslant s\leqslant h}|W_s|\right] \geqslant x\right)dx \\ &\leqslant c\int_0^{+\infty}\mathbb{P}\left(\exp(2\Lambda h)\left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2\sup_{0\leqslant s\leqslant h}|W_s|\right] \geqslant x\right)dx \\ &= c\exp(2\Lambda h)\mathbb{E}\left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2\sup_{0\leqslant s\leqslant h}|W_s|\right]. \end{aligned}$$

By Doob's L^2 inequality,

$$I_1 \leqslant c\exp(2\Lambda h)\big[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 4h^{1/2}\big].$$

Together with (5.9), we deduce that

$$\mathbb{P}(M_t \geqslant 1) \leqslant \exp(2\Lambda h)\big[4(c+2\epsilon^{-1})\Lambda h + 4(c+\epsilon^{-1})h^{1/2} + c\alpha\mathcal{B}_0 h^{1/2}\big].$$

From (5.8), we finally obtain, for $t \leqslant R \wedge h$,

$$\begin{aligned}\mathbb{E}(M_t) &= \sum_{k\geqslant 1}\mathbb{P}(M_t \geqslant k)\\ &\leqslant \exp(2\Lambda h)\big[4(2+c+2\epsilon^{-1})\Lambda h + 4(2+c+\epsilon^{-1})h^{1/2} + c\alpha\mathcal{B}_0 h^{1/2}\big]\\ &\leqslant \exp(2\Lambda h)\big[(8+5c+8\epsilon^{-1})\Lambda h + 4(2+c+\epsilon^{-1})h^{1/2}\big] + c\alpha\mathcal{B}_0 h^{1/2},\end{aligned}$$

provided $\mathcal{B}_0 \exp(2\Lambda h)h^{1/2} \leqslant \epsilon/2 \leqslant 1/2$, which implies

$$c\alpha\mathcal{B}_0\exp(2\Lambda h)h^{1/2} \leqslant c\alpha\mathcal{B}_0 h^{1/2} + c\Lambda h,$$

using the fact that $\exp(2\Lambda h) \leqslant 1 + 2\Lambda h\exp(2\Lambda h)$. Therefore, if $R \leqslant h$, then we can choose $t = R$ in the left-hand side above. By continuity of e on $[0,T]$, it then holds $e(R) = \mathcal{B}_0 h^{1/2}$, so that

$$\begin{aligned}(1-c\alpha)\mathcal{B}_0 h^{1/2} &\leqslant \exp(2\Lambda h)\big[(8+5c+8\epsilon^{-1})\Lambda h + 4(2+c+\epsilon^{-1})h^{1/2}\big]\\ &< \exp(2\Lambda)\big[(8+5c+8\epsilon^{-1})\Lambda + 4(2+c+\epsilon^{-1})\big]h^{1/2},\end{aligned}$$

which is not possible when

$$\mathcal{B}_0 = \frac{\exp(2\Lambda)[(8+5c+8\epsilon^{-1})\Lambda + 4(2+c+\epsilon^{-1})]}{1-c\alpha}.$$

Precisely, with $\mathcal{B}_0$ as above and $\mathcal{B}_0\exp(2\Lambda h)h^{1/2} \leqslant \epsilon/2$ it cannot hold $R \leqslant h$.

□

5.3. **Global Hölder bound.** In this subsection, we shall prove:

Proposition 5.3. *Let $\epsilon \in (0,1)$. Then there exists a positive constant $\alpha_0 \in (0,1]$, only depending upon ϵ, K and Λ, such that: whenever $\alpha < \alpha_0$, there exists a constant $\mathcal{B}$, only depending on α, ϵ, K and Λ, such that, for all positive times $T > 0$ and initial conditions $X_0 = x_0 \leqslant 1-\epsilon$, any solution to (2.1) according to Definition 2.2 satisfies*

$$\left.\begin{array}{l}\mathcal{B}h^{1/2} \leqslant \epsilon/2\\ t_0 + h \leqslant T\end{array}\right\} \Rightarrow e(t_0+h) - e(t_0) \leqslant \mathcal{B}h^{1/2},$$

for any $h \in (0,1)$ and $t_0 \in [0,T]$. Note that $\mathcal{B}$ above may differ from $\mathcal{B}_0$ in the statement of Lemma 5.2. The constant α_0 can be described as follows. Defining T_0 as the largest time less than 1 such that

$$\big(1-\epsilon\big)\exp(\Lambda T_0) \leqslant 1 - 7\epsilon/8, \quad \Lambda T_0\exp(\Lambda T_0) \leqslant \epsilon/8,$$

α_0 can be chosen as the largest (positive) real satisfying (with $B(T_0,\alpha_0,b)$ as in Lemma 5.1)

$$\begin{aligned}&\alpha_0 B(T_0,\alpha_0,b) \leqslant \epsilon/4,\\ &\alpha_0 2^{3/2}(c')^{3/2}\exp\left(-\tfrac{1}{2}\right)\left[\epsilon^{-1} + B(T_0,\alpha_0,b)\right] \leqslant 1,\\ &\alpha_0\left[c' T_0^{-1/2} + 2^{3/2}(c')^{3/2}\exp\left(-\tfrac{1}{2}\right)B(T_0,\alpha_0,b)\right] \leqslant 1.\end{aligned}$$

Here the constant c' is defined by the following property: $c' > 0$, depending on K only, is such that for any diffusion process $(U_t)_{0 \leqslant t \leqslant 1}$ satisfying

$$dU_t = F(t, U_t)dt + dW_t, \quad t \in [0,1],$$

where $U_0 = 0$ and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is K-Lipschitz in x such that $F(t,0) = 0$ for any $t \in [0,1]$, it holds that

$$\frac{1}{dx}\mathbb{P}(U_t \in dx) \leqslant \frac{c'}{\sqrt{t}} \exp\left(-\frac{x^2}{c't}\right), \quad x \in \mathbb{R},\ t \in (0,1].$$

The proof relies on the following:

Lemma 5.4. *Given an initial condition $X_0 = x_0 \leqslant 1 - \epsilon$, with $\epsilon \in (0,1)$, and a solution $(X_t)_{0 \leqslant t \leqslant T}$ to (2.1) on some interval $[0,T]$ according to Definition 2.2, the random variable X_t has a density on $(-\infty, 1]$, for any $t \in (0,T]$. Moreover, defining T_0 as in the statement of Proposition 5.3 and choosing $\alpha \leqslant \alpha_1$ satisfying*

$$\alpha_1 B(T_0, \alpha_1, b) \leqslant \epsilon/4,$$

it holds, for $x \in [1 - \epsilon/4, 1)$,

$$\frac{1}{dx}\mathbb{P}(X_t \in dx) \leqslant 2^{3/2}(c')^{3/2} \exp\left(-\tfrac{1}{2}\right)\left[\epsilon^{-1} + B(T_0, \alpha, b)\right] \quad \text{if } t \leqslant T_0$$
$$\frac{1}{dx}\mathbb{P}(X_t \in dx) \leqslant c' T_0^{-1/2} + 2^{3/2}(c')^{3/2} \exp\left(-\tfrac{1}{2}\right) B(T_0, \alpha, b) \quad \text{if } t > T_0,$$

where the constant c' is also as in the statement of Proposition 5.3.

Before we prove Lemma 5.4, we introduce some materials. As usual, we set $e(t) = \mathbb{E}(M_t)$, for $t \in [0,T]$, the mapping e being assumed to be continuously differentiable on $[0,T]$. Moreover, with $(X_t)_{0 \leqslant t \leqslant T}$, we associate the sequence of hitting times $(\tau_k)_{k \geqslant 0}$ given by (2.2) . We then investigate the marginal distributions of $(X_t)_{0 \leqslant t \leqslant T}$. Given a Borel subset $A \subset (-\infty, 1]$, we write in the same way as in the proof of (4.1)

$$\begin{aligned}\mathbb{P}(X_t \in A) &= \mathbb{P}(X_t \in A, \tau_1 > t) \\ &\quad + \sum_{k \geqslant 1} \int_0^t \mathbb{P}(X_t \in A, \tau_{k+1} > t | \tau_k = s)\mathbb{P}(\tau_k \in ds),\end{aligned} \tag{5.10}$$

where the notation $\mathbb{P}(\cdot|\tau_k = s)$ stands for the conditional law given $\tau_k = s$. Following (5.5) and (5.6), we can shift the system by length $s \in [0,T]$. Precisely, we know that $(X_r^{\sharp_s} := X_{s+r})_{0 \leqslant r \leqslant T-s}$ satisfies

$$X_r^{\sharp_s} = X_s + \int_0^r b\big(X_u^{\sharp_s}\big)du + \alpha e^{\sharp_s}(r) + W_{s+r} - W_s - M_r^{\sharp_s}, \tag{5.11}$$

with $e^{\sharp_s}(r) := e(s+r) - e(s)$, $M_r^{\sharp_s} := M_{s+r} - M_s$ and $\tau_k^{\sharp_s} := \inf\{u > \tau_{k-1}^{\sharp_s} : X_{s+u-} \geqslant 1\}$ for $k \geqslant 1$, $(\tau_0^{\sharp_s} := 0)$. Conditionally on $\tau_k = s$, the law of $(X_r^{\sharp_s})_{0 \leqslant r \leqslant T-s}$ until $\tau_1^{\sharp_s}$ coincides with the law of $(\hat{Z}_r^{\sharp_s,0})_{0 \leqslant r \leqslant T-s}$ until the first time it reaches 1, where, for a given $\mathcal{F}_0$-measurable initial condition ζ with values in $(-\infty, 1)$, $(\hat{Z}_r^{\sharp_s,\zeta})_{0 \leqslant r \leqslant T-s}$ stands for the solution of the SDE:

$$\hat{Z}_r^{\sharp_s,\zeta} = \zeta + \int_0^r b\big(\hat{Z}_u^{\sharp_s,\zeta}\big)du + \alpha e^{\sharp_s}(r) + W_r, \quad r \in [0, T-s]. \tag{5.12}$$

Below, we will write $\hat{Z}_r^{\zeta}$ for $\hat{Z}_r^{\sharp_0,\zeta}$. By (5.10),

$$\begin{aligned}\mathbb{P}(X_t \in A) &\leqslant \mathbb{P}(\hat{Z}_t^{X_0} \in A) + \sum_{k \geqslant 1} \int_0^t \mathbb{P}(\hat{Z}_{t-s}^{\sharp_s,0} \in A)\mathbb{P}(\tau_k \in ds) \\ &= \mathbb{P}(\hat{Z}_t^{X_0} \in A) + \int_0^t \mathbb{P}(\hat{Z}_{t-s}^{\sharp_s,0} \in A)e'(s)ds,\end{aligned} \tag{5.13}$$

for any Borel set $A \subset (-\infty, 1]$, the passage from the first to the second line following from (4.9).

We can now turn to:

Proof of Lemma 5.4. Given an initial condition $x_0 \in (-\infty, 1-\epsilon]$ for $\epsilon \in (0,1)$, we know from Delarue and Menozzi [6] that $\hat{Z}_t^{x_0}$ has a density for any $t \in (0,T]$ (and thus $\hat{Z}_{t-s}^{\sharp_s,0}$ as well for $0 \leqslant s < t$). From (5.13), we deduce that the law of X_t has a density on $(-\infty, 1]$ since $\mathbb{P}(X_t \in A) = 0$ when $|A| = 0$, where $|A|$ stands for the Lebesgue measure of A. Moreover, there exists a constant $c' \geqslant 1$, depending on K only, such that, for any $t \in [0, T \wedge 1]$:

$$\frac{1}{dx}\mathbb{P}(\hat{Z}_t^{x_0} \in dx) \leqslant \frac{c'}{\sqrt{t}} \exp\left(-\frac{[x - \vartheta_t^{x_0}]^2}{c't}\right), \tag{5.14}$$

where $\vartheta_t^{x_0}$ is the solution of the ODE:

$$\frac{d}{dt}\vartheta_t = b(\vartheta_t) + \alpha e'(t), \quad t \in [0,T], \tag{5.15}$$

with $\vartheta_0^{x_0} = x_0$. Above, the function $[0,T] \ni t \mapsto e(t)$ represents $[0,T] \ni t \mapsto \mathbb{E}(M_t)$ given $X_0 = x_0$, which means that the initial condition x_0 of X_0 upon which e depends is fixed once and for all, independently of the initial condition of ϑ. In particular, as the initial condition of ϑ varies, the function e does not. We emphasize that c' is independent of e and can be taken to be that defined in Proposition 5.3. Indeed, we can write $\mathbb{P}(\hat{Z}_t^{x_0} \in dx)$ as $\mathbb{P}(\hat{Z}_t^{x_0} - \vartheta_t^{x_0} \in d(x - \vartheta_t^{x_0}))$, with

$$\begin{aligned} d\big(\hat{Z}_t^{x_0} - \vartheta_t^{x_0}\big) &= F\big(t, \hat{Z}_t^{x_0} - \vartheta_t^{x_0}\big)dt + dW_t, \quad t \in [0,T], \quad \hat{Z}_0^{x_0} - \vartheta_0^{x_0} = 0\ ; \\ F(t,x) &= b\big(x + \vartheta_t^{x_0}\big) - b\big(\vartheta_t^{x_0}\big), \quad t \in [0,T], \quad x \in \mathbb{R}. \end{aligned}$$

We then notice that $F(t,\cdot)$ is K-Lipschitz continuous (since b is) and satisfies $F(t,0) = 0$, so that, referring to [6], all the parameters involved in the definition of the constant c' are independent of e. The fact that c' is independent of e is crucial. As a consequence, we can bound $(1/dx)\mathbb{P}(\hat{Z}_{t-s}^{\sharp_s,0} \in dx)$ in a similar way, that is, with the same constant c' as in (5.14): for any $0 \leqslant s < t \leqslant T$, with $t - s \leqslant 1$,

$$\frac{1}{dx}\mathbb{P}(\hat{Z}_{t-s}^{\sharp_s,0} \in dx) \leqslant \frac{c'}{\sqrt{t-s}} \exp\left(-\frac{[x - \vartheta_{t-s}^{\sharp_s,0}]^2}{c'(t-s)}\right), \tag{5.16}$$

where $\vartheta^{\sharp_s,0}$ is the solution of the ODE:

$$\frac{d}{dt}\vartheta_t^{\sharp_s} = b\big(\vartheta_t^{\sharp_s}\big) + \alpha\frac{d}{dt}e^{\sharp_s}(t), \quad t \in [0, T-s],$$

with $\vartheta_0^{\sharp_s,0} = 0$ as initial condition.

Bound of the density in small time. Keep in mind that $X_0 = x_0 \leqslant 1-\epsilon$. Therefore, by the comparison principle for ODEs, $\vartheta_t^{x_0} \leqslant \vartheta_t^{1-\epsilon}$ for any $t \in [0,T]$, so that by Gronwall's Lemma

$$\vartheta_t^{x_0} \leqslant \vartheta_t^{1-\epsilon} \leqslant \big(1-\epsilon+\Lambda T+\alpha e(T)\big)\exp(\Lambda T).$$

By Lemma 5.1, we know that $e(T) \leqslant B(T,\alpha,b)$, so that

$$\vartheta_t^{x_0} \leqslant \big(1-\epsilon+\Lambda T+\alpha B(T,\alpha,b)\big)\exp(\Lambda T). \tag{5.17}$$

Now choose T_0 as in Proposition 5.3, i.e. $T_0 \leqslant 1$ such that

$$\big(1-\epsilon\big)\exp(\Lambda T_0) \leqslant 1-7\epsilon/8, \quad \Lambda T_0 \exp(\Lambda T_0) \leqslant \epsilon/8,$$

and then take $\alpha_1 \in (0,1)$ such that

$$\alpha_1 B(T_0,\alpha_1,b)\exp(\Lambda T_0) \leqslant \epsilon/4.$$

Then, whenever $\alpha \leqslant \alpha_1$, it holds that

$$\vartheta_t^{x_0} \leqslant 1-\epsilon/2, \quad t \in [0, T_0 \wedge T].$$

Therefore, for $x \geqslant 1-\epsilon/4$,

$$\exp\left(-\frac{[x-\vartheta_t^{x_0}]^2}{c't}\right) \leqslant \exp\left(-\frac{\epsilon^2}{16c't}\right), \quad t \in [0, T_0 \wedge T]. \tag{5.18}$$

Similarly,

$$\vartheta_{t-s}^{\sharp_s,0} \leqslant 3\epsilon/8 \leqslant 3/8, \quad 0 \leqslant s \leqslant t \leqslant T_0 \wedge T.$$

Indeed, $e^{\sharp_s}(T-s) \leqslant e(T)$ for $s \in [0,T]$, so that (5.17) applies to $\vartheta_{t-s}^{\sharp_s,0}$ with $1-\epsilon$ therein being replaced by 0. Therefore, for $x \geqslant 1-\epsilon/4$, it holds that $x - \vartheta_{t-s}^{\sharp_s,0} \geqslant 3/4-3/8 = 3/8 \geqslant 1/4$, so that

$$\exp\left(-\frac{[x-\vartheta_{t-s}^{\sharp_s,0}]^2}{c'(t-s)}\right) \leqslant \exp\left(-\frac{1}{16c'(t-s)}\right), \quad 0 \leqslant s < t \leqslant T_0 \wedge T. \tag{5.19}$$

In the end, for $x \in (1-\epsilon/4, 1)$ and $t \leqslant T_0 \wedge T$, we deduce from (5.13), (5.14), (5.16), (5.18), (5.19) and Lemma 5.1 again, that

$$\frac{1}{dx}\mathbb{P}(X_t \in dx) \leqslant c'\varpi_0\big[\epsilon^{-1}+e(T\wedge T_0)\big] \leqslant c'\varpi_0\big[\epsilon^{-1}+B(T_0,\alpha,b)\big], \tag{5.20}$$

where

$$\varpi_0 = \sup_{t>0}\left[t^{-1/2}\exp\left(-\frac{1}{16c't}\right)\right] = 4\sqrt{c'}\sup_{u>0}\big[u\exp(-u^2)\big] = 2^{3/2}\sqrt{c'}\exp\left(-\frac{1}{2}\right).$$

Bound of the density in long time. We now discuss what happens for $T > T_0$ and $t \in [T_0, T]$. Then,

$$\begin{aligned} \frac{1}{dx}\mathbb{P}(X_t \in dx) &\leqslant \frac{1}{dx}\mathbb{P}\left(X_t \in dx, \tau_1^{\sharp_{t-T_0}} \leqslant T_0\right) + \frac{1}{dx}\mathbb{P}\left(X_t \in dx, \tau_1^{\sharp_{t-T_0}} > T_0\right) \\ &= \pi_1 + \pi_2, \end{aligned} \tag{5.21}$$

with $\tau_1^{\sharp_{t-T_0}} = \inf\{u > 0 : X_{t-T_0+u-} \geqslant 1\} = \inf\{u > 0 : X_{u-}^{\sharp_{t-T_0}} \geqslant 1\}$. The above expression says that we split the event (X_t is in the neighbourhood of x) into two disjoint parts according to the fact that X reaches the threshold or not within the time window $[t - T_0, t]$. We have chosen this interval to be of length T_0 in order to apply the results in small time.

We first investigate π_2. The point is that, on the event that $\tau_1^{\sharp_{t-T_0}} > T_0$ and within the time window $[t - T_0, t]$, X behaves as a standard diffusion process without any jumps, namely as a process with the same dynamics as $\hat{Z}^{\sharp_{t-T_0}, X_{t-T_0}}$. Following (5.14), we then have

$$\begin{aligned} \pi_2 &= \frac{1}{dx}\mathbb{P}\left(\hat{Z}_{T_0}^{\sharp_{t-T_0}, X_{t-T_0}} \in dx, \tau_1^{\sharp_{t-T_0}} > T_0\right) \\ &\leqslant \frac{1}{dx}\mathbb{P}\left(\hat{Z}_{T_0}^{\sharp_{t-T_0}, X_{t-T_0}} \in dx\right) \leqslant \sup_{z \leqslant 1} \frac{1}{dx}\mathbb{P}\left(\hat{Z}_{T_0}^{\sharp_{t-T_0}, z} \in dx\right) \leqslant c' T_0^{-1/2}. \end{aligned} \tag{5.22}$$

We now turn to π_1. Here we write

$$\begin{aligned} \pi_1 &= \frac{1}{dx}\mathbb{P}\left(X_t \in dx, \tau_1^{\sharp_{t-T_0}} \leqslant T_0\right) = \sum_{k \geqslant 1} \frac{1}{dx}\mathbb{P}\left(X_t \in dx, \tau_k^{\sharp_{t-T_0}} \leqslant T_0 < \tau_{k+1}^{\sharp_{t-T_0}}\right) \\ &= \sum_{k \geqslant 1} \int_0^{T_0} \frac{1}{dx}\mathbb{P}\left(X_t \in dx, T_0 < \tau_{k+1}^{\sharp_{t-T_0}} | \tau_k^{\sharp_{t-T_0}} = s\right) \mathbb{P}(\tau_k^{\sharp_{t-T_0}} \in ds) \\ &= \sum_{k \geqslant 1} \int_0^{T_0} \frac{1}{dx}\mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp_{s+t-T_0}, 0} \in dx, T_0 < \tau_{k+1}^{\sharp_{t-T_0}}\right) \mathbb{P}(\tau_k^{\sharp_{t-T_0}} \in ds), \end{aligned}$$

since on the event $\{\tau_k^{\sharp_{t-T_0}} \leqslant T_0 < \tau_{k+1}^{\sharp_{t-T_0}}\}$, given that the k-th (and last) jump of X in the interval $[t - T_0, t]$ occurs at time $t - T_0 + s$ with $s \in [0, T_0]$, we have that the process X_r for $r \in [t - T_0 + s, t]$ coincides with the process $\hat{Z}_u^{\sharp_{s+t-T_0}, 0}$ for $u \in [0, T_0 - s]$. Thus

$$\begin{aligned} \pi_1 &\leqslant \sum_{k \geqslant 1} \int_0^{T_0} \frac{1}{dx}\mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp_{s+t-T_0}, 0} \in dx\right) \mathbb{P}(\tau_k^{\sharp_{t-T_0}} \in ds) \\ &= \int_0^{T_0} \frac{1}{dx}\mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp_{s+t-T_0}, 0} \in dx\right) e'(s + t - T_0) ds. \end{aligned} \tag{5.23}$$

By (5.16), we have

$$\begin{aligned}\int_0^{T_0} \frac{1}{dx}\mathbb{P}\left(\hat{Z}^{\sharp_{s+t-T_0},0}_{T_0-s} \in dx\right) e'(s+t-T_0)ds \\ \leqslant \int_0^{T_0} \frac{c'}{\sqrt{T_0-s}} \exp\left(-\frac{[x-\vartheta^{\sharp_{s+t-T_0},0}_{T_0-s}]^2}{c'(T_0-s)}\right) e'(s+t-T_0)ds.\end{aligned}$$

Recalling that $e^{\sharp_{t-T_0}}(s) = \mathbb{E}(M_{s+t-T_0} - M_{t-T_0})$, it is well seen that the mapping $[0,T_0] \ni s \mapsto e^{\sharp_{t-T_0}}(s)$ satisfies Lemma 5.1, that is

$$\sup_{0 \leqslant s \leqslant T_0} e^{\sharp_{t-T_0}}(s) = \sup_{0 \leqslant s \leqslant T_0} \big[e(s+t-T_0) - e(t-T_0)\big] = e(t) - e(t-T_0) \leqslant B(T_0,\alpha,b).$$

Therefore, we can follow the same strategy as in short time, see (5.19) and (5.20). Indeed, for $\alpha \leqslant \alpha_1$, by the choice of T_0 as before, it holds that

$$\pi_1 \leqslant c'\varpi_0 B(T_0,\alpha,b),$$

for $x \in [1-\epsilon/4, 1)$. Using (5.22) and the above bound, we deduce that, for $t \in [T_0, T]$,

$$\frac{1}{dx}\mathbb{P}\left(X_t \in dx\right) \leqslant c'\big[T_0^{-1/2} + \varpi_0 B(T_0,\alpha,b)\big].$$

□

Proof of Proposition 5.3. Proposition 5.3 follows from the combination of Lemmas 5.2 and 5.4. Indeed, given T_0 and α_0 as defined in Proposition 5.3, then by Lemma 5.4 it follows that $\mathbb{P}(X_t \in A) < (1/\alpha)|A|$ for any Borel subset $A \subset [1-\epsilon/4, 1]$, any $\alpha < \alpha_0$ and any $t \in [0,T]$. The result follows by Lemma 5.2, with $\mathcal{B}$ being given by $\mathcal{B}_0 \exp(2\Lambda)$ with ϵ in $\mathcal{B}_0$ replaced by $\epsilon/4$. □

5.4. **Estimate of the density of the killed process.** In light of the previous subsection, for a solution $(X_t)_{0 \leqslant t \leqslant T}$ to (2.1) such that the mapping $[0,T] \ni t \mapsto e(t) = \mathbb{E}(M_t)$ is continuously differentiable, we here investigate

$$\frac{1}{dx}\mathbb{P}\left(X_t \in dx, t < \tau_1\right), \quad t \in [0,T],\ x \leqslant 1,$$

where $\tau = \inf\{t > 0 : X_{t-} \geqslant 1\}$ as usual. This is the density of the killed process $(X_{t\wedge\tau_1})_{0 \leqslant t \leqslant T}$, which makes sense because of Lemma 4.2.

Here is the main result of this subsection:

Lemma 5.5. *Let $\epsilon \in (0,1)$, $T > 0$ and $\mathcal{B} > 0$. Moreover, let $(\chi_t)_{0 \leqslant t \leqslant T}$ denote the solution to the SDE*

$$d\chi_t = b(\chi_t)dt + \alpha e'(t)dt + dW_t, \quad t \in [0,T]\ ; \quad \chi_0 = x_0,$$

for some continuously differentiable non-decreasing deterministic mapping $[0,T] \ni t \mapsto e(t)$ satisfying

$$e(0) = 0, \qquad e(t) - e(s) \leqslant \mathcal{B}(t-s)^{1/2}, \quad 0 \leqslant s \leqslant t \leqslant T.$$

Then there exist two positive constants μ_T and η_T, only depending upon T, $\mathcal{B}$, ϵ, K and Λ, such that, for any initial condition $x_0 \leqslant 1-\epsilon$,

$$p(t,y) \leqslant \mu_T(1-y)^{\eta_T}, \quad t \in [0,T],\ y \in [1-\epsilon/4, 1], \tag{5.24}$$

where $p(t,y)$ denotes the density of χ_t killed at 1 as in (4.2).

Proof. First Step. The first step is to provide a probabilistic representation for p. For a given $(T,x) \in (0,+\infty) \times (-\infty,1)$, we consider the solution to the SDE:

$$dY_t = -\big[b(Y_t) + \alpha e'(T-t)\big]dt + dW_t, \quad t \in [0,T], \quad Y_0 = y, \tag{5.25}$$

together with some stopping time $\rho \leqslant \rho_0 \wedge T$, where $\rho_0 = \inf\{t \in [0,T] : Y_t \geqslant 1\}$ (with $\inf \emptyset = +\infty$). Then, by Lemma 4.2 and the Itô-Krylov formula (see [11, Chapter II, Section 10]),

$$\begin{aligned}
&d\big(p(T-t,Y_t)\big) \\
&= -\partial_t p(T-t,Y_t)dt - \big[b(Y_t) + \alpha e'(T-t)\big]\partial_y p(T-t,Y_t)dt + \frac{1}{2}\partial^2_{yy} p(T-t,Y_t)dt \\
&\quad + \partial_y p(T-t,Y_t)dW_t \\
&= b'(Y_t)p(T-t,Y_t)dt + \partial_y p(T-t,Y_t)dW_t,
\end{aligned}$$

for $0 \leqslant t \leqslant \rho$. Therefore, the Feynman-Kac formula yields

$$p(T,y) = \mathbb{E}\bigg[p(T-\rho,Y_\rho)\mathbf{1}_{\{Y_\rho \neq 1\}} \exp\bigg(-\int_0^\rho b'(Y_s)ds\bigg)\big|Y_0 = y\bigg], \tag{5.26}$$

the indicator function following from the Dirichlet boundary condition satisfied by $p(\cdot,1)$.

Second Step. We now specify the choice of ρ. Given some free parameters $L \geqslant 1$ and $\delta \in (0,\epsilon/4)$ such that $L\delta \leqslant \epsilon/4$, we assume that the initial condition y in (5.25) is in $(1-\delta,1)$ and then consider the stopping time

$$\rho = \inf\{t \in [0,T] : Y_t \notin (1-L\delta,1)\} \wedge \delta^2. \tag{5.27}$$

Assume that $\delta^2 \leqslant T$. By (5.26), we deduce that

$$p(T,y) \leqslant \exp(K\delta^2)(1-\mathbb{P}(Y_\rho = 1)) \sup_{(t,z)\in\mathcal{Q}(\delta,L)} p(t,z), \tag{5.28}$$

with

$$\mathcal{Q}(\delta,L) = \big\{(t,z) \in [T-\delta^2,T] \times [1-L\delta,1]\big\}.$$

The point is then to give a lower bound for $\mathbb{P}(Y_\rho = 1)$. By assumption, we know that e is $(1/2)$-Hölder continuous on $[0,T]$. Therefore, since $Y_0 = y \in (1-\delta,1)$, we have, for any $t \in [0,\rho]$,

$$Y_t \geqslant 1 - \delta - m\delta^2 - \alpha\mathcal{B}\delta + W_t,$$

with

$$m = \sup_{0 \leqslant z \leqslant 1} |b(z)|. \tag{5.29}$$

Therefore, for $m\delta \leqslant 1$,

$$Y_t \geqslant 1 - 2\delta - \alpha\mathcal{B}\delta + W_t, \quad t \in [0,\rho],$$

so that

$$\big\{Y_\rho = 1\big\} \supset \bigg\{\sup_{0 \leqslant t \leqslant \delta^2} W_t > (2+\alpha\mathcal{B})\delta\bigg\} \cap \bigg\{\inf_{0 \leqslant t \leqslant \delta^2} W_t > (2+\alpha\mathcal{B}-L)\delta\bigg\}. \tag{5.30}$$

Choosing $L = 3 + \alpha\mathcal{B}$ and applying a scaling argument, we deduce that

$$\begin{aligned}&\mathbb{P}\left(\left\{\sup_{0 \leqslant t \leqslant \delta^2} W_t > (2+\alpha\mathcal{B})\delta\right\} \cap \left\{\inf_{0 \leqslant t \leqslant \delta^2} W_t > (2+\alpha\mathcal{B}-L)\delta\right\}\right)\\ &= \mathbb{P}\left(\left\{\sup_{0 \leqslant t \leqslant 1} W_t > (2+\alpha\mathcal{B})\right\} \cap \left\{\inf_{0 \leqslant t \leqslant 1} W_t > -1\right\}\right) =: c'' \in (0,1).\end{aligned} \tag{5.31}$$

We note that the above quantity c'' is independent of δ and T. Moreover, we deduce from (5.30) that $\mathbb{P}(Y_\rho = 1) \geqslant c''$ and therefore, from (5.28), that

$$p(T,y) \leqslant (1-c'')\exp(K\delta^2) \sup_{z \in \mathcal{I}(L\delta)} \sup_{t\in[0,T]} p(t,z),$$

with $\mathcal{I}(r) = [1-r, 1]$, for $r > 0$. Choosing δ small enough such that $(1 - c'')\exp(K\delta^2) \leqslant (1-c''/2)$, we obtain

$$p(T,y) \leqslant \left(1 - \frac{c''}{2}\right) \sup_{z \in \mathcal{I}(L\delta)} \sup_{t\in[0,T]} p(t,z), \quad y \in \mathcal{I}(\delta).$$

Modifying c'' if necessary (c'' being chosen as small as needed), we can summarize the above inequality as follows: for $\delta \leqslant c''$,

$$p(T,y) \leqslant (1-c'') \sup_{z \in \mathcal{I}(L\delta)} \sup_{t\in[0,T]} p(t,z), \quad y \in \mathcal{I}(\delta). \tag{5.32}$$

We now look at what happens when $T \leqslant \delta^2$ in (5.28). In this case we can replace ρ in the previous argument by $\rho \wedge T$. Observing that $p(T - \rho\wedge T, Y_{\rho\wedge T}) = 0$ on the event $\{\rho \geqslant T\} \cup \{Y_{\rho\wedge T} = 1\}$ (since $p(0,\cdot) = 0$ on $[1-\epsilon/4, 1]$) and following (5.28), we obtain, for $y \in \mathcal{I}(\delta)$,

$$p(T,y) \leqslant \exp(K\delta^2)\big[1 - \mathbb{P}\big(\{Y_{\rho\wedge T} = 1\} \cup \{\rho \geqslant T\}\big)\big] \sup_{(t,z)\in\mathcal{Q}'(\delta,L)} p(t,z), \tag{5.33}$$

with $\mathcal{Q}'(\delta, L) = \{(t,z) \in [0,T] \times [1-L\delta, 1]\}$. Now, the right-hand side of (5.30) is included in the event $\{Y_{\rho\wedge T} = 1\} \cup \{\rho \geqslant T\}$ so that (5.31) yields a lower bound for $\mathbb{P}(\{Y_{\rho\wedge T} = 1\} \cup \{\rho \geqslant T\})$. Therefore, we can repeat the previous arguments in order to prove that (5.32) also holds when $T \leqslant \delta^2$, which means that (5.32) holds true in both cases.

Therefore, by replacing T by t in the left-hand side in (5.32) and by letting t vary within $[0,T]$, we have in any case,

$$\sup_{y\in\mathcal{I}(\delta)} \sup_{t\in[0,T]} p(t,y) \leqslant (1-c'') \sup_{z \in \mathcal{I}(L\delta)} \sup_{t\in[0,T]} p(t,z).$$

By induction, for any integer $n \geqslant 1$ such that $L^n\delta \leqslant r_0$, with $r_0 = c'' \wedge (\epsilon/4)$,

$$\sup_{y\in\mathcal{I}(\delta)} \sup_{t\in[0,T]} p(t,y) \leqslant (1-c'')^n \sup_{z \in \mathcal{I}(L^n\delta)} \sup_{t\in[0,T]} p(t,z).$$

Given $\delta \in (0, r_0/L)$, the maximal value for n is $n = \lfloor \ln[r_0/\delta]/\ln L \rfloor$. We deduce that, for any $\delta \in (0, r_0/L)$,

$$\sup_{y\in\mathcal{I}(\delta)} \sup_{t\in[0,T]} p(t,y) \leqslant (1-c'')^{(\ln[r_0/\delta]/\ln L)-1} \sup_{z \in \mathcal{I}(\epsilon/4)} \sup_{t\in[0,T]} p(t,z). \tag{5.34}$$

Following (5.14), we know that

$$\sup_{z\in\mathcal{I}(\epsilon/4)}\sup_{t\in[0,T]} p(t,z) \leqslant \sup_{z\in\mathcal{I}(\epsilon/4)}\sup_{t\in[0,T]}\left[\frac{c_T}{\sqrt{t}}\exp\left(-\frac{[z-\vartheta_t^{x_0}]^2}{c_T t}\right)\right], \tag{5.35}$$

for some constant c_T only depending upon T and K and where $(\vartheta_t^{x_0})_{0\leqslant t\leqslant T}$ stands for the solution of the ODE

$$\frac{d\vartheta}{dt} = b(\vartheta_t) + \alpha e'(t), \quad t\in[0,T]\ ; \quad \vartheta_0 = x_0.$$

Pay attention that we here use the same notation as in (5.15) for the solution of the above ODE but here $e(t)$ is not given as some $\mathbb{E}(M_t)$. Actually, we feel that there is no possible confusion here. Notice also that e is fixed and does not depend upon the initial condition x_0.

By the comparison principle for ODEs and then by Gronwall's Lemma, we deduce from the fact that e is $(1/2)$-Hölder continuous that

$$\vartheta_t^{x_0} \leqslant \vartheta_t^{1-\epsilon} \leqslant \left[1-\epsilon+\Lambda t+\mathcal{B}t^{1/2}\right]\exp(\Lambda t), \quad t\in[0,T].$$

Using the above inequality, we can bound the right-hand side in (5.35). Precisely, the above inequality says that the exponential term in the supremum decays exponentially fast as t tends to 0 so that the term inside the supremum can be bounded when t is small; when t is bounded away from 0, the term inside the supremum is bounded by $c_T/\sqrt{t}$. It is plain to deduce that

$$\sup_{z\in\mathcal{I}(\epsilon/4)}\sup_{t\in[0,T]} p(t,z) \leqslant c_T, \tag{5.36}$$

for a new value of c_T, possibly depending on ϵ as well. Therefore, for $\delta\in(0,r_0/L)$, (5.34) yields

$$\sup_{y\in\mathcal{I}(\delta)}\sup_{t\in[0,T]} p(t,y) \leqslant \frac{c_T}{(1-c'')}\left(\frac{\delta}{r_0}\right)^{\eta},$$

with $\eta = -\ln(1-c'')/\ln L$. This proves (5.24) for $y\in(1-r_0/L,1)$. Note that η is here independent of T, contrary to what is indicated in the statement of Lemma 5.5. However, we feel it is simpler to indicate T in η_T as the constant $\mathcal{B}$ in the sequel will be chosen in terms of T thus making η depend on T. Using (5.36), we can easily extend the bound to any $y\in(1-\epsilon/4,1)$ by modifying if necessary the parameters μ_T and η_T therein. This completes the proof. □

5.5. **Bound for the gradient.** Here is the final step to complete the proof of Theorem 2.3:

Proposition 5.6. *Let $\epsilon\in(0,1)$, $T>0$ and $\mathcal{B}>0$. Moreover, let $(\chi_t)_{0\leqslant t\leqslant T}$ denote the solution to the SDE*

$$d\chi_t = b(\chi_t)dt + \alpha e'(t)dt + dW_t, \quad t\in[0,T], \quad \chi_0 = x_0,$$

for some continuously differentiable non-decreasing deterministic mapping $[0,T]\ni t\mapsto e(t)$ satisfying

$$e(0)=0 \quad ; \quad e(t)-e(s)\leqslant \mathcal{B}(t-s)^{1/2}, \quad 0\leqslant s\leqslant t\leqslant T.$$

Then there exists a constant $\mathcal{M}_T > 0$, only depending upon T, $\mathcal{B}$, ϵ, K and Λ, such that, for any initial condition $x_0 \leqslant 1-\epsilon$ and any integer n such that $n \geqslant \lceil 4/\epsilon \rceil$,

$$|\partial_y p(t,1)| \leqslant \frac{\mathcal{M}_T n^{-\eta_T}}{1-\exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}]}(1+\alpha C_T), \quad t \in [0,T],$$

where $p(t,y)$ is the density of χ_t killed at 1 as in (4.2), η_T is as in Lemma 5.5, and

$$C_T = \sup_{0 \leqslant t \leqslant T} e'(t).$$

Proof. We consider the barrier function

$$q(t,y) = \Theta \exp(Kt)\big[1-\exp(\gamma(y-1))\big], \quad t \geqslant 0,\ y \in \mathbb{R}, \tag{5.37}$$

where γ and Θ are free nonnegative parameters. Then, for $t>0$ and $y<1$,

$$\begin{aligned}&\partial_t q(t,y) + \big(b(y)+\alpha e'(t)\big)\partial_y q(t,y) - \frac{1}{2}\partial^2_{yy} q(t,y)\\ &= \Theta \exp(Kt)\exp(\gamma(y-1))\big(-(b(y)+\alpha e'(t))\gamma + \frac{1}{2}\gamma^2\big) + Kq(t,y).\end{aligned}$$

Keeping in mind that $\sup_{0 \leqslant t \leqslant T} e'(t) = C_T$ and choosing

$$\gamma = 2(\max(m,1)+\alpha C_T), \tag{5.38}$$

where $m = \sup_{0 \leqslant z \leqslant 1} |b(z)|$ as before, we obtain, for $t \in [0,T]$ and $y \in (0,1)$,

$$-\big(b(y)+\alpha e'(t)\big)\gamma + \frac{1}{2}\gamma^2 \geqslant -2(\max(m,1)+\alpha C_T)^2 + 2(\max(m,1)+\alpha C_T)^2 = 0.$$

Thus, for $t \in [0,T]$ and $y \in (0,1)$,

$$\partial_t q(t,y) + \big(b(y)+\alpha e'(t)\big)\partial_y q(t,y) - \frac{1}{2}\partial^2_{yy} q(t,y) \geqslant Kq(t,y) \geqslant -b'(y)q(t,y),$$

which reads

$$\partial_t q(t,y) + \partial_y\big[\big(b(y)+\alpha e'(t)\big)q(t,y)\big] - \frac{1}{2}\partial^2_{yy} q(t,y) \geqslant 0. \tag{5.39}$$

For a given integer $n \geqslant \lceil 4/\epsilon \rceil$, we choose Θ as the solution of

$$\Theta\left[1-\exp\left(-\frac{2(\max(m,1)+\alpha C_T)}{n}\right)\right] = \mu_T n^{-\eta_T}, \tag{5.40}$$

with μ_T and η_T as in the statement of Lemma 5.5. Pay attention that the factor in the left-hand side cannot be 0 as $\max(m,1)>0$. Notice also q thus depends upon n. By Lemma 5.5, we deduce that

$$q\left(t, 1-\frac{1}{n}\right) \geqslant p\left(t, 1-\frac{1}{n}\right), \quad 0 \leqslant t \leqslant T.$$

Now, we can apply the comparison principle for PDEs (see [14, Chap. IX, Thm. 9.7]). Indeed, we also observe that $q(0,y) \geqslant p(0,y) = 0$ for $y \in [1-1/n, 1]$ and $q(t,1) = p(t,1) = 0$ for $t \in [0,T]$. Therefore, by (5.39), we have

$$p(t,y) \leqslant q(t,y), \quad t \in [0,T],\ y \in \left[1-\frac{1}{n}, 1\right]. \tag{5.41}$$

Since $p(t,1) = 0 = q(t,1)$, we deduce

$$|\partial_y p(t,1)| \leqslant |\partial_y q(t,1)| = \frac{2\mu_T(\max(m,1)+\alpha C_T)n^{-\eta_T}}{1-\exp[-2(\max(m,1)+\alpha C_T)/n]}\exp(Kt). \tag{5.42}$$

□

We now complete the proof of Theorem 2.3. We make use of Proposition 4.5. Recall (4.6)

$$e'(t) = -\int_0^t \frac{1}{2}\partial_y p^{(0,s)}(t-s,1)e'(s)ds - \frac{1}{2}\partial_y p(t,1), \quad t \in [0,T],$$

where p represents the density of the process X killed at 1 and $p^{(0,s)}$ represents the density of the process $X^{\sharp_s}$ driven by $e^{\sharp_s} = e(\cdot + s) - e(s)$ (see (5.11)) killed at 1 with $X_0^{\sharp_s} = 0$ as initial condition.

By Proposition 5.3 and Lemma 5.1, we know that, for a given $s \in [0,T)$ and for the prescribed values of α, the mapping $[0,T-s] \ni r \mapsto e^{\sharp_s}(r)$ is 1/2-Hölder continuous, the Hölder constant only depending upon T, α, ϵ, K and Λ (Proposition 5.3 permits to bound the increments of $e^{\sharp_s}$ on small intervals and Lemma 5.1 gives a trivial bound for the increments of $e^{\sharp_s}$ on large intervals). Therefore, by Proposition 5.6, we know that

$$|\partial_y p^{(0,s)}(t-s,1)| \leqslant \frac{\mathcal{M}_T n^{-\eta_T}}{1-\exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}]}(1+\alpha C_T), \quad t \in [s,T], \tag{5.43}$$

for $n \geqslant \lceil 4/\epsilon \rceil$ and for some constant $\mathcal{M}_T$ only depending upon T, α, ϵ, K and Λ. The same bound also holds true for $\partial_y p(t,1)$.

We deduce that, for any $t \in [0,T]$ and any n such that $n \geqslant \lceil 4/\epsilon \rceil$,

$$e'(t) \leqslant \frac{\mathcal{M}_T n^{-\eta_T}}{1-\exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}]}(1+\alpha C_T)\frac{e(T)+1}{2}.$$

By Lemma 5.1, we have a bound for $e(T) = \mathbb{E}(M_T)$, which means that we can bound $(e(T)+1)/2$ in the right-hand side above by modifying the constant $\mathcal{M}_T$. Recalling

$$C_T = \sup_{0 \leqslant t \leqslant T} e'(t),$$

we deduce that

$$C_T\big(1-\exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}]\big) \leqslant \mathcal{M}_T(1+\alpha C_T)n^{-\eta_T}. \tag{5.44}$$

Choosing n large enough such that the right hand side is less than $(1+\alpha C_T)/2$ (so that n depends on T) and multiplying by α, we get (since $\alpha \in (0,1)$):

$$\frac{\alpha C_T}{2} \leqslant \frac{1}{2} + \alpha C_T \exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}].$$

This shows that αC_T must be bounded in terms of $\mathcal{M}_T$ and n. Precisely, we have

$$\alpha C_T \leqslant 1 + 2\sup_{r \geqslant 0}\big[r\exp[-\mathcal{M}_T^{-1}(1+r)n^{-1}]\big] := R < +\infty.$$

By (5.44), we deduce that

$$C_T \leqslant \sup_{0 \leqslant r \leqslant R} \left[\frac{\mathcal{M}_T(1+r)n^{-\eta_T}}{1 - \exp[-\mathcal{M}_T^{-1}(1+r)n^{-1}]} \right],$$

which is independent of α (for $\alpha \in (0, \alpha_0]$), as required. □

6. Proofs of Theorem 2.4

In this section we put everything together to arrive at our goal, which is the proof of Theorem 2.4. We first need the following lemma, which is a corollary of Theorem 2.3. The point is that the result will allow us to re-apply the fixed point result on successive time intervals, since it guarantees that the conditions of the fixed point result are satisfied at the final point of any interval on which we know there is a solution.

Lemma 6.1. *For any $T > 0$, initial condition $X_0 = x_0 < 1$, and $\alpha < \alpha_0$, where $\alpha_0 = \alpha_0(x_0)$ is as in Theorem 2.3, there exists a constant $C_{\mathrm{den}}(T)$ depending only on T, x_0, K and Λ such that any solution to (2.1) on $[0,T]$ satisfies*

$$\frac{1}{dy}\mathbb{P}\left(X_t \in dy\right) \leqslant C_{\mathrm{den}}(T)(1-y),$$

for all $y \in (1 - \epsilon/8, 1)$ and $t \in [0,T]$, with $\epsilon = \min(1, 1-x_0)$.

Proof. We assume that $(X_t)_{0 \leqslant t \leqslant T}$ is a solution to (2.1) with $X_0 = x_0$ up until time T, and set $e(t) = \mathbb{E}(M_t)$. Following the notation of Section 4 (see also the last part of the proof of Theorem 2.3), for $y \leqslant 1$ and $t \leqslant T$, let

$$p(t,y) := \frac{1}{dy}\mathbb{P}\left(X_t \in dy, t < \tau_1\right),$$

$$p^{(0,s)}(t,y) := \frac{1}{dy}\mathbb{P}\left(X_t^{\sharp_s} \in dy, t < \tau_1^{\sharp_s} | X_0^{\sharp_s} = 0\right).$$

By Theorem 2.3, we know that e is $\mathcal{M}_T$-Lipschitz continuous, so that by (5.41),

$$p(t,y) \leqslant q(t,y), \quad t \in [0,T],\ y \in \left[1 - \frac{1}{n}, 1\right],$$

where n stands for $\lceil 4/\epsilon \rceil$ and q is given by (5.37), with γ and Θ being fixed by (5.38) and (5.40), with $C_T = \mathcal{M}_T$. By the specific form of q, this says that there exists a constant C'_T, depending only on T, x_0, K and Λ, such that

$$p(t,y) \leqslant C'_T(1-y), \quad t \in [0,T],\ y \in \left[1 - \frac{\epsilon}{8}, 1\right],$$

using the elementary inequality $1 - \exp(-x) \leqslant x$ for $x \in \mathbb{R}$. Clearly, the same argument applies to $p^{(0,s)}(t-s,y)$, i.e.

$$p^{(0,s)}(t-s,y) \leqslant C'_T(1-y), \quad 0 \leqslant s < t \leqslant T,\ y \in \left[1 - \frac{\epsilon}{8}, 1\right].$$

Now, following the proof of (5.10), we get for $t \in [0,T]$ and $y \in [1-\epsilon/8, 1]$,

$$\frac{1}{dy}\mathbb{P}(X_t \in dy) = p(t,y) + \int_0^t p^{(0,s)}(t-s,y)e'(s)ds \leqslant C'_T(1+e(T))(1-y), \quad (6.1)$$

where we use Lemma 4.2 for justifying the passage to the density in (5.10). By Lemma 5.1, this completes the proof. □

Finally, we can then prove the main result of the present paper:

Proof of Theorem 2.4: We would like a solution up until fixed time $T > 0$. The idea is to iterate the fixed point result (Theorem 4.1), which is possible thanks to Lemma 6.1. Indeed, by Theorem 4.1, we have that there exists a solution to (2.1) with $X_0 = x_0$ up until some small time $T_1 > 0$. By Lemma 6.1, we thus have that

$$\frac{1}{dy}\mathbb{P}(X_{T_1} \in dy) \leqslant C_{\mathrm{den}}(T_1)(1-y), \quad y \in \left[1-\frac{\epsilon}{8}, 1\right], \tag{6.2}$$

where $\epsilon = \min(1-x_0, 1)$. If $T_1 \geqslant T$ we are done. If not, we have the above density bound for $(1/dy)\mathbb{P}(X_{T_1} \in dy)$. We also know from (6.1) and Lemma 4.2 that the density of X_{T_1} is differentiable at $y = 1$. Therefore, we can apply Theorem 4.1 again to see that there exists a solution to (2.1) on some interval $[T_1, T_1+T_2]$ starting from X_{T_1}. As T_2 only depends upon X_{T_1} through ϵ (this is the statement of Theorem 4.1) and $C_{\mathrm{den}}(T_1)$ and as these quantities can be bounded in terms of T, ϵ, K, Λ only, we then see that

$$T_2 \geqslant \phi(T)$$

for some constant $\phi(T)$ that refers to T, α, ϵ, K, Λ only. Now we know that there exists a solution to (2.1) with $X_0 = x_0$ on $[0, T_1 + T_2]$. If $T_1 + T_2 > T$ we are done. If not, by Lemma 6.1 once again,

$$\frac{1}{dy}\mathbb{P}(X_{T_1+T_2} \in dy) \leqslant C_{\mathrm{den}}(T_1+T_2)(1-y), \quad y \in \left[1-\frac{\epsilon}{8}, 1\right],$$

and we can then repeat the argument n times to get a solution up until time $T_1 + \cdots + T_n$, where all $T_k \geqslant \phi(T)$ for $k \geqslant 2$ i.e. each time step is of size at least $\phi(T)$. It is then clear that there exists $n \geqslant 1$ such that $T_1 + \cdots + T_n \geqslant T$, and so we are done for the existence of a solution.

Uniqueness of the solution proceeds in the same way. Given another solution $(X'_t, M'_t)_{0 \leqslant t \leqslant T}$ on the interval $[0, T]$ in the sense of Definition 2.2, it must satisfy the *a priori* estimates in the statements of Theorem 2.3 and Lemmas 5.1 and 6.1. In particular, dividing the interval $[0, T]$ into subintervals of length $\phi(T)$ (except for the last interval the length of which might be less than $\phi(T)$), with the same $\phi(T)$ as above, we can apply the contraction property in Theorem 4.1 on each subinterval iteratively. Precisely, choosing A_1 accordingly in Theorem 4.1, we prove by induction that the two solutions coincide on $[0, \phi(T)]$, $[0, 2\phi(T)]$, and so on. □

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