Periodic motions of coupled impact oscillators

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Summary. We study existence of time-periodic oscillations in a chain of coupled impact oscillators, for rigid impacts without energy dissipation. We formulate the search of periodic solutions as a boundary value problem incorporating unilateral constraints. This problem is solved numerically and different solution branches corresponding to nonlinear localized modes (breathers) and normal modes are computed.

Understanding the dynamics of nonlinear lattices (i.e. large networks of coupled nonlinear oscillators) is a problem of fundamental importance in mechanics, condensed matter physics or biology. One of the major issues concerns the mathematical analysis and numerical computation of special classes of nonlinear time-periodic oscillations that organize the dynamics in many situations. In particular, spatially periodic waves (standing waves or periodic traveling waves) and spatially localized waves (breathers) are the object of intensive research [9, 4]. In this context, many theoretical and numerical works have focused on smooth nonlinear systems, whereas relatively few mathematical results are available for waves in nonsmooth infinite lattices [5, 6, 7]. Developing theoretical and numerical tools for the analysis of nonlinear waves in nonsmooth systems is extremely important for applications, in particular in the context of impact mechanics where unilateral contacts and friction come into play. Spatially discrete lattice models are frequently encountered in this context, in particular for the modeling of waves in multibody mechanical systems (e.g. granular media) or in finite element models of continuum systems. A classical example illustrating the latter case concerns thin oscillating mechanical structures (a string under tension or a clamped beam) contacting rigid obstacles [8, 1]. Such a structure can be described by a one-dimensional finite-element model involving a large number of degrees of freedom. The contact force between the string/beam and a rigid obstacle is either measure-valued (for rebounds with velocity jumps at contact times) or set-valued (if a wrapping of the string on the obstacle occurs). Although nonlinear periodic waves are observed in experiments [2], relatively little is known from a theoretical point of view on their existence and stability. Existence results have been derived in particular cases, for a continuum string model with point-mass or plane obstacle ([3] and references therein). In addition, the existence and stability of time-periodic breathers (spatially localized oscillations) has been analyzed for discrete linear chains with a single node undergoing rigid impacts, both for conservative systems [5] and forced systems with dissipative impacts [6].

In this work, we study the existence and stability of time-periodic oscillations in a chain of linearly coupled impact oscillators reminiscent of a model analyzed in [5], for rigid impacts without energy dissipation. We introduce a numerical method allowing to compute branches of time-periodic solutions *when an arbitrary number of nodes undergo rigid impacts*. For this purpose, we reformulate the search of periodic solutions as a boundary value problem incorporating unilateral constraints. We illustrate this numerical approach by computing some families of nonlinear spatially localized modes (breathers) and extended modes.
We consider an infinite chain of impact ocillators with positions described by an infinite vector $y(t) \in \ell_\infty(\mathbb{Z})$ (the space of bounded sequences on $\mathbb{Z}$). The dynamics is described by the following complementarity system

$$\ddot{y}_n + y_n - \gamma\,(\Delta y)_n = \lambda_n, \quad n \in \mathbb{Z}, \tag{1}$$

$$0 \leq \lambda \perp (y + \mathbb{1}) \geq 0, \tag{2}$$

$$\text{if } \dot{y}_n(t^-) < 0 \text{ and } y_n(t) = -1 \text{ then } \dot{y}_n(t^+) = -\dot{y}_n(t^-), \tag{3}$$

where $(\Delta y)_n = y_{n+1} - 2\,y_n + y_{n-1}$ defines a discrete Laplacian operator, $\mathbb{1}$ denotes the constant sequence with all terms equal to unity and $\gamma \geq 0$ is a parameter. Non-dissipative impacts occur for $y(t) = -1$ and give rise to impulsive reaction forces $\lambda(t)$. We look for T-periodic solutions even in time, and assume each particle undergoes at most one impact during each period of oscillation. Consequently, for a given particle, impacts either occur at half-period multiples or do not occur at all. We denote by $I_k \subset \mathbb{Z}$ with $k = 1$ or 2 the index sets of particles impacting at $t = (2p + k)\,T/2$ for all $p \in \mathbb{Z}$, and by $I_0 := \mathbb{Z} \setminus (I_1 \cup I_2)$ the index set corresponding to non-impacting particles. We have thus $\lambda_n = 0$ for all $n \in I_0$ and

$$\lambda_n = 2\,\dot{y}_n(\frac{kT^+}{2}) \sum_{p \in \mathbb{Z}} \delta_{(p+\frac{k}{2})T} \quad \text{for all } n \in I_k.$$

Introducing the splitting $y = (y^{(0)}, y^{(1)}, y^{(2)})$ corresponding to $\mathbb{Z} = I_0 \cup I_1 \cup I_2$, the above system can be reformulated as a boundary value problem on a half-period interval $(0, T/2)$,

$$\ddot{y}_n + y_n - \gamma\,(\Delta y)_n = 0, \quad n \in \mathbb{Z}, \quad t \in (0, T/2), \tag{4}$$

with boundary conditions

$$\dot{y}^{(i)}(0) = 0 \text{ for } i \in I_0 \cup I_1, \quad y^{(2)}(0) = -\mathbb{1}, \quad \dot{y}^{(i)}(T/2) = 0 \text{ for } i \in I_0 \cup I_2, \quad y^{(1)}(T/2) = -\mathbb{1}, \tag{5}$$

and constraint

$$y(t) + \mathbb{1} > 0, \quad t \in (0, T/2). \tag{6}$$

We solve this problem numerically for a chain of $N = 100$ oscillators with periodic boundary conditions. We use a shooting method, i.e. determine $z = (y^{(0)}(0), y^{(1)}(0), \dot{y}^{(2)}(0)) \in \mathbb{R}^N$ such that the three boundary conditions of (5) at $t = T/2$ are satisfied. This requires to solve a linear system for z obtained through time-integration of the linear ODE (4) (the case of nonlinear local or interaction potentials could be addressed similarly using a Newton method). The constraint (6) is checked *a posteriori*. Solution branches are continued for fixed values of T, varying the linear stiffness γ and starting from the uncoupled (or "anticontinuum") limit $\gamma = 0$ [4]. In this limit, for all fixed $T \in (\pi, 2\pi)$, a choice of impacting particles and phases (determined by I_1, I_2) selects a unique solution which can be continued up to some maximal value of γ. The linear stability of periodic solutions is analyzed through the eigenvalues of an associated monodromy matrix. To perform this computation, we integrate (1)-(2)-(3) numerically using the Siconos software for nonsmooth dynamical systems [10]. As an example, we describe in figure 1 three families of periodic solutions obtained with this method, namely two breather solutions (site-centered or bond-centered) and a spatially extended solution (nonlinear normal mode).

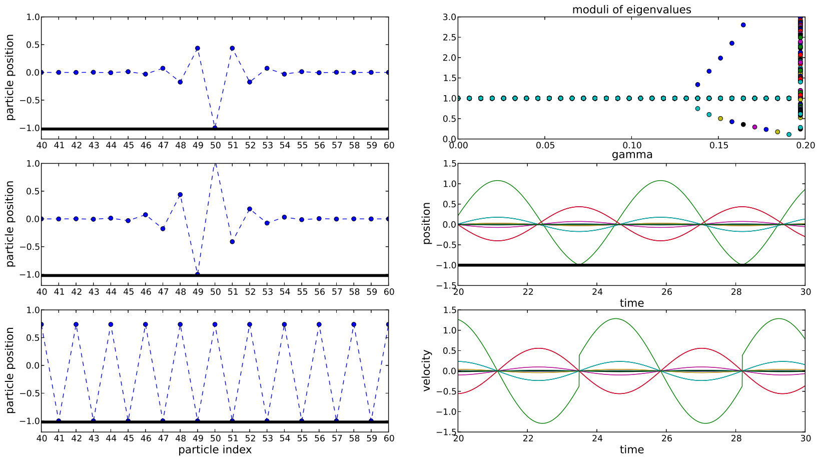

Figure 1: Computation of different periodic solutions for $T \approx 4.7$. The left column displays particle positions at $t = 0$ for $\gamma = 0.16$, for two breather solutions with $I_2 = \{50\}, I_1 = \emptyset$ (site-centered breather, top panel) and $I_2 = \{49\}, I_1 = \{50\}$ (bond-centered breather, middle panel), and for a nonlinear normal mode with spatial period two and $I_0 = 2\mathbb{Z}, I_1 = \emptyset$ (bottom panel). These solutions can be continued for $\gamma \in [0, \gamma_{\text{max}})$ with $\gamma_{\text{max}} \approx 0.19$. The site-centered breather is linearly stable for $\gamma < \gamma_c \approx 0.13$, after which it becomes unstable (the top right panel displays the moduli of the corresponding Floquet eigenvalues). The time evolution of the position and velocity of the impacting particle ($n = 50$) is illustrated over a few periods for $\gamma = 0.16$ (right column, middle and bottom panels). The bond-centered breather and nonlinear normal mode are both unstable for all values of γ.

The computation of periodic solutions based on the above approach is much more effective than numerical continuation of periodic solutions based on compliant models. In the latter case, impacts are described by smooth nonlinear Hertzian type potentials leading to stiff ODE and costly numerical continuation. Future extensions of this work will include an analytical continuation and stability analysis based on the same approach, the inclusion of dissipative impacts and forcing, and the application of the method to more complex finite-element models of continuous systems under impacts.

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