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PROOF OF WIENER-LIKE LINEAR REGRESSION OF ISOTROPIC COMPLEX SYMMETRIC ALPHA-STABLE RANDOM VARIABLES

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This document features supplementary materials to the reference paper [1]. It provides the proof of equation (8) in [1]. This proof concerns a particular regression property of complex isotropic symmetric α-stable random variables (SαS, see [2]). In [1], this property is shown paramount in building efficient filters for separating SαS processes. Such processes exhibit very large dynamic ranges while being locally stationary, and have been shown appropriate for audio modeling.

Proposition 1 (Wiener-like linear regression of isotropic complex SαS random variables)**.** *Let $\alpha \in]0, 2]$. Let s_1 and s_2 be two independent isotropic complex SαS random variables of scale parameters σ_1 and σ_2, respectively. Let $x = s_1 + s_2$. Then the conditional expectation of s_1 given x is expressed as follows:*

$$\mathbb{E}[s_1|x] = \frac{\sigma_1^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}\, x. \tag{1}$$

Proof. Let $s_1^r = Re(s_1)$, $s_1^i = Im(s_1)$, $s_2^r = Re(s_2)$, $s_2^i = Im(s_2)$, $x^r = Re(x)$, and $x^i = Im(x)$, where $Re(.)$ and $Im(.)$ denote the real and imaginary parts of complex numbers, respectively. For $j \in \{1, 2\}$, the characteristic function of the isotropic complex SαS random variable s_j is $\phi_{s_j}(\theta_x^r, \theta_x^i) = e^{-\sigma_j^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}}$ [2]. Therefore the characteristic function of the random vector $(s_1^r, x^r, x^i) \in \mathbb{R}^3$ is

$$\begin{aligned}
\phi_{(s_1^r,x^r,x^i)}(\theta_1^r, \theta_x^r, \theta_x^i) &= \mathbb{E}\big[e^{i(\theta_1^r s_1^r + \theta_x^r x^r + \theta_x^i x^i)}\big] \\
&= \mathbb{E}\big[e^{i(\theta_1^r s_1^r + \theta_x^r (s_1^r + s_2^r) + \theta_x^i (s_1^i + s_2^i))}\big] \\
&= \mathbb{E}\big[e^{i((\theta_1^r + \theta_x^r) s_1^r + \theta_x^i s_1^i)}\big]\mathbb{E}\big[e^{i(\theta_x^r s_2^r + \theta_x^i s_2^i)}\big] \\
&= \phi_{s_1}(\theta_1^r + \theta_x^r, \theta_x^i)\, \phi_{s_2}(\theta_x^r, \theta_x^i) \\
&= e^{-\left(\sigma_1^\alpha (|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}} + \sigma_2^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}\right)}.
\end{aligned}$$

Using [2, eq. (5.1.7) p. 226], the conditional characteristic function of s_1^r given x^r and $x^i \in \mathbb{R}$ is expressed as

$$\begin{aligned}
\phi_{(s_1^r|x^r,x^i)}(\theta_1^r) &= \frac{\int_{\mathbb{R}^2} \phi_{(s_1^r,x^r,x^i)}(\theta_1^r, \theta_x^r, \theta_x^i) e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2} \phi_{(s_1^r,x^r,x^i)}(0, \theta_x^r, \theta_x^i) e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i} \\
&= \frac{\int_{\mathbb{R}^2} e^{-\left(\sigma_1^\alpha (|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}} + \sigma_2^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}\right)} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2} e^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}.
\end{aligned} \tag{2}$$

We know that $\mathbb{E}[s_1^r|x^r, x^i]$ is defined if and only if $\phi_{(s_1^r|x^r,x^i)}(\theta_1^r)$ is differentiable at $\theta_1^r = 0$, and that in that case $\frac{d\phi_{(s_1^r|x^r,x^i)}}{d\theta_1^r}(0) = i\mathbb{E}[s_1^r|x^r, x^i]$. Firstly, it is easy to prove that the first order derivative of $\phi_{(s_1^r|x^r,x^i)}$ is well-defined and has the following expression:

$$\frac{d\phi_{(s_1^r|x^r,x^i)}}{d\theta_1^r}(\theta_1^r) = \frac{\int_{\mathbb{R}^2} -\alpha\sigma_1^\alpha (\theta_1^r + \theta_x^r)(|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{(\frac{\alpha}{2} - 1)} e^{-\left(\sigma_1^\alpha (|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}} + \sigma_2^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}\right)} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2} e^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}. \tag{3}$$

Indeed, differentiating under the $\int$ sign in the numerator of the last member of (2) is allowed, because the term following the $\int$ sign in the numerator of (3) can be upper bounded by an integrable function independently of $\theta_1^r \in \mathbb{R}$:

$$\begin{aligned}
&\left|-\alpha\sigma_1^\alpha(\theta_1^r+\theta_x^r)(|\theta_1^r+\theta_x^r|^2+|\theta_x^i|^2)^{(\frac{\alpha}{2}-1)}e^{-\left(\sigma_1^\alpha(|\theta_1^r+\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}+\sigma_2^\alpha(|\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}\right)}e^{-i(\theta_x^r x^r+\theta_x^i x^i)}\right| \\
&\leq \alpha\sigma_1^\alpha(|\theta_1^r+\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha-1}{2}}e^{-\left(\sigma_1^\alpha(|\theta_1^r+\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}+\sigma_2^\alpha(|\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}\right)} \\
&\leq g(|\theta_1^r+\theta_x^r|^2+|\theta_x^i|^2)\,h(\theta_x^r,\theta_x^i) \\
&\leq \|g\|_\infty\,h(\theta_x^r,\theta_x^i),
\end{aligned}$$

where the nonnegative functions $g \in L^\infty(\mathbb{R})$ and $h \in L^1(\mathbb{R}^2)$ are defined according to the value of α:

- if $1 \leq \alpha \leq 2$, $g(t) = \alpha\sigma_1^\alpha|t|^{\frac{\alpha-1}{2}}e^{-\sigma_1^\alpha|t|^{\frac{\alpha}{2}}}$ and $h(\theta_x^r,\theta_x^i) = e^{-\sigma_2^\alpha(|\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}}$;
- if $0 < \alpha < 1$, $g(t) = \alpha\sigma_1^\alpha e^{-\sigma_1^\alpha|t|^{\frac{\alpha}{2}}}$ and $h(\theta_x^r,\theta_x^i) = \frac{e^{-\sigma_2^\alpha(|\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}}}{|\theta_x^i|^{1-\alpha}}$.

Then applying equation (3) to $\theta_1^r = 0$ yields

$$\begin{aligned}
\frac{d\phi_{(s_1^r|x^r,x^i)}}{d\theta_1^r}(0) &= \frac{\int_{\mathbb{R}^2}-\alpha\sigma_1^\alpha\theta_x^r(|\theta_x^r|^2+|\theta_x^i|^2)^{(\frac{\alpha}{2}-1)}e^{-(\sigma_1^\alpha+\sigma_2^\alpha)(|\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}}e^{-i(\theta_x^r x^r+\theta_x^i x^i)}d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2}e^{-(\sigma_1^\alpha+\sigma_2^\alpha)(\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}}e^{-i(\theta_x^r x^r+\theta_x^i x^i)}d\theta_x^r d\theta_x^i} \\
&= \frac{\sigma_1^\alpha}{\sigma_1^\alpha+\sigma_2^\alpha}\frac{\int_{\mathbb{R}^2}\frac{de^{-(\sigma_1^\alpha+\sigma_2^\alpha)(|\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}}}{d\theta_x^r}e^{-i(\theta_x^r x^r+\theta_x^i x^i)}d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2}e^{-(\sigma_1^\alpha+\sigma_2^\alpha)(\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}}e^{-i(\theta_x^r x^r+\theta_x^i x^i)}d\theta_x^r d\theta_x^i} \\
&= -\frac{\sigma_1^\alpha}{\sigma_1^\alpha+\sigma_2^\alpha}\frac{\int_{\mathbb{R}^2}e^{-(\sigma_1^\alpha+\sigma_2^\alpha)(|\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}}\frac{de^{-i(\theta_x^r x^r+\theta_x^i x^i)}}{d\theta_x^r}d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2}e^{-(\sigma_1^\alpha+\sigma_2^\alpha)(\theta_x^r|^2+|\theta_x^i|^2)^{\frac{\alpha}{2}}}e^{-i(\theta_x^r x^r+\theta_x^i x^i)}d\theta_x^r d\theta_x^i} \\
&= i\frac{\sigma_1^\alpha}{\sigma_1^\alpha+\sigma_2^\alpha}x^r.
\end{aligned}$$

This proves that $\mathbb{E}[s_1^r|x^r,x^i] = \frac{\sigma_1^\alpha}{\sigma_1^\alpha+\sigma_2^\alpha}\,x^r$. In exactly the same way, it is proved that $\mathbb{E}[s_1^i|x^r,x^i] = \frac{\sigma_1^\alpha}{\sigma_1^\alpha+\sigma_2^\alpha}\,x^i$, which finally proves equation (1). □

1. REFERENCES

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[2] G. Samoradnitsky and M. Taqqu, *Stable non-Gaussian random processes: stochastic models with infinite variance*, vol. 1, CRC Press, 1994.