EARTH MOTION BENEATH A PRESCRIBED BOUNDARY DISPLACEMENT

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by

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Joint Highway Research Project

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LAFAYETTE INDIANA

TECHNICAL PAPER

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BOUNDARY DISPLACEMENT

TO: K. B. Woods, Director
Joint Highway Research Project

February 28, 1958

FROM: H. L. Michael, Assistant Director
Joint Highway Research Project

File: 9-7-1
Project: C-36-A

Attached is a technical paper entitled, "Earth Motion Beneath A Prescribed Boundary Displacement." This paper has been prepared by R. C. Geldmacher, J. W. Dunkin, and R. L. Anderson. It has been submitted to the Journal of Applied Mechanics for possible publication.

The research reported in this paper was performed for the Pavement Deflection Research Project. This project was a cooperative project between the State Highway Department of Indiana (JHRP) and the Bureau of Public Roads. Approval for this publication is being obtained from the Bureau of Public Roads.

The paper is submitted for the record.

Respectfully submitted,

Harold L. Michael

Harold L. Michael, Assistant Director
Joint Highway Research Project

HLM:acc

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Project C36A

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EARTH MOTION BENEATH A PRESCRIBED BOUNDARY DISPLACEMENT

ABSTRACT

Earth deflections beneath a prescribed boundary displacement were obtained theoretically and experimentally.

The problem was stated in terms of a two-dimensional model, and an attempt was made to design a corresponding experiment.

Relative deflections between the surface of the earth and points of increasing depths within the earth were measured. The P.C. maximum depth was 42' 7" below the earth's surface.

Theoretical and experimental results were compared.

Introduction

The material presented in this paper is the result of a theoretical and experimental study of earth motion beneath a prescribed boundary deformation done as a phase of an investigation of the support characteristics of sub-base treatments beneath rigid pavements.[1] It should be of interest to anyone engaged in measuring the absolute deflection of a highway, landing strip, or similar structure by means of a device that requires a reference point in the earth.

The study was prompted by the results of a series of exploratory measurements of earth motion beneath a pavement deflected by vehicular loads. These measurements indicated that significant vertical motion of the earth occurred at much greater depths than had been imagined, and while made under relatively uncontrolled conditions, they suggested that a theoretical analysis of the movement of the earth beneath a prescribed boundary displacement and an experiment planned in terms of such an analysis might be fruitful.

In the exploratory experiment, the earth motion was obtained by measuring the relative displacement between the surface of the deflected pavement and a rod driven into the bottom of a small diameter cylindrical hole augered to progressively greater depths. The results of these measurements are shown in Figure 1. The flat spot occurring in the graph was of special interest because of the discovery of a layer of hard earth at a depth of six feet.

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[1] Floating numbers refer to bibliography in back.

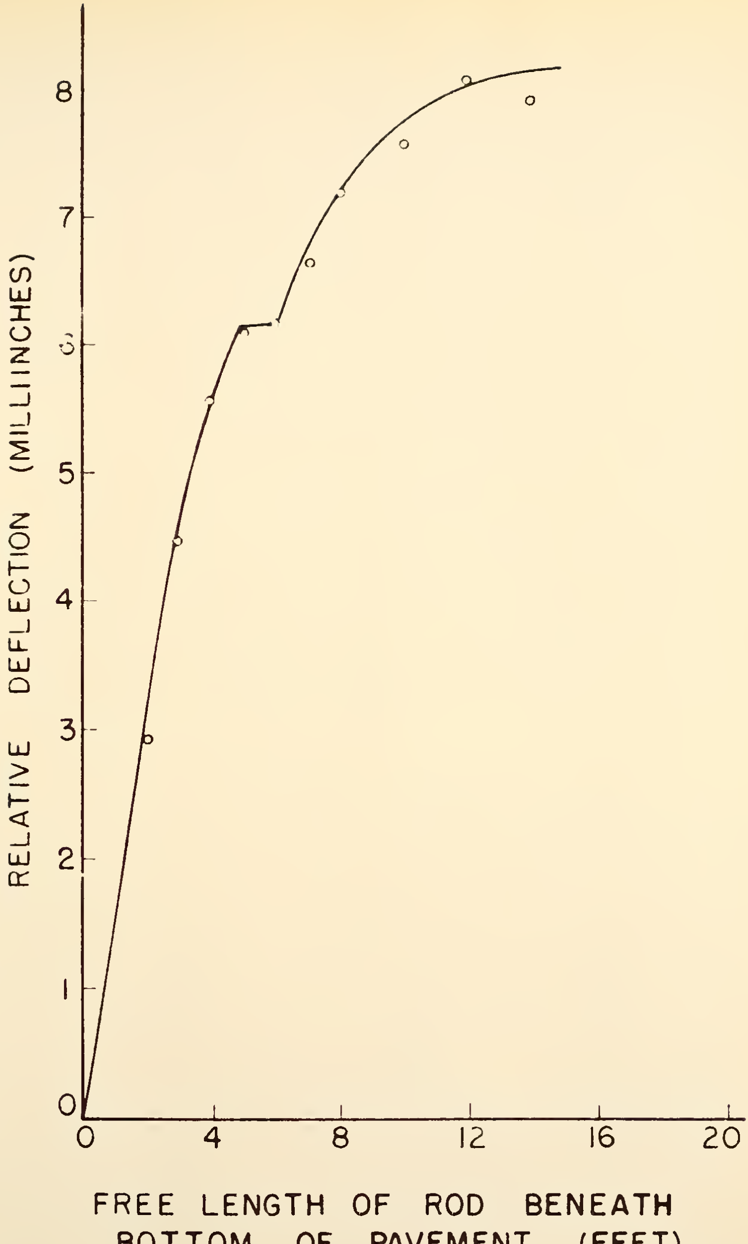

FIGURE 1: RELATIVE DEFLECTION BETWEEN EARTH AND PAVEMENT, EXPLORATORY MEASUREMENT.

After a few preliminary skirmishes with various kinds of theoretical models of the system to be studied, it was decided to formulate the problem in terms of a linear plane strain model because of the simplicity of the theory in comparison to that of a three-dimensional model. However, the difficulties encountered in trying to design a two-dimensional experiment suggest that any subsequent treatment of the problem should be three-dimensional.

Experiment

For a truly two-dimensional model, it would have been necessary to have a transverse load extending infinitely far in each transverse direction. Such a load could be visualized as consisting of many equal concentrated loads closely spaced at equal increments along an infinitely long line. The resulting deflection would be an infinitely long transverse trough.

Two methods for approximating a load which would give the desired result were considered. One was to obtain a vehicle, or vehicles, sufficiently wide and with wheels closely spaced; the other was to superpose the deflections produced by successive passes of a single vehicle such that the resultant load would be uniformly distributed in the transverse direction. Of the two methods considered, the second was chosen because it could be more easily realized and because preliminary studies had shown the pavement-earth system to be quite linear in its elastic behavior and thus justified the use of superposition.[1]

The experimental measurements were made on a section of U.S. Highway 52. The section chosen was relatively new, had no visible cracks, and was placed on an earth cut. It was approximately fifty feet long and was comprised of two lanes of the four-lane highway. The vehicle used was an especially prepared one belonging to the State of Indiana. The distance from its front axle to a point midway between its rear tandem wheels was fourteen feet, and the total load carried by the rear axles was 19,330 pounds. This truck had been originally built for another highway testing program and was made available through the Indiana State Highway Department.

Preliminary measurements had shown that when a test vehicle was near the center of the section, the range of a significant influence did not extend to the ends; and, hence, the slab could be regarded as extending infinitely far to the front and rear. It was reasoned, therefore, that the time dependent deflections of the middle of the section, for slowly moving loads, could, after an appropriate correlation of time and longitudinal distance, be used as the instantaneous profile of the top surface in the longitudinal direction.

Differential transformers were used as deflection transducers.[1] Six transformers were placed one foot apart in a longitudinal row in a manner which measured the relative motion between the pavement and positions in the earth 1'10", 5' 4", 9' 5", 14' 11", 27', and 42' 7" beneath the bottom of the pavement.

The positions of the transducers in the pavement and the vehicle loading pattern are shown in Figure 2. Vehicle positions are identified by number and give the location of the left rear duals relative to the line along which the transducers were placed. The letters "L" and "R" indicate left and right rear duals respectively, and the arrows indicate the position of the midpoint of the tires. Thus, when in position one, the center of the outer left rear dual of the vehicle was .76' from the line of centers of the transducers. The successive positions were arranged so that after algebraically summing, the resultant load consisted of a regular distribution of the load caused by the left rear dual. Twelve successive passes were made at creep speed at each of the prescribed positions. A reproduction of the recorded deflections for one pass of the vehicle at position one is shown in Figure 3. An estimate of the distribution of the twelve measurements made at each vehicle position may be inferred from Table 1, which gives the deflections occurring when the vehicle passed along position one.

The measured motion of the pavement with respect to the earth for the six depths chosen is shown in Figure 4. Each point on this curve represents the proper superposition of the means of the deflections resulting from the passage of the vehicle over the prescribed separate paths.

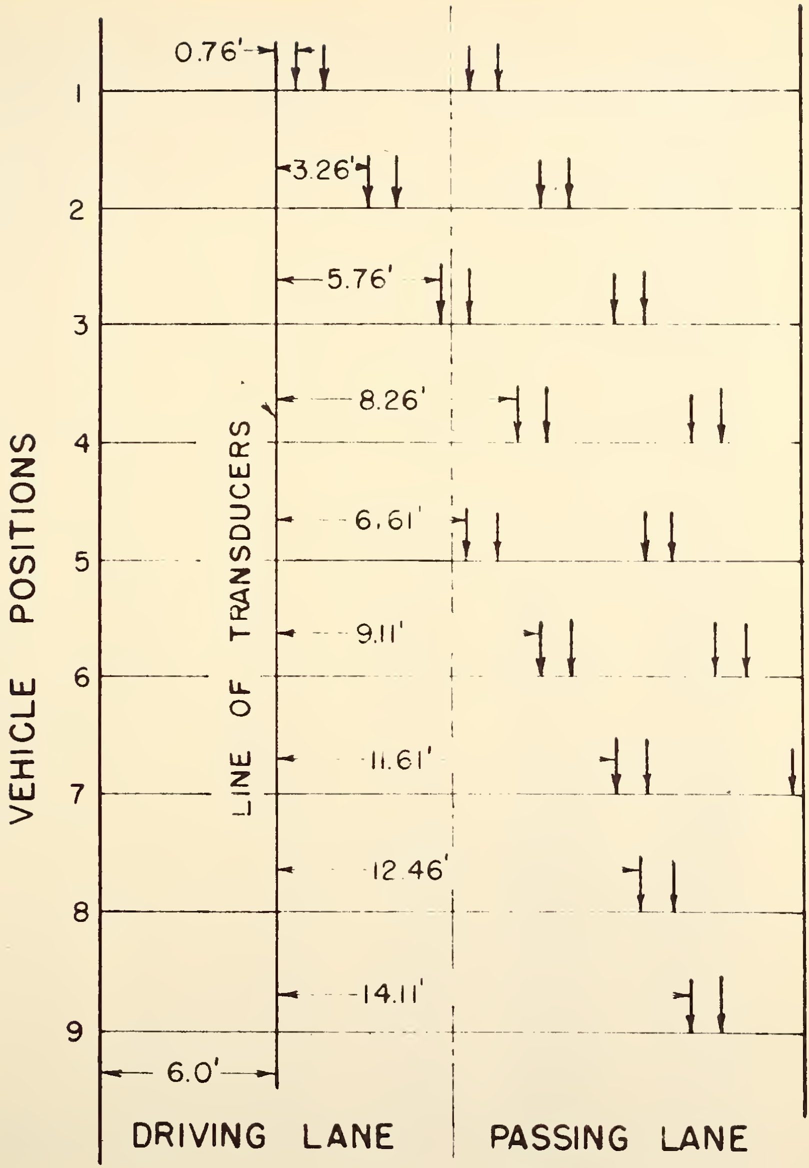

FIGURE 2: RELATIVE POSITION OF TRANSDUCER ROW AND REAR DUALS OF VEHICLE (ARROWS).

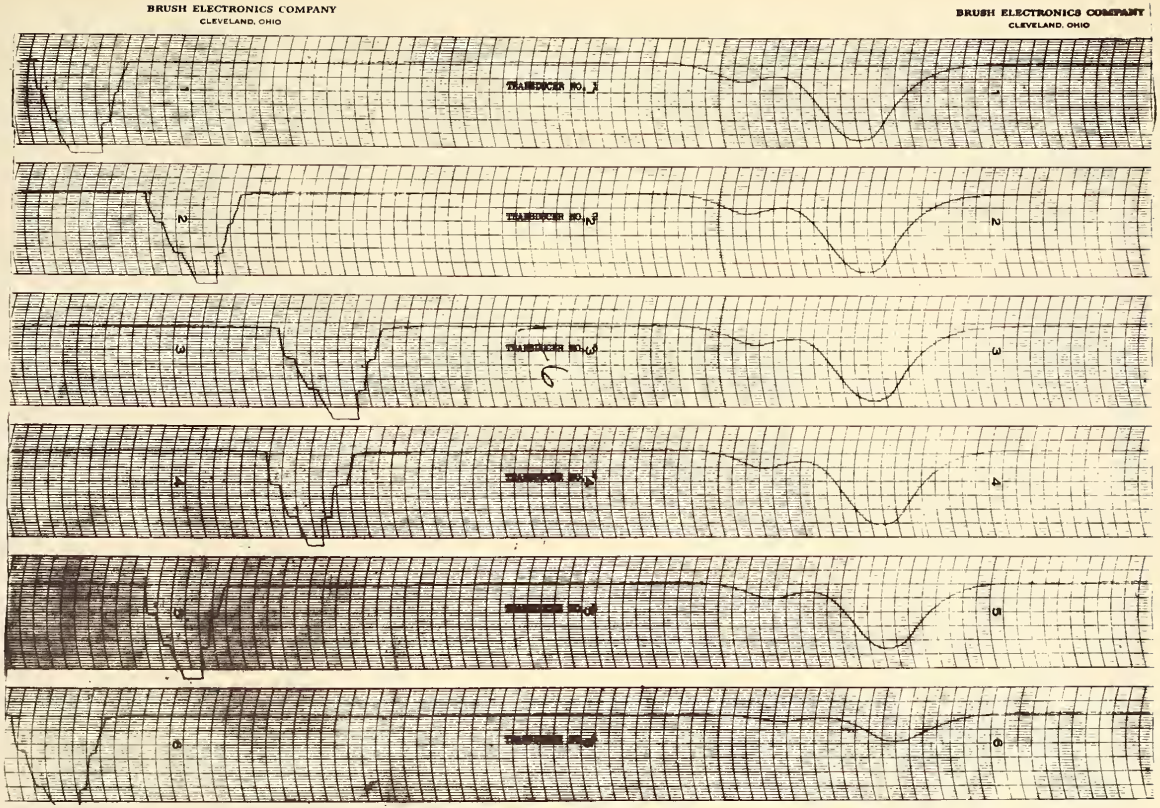

FIGURE 3 RELATIVE DEFLECTION BETWEEN EARTH AND PAVEMENT VERSUS TIME FOR SIX EARTH DEPTHS. RUN NUMBER SIX, VEHICLE TRAVELING AT CREEP SPEED. CHART SPEED: 5 mm PER SECOND.

TABLE 1

DEFLECTION OF PAVEMENT IN MILLINCHES WITH RESPECT TO SIX DEPTHS IN EARTH FOR TWELVE RUNS AT VEHICLE POSITION ONE

Run	TRANSDUCER NUMBER AND DEPTH					
Run	1	2	3	4	5	6
Number	(42'7")	(27'0")	(14'11")	(9'5")	(5'4")	(1'10")
1.	6.8	6.8	6.6	6.3	5.6	2.1
2.	6.6	6.6	6.1	5.9	5.0	1.9
3.	6.4	6.6	6.2	5.9	5.8	1.9
4.	6.7	6.7	6.2	6.0	5.6	2.1
5.	6.4	6.6	6.2	5.8	5.3	1.9
6.	6.3	6.4	6.1	5.8	5.4	1.8
7.	6.5	6.5	6.1	6.0	5.5	2.0
8.	6.2	6.4	6.1	5.8	5.4	1.9
9.	6.5	6.6	6.1	5.8	5.2	2.0
10.	6.1	6.9	6.1	6.0	5.3	1.9
11.	6.6	6.8	6.2	5.7	5.3	2.0
12.	6.2	6.5	6.1	5.8	5.4	2.1
AVERAGE	6.44	6.62	6.17	5.90	5.40	1.98

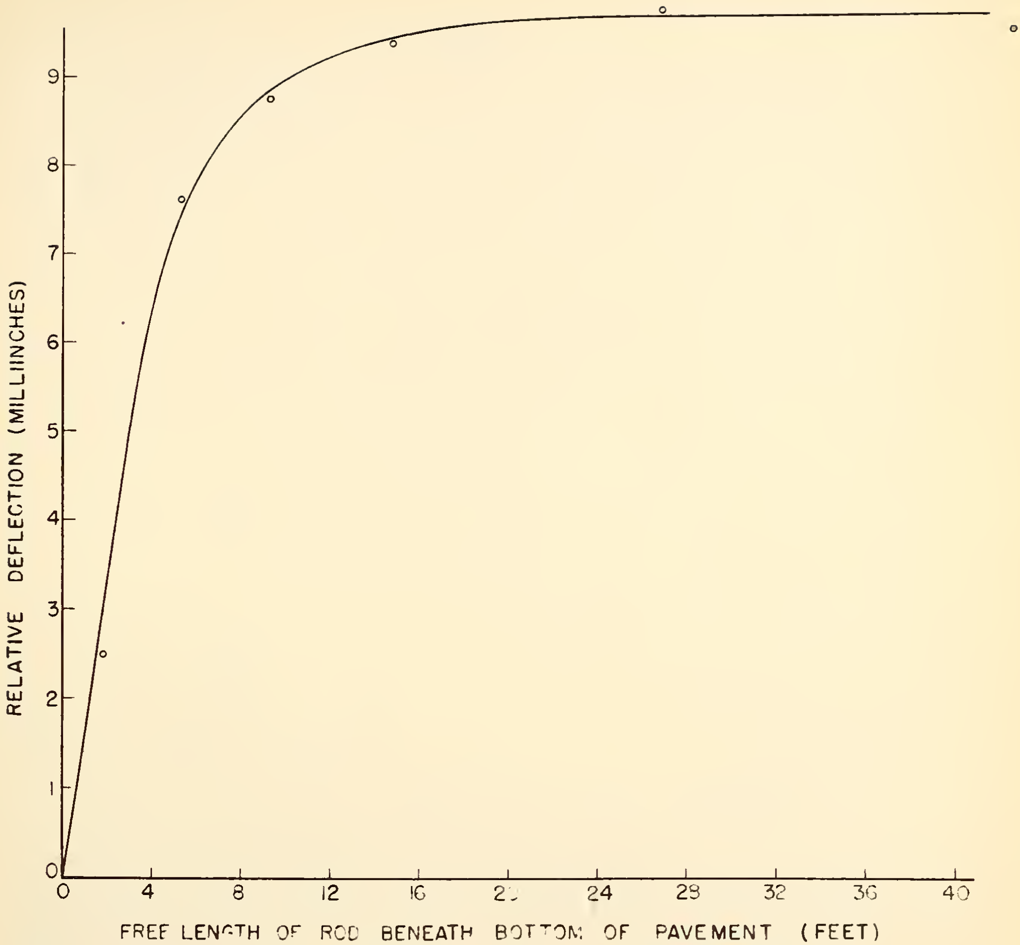

FIGURE 4 RELATIVE DEFLECTION RESULTING FROM SUPERPOSITION PROCEDURE.

Theory

The theoretical model was chosen as a semi-infinite, homogeneous, isotropic, elastic medium with its upper surface displaced in the form of an infinitely long trench, thereby providing a condition of plane strain. The analysis is that found in the works of Muskhelishvili.

To find the horizontal displacement u and the vertical displacement v of a point inside such a system, the following boundary value problem must be solved.

$$(\lambda + \mu) \frac{\partial \theta}{\partial x} + \mu \Delta u = 0 \qquad (1)$$

$$(\lambda + \mu) \frac{\partial \theta}{\partial y} + \mu \Delta v = 0 \qquad (2)$$

under the boundary conditions:

(1) u, v tend to 0 as x approaches $\pm\infty$ and y approaches $-\infty$;

(2) v(x,0) and u(x,0) are specified functions;

where λ and μ are the Lamé constants, $\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, and Δ is the Laplacian operator.

The complex variable solution presented by Muskhelishvili employs the complex coordinates shown in Figure 5. For y = 0, the complex displacement u + iv reduces to the boundary condition:

$$2\mu \; [u(x,0) + iv(x,0)] = 2\mu \; [g_1(t) + ig_2(t)] \qquad (3)$$

where t represents the distance along the boundary; $g_1(t)$ is the horizontal displacement of the boundary surface; and $g_2(t)$ is the vertical displacement of the boundary surface.

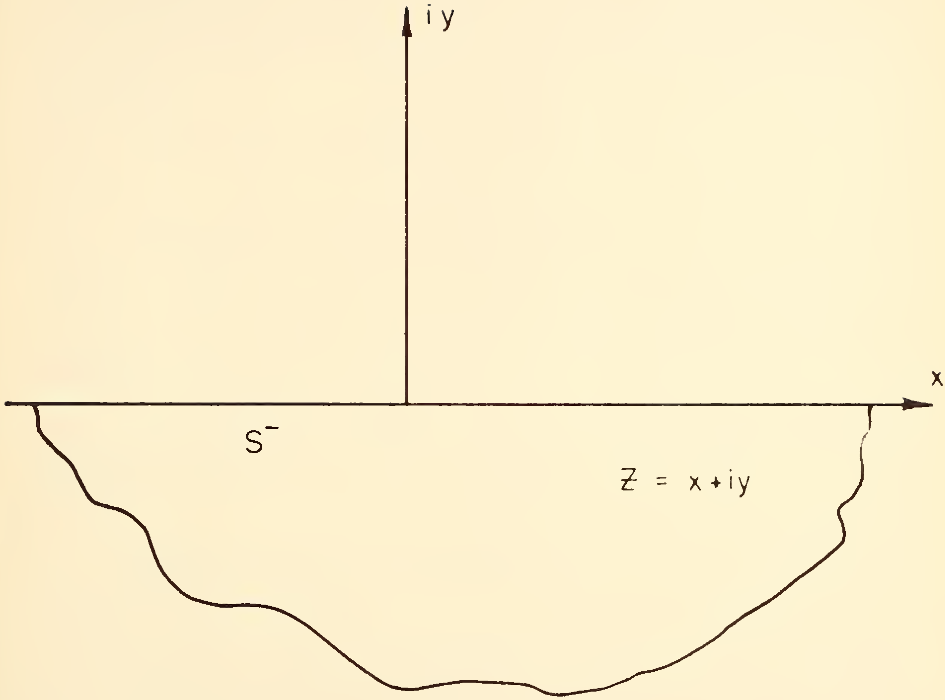

FIGURE 5 COMPLEX COORDINATES.

The displacement (u + iv) of a point z is expressed in terms of two functions, holomorphic throughout the lower half plane (designated as S^-) including the point at infinity. Thus, the problem becomes that of finding these two holomorphic functions under the given boundary conditions.

The displacement is defined as:[2]

$$2\mu(u + iv) = k\phi(z) - \overline{z\phi'(z)} - \overline{\psi(z)} \qquad (4)$$

where $k = \frac{+3\mu}{+\mu}$, $\phi(z)$ and $\psi(z)$ are the two holomorphic functions; $\phi'(z)$ is the derivative of $\phi(z)$, and $\overline{\phi'(z)}$ and $\overline{\psi(z)}$ are the complex conjugates of $\phi'(z)$ and $\psi(z)$.

The functions $\phi(z)$ and $\psi(z)$ may be expressed in terms of the boundary conditions as :

$$k\phi(z) = -\frac{\mu}{\pi i} \int_{-\infty}^{\infty} \frac{g_1 + ig_2}{t - z}\, dt \qquad (5)$$

$$\psi(z) = \frac{\mu}{\pi i} \int_{-\infty}^{\infty} \frac{g_1 - ig_2}{t - z}\, dt - z\phi'(z) \qquad (6)$$

Using (5) and the Cauchy integral relation for $\phi'(z)$,

$$\phi'(z) = -\frac{\mu}{k\pi i} \int_{-\infty}^{\infty} \frac{g_1 + ig_2}{(t-z)^2}\, dt \qquad (7)$$

and from (4), (5), (6), and (7),

$$u + iv = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g_2 - ig_1}{(t - z)}dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g_2 - ig_1}{(t-\bar{z})}\, dt$$

$$- \frac{y}{k\pi}\int_{-\infty}^{\infty} \frac{g_1 - ig_2}{(t - \bar{z})^2}\, dt \qquad (8)$$

After separating into real and imaginary parts,

$$u = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yg_1}{(t-x)^2+y^2}\,dt - \frac{y}{k\pi}\int_{-\infty}^{\infty}\frac{[(t-x)^2-y^2]\,g_1-2y(t-x)g_2}{[(t-x)^2+y^2]^2}\,dt \qquad (9)$$

$$v = -\frac{1}{\pi}\quad\frac{yg_2}{(t-x)^2+y^2}\,dt - \frac{y}{k\pi}\quad\frac{[y^2-(t-x)^2]\,g_2-2y(t-x)g_1}{[(t-x)^2+y^2]^2}\,dt \qquad (10)$$

Relations (9) and (10) give u and v for a point (x,y) in terms of the boundary deformations $g_1(t)$ and $g_2(t)$. It can be seen that no great trouble will be involved in using measured values of g_1 and g_2 and numerically performing the integrations indicated in (9) and (10).

Application of Theory

Since only the vertical component of deflection was measured, it was necessary to assume the boundary function $g_1(t)$ in relations (9) and (10). This function was chosen to be identically zero.

In order to throw some light on the significance of this choice, one may examine the vertical component of displacement due to $g_1(t)$. From (10), it is found to be

$$v_{g_1} = \frac{2y^2}{k\pi}\int_{-\infty}^{\infty}\frac{(t-x)\;g_1(t)}{[(t-x)^2+y^2]^2}\,dt \qquad (11)$$

For $x = 0$, this relation will be positive for odd functions of $g_1(t)$ which are non-negative in the interval $t > 0$ and negative for odd functions of $g_1(t)$ which are non-negative in the interval $t < 0$. It can be seen that functions of the first kind will be associated with an outward horizontal movement of the surface while functions of the second kind will be associated with an inward horizontal movement of the surface. Of the two functions, the first would seem to be the more realistic for the case under consideration; and, as consequence, the deflection obtained from (10) would be less in magnitude than if $g_1(t)$ were identically zero. This observation is of importance in interpreting the results obtained.

After setting $g_1(t) = 0$, relations (9) and (10) become

$$u = \frac{y}{k\pi}\int_{-\infty}^{\infty} \frac{2y(t-x)\, g_2(t)}{[(t-x)^2 + y^2]^2}\,dt \tag{12}$$

$$v = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\, g_2(t)}{[(t-x)^2 + y^2]}\,dt - \frac{y}{k\pi}\int_{-\infty}^{\infty} \frac{[y^2-(t-x)^2]\, g_2(t)}{[(t-x)^2 + y^2]^2}\,dt \tag{13}$$

The measured function $g_2(t)$ is the top curve in Figure 6. The integration of (12) and (13) was performed numerically using Simpson's rule. The integrand was evaluated at intervals of approximately 2.25 feet and the values were appropriately summed. Computations were made for positions defined by values of $x = 0, \pm\frac{2a}{4}, \ldots, \pm\frac{6a}{4}$ and values of $y = 0, -\frac{a}{4}, -\frac{2a}{4}, \ldots, -\frac{6a}{4}$ (where 2a equals the length of the surface indentation, in this case 36'). The results are shown in Figure 6.

It should be noted that Poisson's ratio is contained in the deflection relations. The effect of selected values of Poisson's ratio is shown in Figure 7. The value used in the computation of deflections shown in Figure 7. The value used in the computation of deflections shown in Figure 6 was 0.25.

Discussion of Results

A comparison of the measured and theoretical results may be made by referring to Figures 4 and 8. Here the theoretical results are presented for the earth deflections immediately below the point of maximum

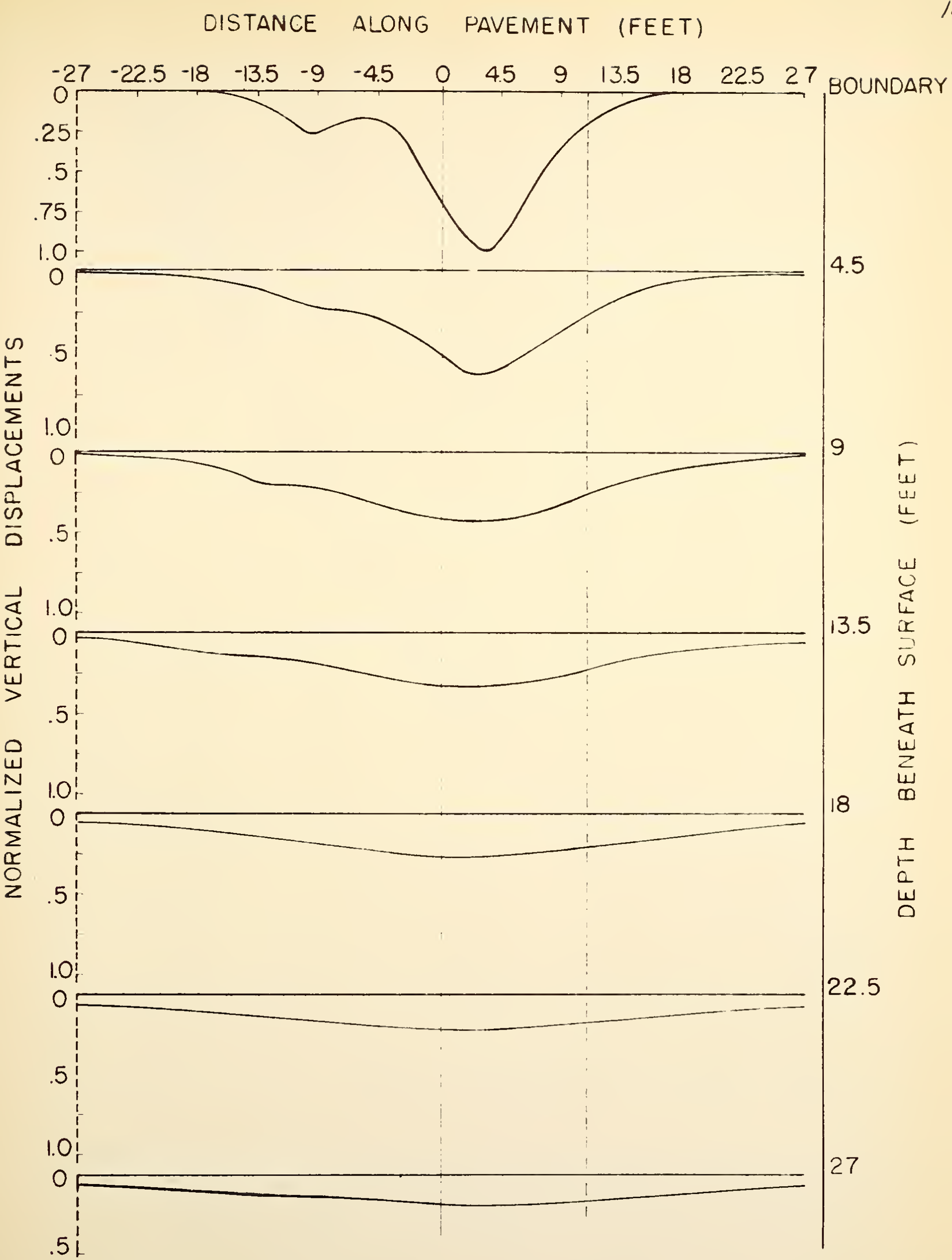

FIGURE 6 DEFLECTION OF EARTH PREDICTED BY TWO-DIMENSIONAL THEORY.

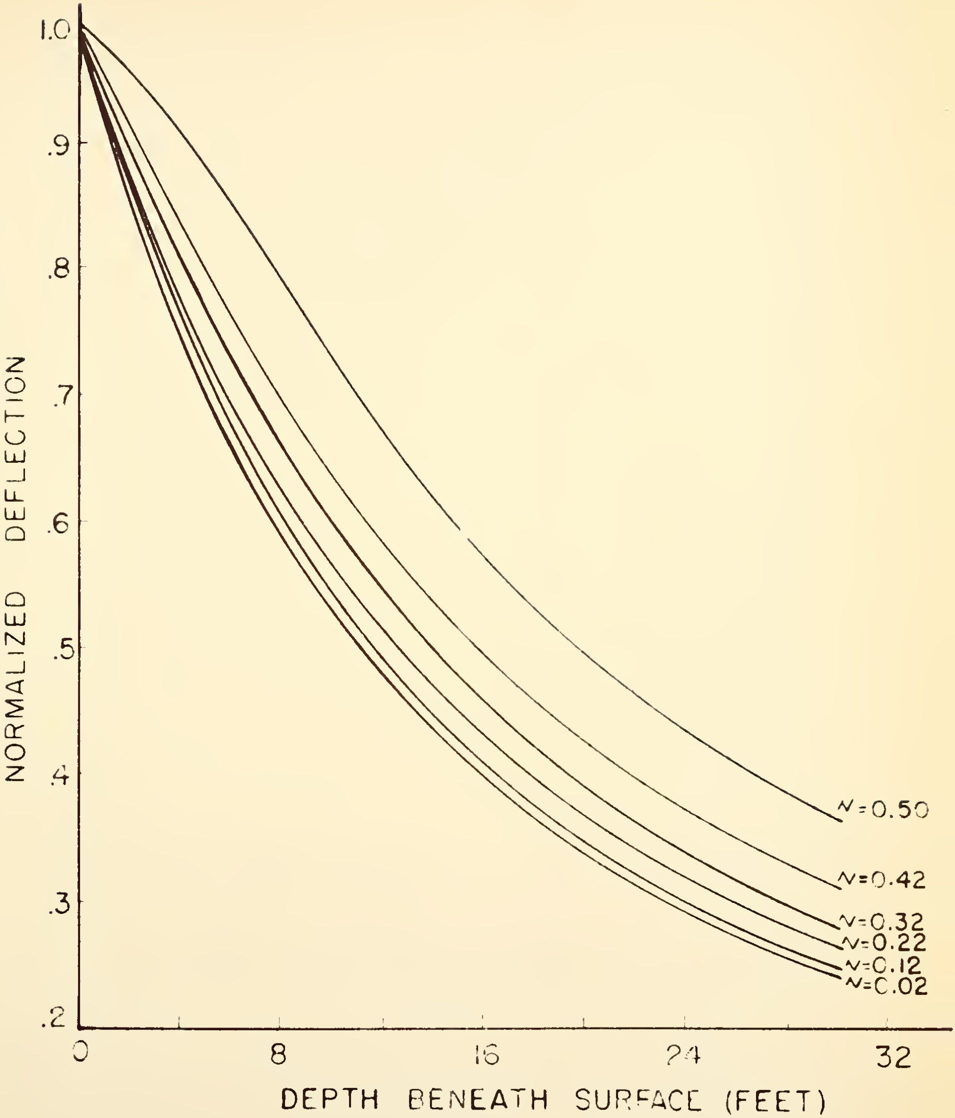

FIGURE 7 EFFECT OF POISSON'S RATIO UPON EARTH DEFLECTION BENEATH POINT OF MAXIMUM BOUNDARY DEFLECTION.

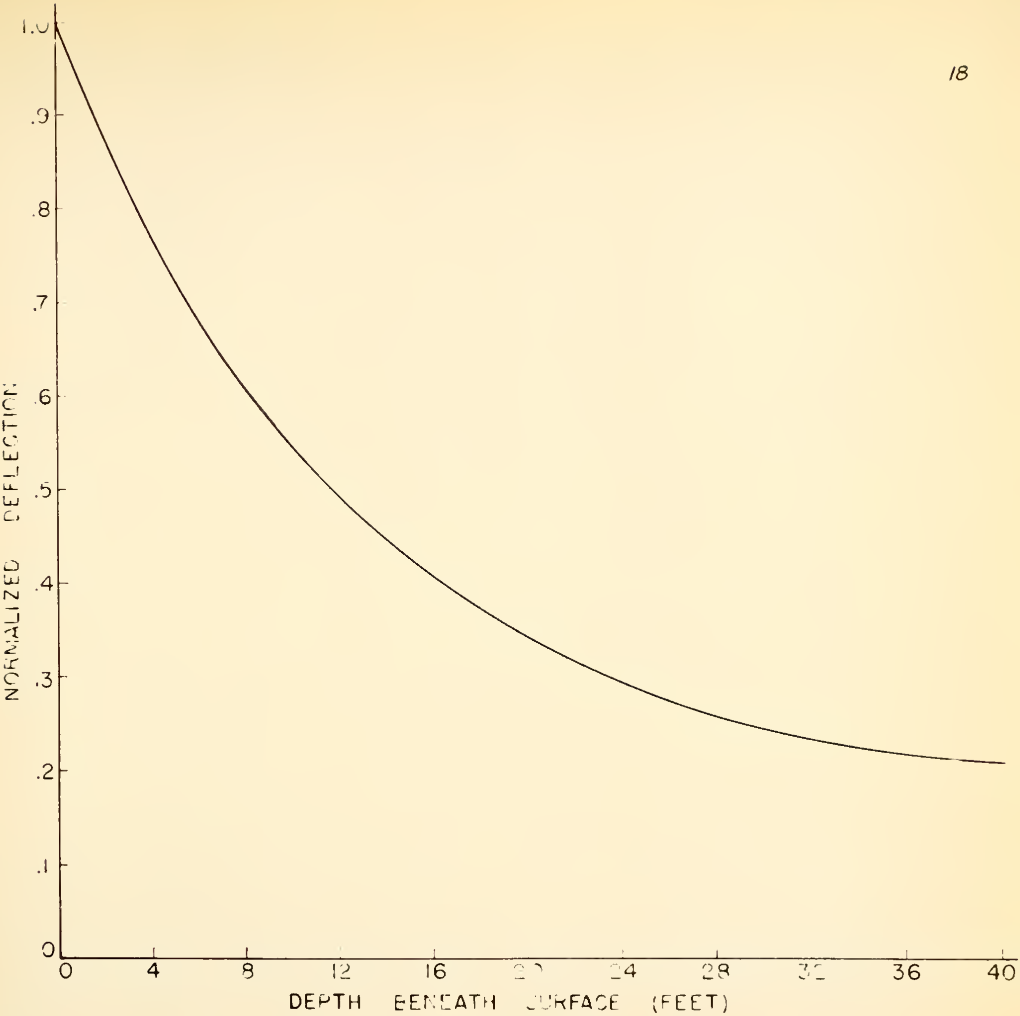

FIGURE 8 THORETICAL DEFLECTION VERSUS DEPTH FOR POINTS BENEATH POINT OF MAXIMUM BOUNARY DEFLECTION

deflection of the boundary and are normalized to this deflection at the boundary. The experimental results have not been put in terms of absolute deflections since it was not possible to obtain a measure of the absolute motion of the earth. In order to convert the relative measurements of earth motion to absolute motion, it would be necessary to obtain the absolute motion of one point in the earth.

One might suppose that the flatness of the relative motion curve (experimental curve) at large depths would indicate that the absolute motion is zero here. However, such a conclusion could be in error since the theoretical curve suggests that even though the slope of the curve may be quite small, a significant absolute deflection may be present. Furthermore, to obtain the slope of the relative deflection curve accurately at greater depths, it would be necessary to measure small differences occurring in large relative deflections; this is precluded by the limit of precision of the instrument.[3]

A great deal of effort was expended in an attempt to obtain an absolute reference but with no success. Several attempts were made to construct a long bridge across the pavement for this purpose, but the instability of the stuctures (motion on the order of a few ten thousandths of an inch could not be tolerated) made the scheme unworkable. However, had the structures been stable, the validity of the measurements would still have been uncertain since it was subsequently observed that the bridge supports could have experienced significant motion. The establishment of a suitable reference remains a major problem in work of this sort.

Another significant difference between the experimental and theoretical results is that the experimental curve approaches an asymptote much more rapidly than does the theoretical. This difference is thought to be due to one or more of the following conditions:

(1) the use of a two-dimensional theory and the subsequent inability to provide an experiment that was truly two-dimensional.

(2) the assumption in the application of the theory that the horizontal deflection was zero

(3) non-homogeniety of the earth

(4) lack of knowledge of the magnitude of Poisson's ratio.

The two-dimensional theory does not bring in stresses, in particular shear stresses, that exist in the three-dimensional problem; as a result, it could be anticipated that the normalized deflections for the three-dimensional case would reach an asymptote more rapidly than does the two dimensional result.

The difficulty in providing a two-dimensional experiment stemmed from the facts that absolute measurements could not be made and that relative deflections were not significant when the vehicle was at position 6 or beyond; thus, deflections that could conceivably have summed to substantial amounts may not have been included in the superposition procedure. For example, referring to Figure 6 and comparing the absolute deflection at the surface with that predicted by the theory for a depth of 18 feet, one observes that at a distance of approximately 11 feet from the origin of coordinates (position of dotted line), the elastic deflections are the same, and, as a consequence, the relative motion between these points is zero. Furthermore, for points further away from the origin of coordinates, the theory suggests that the surface would appear to rise if measured relative to the 18 foot depth. Examination of the record for transducer number one shown in Figure 3 shows a small upward relative movement of the pavement beginning when the vehicle was approximately 56 feet in front of the transducers. This movement could be explained on the basis of the preceeding observation drawn from the theory; however, the relationship is not conclusive since the measured results were used as boundary conditions to obtain the theoretical deflections, and it is thus conceivable that the pavement actually moved upward relative to an absolute reference.

In the section on the application of the theory, it was pointed out that the assumption of a realistic non-zero horizontal displacement of the boundary would lead to smaller vertical deflections of the earth beneath the boundary and, as a result, the normalized deflection curve would reach an asymptote more rapidly. Since the horizontal boundary condition used in the application of the theory was that of zero motion, the predicted result would be modified by the use of a measured boundary deflection. Thus, a significant refinement might result if the horizontal displacement of the boundary was measured and introduced into the integrals from which the earth deflection is computed.

Conclusions

Significant motions of the earth exist at greater depths than has been previously imagined and any experiments planned for measuring the relative deflections of highways, landing strips, or similar structures could give grossly incorrect results if the existence of this motion is not recognized.

In order to provide a more valid basis for comparing theory and experiment, effort should be directed toward solving the three-dimensional theoretical problem; thus, bringing an accompanying simplification in the experiment.

The experimental results obtained have the general character of the theoretical results. However, the plotted experimental data approach an asymptote much more rapidly than do the theoretical.

The theory predicts that even though the slope of the normalized displacement curve may be quite small, a significant absolute deflection may be present. The experimental verification of this deflection remains to be obtained.

The theory predicts that the magnitude of Poisson's ratio has a strong influence on the deflections, smaller values of Poisson's ratio causing the deflection curve to become asymptotic more rapidly.

A significant refinement in the application of the theory might result if the horizontal deflection of the boundary was obtained and introduced into the integrals from which the vertical deflection was computed.

An important experimental problem to be solved is that of making an absolute measure of earth displacement.

References

1. Geldmacher, R. C., Anderson, R.L., Dunkin, J.W., Partridge, G.R., et al; Subgrade Support Characteristics, Report No. 38, Dec., 1956, Joint Highway Research Project, Purdue University, Lafayette, Indiana

2. Muskhelishvili, N.I., Some Basic Problems in the Mathematical Theory of Elasticity, P. Noordhoff, Groningen, Holland, 1953.

3. Partridge, G.R., Dunkin, J.W., Anderson, R.L., Geldmacher, R.C.,; "A 14-Channel Displacement Measuring Device Utilizing Magnetic and Paper Tape Recording", Communications and Electronics, September, 1957 (AIEE Transactions).

4. Geldmacher, R.C., Anderson, R.L., Dunkin, J.W., Partridge, G.R., Harr, M.E., and Wood, L.E.; "Subgrade Support Characteristics as Indicated by Measurements of Deflection and Strain", Proceedings, Thirty-Sixth Annual Meeting, Highway Research Board 1957, pp. 479-496.

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