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Mathematical Modelling and Numerical Analysis Will be set by the publisher
Modélisation Mathématique et Analyse Numérique

SPLINE DISCRETE DIFFERENTIAL FORMS AND A NEW FINITE DIFFERENCE DISCRETE HODGE OPERATOR.

Aurore Back[1] and Eric Sonnendrücker[2]

Abstract. We construct a new set of discrete differential forms based on B-splines of arbitrary degree as well as an associated Hodge operator. The theory is first developed in 1D and then extended to multi-dimension using tensor products. We link our discrete differential forms with the theory of chains and cochains. The spline discrete differential forms are then applied to the numerical solution of Maxwell's equations.

AMS Subject Classification. — Please, give AMS classification codes —.

30 November 2012.

1. Introduction

The equations of physics are mathematical models consisting of geometric objects and relationships between them. There are many methods to discretize equations, but few maintain the physical nature of objects that constitute them. To respect the geometrical nature of physics, it is necessary to change the point of view and use differential geometry, also for the numerical study. In differential geometry and tensor calculus, differential forms are an approach to multivariable calculus that is independent of coordinates. The operators such as divergence, curl or gradient are replaced by the exterior derivative d. The exterior derivative acts on a k-form to produce a (k+1)-form. So the fundamental theorem of calculus, the divergence theorem, Green's theorem, and Stokes' theorem are also well defined in differential geometry and we also have the de Rham cohomology. Another advantage to use this approach is to know where, in the equations, there is a coordinates dependance i.e. a metric dependance. These notions can be determined by using the Hodge star operator. There have been several articles on the subject because there are many problems such as the discretization of Hodge star operator [2, 3, 12] (an important notion which contains all the metric of our domain), and the interpolation of differential forms [1, 4–6]. The first who used this point of view to discretize equations is Alain Bossavit [4]. He uses Whitney elements [5] to discretize differential forms and hence, discretize Maxwell equations in the language of differential geometry and until now, the basis functions used for interpolation have been Whitney forms. In this paper we propose to define a new class of discrete differential forms using B-splines and so a new discrete Hodge operator. This new approach proves to have many advantages. It allows to define high order approximation and higher degree B-splines are computed by recurrence with de Boor algorithm [8] so its easy and efficient to implement them; discrete differential forms verify the same properties as "continuous" differential forms especially they preserve the de Rham diagram. Moreover, the new discrete Hodge star operator is represented by a banded matrix and respect some properties of continuous approach when the degree of B-splines tends to infinite. It also appears that in the Finite Element context our B-spline discrete differential forms are naturally related to the B-spline finite elements appearing in isogeometric analysis [7, 14].

Keywords and phrases: Discrete differential forms, Discrete Hodge operator, B-splines, Maxwell, Numerical simulation.

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In the sequel, we recall the construction of B-splines and their properties (for this part the reader is refered to the book of de Boor [8]) then we explain how to construct discrete differential forms based on B-splines for uniform and non uniform meshes and for periodic and perfect conductors boundary conditions. Our objects are constructed in detail in the $1D$ case and are extended to the $3D$ case by tensor product using the $1D$ forms. We explain also the construction of applications acting on differential forms such as the exterior derivative and so the Hodge operator. For discretizing the last operator, we adapt the technique of T. Tarhasaari, L. Kettunen and A. Bossavit [3] for discretizing the Hodge star operator to the case of B-splines. The Hodge star operator $\star$ must verify the fact that $\star\star^{-1}$ applied on a k-differential form must be equal to $(-1)^{k(n-k)}$ (where n corresponds to the dimension of our space). We will see, numerically, that these property is verify when the degree of B-splines tends to infinite. Then, in the fourth part, we show the link between the k-cochains or differential k-forms and k-chains but with a discrete point of view. It is the discrete version of integration of k-forms. This part proves the coherence of discrete differential forms based on B-splines because we obtain the same link that we can find in the continuous case and we show that the discrete de Rham diagram is also preserved. Finally, we apply this theory to the Maxwell equations and we test it on uniform meshes with periodic conditions and on non uniform meshes with perfect conductors boundary conditions. Moreover, since this point of view provides a geometric formulation of Lagrangian equations, this means no reference coordinate system and so the construction of approximation schemes remains valid in case of continuous deformation, so we apply a change of variables on non uniform mesh with perfect conductors boundary conditions.

2. A short overview of B-splines

Let us denote by $T = (t_i)_{0 \leqslant i \leqslant M}$ a non uniform set of increasing knots who can contains multiplicity. B-splines on T can be defined recursively. We remark that we make the difference between nodes of mesh and knots using for constructing the B-spline. Later, we will see that this difference allows to define some kind of boundary conditions. In particular natural boundary conditions (vanishing second derivative), Hermite boundary condition (given derivative), or periodic boundary conditions can be used. Let us denote by B_i^α the B-spline of degree α with support in the interval $[t_i, t_{i+\alpha+1}]$. Then B_i^α is defined recursively by

$$B_i^0(x) = \left| \begin{array}{l} 1 \text{ if } t_i \leq x < t_{i+1} \\ 0 \text{ else,} \end{array} \right.$$

and for $\alpha \geq 1$

$$B_i^\alpha(x) = \frac{x - t_i}{t_{i+\alpha} - t_i} B_i^{\alpha-1}(x) + \frac{t_{i+\alpha+1} - x}{t_{i+\alpha+1} - t_{i+1}} B_{i+1}^{\alpha-1}(x) \tag{1}$$

The B-splines verify the following properties:

(1) The B-spline B_i^α is a polynomial of degree α between two consecutive knots,
(2) We can have knots with a multiplicity m.
(3) The B-spline B_i^α is of class $C^{\alpha+1-m}$ on knot t with multiplicity m.
(4) Partition of unity: for any point x, we have $\sum_{i=0}^{N+\alpha} B_i^\alpha(x) = 1$.

We shall also need the recursion formula for the derivatives:

$$B_i^{\alpha\prime}(x) = \alpha \left(\frac{B_i^{\alpha-1}(x)}{t_{i+\alpha} - t_i} - \frac{B_{i+1}^{\alpha-1}(x)}{t_{i+\alpha+1} - t_{i+1}} \right). \tag{2}$$

For details, the reader is refered to the book of de Boor [8]

3. Construction of discrete differential forms based on B-splines in 1D case

3.1. **Uniform periodic mesh**

3.1.1. *Construction of discrete 0-form*

Let us define by $x_0 < x_1 < \cdots < x_{N-1} < x_N$ the nodes of our 1D mesh and by $N + 1$ the number of nodes. Since we assume a 1D periodic domain with $x_N - x_0$ as periodicity, all functions will be equal at

x_0 and x_N. Then x_N will not be part of the mesh. In these conditions, the B-splines of degree α, B_i^α are defined on knots equal to $t_i = x_0 + i\Delta_x$ for $i = -[\frac{\alpha+1}{2}], \dots, N+\alpha+1$ (there are no multiplicity and $[\frac{\alpha+1}{2}]$ corresponds to the entire part of $\frac{\alpha+1}{2}$) and $\Delta_x = (x_0 - x_N)/N$. So we have $N+\alpha+1$ knots, N nodes and N B-splines (because we consider that for $i \geqslant N - [\frac{\alpha+1}{2}]$, we have B_i is equal to B_{i-N}^α).

In the 1D case, we need to define discrete 0-forms and 1-forms that will be constructed using basis functions denoted respectively by $w_i^{0,\alpha}$ and $w_i^{1,\alpha}$.

Let us start with the discrete 0-form. We define the basis functions $w_i^{0,\alpha} = B_i^\alpha$ and the space of linear spline 0-forms $\mathcal{S}_0^\alpha$ will be the vector space generated by these basis functions. Any function $C^0 \in \mathcal{S}_0^\alpha$ writes

$$C^0(x) = \sum_{j=-[\frac{\alpha+1}{2}]}^{N-[\frac{\alpha+1}{2}]} c_j^0 B_j^\alpha(x) = \sum_{j=0}^{N-1} c^0_{j-[\frac{\alpha+1}{2}]} B^\alpha_{j-[\frac{\alpha+1}{2}]}(x),$$

For more simplicity, in the following, we will denote by $B_j^\alpha(x)$ the B-spline $B^\alpha_{j-[\frac{\alpha+1}{2}]}(x)$ and by c_j^0 the coefficients $c^0_{j-[\frac{\alpha+1}{2}]}$ i.e

$$C^0(x) = \sum_{j=0}^{N-1} c_j^0 B_j^\alpha(x).$$

The c_j^0 defined by the interpolation conditions $C^0(x_i) = \sum_{j=0}^{N-1} c_j^0 B_j^\alpha(x_i)$ for $0 \le i \le N-1$ which is a linear system that can be written in matrix form $M_\alpha^0 c^0 = \mathbb{C}^0$, with $\mathbb{C}^0 = (C^0(x_0), \dots, C^0(x_{N-1}))^T$, $c^0 = (c_0^0, \dots, c_{N-1}^0)^T$ and M_α^0 the square matrix whose components are $m_{ij}^0 = B_j^\alpha(x_i)$ for $i, j = 0 \dots N-1$.

Lemma 3.1. *On a uniform set of nodes, we have*

- *for all odd α, the matrix $(M^0_{\alpha,ij})_{0\le i,j\le N-1}$ is non singular.*
- *for all even α and number of mesh points N an odd number, the matrix $(M^0_{\alpha,ij})_{0\le i,j\le N-1}$ is non singular.*

Proof. We denote $B_{i,j}^\alpha = B_i^\alpha(x_j) = M^0_{\alpha,ij}$. We can easily show by induction, with help of the formula (1), that $B_{i,j}^\alpha = 0$ for $j \notin \{i+1, \dots, i+\alpha\}$, $B_{i,i+1}^\alpha = B_{i,i+\alpha}^\alpha = 1$ and $B_{i+1,j}^\alpha = B_{i,j-1}^\alpha$. So the formula (1) becomes:

$$B_{i,j}^\alpha = \frac{j-i}{\alpha} B_{i,j}^{\alpha-1} + \frac{i+\alpha+1-j}{\alpha} B_{i,j-1}^{\alpha-1}.$$

We notice that M_α^0 is a circulant matrix and since $B_{i,j}^\alpha = 0$ for $j \notin \{i+1, \dots, i+\alpha\}$, it can be written:

$$M_\alpha^0 = \sum_{j=1}^{\alpha} B_{0,j}^\alpha J_N^j, \tag{3}$$

where $J_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}$ with $N \times N$ its size. We can factorize (3) by $\frac{J_N}{\alpha!}$ and we obtain:

$$\begin{aligned} M_\alpha^0 &= \frac{J_N}{\alpha!} \sum_{j=1}^{\alpha} \left(j(\alpha-1)! B_{0,j}^{\alpha-1} + (\alpha+1-j)(\alpha-1)! B_{0,j-1}^{\alpha-1} \right) J_N^{j-1}, \\ &= \frac{J_N}{\alpha!} E^{\alpha-1}(J_N), \end{aligned}$$

where $E^{\alpha-1}(X) = \sum_{j=1}^{\alpha} \left(j(\alpha-1)! B_{0,j}^{\alpha-1} + (\alpha+1-j)(\alpha-1)! B_{0,j-1}^{\alpha-1} \right) X^{j-1}$ are Eulerian polynomials. Their coefficients are positive and symmetric (symmetric means that if $P(X) = a_n X^n + \dots + a_0$, $a_{n-k} = a_k$). Moreover, the eigenvalues of J_N are the Nth roots of unity: $\{\omega^0, \omega^1, \dots\}$ with $\omega = \exp(\frac{2i\pi}{N})$. So, the eigenvalues of M_α^0 are $\{\omega^0 E^{\alpha-1}(\omega^0), \omega^1 E^{\alpha-1}(\omega^1), \dots\}$. Since the determinant is the product of eigenvalues, if an Eulerian polynomial has a root which is a root of unity, M^0_α is singular. But we know [13] that Eulerian polynomials have real, negative and distinct roots. So, if -1 is the root of Eulerian polynomials, M^0_α is singular. We can observe that if α is an odd number, $E^{\alpha-1}(X)$ is a polynomial with even degree and positive, symmetric coefficients. So $E^{\alpha-1}(-1) \neq 0$ if α is an odd number that implies M^0_α is non singular. If α is an even number we have $E^{\alpha-1}(-1) = 0$ and so -1 must not be a root of J_N. However, $\exp(\frac{2ik\pi}{N}) = -1$ if and only if N is an even number and $k = \frac{N}{2}$. So $E^{\alpha-1}(-1) \neq 0$ if α is an even number and N an odd number that implies M^0_α is non singular. □

3.1.2. *Construction of the exterior derivative and discrete 1-form*

The exterior derivative is an application who acts on k-form and return a $(n-k)$-form where n is the dimension of our space. So, in the 1D case if we apply on discrete 0-form the exterior derivative, to be coherent, we must be have a 1-form. That is help us to construction the 1-form and the discrete exterior derivative. To define them we apply the exterior derivative on discrete 0-form. We obtain

$$\begin{aligned} dC^0(x) &= \sum_{i=0}^{N-1} c_i^0 d\left(B_i^\alpha(x)\right) = \sum_{i=0}^{N-1} c_i^0 (D_i^\alpha(x) - D_{i+1}^\alpha(x))\, dx \\ &= \sum_{i=1}^{N} (c_i^0 - c_{i-1}^0) D_i^\alpha(x)\, dx \\ &= \sum_{i=1}^{N} (c_i^0 - c_{i-1}^0)\, {}^1\mathbf{w}_i^\alpha(x) \end{aligned}$$

where $c_N^0 = c_0^0$. We deduce that in order to define the 1-forms we shall need the notation

$$D_i^\alpha(x) = \frac{\alpha}{t_{i+\alpha} - t_i} B_i^{\alpha-1}(x),$$

for this function linked to the derivative of the B-spline $B_i^{\alpha-1}$. We can now define the basis functions for the discrete 1-forms by

$$w_i^{1,\alpha}(x) = D_i^\alpha(x)\, dx.$$

The discrete exterior derivative d is defined by a matrix contains $0, 1$ and -1 and acting on splines coefficients such as $d(c_i^0)_{0\leqslant i\leqslant N-1} = (c_i^0 - c_{i-1}^0)_{1\leqslant i\leqslant N}$. It corresponds to the incidence matrix.

The space of linear spline 1-forms $\mathcal{S}_1^\alpha$ will be the vector space generated by these basis functions. Any 1-form $C^1 \in \mathcal{S}_1^\alpha$ writes

$$C^1(x) = \sum_{j=0}^{N-1} c_j^1 D_j^\alpha(x)\, dx,$$

the coefficients c_j^1 being defined by the relations

$$\int_{x_i}^{x_{i+1}} C^1(x) = \sum_{j=0}^{N-1} c_j^1 \int_{x_i}^{x_{i+1}} D_j^\alpha(x)\, dx \quad \text{for } 0 \leq i \leq N-1,$$

this also defines a linear system that can be written in matrix form $M_\alpha^1 c^1 = \mathbb{C}^1$, with

$$\mathbb{C}^1 = (\int_{x_0}^{x_1} C^1(x), \ldots, \int_{x_{N-1}}^{x_N} C^1(x))^T,$$

$c^1 = (c_0^1, \ldots, c_{N-1}^1)^T$ and M_α^1 the square matrix whose components are $m_{ij}^1 = \int_{x_i}^{x_{i+1}} D_j^\alpha(x)\, dx$.

Lemma 3.2. *Under the conditions of the previous lemma, the matrix M_α^1 is non singular.*

Proof. The basis functions $\mathbf{w}_i^{1,\alpha}$ have been chosen using the recursion formula for the spline derivatives (2) and verify

$$\mathbf{w}_i^{1,\alpha}(x) - \mathbf{w}_{i+1}^{1,\alpha}(x) = B_i^{\alpha\prime}(x)\,dx.$$

Using this relation and the fact that $B_i^\alpha(x)$ vanishes for $x \notin [t_i, t_{i+\alpha}]$, it follows easily by recurrence on the degree α that

$$\int_{t_{i+\nu}}^{t_{i+\nu+1}} \mathbf{w}_i^{1,\alpha}(x) = \sum_{k=0}^{\nu} B_{i+k}^\alpha(t_{i+\nu+1}) - \sum_{k=0}^{\nu-1} B_{i+k}^\alpha(t_{i+\nu}).$$

In the case of uniform meshes, we have the property $B_{i+k}^\alpha(t_{i+\nu+1}) = B_{i+k+\nu+1}^\alpha(t_i)$. So, we deduce that $(M_\alpha^1)_{i,j} = \int_{x_i}^{x_{i+1}} D_j^\alpha(x)\,dx = B_{j+1}^\alpha(x_{i+1})$. □

3.1.3. *The discrete Hodge Star*

The Hodge Star is an application who associates a k-form to a $(n-k)$-form in n-dimensionnal space. That says, in 3-dimensionnal space, it associates a 0-form to a 3-form and a 1-form to a 2-form. Indeed, for the mesh that's means the number of nodes must be equal to the number of volumes and the number of edges must be equal to the number of surfaces. No mesh verify these assumptions so we must consider two meshes: a primal mesh where nodes are denoted by x_i and a dual mesh where the dual nodes are the middle of the primal nodes, $x_{i+1/2} = (x_i + x_{i+1})/2$. With these construction we obtain that the number of nodes is equal to the number of dual volumes, the number of edges is equal to the number of dual surfaces and vice versa, the number of dual nodes is equal to the number of volumes and the number of dual edges is equal to the number of surfaces.
So we deduce that we must consider two types of differential forms: the primal forms, constructed on the primal mesh and dual forms on the dual mesh.

Denoting by $B_{j+1/2}^\alpha$ the splines whose knots are based on the dual mesh, the discrete 0-forms and 1-forms on the dual mesh are defined in the same way by

$$\tilde{C}^0(x) = \sum_{j=0}^{N-1} \tilde{c}_{j+1/2}^0 B_{j+1/2}^\alpha(x),$$

with the $\tilde{c}_{j+1/2}^0$ defined by the interpolation conditions $\tilde{C}^0(x_{i+1/2}) = \sum_{j=0}^{N-1} \tilde{c}_{j+1/2}^0 B_{j+1/2}^\alpha(x_{i+1/2})$ for $0 \le i \le N-1$ which is a linear system that can be written in matrix form $\tilde{M}_\alpha^0 \tilde{c}^0 = \tilde{\mathbb{C}}^0$, with

$$\tilde{\mathbb{C}}^0 = (\tilde{C}^0(x_{1/2}), \dots, \tilde{C}^0(x_{N-1/2}))^T,$$

$\tilde{c}^0 = (\tilde{c}_{1/2}^0, \dots, \tilde{c}_{N-1/2}^0)^T$ and $\tilde{M}_\alpha^0$ the square matrix whose components are $\tilde{m}_{ij}^0 = B_{j+1/2}^\alpha(x_{i+1/2})$. We have that $\tilde{M}_\alpha^0$ meet the conditions of the first lemma and so the square matrix is non singular.

A discrete 1-form on the dual mesh is defined by

$$\tilde{C}^1(x) = \sum_{j=0}^{N-1} \tilde{c}_{j+1/2}^1 D_{j+1/2}^\alpha(x)\,dx,$$

the coefficients $\tilde{c}_{j+1/2}^1$ being defined by the relations

$$\int_{x_{i+1/2}}^{x_{i+3/2}} \tilde{C}^1(x) = \sum_{j=0}^{N-1} \tilde{c}_{j+1/2}^1 \int_{x_{i+1/2}}^{x_{i+3/2}} D_{j+1/2}^\alpha(x)\,dx \quad \text{for } 0 \le i \le N-1,$$

this also defines a linear system that can be written in matrix form $\tilde{M}_\alpha^1 \tilde{c}^1 = \tilde{\mathbb{C}}^1$, with

$$\tilde{\mathbb{C}}^1 = (\int_{x_{1/2}}^{x_{3/2}} \tilde{C}^1(x), \dots, \int_{x_{N-1/2}}^{x_{N+1/2}} \tilde{C}^1(x))^T,$$

$\tilde{c}^1 = (\tilde{c}^1_{1/2}, \ldots, \tilde{c}^1_{N-1/2})^T$ and $\tilde{M}^1_\alpha$ the square matrix whose components are $\tilde{m}^1_{ij} = \int_{x_{i+1/2}}^{x_{i+3/2}} D^\alpha_{j+1/2}(x)\,dx$. For the same reason as M^1, $\tilde{M}^1_\alpha$ is non singular. Note that due to the periodicity hypothesis $\int_{x_{N-1/2}}^{x_{N+1/2}} \tilde{C}^1(x) = \int_{x_{N-1/2}}^{x_N} \tilde{C}^1(x) + \int_{x_0}^{x_{1/2}} \tilde{C}^1(x)$.

Having defined discrete 0-forms and 1-forms on both grids, we can now define in a natural way the discrete Hodge operators [2,3], mapping primal 0-forms to dual 1-forms, primal 1-forms to dual 0-forms and the other way round.

As discrete differential forms are defined by their coefficients in the appropriate basis, the discrete Hodge operator should map those coefficients to those on the image basis. Let us start with the discrete Hodge mapping primal 0-forms to dual 1-forms. Given a discrete 0-form on the primal mesh

$$C^0(x) = \sum_{j=0}^{N-1} c^0_j B^\alpha_j(x),$$

we can apply the continuous Hodge operator to it, as $\star 1 = dx$, we get

$$\star C^0(x) = \sum_{j=0}^{N-1} c^0_j B^\alpha_j(x)\,dx.$$

Now, as B^α_j are not splines on the dual mesh, this does not define a discrete differential form on the dual mesh. We need an additional projection step. Denoting by πC^0 the projection of $\star C^0$ on the space of discrete differential forms of the same order on the dual mesh, we can write

$$\pi C^0(x) = \sum_{j=0}^{N-1} \tilde{c}^1_{j+1/2} D^\alpha_{j+1/2}(x)\,dx,$$

with

$$\int_{x_{i+1/2}}^{x_{i+3/2}} \star C^0(x) = \sum_{j=0}^{N-1} \tilde{c}^1_{j+1/2} \int_{x_{i+1/2}}^{x_{i+3/2}} D^\alpha_{j+1/2}(x)\,dx \quad \text{for } 0 \le i \le N-1.$$

Now defining $\tilde{S}^1$ the matrix whose i,j coefficient is $\int_{x_{i+1/2}}^{x_{i+3/2}} B^\alpha_j(x)\,dx$, this relation becomes in matrix form

$$\tilde{S}^1 c^0 = \tilde{M}^1_\alpha \tilde{c}^1,$$

so that the discrete Hodge operator mapping c^0 to $\tilde{c}^1$ is

$$(\tilde{M}^1_\alpha)^{-1} \tilde{S}^1 \text{ with } \tilde{S}^1_{i,j} = \int_{x_{i+1/2}}^{x_{i+3/2}} B^\alpha_j(x)\,dx.$$

In order to define the Hodge operator mapping discrete 1-forms on the primal grid to discrete 0-forms on the dual grid, we apply the continuous Hodge operator ($\star dx = 1$) to a discrete 1-form on the primal grid

$$\star C^1(x) = \sum_{j=0}^{N-1} c^1_j D^\alpha_j(x).$$

Its projection on the space of discrete 0-forms on the dual grid is defined by the point values $\star C^1(x_{i+1/2})$. Hence in the same way as before the discrete Hodge in this case is defined by

$$(\tilde{M}^0_\alpha)^{-1} \tilde{S}^0 \text{ with } \tilde{S}^0_{i,j} = D^\alpha_j(x_{i+1/2}).$$

The Hodge operators mapping from the dual grid to the primal grid are naturally defined in the same way by

$$(M^1_\alpha)^{-1} S^1 \text{ with } S^1_{i,j} = \int_{x_i}^{x_{i+1}} B^\alpha_{j+1/2}(x)\,dx,$$

$$(M^0_\alpha)^{-1}S^0 \text{ with } S^0_{i,j} = D^\alpha_{j+1/2}(x_i).$$

Let us finally explicit the different matrices involved in the Hodge operators for the case of uniform periodic linear and cubic splines. In the case of a uniform mesh, due to the recurrence relation on spline derivatives we have

$$D^\alpha_j(x) - D^\alpha_{j+1}(x) = {B^\alpha_{j+1}}'(x).$$

Integrating between i an $i+1$ and using that $B^\alpha_j(x) = B^\alpha_0(x - x_j)$ yields

$$\int_{x_i}^{x_{i+1}} (D^\alpha_0(x - x_j) - D^\alpha_0(x - x_j - \Delta x))\, dx = B^\alpha_0((i-j)\Delta x) - B^\alpha_0((i-j-1)\Delta x).$$

So that

$$\int_{x_i}^{x_{i+1}} D^\alpha_j(x)\, dx = B^\alpha_{j+1}(x_{i+1}).$$

From this it follows that on a uniform grid $M^0_\alpha = M^1_\alpha = \tilde{M}^0_\alpha = \tilde{M}^1_\alpha$ are all the usual degree α periodic spline interpolation matrix.

For linear splines ($\alpha = 1$) the matrices $M^0_1 = M^1_1 = \tilde{M}^0_1 = \tilde{M}^1_1 = \mathbb{I}$ are all the identity matrix. For cubic splines ($\alpha = 3$) these matrices are the circulant matrices with $2/3$ on the diagonal and $1/6$ on the upper and lower diagonal.

Let us now come to the Hodge matrices. Due to their expressions the matrices are also constant circulant matrices with $S^0_\alpha = \tilde{S}^0_\alpha$ and $S^1_\alpha = \tilde{S}^1_\alpha$. In the case $\alpha = 1$, $D^\alpha_i(x) = \frac{1}{\Delta x}$ for $x_{i-1} \le x \le x_i$ and 0 elsewhere. Hence

$$S^0_1 = \tilde{S}^0_1 = \frac{1}{\Delta x}\mathbb{I}.$$

And a simple computation yields that $S^1_1 = \tilde{S}^1_1$ are circulant matrices with three diagonals that read

$$S^1_1 = \tilde{S}^1_1 = \Delta x\, \mathrm{circ}[\frac{1}{8}, \frac{3}{4}, \frac{1}{8}],$$

where

$$\mathrm{circ}[\frac{1}{8}, \frac{3}{4}, \frac{1}{8}] = \begin{pmatrix} \frac{1}{8} & 0 & \cdots & \frac{1}{8} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{8} & 0 & \cdots & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \end{pmatrix}.$$

Notice that S^0_1 and $\tilde{S}^1_1$ are not exactly the inverse of each other as is the case for their continuous counterparts and the same for S^1_1 and $\tilde{S}^0_1$, but for example $\tilde{S}^1_1$ and $(S^0_1)^{-1}$ can be used as approximations for the discrete Hogde operator.

In the case of cubic splines we have, the following circulant matrices

$$S^0_3 = \tilde{S}^0_3 = \frac{1}{\Delta x}\, \mathrm{circ}[\frac{1}{8}, \frac{3}{4}, \frac{1}{8}],$$

$$S^1_3 = \tilde{S}^1_3 = \Delta x\, \mathrm{circ}[\frac{1}{384}, \frac{19}{96}, \frac{115}{192}, \frac{19}{96}, \frac{1}{384}].$$

In theory, we have $\star\star\,\omega^k = (-1)^{k(n-k)}\omega^k$ for all differential k-forms in n-dimensional Riemann manifold. We do not have this property with discrete differential forms but $(M^{n-k})^{-1}S^{n-k}(\tilde{M}^{n-k})^{-1}\tilde{S}^{n-k}$ tends to $(-1)^{k(n-k)}Id$ when we increase the order of spline for interpolation. We remark that it does not depend on the number of mesh points. In the case of periodic domain in 1-dimension, we study $(M^{n-k})^{-1}S^{n-k}(\tilde{M}^{n-k})^{-1}\tilde{S}^{n-k} - (-1)^{k(n-k)}Id$ and we obtain the following table:

Degree of Spline	Frobenius norm	2-norm	1-norm
$\alpha = 1$	3.66358772244	1.5246887983	1.75
$\alpha = 3$	0.0723421421156	0.0625	0.0625
$\alpha = 5$	0.0145136394546	0.00821424978684	0.0137692232972

3.2. Non uniform mesh with perfect conductors boundary conditions

For perfect conductors boundary conditions the main idea is the same. We have a primal and dual meshes, primal and dual forms construct respectively on primal and dual mesh, the same construction for discrete exterior derivative and the same idea for the construction of Hodge Star. The only difference is the set of knots who determine the construction of B-spline.

Remark 3.1. *We have three types of points. We called by nodes the points describing our mesh, by knots the points helping to construct B-splines, and by points the points used for interpolation.*
In the case of periodic boundary conditions, the notions of nodes, knots and points are equivalent; In the case of perfect conductors boundary conditions these elements are all different.

3.2.1. *Construction of B-splines*

The primal $1D$ mesh of our domain will be a non uniform set of nodes $x_0 < x_1 < \cdots < x_{N-1} < x_N$. For perfect conductors boundary conditions, using a degree α, the set of knots is corresponding to the set of nodes $x_0 < x_1 < \cdots < x_{N-1} < x_N$ but we duplicate $\alpha+1$ times the knots in the boundary to have a set of B-splines C^0 on the boundary x_0 and x_N and $C^{\alpha+1}$ otherwise. So, the set of knots constructing the set of B-splines is $\underbrace{x_0,\dots,x_0}_{\alpha+1}, x_1,\dots,x_{N-1}, \underbrace{x_N,\dots,x_N}_{\alpha+1}$. In this case, we have to contend a problem because we have $N+\alpha$ splines functions and $N+1$ different points for interpolation in the primal mesh. So we must add interpolation's points on primal mesh. So as α is odd, we add the middle of $\frac{(\alpha-1)}{2}$ first primal cells and $\frac{(\alpha-1)}{2}$ last primal cells. For example, if α equals 3, we must add 2 points, so we take the middle of $[x_0, x_1]$ and $[x_{N-1}, x_N]$.
The primal interpolation points will be

$$\{x_0, \cdots, x_N\} \cup \left\{ \frac{(x_{i+1}+x_i)}{2} | i \in \{0,\cdots,\frac{(\alpha-1)}{2}-1\} \right\} \cup \left\{ \frac{(x_{i-1}+x_i)}{2} | i \in \{N-\frac{(\alpha-1)}{2}+1,\cdots,N\} \right\}.$$

Denoting by $p_0 < p_1 < \cdots < p_{N+\alpha-1}$ the interpolation nodes of our primal mesh in increasing order.

For the case of dual mesh, we proceed with the same idea. The dual mesh will consist of the middle points of the primal mesh with the extremal points x_0 and x_N, i.e. $N+2$ nodes: $x_0 < x_{1/2} < \cdots < x_{N-1/2} < x_N$. And so the set of knots is $\underbrace{x_0,\dots,x_0}_{\alpha+1} < x_{1/2} < \cdots < x_{N-1/2} < \underbrace{x_N,\dots,x_N}_{\alpha+1}$, and constructs $N+\alpha+1$ B-splines. The same previous problem appears, we have $N+\alpha+1$ B-splines and $N+2$ different nodes or interpolation's points. So we consider α is odd and we add the middle of $\frac{(\alpha-1)}{2}$ first dual cells and $\frac{(\alpha-1)}{2}$ last dual cells. For example, if α equals 3, we must add 2 points, so we take the middle of $[x_0, x_{1/2}]$ and $[x_{N-1/2}, x_N]$.
The dual interpolation points of our dual mesh will be

$$\{x_{-1/2} = x_0, x_{1/2} \cdots, x_{N-1/2}, x_N = x_{N+1/2}\} \cup \left\{ \frac{(x_{i-1/2}+x_{i+1/2})}{2} | i \in \{0,\cdots,\frac{(\alpha-1)}{2}-1\} \cup \{N-\frac{(\alpha-1)}{2}+1,\cdots,N\} \right\}$$

and denoting by $\tilde{p}_0 < \tilde{p}_1 < \cdots < \tilde{p}_{N+\alpha}$ in increasing order.

3.2.2. *Construction of discrete differential forms*

With this construction, we can interpolate primal and dual 0 and 1-forms. Let us start with the discrete 0-form on the primal mesh. Any function $C^0 \in \mathcal{S}_0^\alpha$ writes

$$C^0(x) = \sum_{j=0}^{N+\alpha-1} c_j^0 B_j^\alpha(x),$$

with the c_j^0 defined by the interpolation conditions $C^0(p_i) = \sum_{j=0}^{N+\alpha-1} c_j^0 B_j^\alpha(p_i)$ for $i \in \{0, \cdots N+\alpha-1\}$ on the primal mesh which is a linear system that can be written in matrix form $M_\alpha^0 c^0 = \mathbb{C}^0$, with

$$\mathbb{C}^0 = \left(C^0(p_0), \dots, C^0(p_{N+\alpha-1})\right)^T,$$

$c^0 = \left(c^0_0, \dots, c^0_{N+\alpha-1}\right)^T$ and M^0_α the square matrix with size $N+\alpha$ whose components are $m^0_{i,j} = B^\alpha_j(p_i)$ for $j = 0, \cdots, N+\alpha-1$ and $i = 0, \cdots N+\alpha-1$.

Lemma 3.3. *The matrix M^0_α is non singular.*

Proof. M^0_α meets the conditions of the Schoenberg-Whitney Theorem [8] and so the square matrix is non singular. □

The space of linear spline 1-forms $\mathcal{S}^\alpha_1$ will be the vector space generated by these basis functions. Any 1-form $C^1 \in \mathcal{S}^\alpha_1$ writes

$$C^1(x) = \sum_{j=0}^{N+\alpha-2} c^1_j D^\alpha_j(x)\,dx,$$

the coefficients c^1_j being defined by the relations

$$\int_{p_i}^{p_{i+1}} C^1(x) = \sum_{j=0}^{N+\alpha-2} c^1_j \int_{p_i}^{p_{i+1}} D^\alpha_j(x)\,dx \quad \text{for } 0 \le i \le N+\alpha-2,$$

this also defines a linear system that can be written in matrix form $M^1_\alpha c^1 = \mathbb{C}^1$, with $\mathbb{C}^1 = \left(\int_{p_0}^{p_1} C^1(x), \dots, \int_{p_{N+\alpha-2}}^{p_{N+\alpha-1}} C^1(x)\right)^T$, $c^1 = \left(c^1_0, \dots, c^1_{N+\alpha-2}\right)^T$ and M^1_α the square matrix with size $N+\alpha-1$ whose components are $m^1_{i,j} = \int_{p_i}^{p_{i+1}} D^\alpha_j(x)\,dx$ for $j, i = 0, \cdots N+\alpha-2$.

Lemma 3.4. *The matrix M^1_α is non singular.*

Proof. Using the relation $\mathbf{w}^{1,\alpha}_i(x) - \mathbf{w}^{1,\alpha}_{i+1}(x) = B^{\alpha\prime}_i(x)\,dx$ and the fact that $B^\alpha_i(x)$ vanishes for $x \notin [x_i, x_{i+\alpha}]$, it follows easily by recurrence on the degree α that

$$\int_{p_i}^{p_{i+1}} \mathbf{w}^{1,\alpha}_i(x) = \sum_{k=j}^{N-1} B^\alpha_k(p_{i+1}) - \sum_{k=j}^{N-1} B^\alpha_k(p_i) =: A^\alpha_j(p_{i+1}) - A^\alpha_j(p_i).$$

We can observe that M^1_α is the principal minor (1,1) of the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} M^0_\alpha \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} = \begin{pmatrix} A^\alpha_{-\alpha}(p_0) & A^\alpha_{-\alpha+1}(p_0) & \cdots & A^\alpha_{N-1}(p_0) \\ \int_{p_0}^{p_1} D_{-\alpha} & & & \\ \int_{p_1}^{p_2} D_{-\alpha} & & & \\ \vdots & & M^1_\alpha & \\ \int_{p_{N+\alpha-2}}^{p_{N+\alpha-1}} D_{-\alpha} & & & \end{pmatrix}.$$

The matrix $\mathcal{A}$ on the left hand side is non singular because it's the product of non singular matrices. Since for all i, $p_i \geqslant x_0$ and the support of $D_{-\alpha}$ is $]x_{-\alpha}, x_0[$, we have $\int_{p_i}^{p_{i+1}} D_{-\alpha} = 0$. Furthermore, $A^\alpha_{-\alpha}(p_0) \neq 0$ because $B_i(x) \geqslant 0$ and $B_{-\alpha}(p_0) \neq 0$. So, $0 \neq det(\mathcal{A}) = A^\alpha_{-\alpha}(p_0) det(M^1_\alpha)$ that implies $det(M^1_\alpha) \neq 0$. □

The discrete 0-forms and 1-forms on the dual mesh are defined in the same way by

$$\tilde{C}^0(x) = \sum_{j=0}^{N+\alpha} \tilde{c}^0_{j+1/2} B^\alpha_{j+1/2}(x),$$

with the $\tilde{c}^0_{j+1/2}$ defined by the interpolation conditions

$$\tilde{C}^0(\tilde{p}_i) = \sum_{j=0}^{N+\alpha} \tilde{c}^0_{j+1/2} B^\alpha_{j+1/2}(\tilde{p}_i) \qquad \text{for } i \in \{0, \cdots, N+\alpha\}$$

on the dual mesh which is a linear system that can be written in matrix form $\tilde{M}^0_\alpha \tilde{c}^0 = \tilde{\mathbb{C}}^0$, with

$$\tilde{\mathbb{C}}^0 = \left(\tilde{C}^0(\tilde{p}_0), \tilde{C}^0(\tilde{p}_1), \cdots, \tilde{C}^0(\tilde{p}_{N+\alpha})\right)^T,$$

$\tilde{c}^0 = \left(\tilde{c}^0_{1/2}, \ldots, \tilde{c}^0_{N+\alpha+1/2}\right)^T$ and $\tilde{M}^0_\alpha$ the square matrix with size $N+\alpha+1$ whose components are $\tilde{m}^0_{i,j} = B^\alpha_{j+1/2}(\tilde{p}_i)$ and is nonsingular with help Schoenberg-Whitney Theorem.

A discrete 1-form on the dual mesh is defined by

$$\tilde{C}^1(x) = \sum_{j=0}^{N+\alpha-1} \tilde{c}^1_{j+1/2} D^\alpha_{j+1/2}(x)\, dx,$$

the coefficients $\tilde{c}^1_{j+1/2}$ being defined by the relations

$$\int_{\tilde{p}_i}^{\tilde{p}_{i+1}} \tilde{C}^1(x) = \sum_{j=0}^{N+\alpha-1} \tilde{c}^1_{j+1/2} \int_{\tilde{p}_i}^{\tilde{p}_{i+1}} D^\alpha_{j+1/2}(x)\, dx \quad \text{for } 0 \le i \le N+\alpha-1,$$

this also defines a linear system that can be written in matrix form $\tilde{M}^1_\alpha \tilde{c}^1 = \tilde{\mathbb{C}}^1$, with $\tilde{\mathbb{C}}^1 = \left(\int_{\tilde{p}_0}^{\tilde{p}_1} \tilde{C}^1(x), \cdots, \int_{\tilde{p}_{N+\alpha-2}}^{\tilde{p}_{N+\alpha-1}} \tilde{C}^1(x)\right)^T$, $\tilde{c}^1 = \left(\tilde{c}^1_{1/2}, \ldots, \tilde{c}^1_{N+\alpha-1/2}\right)^T$ and $\tilde{M}^1_\alpha$ the square matrix with size $N+\alpha$ whose components are $\tilde{m}^1_{i,j} = \int_{\tilde{p}_i}^{\tilde{p}_{i+1}} D^\alpha_{j+1/2}(x)\, dx$ and is nonsingular thanks to the previous lemma.

3.2.3. *The discrete Hodge operator:*

Let us start with the discrete Hodge [2, 3] mapping primal 0-forms to dual 1-forms. Given a discrete 0-form on the primal mesh

$$C^0(x) = \sum_{j=0}^{N+\alpha} c^0_j B^\alpha_j(x),$$

and so

$$\star C^0(x) = \sum_{j=0}^{N+\alpha-1} c^0_j B^\alpha_j(x)\, dx.$$

Denoting by πC^0 the projection of $\star C^0$ on the space of discrete differential forms of the same order on the dual mesh, we can write

$$\pi C^0(x) = \sum_{j=0}^{N+\alpha-1} \tilde{c}^1_{j+1/2} D^\alpha_{j+1/2}(x)\, dx,$$

with

$$\int_{\tilde{p}_i}^{\tilde{p}_{i+1}} \star C^0(x) = \sum_{j=0}^{N+\alpha-1} \tilde{c}^1_{j+1/2} \int_{\tilde{p}_i}^{\tilde{p}_{i+1}} D^\alpha_{j+1/2}(x)\, dx \quad \text{for } 0 \le i \le N+\alpha-1.$$

Now defining $\tilde{S}^1$ the square matrix with a size $N+\alpha$ whose i,j coefficient is $\int_{\tilde{p}_i}^{\tilde{p}_{i+1}} B^\alpha_j(x)\, dx$, this relation becomes in matrix form

$$\tilde{S}^1 c^0 = \tilde{M}^1_\alpha \tilde{c}^1,$$

so that the discrete Hodge operator mapping c^0 to $\tilde{c}^1$ is

$$(\tilde{M}^1_\alpha)^{-1} \tilde{S}^1 \text{ with } \tilde{S}^1_{i,j} = \int_{n_{i-1/2}}^{n_{i+1/2}} B^\alpha_j(x)\, dx.$$

In order to define the Hodge operator mapping discrete 1-forms in the primal grid to discrete 0-forms on the dual grid, we apply the continuous Hodge operator to a discrete 1-form on the primal grid

$$\star C^1(x) = \sum_{j=0}^{N+\alpha-2} c^1_j D^\alpha_j(x).$$

Its projection on the space of discrete 0-forms on the dual grid is defined by the point values $\star C^1(\tilde{p}_i)$ for $i \in \{0, \cdots, N+\alpha\}$. Hence in the same way as before the discrete Hodge in this case is defined by

$$(\tilde{M}^0_\alpha)^{-1}\tilde{S}^0 \text{ with } \tilde{S}^0_{i,j} = D^\alpha_j(\tilde{p}_i).$$

We can notice that this matrix is not square because its size is $(N+\alpha+1)\times(N+\alpha-1)$.

The Hodge operators mapping from the dual grid to the primal grid are naturally defined in the same way by

$$(M^1_\alpha)^{-1}S^1 \text{ with } S^1_{i,j} = \int_{p_i}^{p_{i+1}} B^\alpha_{j+1/2}(x)\,dx.$$

This matrix is not square, its size is $(N+\alpha-1)\times(N+\alpha+1)$, but the matrix S^0 is square, with a size $(N+\alpha)\times(N+\alpha)$:

$$(M^0_\alpha)^{-1}S^0 \text{ with } S^0_{i,j} = D^\alpha_{j+1/2}(p_i).$$

In the case of perfect conductors boundary $\star\star\,\omega^k = (-1)^{k(n-k)}\omega^k$ for all differential k-forms in n-dimensional Riemann manifold. In 1-dimension, $(M^{n-k})^{-1}S^{n-k}(\tilde{M}^{n-k})^{-1}\tilde{S}^{n-k}$ tends to $(-1)^{k(n-k)}Id$ when we increase the order of spline for interpolation.
We study the different norm for $(M^{n-k})^{-1}S^{n-k}(\tilde{M}^{n-k})^{-1}\tilde{S}^{n-k} - (-1)^{k(n-k)}Id$, we obtain the following table:

Degree of Spline	Frobenius norm	2-norm	1-norm
$\alpha = 1$	2.34520787991	0.499657383689	0.5
$\alpha = 3$	0.177424498932	0.0622720895077	0.0754437671418
$\alpha = 5$	0.0490270024226	0.0345996163058	0.0209297727957

4. Construction of discrete differential forms based on B-splines in 3D case

4.1. The basis

We are now going to define the 3D discrete differential forms on a cartesian grid, which will be needed for Maxwell's equations, by tensor product using the 1D form. This procedure can be generalized in a natural way to any number of dimensions.

The set of 3D discrete differential forms will be defined as the span of the following basis functions:

- The basis functions for the 0-forms are

$$^0w^\alpha_{i,j,k}(x,y,z) = B^\alpha_i(x)B^\alpha_j(y)B^\alpha_k(z).$$

- The basis functions for the 1-forms are

$$\begin{aligned}
^1\mathbf{w}^{\alpha,x}_{i,j,k}(x,y,z) &= D^\alpha_i(x)B^\alpha_j(y)B^\alpha_k(z)\,dx,\\
^1\mathbf{w}^{\alpha,y}_{i,j,k}(x,y,z) &= B^\alpha_i(x)D^\alpha_j(y)B^\alpha_k(z)\,dy,\\
^1\mathbf{w}^{\alpha,z}i,j,k(x,y,z) &= B^\alpha_i(x)B^\alpha_j(y)D^\alpha_k(z)\,dz.
\end{aligned}$$

- The basis functions for the 2-forms are

$$\begin{aligned}
^2\mathbf{w}^{\alpha,x}_{i,j,k}(x,y,z) &= B^\alpha_i(x)D^\alpha_j(y)D^\alpha_k(z)\,dy\wedge dz,\\
^2\mathbf{w}^{\alpha,y}_{i,j,k}(x,y,z) &= D^\alpha_i(x)B^\alpha_j(y)D^\alpha_k(z)\,dz\wedge dx,\\
^2\mathbf{w}^{\alpha,z}_{i,j,k}(x,y,z) &= D^\alpha_i(x)D^\alpha_j(y)B^\alpha_k(z)\,dx\wedge dy.
\end{aligned}$$

- The basis functions for the 3-forms are

$$^3w^\alpha_{i,j,k}(x,y) = D^\alpha_i(x)D^\alpha_j(y)D^\alpha_k(z)\,dx\wedge dy\wedge dz.$$

This construction will yield the same basis functions as in [7] and [14] where vector calculus is used.

4.2. **Example: the Maxwell equations**

The 3D Maxwell equations can be written in terms of differential forms [4] in the following way

$$-\partial_t\,{}^2\mathbf{D} + d\,{}^1\mathbf{H} = {}^2\mathbf{J}, \tag{4}$$

$$\partial_t\,{}^2\mathbf{B} + d\,{}^1\mathbf{E} = 0, \tag{5}$$

$$d\,{}^2\mathbf{D} = {}^3\rho, \tag{6}$$

$$d\,{}^2\mathbf{B} = 0, \tag{7}$$

where ${}^2\mathbf{D}$, ${}^2\mathbf{B}$, ${}^2\mathbf{J}$ are 2-forms, ${}^1\mathbf{E}$, ${}^1\mathbf{H}$ are 1-forms and ${}^3\rho$ is a 3-form.

For the purpose of numerical validation, let us consider the 2D case, where all functions depend only on the x and y variables. In this case, the Maxwell equations keep their three components, we therefore still need to consider the differential forms in a 3D space but restrict them to coefficients depending only on x and y. For the discretization of Maxwell's equations with our spline discrete differential forms, we can define two dual uniform cartesian grids of $[0,1]^3$. The mesh we shall consider will be a the cartesian product of a 2D mesh with one cell of length one in the z direction. The primal 2D grid is based on the points $x_i = i/\Delta x$, $y_j = j/\Delta y$, with $(i,j)\in[0,N_x]\times[0,N_y]$ and $N_x\Delta x = N_y\Delta y$. In case of periodic boundary conditions in the x direction, the point x_{N_x} corresponds to x_0 and is omitted from the grid. Periodic boundary conditions in the other direction are dealt with in the same manner.

The points of the dual grid are $x_{i+1/2} = (i+1/2)/\Delta x$, $y_{j+1/2} = (j+1/2)/\Delta y$, with $(i,j,k)\in[0,N_x-1]\times[0,N_y-1]$.

The basis functions for our spline discrete differential forms in this case will be

- for the 0-forms

$$^0w^\alpha_{i,j}(x,y) = B^\alpha_i(x)B^\alpha_j(y),$$

- for the 1-forms

$$^1\mathbf{w}^{\alpha,x}_{i,j}(x,y) = D^\alpha_i(x)B^\alpha_j(y)\,dx,\ \ {}^1\mathbf{w}^{\alpha,y}_{i,j}(x,y) = B^\alpha_i(x)D^\alpha_j(y)\,dy,\ \ {}^1\mathbf{w}^{\alpha,z}_{i,j}(x,y) = B^\alpha_i(x)B^\alpha_j(y)\,dz,$$

- for the 2-forms

$$^2\mathbf{w}^{\alpha,x}_{i,j}(x,y) = B^\alpha_i(x)D^\alpha_j(y)\,dy\wedge dz,\ \ {}^2\mathbf{w}^{\alpha,y}_{i,j}(x,y) = D^\alpha_i(x)B^\alpha_j(y)\,dz\wedge dx,$$

$$^2\mathbf{w}^{\alpha,z}_{i,j}(x,y) = D^\alpha_i(x)D^\alpha_j(y)\,dx\wedge dy,$$

- for the 3-forms

$$^3w^\alpha_{i,j}(x,y) = D^\alpha_i(x)D^\alpha_j(y)\,dx\wedge dy\wedge dz.$$

The discrete differential forms on the dual mesh are defined in the same way with their indices on the dual mesh.

Let us now introduce approximations of the unknowns of Maxwell's equations as linear combinations of these basis functions for the corresponding p-forms. Let us denote by ${}^1\tilde{\mathbf{w}}^\alpha$ and ${}^2\tilde{\mathbf{w}}^\alpha$, the discrete spline one form and two form basis functions for the splines with knots being the points of the dual mesh. In 2D, Maxwell's equations decouple into two systems, one linking the x and y components of ${}^2\mathbf{D}$ and ${}^1\mathbf{E}$ and the z component of ${}^2\mathbf{B}$ and ${}^1\mathbf{H}$, and the other one linking the x and y components of ${}^2\mathbf{B}$ and ${}^1\mathbf{H}$ and the z component of ${}^2\mathbf{D}$ and ${}^1\mathbf{E}$. We shall only consider the first one here. The other can be dealt with similarly.

Let us now express the relevant components of our electromagnetic field in the appropriate basis of discrete differential forms

$$
\begin{aligned}
{}^2\mathbf{D}_h^x(t,x,y) &= \sum_{i,j} d^x_{i+1/2,j+1/2}(t)\,{}^2\tilde{\mathbf{w}}^{\alpha,x}_{i+1/2,j+1/2}(x,y), \quad {}^1\mathbf{E}_h^x(t,x,y) = \sum_{i,j} e^x_{i,j}(t)\,{}^1\mathbf{w}^{\alpha,x}_{i,j}(x,y),\\
{}^2\mathbf{D}_h^y(t,x,y) &= \sum_{i,j} d^y_{i+1/2,j+1/2}(t)\,{}^2\tilde{\mathbf{w}}^{\alpha,y}_{i+1/2,j+1/2}(x,y) \quad {}^1\mathbf{E}_h^y(t,x,y) = \sum_{i,j} e^y_{i,j}(t)\,{}^1\mathbf{w}^{\alpha,y}_{i,j}(x,y),\\
{}^1\mathbf{H}_h^z(t,x,y) &= \sum_{i,j} h^z_{i+1/2,j+1/2}(t)\,{}^1\tilde{\mathbf{w}}^{\alpha,z}_{i+1/2,j+1/2}(x,y), \quad {}^2\mathbf{B}_h(t,x,y) = \sum_{i,j} b^z_{i,j}(t)\,{}^2\mathbf{w}^{\alpha,z}_{i,j}(x,y).
\end{aligned}
$$

In order to obtain equations relating the coefficients of this discrete differential forms, we inject these expressions into the Maxwell equations (4)-(7), and take the De Rahm maps for two forms on each facet of the the dual mesh for (4) and (6) and of primal mesh for (5) and (7) .

Let us first compute the exterior derivatives of ${}^1\mathbf{H}_h$ and ${}^1\mathbf{E}_h$:

$$
\begin{aligned}
d\,{}^1\mathbf{H}_h^z(t,x,y) &= \sum_{i,j} h^z_{i+1/2,j+1/2}(t) d\,{}^1\tilde{\mathbf{w}}^{\alpha,z}_{i+1/2,j+1/2}(x,y)\\
&= \sum_{i,j} h^z_{i+1/2,j+1/2}(t)\left(-{B^\alpha_{i+1/2}}'(x)B^\alpha_{j+1/2}(y)\,dz\wedge dx + B^\alpha_{i+1/2}(x){B^\alpha_{j+1/2}}'(y)\,dy\wedge dz\right),
\end{aligned}
$$

using the formula (2) for the derivative of the spline functions and the definition of D_i^α, (3.1.2), we get

$$
\begin{aligned}
d\,{}^1\mathbf{H}_h^z(t,x,y) &= \sum_{i,j} h^z_{i+1/2,j+1/2}(t)\Big((D^\alpha_{i+3/2}(x)-D^\alpha_{i+1/2}(x))B^\alpha_{j+1/2}(y)\,dz\wedge dx\\
&\quad + B^\alpha_{i+1/2}(x)(D^\alpha_{j+1/2}-D^\alpha_{j+3/2})(y)\,dy\wedge dz\Big)\\
&= \sum_{i,j}(h^z_{i-1/2,j+1/2}(t)-h^z_{i+1/2,j+1/2}(t))D^\alpha_{i+1/2}(x)B^\alpha_{j+1/2}(y)\,dz\wedge dx\\
&\quad + \sum_{i,j}(h^z_{i+1/2,j+1/2}(t)-h^z_{i+1/2,j-1/2}(t))B^\alpha_{i+1/2}(x)D^\alpha_{j+1/2}(y)\,dy\wedge dz.
\end{aligned}
\tag{8}
$$

In the same way

$$
\begin{aligned}
d\,{}^1\mathbf{E}_h^x(t,x,y) &= \sum_{i,j} e^x_{i,j}(t) d\,{}^1\tilde{\mathbf{w}}^{\alpha,x}_{i,j}(x,y)\\
&= -\sum_{i,j} e^x_{i,j}(t) D^\alpha_i(x){B^\alpha_j}'(y)\,dx\wedge dy\\
&= -\sum_{i,j}(e^x_{i,j}(t)-e^x_{i,j-1}(t))D^\alpha_i(x)D^\alpha_j(y)\,dx\wedge dy,
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
d\,{}^1\mathbf{E}_h^y(t,x,y) &= \sum_{i,j} e^y_{i,j}(t) d\,{}^1\tilde{\mathbf{w}}^{\alpha,y}_{i,j}(x,y)\\
&= \sum_{i,j} e^y_{i,j}(t){B^\alpha_i}'(x)D^\alpha_j(y)\,dx\wedge dy\\
&= \sum_{i,j}(e^y_{i,j}(t)-e^y_{i-1,j}(t))D^\alpha_i(x)D^\alpha_j(y)\,dx\wedge dy,
\end{aligned}
\tag{10}
$$

Ampere's law (4), without current, for the first two components can be written

$$\partial_t\,{}^2\mathbf{D}^x + \partial_t\,{}^2\mathbf{D}^y - d\,{}^1\mathbf{H} = 0.$$

Then using expression (8) and identifying the components on the basis vectors of the discrete differential forms we get the following relation between the spline coefficients

$$d^x_{i+1/2,j+1/2}{}'(t) + h^z_{i+1/2,j+1/2}(t) - h^z_{i+1/2,j-1/2}(t) = 0,$$
$$d^y_{i+1/2,j+1/2}{}'(t) - h^z_{i+1/2,j+1/2}(t) + h^z_{i-1/2,j+1/2}(t) = 0.$$

On the other hand, Faraday's law (5), for the third component can be written

$$\partial_t\, {}^2\mathbf{B}^z + d\, {}^1\mathbf{E} = 0.$$

This becomes using (9) and (10) and identifying the components on the basis vectors of the discrete differential forms

$$b^z_{i,j}{}'(t) + (e^y_{i,j}(t) - e^y_{i-1,j}(t)) - (e^x_{i,j}(t) - e^x_{i,j-1}(t)) = 0.$$

4.2.1. *Discrete Hodge operators:*

Let us denote by

$$\mathbf{b}^z = ((b^z_{i,j}))_{1\le i\le N_x-1, 1\le j\le N_y-1},\ \mathbf{e}^x = ((e^x_{i,j}))_{1\le i\le N_x-1, 1\le j\le N_y-1},\ \mathbf{e}^y = ((e^y_{i,j}))_{1\le i\le N_x-1, 1\le j\le N_y-1},$$

$$\mathbf{h}^z = ((h^z_{i+1/2,j+1/2}))_{1\le i\le N_x-1, 1\le j\le N_y-1},\ \mathbf{d}^x = ((d^x_{i+1/2,j+1/2}))_{1\le i\le N_x-1, 1\le j\le N_y-1},$$
$$\mathbf{d}^y = ((d^y_{i+1/2,j+1/2}))_{1\le i\le N_x-1, 1\le j\le N_y-1},$$

the matrices of spline coefficients on the discrete grids.

We now need to define the discrete Hodge operators mapping $\mathbf{d}^x$ to $\mathbf{e}^x$, $\mathbf{d}^y$ to $\mathbf{e}^y$ and $\mathbf{b}^z$ to $\mathbf{h}^z$. The same procedure as for the 1D case will be used. We have

$${}^2\mathbf{D}^x_h(t,x,y) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(x) D^\alpha_{j+1/2}(y)\, dy\wedge dz,$$

defines a 2-form to which we can apply the continuous Hodge operator, yielding the one form

$$\star\, {}^2\mathbf{D}^x_h(t,x,y) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t) D^\alpha_{i+1/2}(x) B^\alpha_{j+1/2}(y)\, dx.$$

We now define the image of ${}^2\mathbf{D}$ by the discrete Hodge operator as the projection of this 1-form onto the primal grid. Denoting ${}^1\mathbf{E}_h = \sum_{i,j} e^x_{i,j}(t) D^\alpha_i(x) B^\alpha_j(y)\, dx$ this image. Then we have for any $(k,l) \in [0, N_x-1]\times[0,N_y-1]$

$$\int_{x_k}^{x_{k+1}} \star\, {}^2\mathbf{D}^x_h(t,x,y_l) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t) \int_{x_k}^{x_{k+1}} B^\alpha_{i+1/2}(x)\, dx\, D^\alpha_{j+1/2}(y_l) = \sum_{i,j} e^x_{i,j}(t) \int_{x_k}^{x_{k+1}} D^\alpha_i(x)\, dx\, B^\alpha_j(y_l).$$

Recalling that $\int_{x_k}^{x_{k+1}} B^\alpha_i(x)\, dx$ is the term at position (k,i) of matrix M^1_α, that $B^\alpha_j(y_l)$ is the term at position (l,j) of matrix M^0_α and denoting by S^1 the matrix whose (k,i) component is $\int_{x_k}^{x_{k+1}} B^\alpha_{i+1/2}(x)\, dx$ and by S^0 the matrix whose (l,j) component is $D^\alpha_{j+1/2}(y_l)$, the above relation can be written in matrix form

$$S^1\mathbf{d}^x(S^0)^T = M^1_\alpha \mathbf{e}^x (M^0_\alpha)^T.$$

This defines the discrete Hodge operator mapping $\mathbf{d}$ to $\mathbf{e}$. Note that on a uniform grid all these matrices are symmetric so that the transpose can be omitted. For the y component, the D^α_i and B^α_j are interchanged so that we get

$$S^0\mathbf{d}^y(S^1)^T = M^0_\alpha \mathbf{e}^y (M^1_\alpha)^T.$$

And the same computation gives us the discrete Hodge operator for the z component

$$S^1\mathbf{b}^z(S^1)^T = M^0_\alpha \mathbf{h}^z (M^0_\alpha)^T.$$

5. Link with the theory of chains and cochains

We now point out the link of our discrete differential forms with the theory of chains and cochains [4,6,9,15].

5.1. Notion of chain

Hypercubes [16,17]: Let us denote by Ω a domain and by T its boundary. We pave Ω with squares and we denote by m this mesh. We call 0-cubes, the nodes of our mesh, 1-cubes, the edges, 2-cubes, the squares and 3-cubes, the cubes (4-cubes, the tesseract, 5-cubes for the penteract, ..). We define an orientation for all of elements of m and denote by $H_{\alpha,p}(m)$ the set of p-cubes (we will see that in the case of a non periodic domain the number of p-cubes depends on the degree of B-splines α). Now, we define the boundary operator ∂ that maps a p-cube to a $(p-1)$-cube. For example: let e_i a vector who has for origin x_i and for extremity point x_{i+1}, so the boundary of e_i is $\partial e_i = x_{i+1} - x_i$. In fact, we can define the boundary operator ∂^p, mapping a p-cube to a $(p-1)$-cube, by a sparse matrix containing only $-1, 1$ or 0 with size $|H_{\alpha,p-1}(m)| \times |H_{\alpha,p}(m)|$. These matrices are called the transpose of incidence matrices and verify $\partial^p\,\partial^{p+1} = 0$.
Notion of Chains and discrete manifolds: A p-chain $\mathbf{c}_p$ is a linear combination of all p-cubes on m i.e.:

$$\mathbf{c}_p = \sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s}\, h^{p,s},$$

where $c_{p,s} \in \mathbb{R}$. We will denote the set of all p-chains as $C_{\alpha,p}(m)$. We define that a p-dimensional manifold is discretized by a p-chain and the boundary operator acts on p-chains by linearity as:

$$\partial \mathbf{c}_p = \partial \left(\sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s}\, h^{p,s} \right) = \sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s}\, \partial\, h^{p,s},$$

and when we collect the s-th $(p-1)$-cube we obtain:

$$\partial \mathbf{c}_p = \sum_{h^{p-1,s} \in H_{\alpha,p-1}(m)} (\partial^p c_p)_s\, h^{p-1,s},$$

where $c_p = (c_{p,s})_s$ is a column vector containing the coefficients of a p-chain. And so, applying the boundary operator on a p-chain is equivalent to applying the operator ∂^p on coefficients of a p-chain. Furthermore, since all p-chains are defined by their $|H_{\alpha,p}(m)|$ coefficients $c_{p,s}$, we can find a bijection mapping $C_{\alpha,p}(m)$ to $\mathbb{R}^{|H_{\alpha,p}(m)|}$. But, before, we will define the mapping that determines the coefficients $c_{p,s}$.
A p-chain $\mathbf{c}_p$ represents a discrete p-manifold, so for all basis functions ${}^p\mathbf{w}_i^\alpha \in W^{\alpha,p}(m)$ we can compute the integral of ${}^p\mathbf{w}_i^\alpha$ over $\mathbf{c}_p$. So, coefficient $c_{p,s}$ is defined by the relations, for all $i \in |H_{\alpha,p}(m)|$:

$$\int_{\mathbf{c}_p} {}^p\mathbf{w}_i^\alpha = \sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s} \int_{h^{p,s}} {}^p\mathbf{w}_i^\alpha.$$

This yields a linear system. For example, in one dimension, we must solve

- for a 0-chain
$$\mathbb{C}_0 = (M_\alpha^0)^t\, c_0,$$
where $\mathbb{C}_0 = (B_0^\alpha(\mathbf{c}_0), \ldots, B_{|H_{\alpha,0}(m)|-1}^\alpha(\mathbf{c}_0))^t$, $c_0 = (c_{0,0}, \ldots, c_{0,|H_{\alpha,0}(m)|-1})^t$ and M_α^0 is the square matrix we constructed for discrete differential 0-forms.
- for a 1-chain
$$\mathbb{C}_1 = (M_\alpha^1)^t\, c_1,$$
where $\mathbb{C}_1 = (\int_{\mathbf{c}_1} D_0^\alpha(x)dx, \ldots, \int_{\mathbf{c}_1} D_{|H_{\alpha,1}(m)|-1}^\alpha(x)dx)^t$, $c_1 = (c_{1,0}, \ldots, c_{1,|H_{\alpha,1}(m)|-1})^t$ and M_α^1 is the square matrix we constructed for discrete differential 1-forms.

In the same way, in two or three dimension, we can remark that we also must solve linear systems involving tensor products of matrices $(M_\alpha^1)^t$ and $(M_\alpha^0)^t$. More generally, for all dimensions, we denote by $\mathbb{M}_\alpha^p$ the square matrix that we use for finding the coefficients of p-chain $\mathbf{c}_p$ i.e. for solving the linear system $\mathbb{C}_p = (\mathbb{M}_\alpha^p)^t c_p$. Now, we can define $\mathcal{P}_\alpha^t$ mapping $C_{\alpha,p}(m)$ to $\mathbb{R}^{|H_{\alpha,p}(m)|}$ and $\mathcal{R}_\alpha^t$ mapping $\mathbb{R}^{|H_{\alpha,p}(m)|}$ to $C_{\alpha,p}(m)$ such that

$$\begin{aligned}\mathcal{P}_\alpha^t : \quad & C_{\alpha,p}(m) \to \mathbb{R}^{|H_{\alpha,p}(m)|} \\ & \mathbf{c}_p \mapsto \{c_p \mid \mathbb{C}_p = (\mathbb{M}_\alpha^p)^t\, c_p\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_\alpha^t : \quad & \mathbb{R}^{|H_{\alpha,p}(m)|} \to C_{\alpha,p}(m) \\ & c_p = (c_{p,s})_s \mapsto \sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s}\, h^{p,s}.\end{aligned}$$

Now we can show that we have a de Rham complex:

$$\begin{array}{ccccccc} C_{\alpha,3}(m) & \xrightarrow{\partial} & C_{\alpha,2}(m) & \xrightarrow{\partial} & C_{\alpha,1}(m) & \xrightarrow{\partial} & C_{\alpha,0}(m) \\ \mathcal{P}_\alpha^t \uparrow\downarrow \mathcal{R}_\alpha^t & & \mathcal{P}_\alpha^t \uparrow\downarrow \mathcal{R}_\alpha^t & & \mathcal{P}_\alpha^t \uparrow\downarrow \mathcal{R}_\alpha^t & & \mathcal{P}_\alpha^t \uparrow\downarrow \mathcal{R}_\alpha^t \\ \mathbb{R}^{|H_{\alpha,3}(m)|} & \xrightarrow{\partial^3} & \mathbb{R}^{|H_{\alpha,2}(m)|} & \xrightarrow{\partial^2} & \mathbb{R}^{|H_{\alpha,1}(m)|} & \xrightarrow{\partial^1} & \mathbb{R}^{|H_{\alpha,0}(m)|} \end{array}$$

with $\partial^2\partial^3 = 0$ and $\partial^1\partial^2 = 0$.

Lemma 5.1. *This diagram is commutative.*

Proof. Remember a property of B-splines $B_j^\alpha(x_{i+1}) - B_j^\alpha(x_i) = \int_{x_i}^{x_{i+1}} D_j(x)\,dx - \int_{x_i}^{x_{i+1}} D_{j+1}(x)\,dx$, this becomes in matrix from $(\mathbb{M}_\alpha^0)^t\,\partial^1 = \partial^1\,(\mathbb{M}_\alpha^1)^t$ and we have also $(\mathbb{M}_\alpha^1)^t\,\partial^2 = \partial^2\,(\mathbb{M}_\alpha^2)^t$ and $(\mathbb{M}_\alpha^2)^t\,\partial^3 = \partial^3\,(\mathbb{M}_\alpha^3)^t$. Then, remember also that applying the boundary operator on a p-chain its equivalent to applying the operator ∂^p on the coefficients of the p-chain. So, for a 1-chain $\mathbf{c}_1$, $\partial\,\mathbf{c}_1 = \sum_{h^{0,s} \in H_{\alpha,0}(m)} (\partial^1 c_1)_s\, h^{0,s}$ and so $\mathcal{P}_\alpha^t\,\partial\,\mathbf{c}_1 = ((\mathbb{M}_\alpha^0)^t)^{-1}\,\partial^1\,c_1$ and $\partial^1\,\mathcal{P}_\alpha^t\,\mathbf{c}_1 = \partial^1\,((\mathbb{M}_\alpha^1)t)^{-1}\,c_1$. We can deduce that $\partial^1\,(\mathcal{P}_\alpha)^t\,\mathbf{c}_1 = \mathcal{P}_\alpha^t\,\partial\,\mathbf{c}_1$. Proceeding in the same way, we also obtain that $\partial^2\,\mathcal{P}_\alpha^t\,\mathbf{c}_2 = \mathcal{P}_\alpha^t\,\partial\,\mathbf{c}_2$ and $\partial^3\,\mathcal{P}_\alpha^t\,\mathbf{c}_3 = \mathcal{P}_\alpha^t\,\partial\,\mathbf{c}_3$. □

In the dual mesh, denoting by $H_{\alpha,p}^\star(m)$ the p-cell, dual of $(n-p)$-cubes and $C_{\alpha,p}^\star(m)$ the set of p-chain in the dual mesh. Similarly, we obtain a de Rham complex, also commutative:

$$\begin{array}{ccccccc} C^\star_{\alpha,3}(m) & \xrightarrow{\partial} & C^\star_{\alpha,2}(m) & \xrightarrow{\partial} & C^\star_{\alpha,1}(m) & \xrightarrow{\partial} & C^\star_{\alpha,0}(m) \\ \mathcal{P}_\alpha^t \uparrow\downarrow \mathcal{R}_\alpha^t & & \mathcal{P}_\alpha^t \uparrow\downarrow \mathcal{R}_\alpha^t & & \mathcal{P}_\alpha^t \uparrow\downarrow \mathcal{R}_\alpha^t & & \mathcal{P}_\alpha^t \uparrow\downarrow \mathcal{R}_\alpha^t \\ \mathbb{R}^{|H_{\alpha,0}(m)|} & \xrightarrow{(\partial^1)^t} & \mathbb{R}^{|H_{\alpha,1}(m)|} & \xrightarrow{(\partial^2)^t} & \mathbb{R}^{|H_{2,\alpha}(m)|} & \xrightarrow{(\partial^3)^t} & \mathbb{R}^{|H_{\alpha,3}(m)|} \end{array}$$

5.2. **Notion of cochain**

Cochains: A p-cochain $\mathbf{w}^p$ is the dual of a p-chain. That is to say $\mathbf{w}^p$ is a linear mapping that takes p-chains to $\mathbb{R}$:

$$\begin{aligned}\mathbf{w}^p : \quad & C_{\alpha,p}(m) \to \mathbb{R} \\ & \mathbf{c}_p \mapsto \mathbf{w}^p(\mathbf{c}_p)\end{aligned}$$

We denote the set of p-cochains or p-forms by $W^{\alpha,p}(m)$. By duality, this space has a finite dimension $|H_{\alpha,p}(m)|$. A p-cochain $\mathbf{w}^p$ operates on a p-chain $\mathbf{c}_p$ and returns a linear combination of the values of the cochain on each p-cube.

Link with differentials forms: p-cochains are discrete analogs of differential forms. So, a discrete differential p-form, $\mathbf{w}^p$, is a linear mapping that takes p-chains to $\mathbb{R}$. We have seen, during the construction of discrete differential p-forms, or p-cochains, that they can be written as:

$$\mathbf{w}^p = \sum_{{}^p\mathbf{w}_s^\alpha \in W^{\alpha,p}(m)} w^{p,s}\, {}^p\mathbf{w}_s^\alpha,$$

where the coefficients $w^{p,s}$ are defined as the solution of the linear system:

$$\mathbb{W}^p = \mathbb{M}_\alpha^p\, w^p.$$

Now, we can define the non degenerate bilinear form:

$$\begin{aligned} \langle \quad , \quad \rangle : \quad & W^{\alpha,p}(m) \times C_{\alpha,p}(m) \to \mathbb{R} \\ & (\mathbf{w}^p, \mathbf{c}_p) \longmapsto \langle \mathbf{w}^p, \mathbf{c}_p \rangle = \int_{\mathbf{c}_p} \mathbf{w}^p. \end{aligned}$$

Let $\mathbf{c}_p$ a p-chain and $\mathbf{w}^p$ a differential p-form, we obtain by linearity:

$$\int_{\mathbf{c}_p} \mathbf{w}^p = \sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s} \int_{h^{p,s}} \mathbf{w}^p.$$

So, the bilinear mapping acts on the spline coefficients and on the coefficients of p-chains as:

$$\langle \mathbf{w}^p, \mathbf{c}_p \rangle = c_p^t\, \mathbb{M}_\alpha^p\, w^p,$$

where c_p and w^p are column vectors of coefficients of the p-chain $\mathbf{c}_p$ and the p-cochain $\mathbf{w}^p$ respectively. Exterior derivative, Stokes theorem and de Rham complex: The exterior derivative maps a $(p-1)$-form to p-form. For example, in the $3D$ case and $p = 1$:

$$\begin{aligned} d\mathbf{w}^0 &= d\left(\sum w^{i,j,k} B_i^\alpha(x) B_j^\alpha(y) B_k^\alpha(z)\right) \\ &= \sum w^{i,j,k} d\left(B_i^\alpha(x) B_j^\alpha(y) B_k^\alpha(z)\right) \\ &= \sum w^{i,j,k} (D_i^\alpha(x) - D_{i+1}^\alpha(x)) B_j^\alpha(y) B_k^\alpha(z)\, dx \\ &\quad + w^{i,j,k} (D_j^\alpha(y) - D_{j+1}^\alpha(y)) B_i^\alpha(x) B_k^\alpha(z)\, dy \\ &\quad + w^{i,j,k} (D_k^\alpha(z) - D_{k+1}^\alpha(z)) B_i^\alpha(x) B_j^\alpha(y)\, dz \\ &= \sum (w^{i,j,k} - w^{i-1,j,k}) D_i^\alpha(x) B_j^\alpha(y) B_k^\alpha(z)\, dx \\ &\quad + (w^{i,j,k} - w^{i,j-1,k}) B_i^\alpha(x) D_j^\alpha(y) B_k^\alpha(z)\, dy \\ &\quad + (w^{i,j,k} - w^{i,j,k-1}) B_i^\alpha(x) B_j^\alpha(y) D_k^\alpha(z)\, dz \\ &= \sum (w^{i,j,k} - w^{i-1,j,k})\, {}^1\mathbf{w}_{i,j,k}^{\alpha,x}(x,y,z) \\ &\quad + (w^{i,j,k} - w^{i,j-1,k})\, {}^1\mathbf{w}_{i,j,k}^{\alpha,y}(x,y,z) \\ &\quad + (w^{i,j,k} - w^{i,j,k-1})\, {}^1\mathbf{w}_{i,j,k}^{\alpha,z}(x,y,z). \end{aligned}$$

We remember that we have an equivalence between applying the boundary operator on a p-chain and applying ∂^p on coefficients of a p-chain for $p = 1, 2, 3$ respectively. Here, we have the dual property. We can see that applying the exterior derivative on a differential p-form is equivalent to applying the incidence matrix $(\partial^p)^t$ on the spline coefficients.
Now, construct a diagram. Denoting by $F^p(\Omega)$ the set of differential p-forms on Ω (not discrete). We can pass from p-form to a $(p+1)$-form by exterior derivative and the sequence

$$F^0(\Omega) \xrightarrow{\ d\ } F^1(\Omega) \xrightarrow{\ d\ } F^2(\Omega) \xrightarrow{\ d\ } F^3(\Omega)\ ,$$

is exact when Ω is star-shaped [10] and we have, in the same way, for discrete differential p-forms,

$$W^{\alpha,0}(m) \xrightarrow{d} W^{\alpha,1}(m) \xrightarrow{d} W^{\alpha,2}(m) \xrightarrow{d} W^{\alpha,3}(m) \ ,$$

is exact when m is the mesh of a star-shaped Ω. Seeing that $W^{\alpha,p}(m)$ was constructed as a finite dimensional subspace of $F^p(\Omega)$, we can project the set of differential p-forms on $W^{p,\alpha}(m)$. We obtain the de Rham complex:

$$\begin{array}{ccccccc}
F^0 & \xrightarrow{d} & F^1 & \xrightarrow{d} & F^2 & \xrightarrow{d} & F^3 \\
\downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} \\
W^{\alpha,0}(m) & \xrightarrow{d} & W^{\alpha,1}(m) & \xrightarrow{d} & W^{\alpha,2}(m) & \xrightarrow{d} & W^{\alpha,3}(m) \\
{\scriptstyle \mathcal{P}_\alpha}\uparrow\downarrow{\scriptstyle \mathcal{R}_\alpha} & & {\scriptstyle \mathcal{P}_\alpha}\uparrow\downarrow{\scriptstyle \mathcal{R}_\alpha} & & {\scriptstyle \mathcal{P}_\alpha}\uparrow\downarrow{\scriptstyle \mathcal{R}_\alpha} & & {\scriptstyle \mathcal{P}_\alpha}\uparrow\downarrow{\scriptstyle \mathcal{R}_\alpha} \\
\mathbb{R}^{|H_{\alpha,0}(m)|} & \xrightarrow{(\partial^1)^t} & \mathbb{R}^{|H_{\alpha,1}(m)|} & \xrightarrow{(\partial^2)^t} & \mathbb{R}^{|H_{\alpha,2}(m)|} & \xrightarrow{(\partial^3)^t} & \mathbb{R}^{|H_{\alpha,3}(m)|}
\end{array}$$

where $proj$ is a projection and mapping $\mathcal{P}_\alpha$ and $\mathcal{R}_\alpha$ are the dual mapping of $\mathcal{P}^t_\alpha$ and $\mathcal{R}^t_\alpha$ defined by :

$$\begin{aligned} \mathcal{P}_\alpha : \quad & W^{\alpha,p}(m) \to \mathbb{R}^{|H_{\alpha,p}(m)|} \\ & \mathbf{w}^p \mapsto \{w^p | \mathbb{W}^p = \mathbb{M}^p_\alpha \, w^p\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_\alpha : \quad & \mathbb{R}^{|H_{\alpha,p}(m)|} \to W^{\alpha,p}(m) \\ & w^p \mapsto \sum_{{}^p\mathbf{w}^\alpha_s \in W^{\alpha,p}(m)} w^{p,s} \; {}^p\mathbf{w}^\alpha_s . \end{aligned}$$

Lemma 5.2. *This diagram is commutative.*

Proof. First, let us show that the part involving discrete differential p-forms and spline coefficients, a commutative diagram.
With help of a property of splines, we have seen that $(\mathbb{M}^0_\alpha)^t \, \partial^1 = \partial^1 \, (\mathbb{M}^1_\alpha)^t$, $(\mathbb{M}^1_\alpha)^t \, \partial^2 = \partial^2 \, (\mathbb{M}^2_\alpha)^t$ and $(\mathbb{M}^2_\alpha)^t \, \partial^3 = \partial^3 \, (\mathbb{M}^3_\alpha)^t$. We take the tranpose and we have $(\partial^1)^t \, \mathbb{M}^0_\alpha = \mathbb{M}^1_\alpha \, (\partial^1)^t$, $(\partial^2)^t \, \mathbb{M}^1_\alpha = \mathbb{M}^2_\alpha \, (\partial^2)^t$ and $(\partial^3)^t \, \mathbb{M}^2_\alpha = \mathbb{M}^3_\alpha \, (\partial^3)^t$. Furthermore, applying the exterior derivative on a differential p-form is equivalent to applying the incidence matrix on the spline coefficients. We have $\mathcal{P}_\alpha \, d \, \mathbf{w}^0 = (\mathbb{M}^1_\alpha)^{-1} \, (\partial^1)^t \, w^0$ and $(\partial^1)^t \, \mathcal{P}_\alpha \, \mathbf{w}^0 = (\partial^1)^t \, (\mathbb{M}^0_\alpha)^{-1} \, w^0$ and so $(\partial^1)^t \, \mathcal{P}_\alpha \, \mathbf{w}^0 = \mathcal{P}_\alpha \, d \, \mathbf{w}^0$. In the same way, we obtain $(\partial^2)^t \, \mathcal{P}_\alpha \, \mathbf{w}^1 = \mathcal{P}_\alpha \, d \, \mathbf{w}^1$, $(\partial^3)^t \, \mathcal{P}_\alpha \, \mathbf{w}^2 = \mathcal{P}_\alpha \, d \, \mathbf{w}^2$.

Secondly, let us show that the part concerning differential p-forms and discrete differential p-forms is a commutative diagram.
On the one hand, let $\omega^{p-1} \in F^{p-1}(\Omega)$ a differential $(p-1)$-form. The exterior derivative of a $(p-1)$-form $d\omega^{p-1}$ is p-form. So, with help of the Leibniz property , we obtain that

$$proj \; d \, \omega^{p-1} = \sum_{{}^p\mathbf{w}^\alpha_s \in W^{\alpha,p}(m)} \left((\mathbb{M}^p_\alpha)^{-1} \; (\partial^p)^t \mathbb{W}^{p-1} \right)_s \, {}^p\mathbf{w}^\alpha_s .$$

On other hand, with help of exterior derivative on splines coefficients

$$d \; proj \; \omega^{p-1} = \sum_{{}^p\mathbf{w}^\alpha_s \in W^{\alpha,p}(m)} \left((\partial^p)^t \; \left(\mathbb{M}^{p-1}_\alpha\right)^{-1} \; \mathbb{W}^{p-1} \right)_s \, {}^p\mathbf{w}^\alpha_s .$$

And so, showing that $d \; proj \; \omega^{p-1} = proj \; d \, \omega^{p-1}$ is equivalent to showing that $(\partial^p)^t \; \left(\mathbb{M}^{p-1}_\alpha\right)^{-1} = (\mathbb{M}^p_\alpha)^{-1} \; (\partial^p)^t$, that we have. □

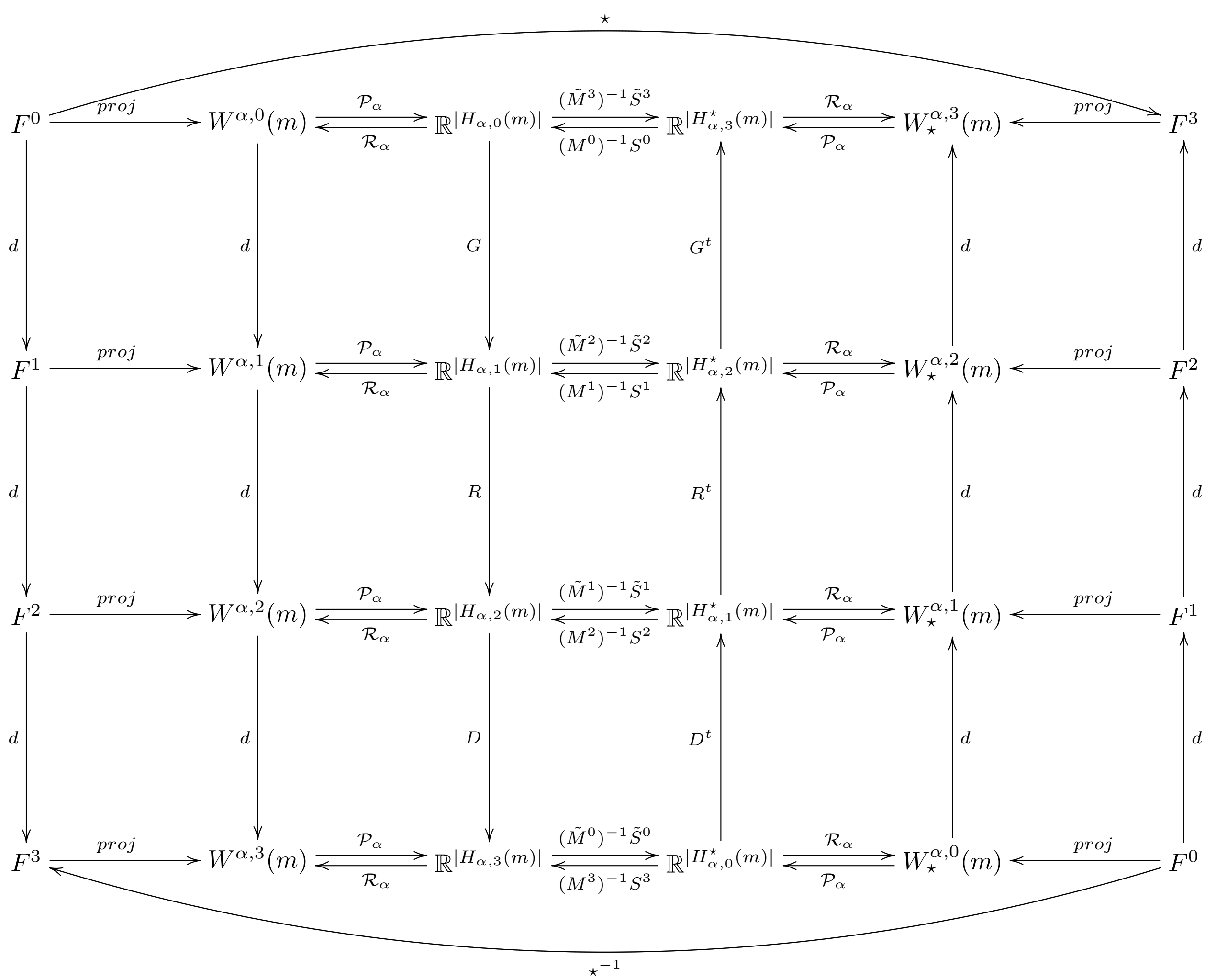
Remark 5.1. *Since splines have a compact support, we can observe that* $W^{\alpha,0}(m) \subset H(grad, \Omega)$*,* $W^{\alpha,1}(m) \subset H(curl, \Omega)$*,* $W^{\alpha,2}(m) \subset H(div, \Omega)$ *and* $W^{\alpha,3}(m) \subset L_2(\Omega)$ *in the sense of differential forms. That is, if* $\mathbf{w}^p \in W^{\alpha,p}(m)$*,* $\mathbf{w}^p \wedge \star \mathbf{w}^p \in L_2(\Omega)$ *and* $d\mathbf{w}^p \wedge \star d\mathbf{w}^p \in L_2(\Omega)$*.*

Also, for p-cochains on the dual mesh $W_\star^{\alpha,p}$, we have similarly a commutative diagram:

$$\begin{array}{ccccccc}
F^0 & \xrightarrow{d} & F^1 & \xrightarrow{d} & F^2 & \xrightarrow{d} & F^3 \\
\downarrow proj & & \downarrow proj & & \downarrow proj & & \downarrow proj \\
W_\star^{\alpha,0}(m) & \xrightarrow{d} & W_\star^{\alpha,1}(m) & \xrightarrow{d} & W_\star^{\alpha,2}(m) & \xrightarrow{d} & W_\star^{\alpha,3}(m) \\
\mathcal{P}_\alpha \uparrow\downarrow \mathcal{R}_\alpha & & \mathcal{P}_\alpha \uparrow\downarrow \mathcal{R}_\alpha & & \mathcal{P}_\alpha \uparrow\downarrow \mathcal{R}_\alpha & & \mathcal{P}_\alpha \uparrow\downarrow \mathcal{R}_\alpha \\
\mathbb{R}^{|H^\star_{\alpha,0}(m)|} & \xrightarrow{\partial^3} & \mathbb{R}^{|H^\star_{\alpha,1}(m)|} & \xrightarrow{\partial^2} & \mathbb{R}^{|H^\star_{\alpha,2}(m)|} & \xrightarrow{\partial^1} & \mathbb{R}^{|H^\star_{\alpha,3}(m)|}
\end{array}$$

5.3. A broader view

In parallel with gradient, curl and divergence, we denote by G, R, D the matrix $(\partial^p)^t$ for $p = 1, 2, 3$ respectively. Furthermore, using Hodge star operator $\star$ mapping a p-form to a $(n-p)$-form, where n is the dimension of space, we have



6. Numerical results

6.1. Test case in $2D$ with periodic boundary conditions

In 2 dimension with periodic boundary condition, let us consider the following solution of Maxwell's equations: the electric field, a 1-form, is given by ${}^1\mathbf{E} = -k_y \sin(k_x\, x + k_y\, y - \omega\, t) dx - k_x \sin(k_x\, x + k_y\, y - \omega\, t) dy$, and the magnetic field, a 2- form, is given by ${}^2\mathbf{B} = \omega \cos(k_x\, x + k_y\, y - \omega\, t) dx\, dy$. Then, with help of the Hodge star operator we obtain the formula for the electric displacement field: ${}^1\mathbf{D} = -k_y \sin(k_x\, x + k_y\, y - \omega\, t) dy - k_x \sin(k_x\, x + k_y\, y - \omega\, t) dx$ and for the magnetizing field: ${}^0\mathbf{H} = \omega \cos(k_x\, x + k_y\, y - \omega\, t)$. Constants are given by $k_x = \frac{2\,\pi}{L_x}, k_y = \frac{2\,\pi}{L_y}$, where L_x and L_y are the length of our domain in x and y respectively and $\omega = \sqrt{k_x^2 + k_y^2}$. We test our code with $L_x = L_y = 1$ and with a time scheme of order 4 and we observe that the order of our scheme is given by the spline order.

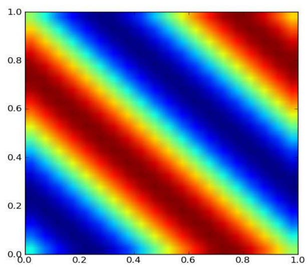


Figure 1: The magnetizing field H in $2D$ with periodic boundary conditions.

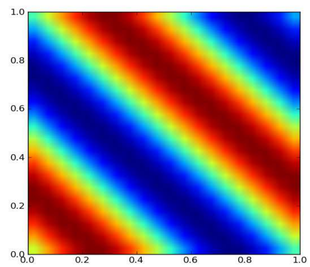


Figure 2: The first component of the electric displacement field Dx in $2D$ with periodic boundary conditions.

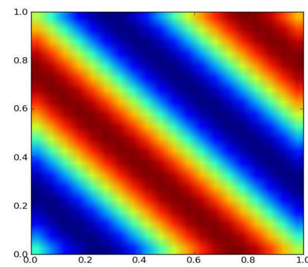


Figure 3: The second component of the electric displacement field Dy in $2D$ with periodic boundary conditions.

spline degree $\alpha = 1$		
Number of points	L^2 errors for D_x	conv. order in D_x
10	0.963466687612	
20	0.244961611114	1.97567910882
40	0.0617111800439	1.98895188936
80	0.015439432949	1.99891211506

spline degree $\alpha = 3$		
Number of points	L^2 errors for D_x	conv. order in D_x
10	0.0346451065418	
20	0.00221920090229	3.96453940841
40	0.000139584011258	3.99083467768
80	8.73723793614e-06	3.99781260726

6.2. Test case in $1D$ with perfect electric conductor boundary conditions

In one dimension with perfect electric conductor boundary conditions, we have a solution where the electric field, a 0-form, is given by ${}^0\mathbf{E} = \frac{k}{\omega} \sin(k\, x) \cos(\omega\, t)$, and the magnetic field, a 1- form, has the form, ${}^1\mathbf{B} = -\cos(k_x\, x) \sin(\omega\, t) dx$. Then, with help of the Hodge star operator we obtain the formula for the electric displacement field: ${}^1\mathbf{D} = \frac{k}{\omega} \sin(k\, x) \cos(\omega\, t) dx$ and for the magnetizing field : ${}^0\mathbf{H} = -\cos(k_x\, x) \sin(\omega\, t)$. Constants are given by $k = \frac{2\,\pi}{L}$, where L are the length of our domain in x and $\omega = |k_x|$. We test our code with $L_x = 1$ and with a time scheme of order 4 and we observe that the order of our scheme is given by the spline order.

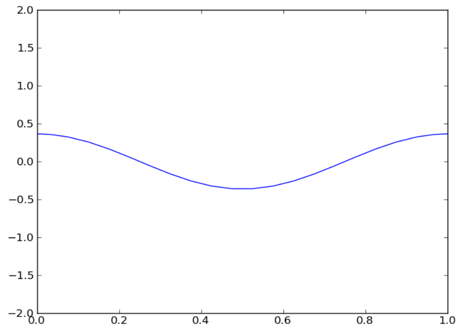


Figure 4: The magnetizing field H in $1D$ with perfect electric conductor boundary conditions.

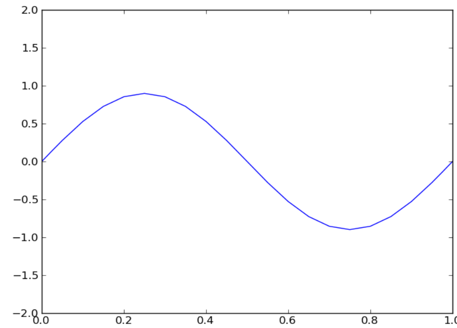


Figure 5: The electric field D in $1D$ with perfect electric conductor boundary conditions.

spline degree $\alpha = 1$		
Number of points	L^2 errors for H	conv. order in H
10	0.14405968937	
20	0.0365895785828	1.9767117931
40	0.0091869581337	1.9927326542
80	0.00229392316474	2.000650123

spline degree$\alpha = 3$		
Number of points	L^2 errors for H	conv. order in H
10	0.00291782099873	
20	0.000191704646607	3.93368451201
40	1.22331803519e-05	3.97851382897
80	7.61847256487e-07	4.00012583133

We are going up to time $T = 2$. These convergence order are obtain with a time step $\Delta t = \Delta x/0.5$ when we use B-spline of degree 1 and $\Delta t = \Delta x/3.5$ when we use B-spline of degree 3.

6.3. **Change of variables in two dimensions**

6.3.1. *General case*

Let us now consider a computational domain defined by a mapping f from a square. The change of variables $f : (r, s) \to (f_1(r, s), f_2(r, s)) = (x, y)$ is a diffeomorphism. Let us denote J the jacobian matrix of f

$$J(r, s) = \begin{pmatrix} \frac{\partial f_1}{\partial r}(r, s) & \frac{\partial f_1}{\partial s}(r, s) \\ \frac{\partial f_2}{\partial r}(r, s) & \frac{\partial f_2}{\partial s}(r, s) \end{pmatrix},$$

and $|J|$ its determinant. By definition we also have that the jacobian matrix of the inverse of f at point $(f_1(r, s), f_2(r, s))$ is the inverse of the jacobian matrix of f,

$$J^{-1}(r, s) = \begin{pmatrix} \frac{\partial f_1^{-1}}{\partial x}(f_1(r, s), f_2(r, s)) & \frac{\partial f_1^{-1}}{\partial y}(f_1(r, s), f_2(r, s)) \\ \frac{\partial f_2^{-1}}{\partial x}(f_1(r, s), f_2(r, s)) & \frac{\partial f_2^{-1}}{\partial y}(f_1(r, s), f_2(r, s)) \end{pmatrix} = \frac{1}{|J|} \begin{pmatrix} \frac{\partial f_2}{\partial s}(r, s) & -\frac{\partial f_1}{\partial s}(r, s) \\ -\frac{\partial f_2}{\partial r}(r, s) & \frac{\partial f_1}{\partial r}(r, s) \end{pmatrix}.$$

We discretize the cartesian domain parametrized by (r, s) and we will work only on this domain. The primal 2D mesh of our domain will be $r_0 < r_1 < \cdots < r_{Nr-1} < r_{Nr}$ in the r-direction and $s_0 < s_1 < \cdots < s_{Ns-1} <$

s_{Ns} in the s-direction. In $2D$, Maxwell's equations use the following differential forms

$$\begin{aligned}
{}^1\mathbf{H}(x,y) &= h_z(x,y)dz,\\
{}^2\mathbf{B}(x,y) &= b_z(x,y)dx\wedge dy,\\
{}^1\mathbf{E}(x,y) &= e_x(x,y)dx + e_y(x,y)dy,\\
{}^2\mathbf{D}(x,y) &= d_x(x,y)dy\wedge dz + d_y(x,y)dz\wedge dx.
\end{aligned}$$

When we do the change of coordinates, we obtain

$$\begin{aligned}
f^\star\,{}^1\mathbf{H}(r,s) &= h_z(f_1(r,s),f_2(r,s))dz,\\
f^\star\,{}^2\mathbf{B}(r,s) &= b_z(f_1(r,s),f_2(r,s))|J|dr\wedge ds,\\
f^\star\,{}^1\mathbf{E}(r,s) &= (e_x(f_1(r,s),f_2(r,s))\frac{\partial f_1}{\partial r}(r,s) + e_y(f_1(r,s),f_2(r,s))\frac{\partial f_2}{\partial r}(r,s))dr\\
&\quad +(e_x(f_1(r,s),f_2(r,s))\frac{\partial f_1}{\partial s}(r,s) + e_y(f_1(r,s),f_2(r,s))\frac{\partial f_2}{\partial s}(r,s))ds,\\
f^\star\,{}^2\mathbf{D}(r,s) &= (d_x(f_1(r,s),f_2(r,s))\frac{\partial f_2}{\partial r}(r,s) - d_y(f_1(r,s),f_2(r,s))\frac{\partial f_1}{\partial r}(r,s))dr\wedge dz\\
&\quad +(d_x(f_1(r,s),f_2(r,s))\frac{\partial f_2}{\partial s}(r,s) - d_y(f_1(r,s),f_2(r,s))\frac{\partial f_1}{\partial s}(r,s))ds\wedge dz.
\end{aligned}$$

Let us now express differentials forms in the new coordinates in the appropriate basis of discrete differential forms

$$\begin{aligned}
f^\star\,{}^2\mathbf{D}^r_h(t,r,s) &= \sum_{i,j} d^r_{i+1/2,j+1/2}(t)\,{}^2\tilde{\mathbf{w}}^{\alpha,r}_{i+1/2,j+1/2}(r,s),\ f^\star\,{}^1\mathbf{E}^r_h(t,r,s) = \sum_{i,j} e^r_{i,j}(t)\,{}^1\mathbf{w}^{\alpha,r}_{i,j}(r,s),\\
f^\star\,{}^2\mathbf{D}^s_h(t,r,s) &= \sum_{i,j} d^s_{i+1/2,j+1/2}(t)\,{}^2\tilde{\mathbf{w}}^{\alpha,s}_{i+1/2,j+1/2}(r,s)\ f^\star\,{}^1\mathbf{E}^s_h(t,r,s) = \sum_{i,j} e^s_{i,j}(t)\,{}^1\mathbf{w}^{\alpha,s}_{i,j}(r,s),\\
f^\star\,{}^1\mathbf{H}^z_h(t,r,s) &= \sum_{i,j} h^z_{i+1/2,j+1/2}(t)\,{}^1\tilde{\mathbf{w}}^{\alpha,z}_{i+1/2,j+1/2}(r,s),\ f^\star\,{}^2\mathbf{B}_h(t,r,s) = \sum_{i,j} b^z_{i,j}(t)\,{}^2\mathbf{w}^{\alpha,z}_{i,j}(r,s).
\end{aligned}$$

Coefficients associated to splines are calculated with help of degree of freedom on the cartesian grid (r,s,z). The exterior derivative d does not depend on the coordinate system, we have $f^\star d = df^\star$. This proprety can be verified on discrete differential forms with help of the discrete exterior derivative acting on spline coefficients. So Ampere's law (4), without current, for the first two components can be written

$$\partial_t f^\star\,{}^2\mathbf{D}^r + \partial_t f^\star\,{}^2\mathbf{D}^s - df^\star\,{}^1\mathbf{H} = 0.$$

On the other hand, Faraday's law (5), without current, for the third component can be written

$$\partial_t f^\star\,{}^2\mathbf{B}^z + df^\star\,{}^1\mathbf{E} = 0.$$

Discrete Hodge operators: We must be careful since $f^\star$ does not commute with the Hodge operator, so we must define discrete Hodge operators in the new coordinate system such that the following equalities are true:

$$\begin{aligned}
f^\star\,{}^2\mathbf{B} &= f^\star\left(\star\,{}^1\mathbf{H}\right),\\
f^\star\,{}^1\mathbf{E} &= f^\star\left(\star\,{}^2\mathbf{D}\right).
\end{aligned}$$

We know $f^\star\,{}^1\mathbf{H}, f^\star\,{}^2\mathbf{D}$ and we want calculate the spline coeficients for $f^\star\,{}^2\mathbf{B}, f^\star\,{}^1\mathbf{E}$. For this, we apply the pullback $f^{-1\star}$ on $f^\star\,{}^1\mathbf{H}, f^\star\,{}^2\mathbf{D}$ to come back in the old variables (x,y,z) to have ${}^1\mathbf{H}$, ${}^2\mathbf{D}$ then we apply the Hodge Star operator followed by the pullback $f^\star$. After simplication, we obtain

$$f^\star\left(\star\,{}^1\mathbf{H}\right) = \sum_{i,j} d^z_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(r) B^\alpha_{j+1/2}(s) |J(r,s)| dr \wedge ds,$$

$$(f^\star(\star\,{}^2\mathbf{D}_h))^r(t,r,s) = \sum_{i,j}\left(d^r_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(r) D^\alpha_{j+1/2}(s) \frac{(J^tJ(r,s))_{1,1}}{|J(r,s)|} + d^s_{i+1/2,j+1/2}(t) D^\alpha_{i+1/2}(r) B^\alpha_{j+1/2}(s) \frac{(J^tJ(r,s))_{2,1}}{|J(r,s)|}\right) dr,$$

$$(f^\star(\star\,{}^2\mathbf{D}_h))^s(t,r,s) = \sum_{i,j}\left(d^r_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(r) D^\alpha_{j+1/2}(s) \frac{(J^tJ(r,s))_{1,2}}{|J(r,s)|} + d^s_{i+1/2,j+1/2}(t) D^\alpha_{i+1/2}(r) B^\alpha_{j+1/2}(s) \frac{(J^tJ(r,s))_{2,2}}{|J(r,s)|}\right) ds,$$

where $(J^tJ(r,s))_{i,j}$ is the (i,j) coefficient of the matrix $J^tJ(r,s)$. We now define the image of ${}^1\mathbf{H}$ by $f^\star\star$ as the projection of this 2-form, denoting by $f^\star\,{}^2\mathbf{B}_h(t,r,s) = \sum_{i,j} b^z_{i,j}(t)\,{}^2\mathbf{w}^{\alpha,z}_{i,j}(r,s)$, onto primal grid. Then we have for any $(k,l) \in [0, N_r - 1] \times [0, N_s - 1]$

$$\begin{aligned}\int_{r_k}^{r_{k+1}}\int_{s_l}^{s_{l+1}} f^\star(\star\,{}^1\mathbf{H}_h)(t,r,s) &= \sum_{i,j} d^z_{i+1/2,j+1/2}(t) \int_{r_k}^{r_{k+1}}\int_{s_l}^{s_{l+1}} B^\alpha_{i+1/2}(r) B^\alpha_{j+1/2}(s)|J(r,s)| dr\wedge ds \\ &= \int_{r_k}^{r_{k+1}}\int_{s_l}^{s_{l+1}} f^\star\,{}^2\mathbf{B}_h(t,r,s) = \sum_{i,j} b^z_{i,j}(t) \int_{r_k}^{r_{k+1}}\int_{s_l}^{s_{l+1}} {}^2\mathbf{w}^{\alpha,z}_{i,j}(r,s).\end{aligned}$$

Also, we denote by $f^\star\,{}^1\mathbf{E}_h = f^\star\,{}^1\mathbf{E}^r_h(t,r,s) + f^\star\,{}^1\mathbf{E}^s_h(t,r,s) = \sum_{i,j} e^r_{i,j}(t)\,{}^1\mathbf{w}^{\alpha,r}_{i,j}(r,s) + \sum_{i,j} e^s_{i,j}(t)\,{}^1\mathbf{w}^{\alpha,s}_{i,j}(r,s)$ the projection of ${}^2\mathbf{D}$ and we have

$$\begin{aligned}\int_{r_k}^{r_{k+1}} (f^\star\star\,{}^2\mathbf{D}_h)^r(t,r,s_l) &= \sum_{i,j} d^r_{i+1/2,j+1/2}(t) D^\alpha_{j+1/2}(s_l) \int_{r_k}^{r_{k+1}} \left(B^\alpha_{i+1/2}(r) \frac{(J^tJ)_{1,1}(r,s_l)}{|J(r,s_l)|}\right) dr \\ &\quad + d^s_{i+1/2,j+1/2}(t) B^\alpha_{j+1/2}(s_l) \int_{r_k}^{r_{k+1}} \left(D^\alpha_{i+1/2}(r) \frac{(J^tJ)_{2,1}(r,s_l)}{|J(r,s_l)|}\right) dr \\ &= \int_{r_k}^{r_{k+1}} f^\star\,{}^1\mathbf{E}^r_h(t,r,s_l) = \sum_{i,j} e^r_{i,j}(t) \int_{r_k}^{r_{k+1}} {}^1\mathbf{w}^{\alpha,r}_{i,j}(r,s_l), \\ \int_{s_l}^{s_{l+1}} (f^\star(\star\,{}^2\mathbf{D}_h))^s(t,r_k,s) &= \sum_{i,j} d^r_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(r_k) \int_{s_l}^{s_{l+1}} \left(D^\alpha_{j+1/2}(s) \frac{(J^tJ)_{1,2}(r_k,s)}{|J(r_k,s)|}\right) ds \\ &\quad + d^s_{i+1/2,j+1/2}(t) D^\alpha_{i+1/2}(r_k) \int_{s_l}^{s_{l+1}} \left(B^\alpha_{j+1/2}(s) \frac{(J^tJ)_{2,2}(r_k,s)}{|J(r_k,s)|}\right) ds \\ &= \int_{s_l}^{s_{l+1}} f^\star\,{}^1\mathbf{E}^s_h(t,r_k,s) = \sum_{i,j} e^s_{i,j}(t) \int_{s_l}^{s_{l+1}} {}^1\mathbf{w}^{\alpha,s}_{i,j}(r_k,s).\end{aligned}$$

We proceed in the same way if we must solve $f^\star\,{}^1\mathbf{H} = f^\star\left(\star\,{}^2\mathbf{B}\right)$ knowing $f^\star\,{}^2\mathbf{B}$. We apply the pullback $f^{-1\star}$ on $f^\star\,{}^2\mathbf{B}$ to come back in the old variables (x,y,z) to have ${}^2\mathbf{B}$ then we apply the Hodge Star operator followed by the pullback $f^\star$. After simplification, we obtain

$$f^\star\left(\star\,{}^2\mathbf{B}\right) = \sum_{i,j} b^z_{i,j}(t) D^\alpha_i(r) D^\alpha_j(s) \frac{1}{|J(r,s)|} dz.$$

We denote $f^\star\, {}^1\mathbf{H}_h^z(t,r,s) = \sum_{i,j} h^z_{i+1/2,j+1/2}(t)\, {}^1\tilde{\mathbf{w}}^{\alpha,z}_{i+1/2,j+1/2}(r,s)$ the projection onto the dual grid and so we obtain

$$\begin{aligned} \int_0^1 f^\star \star\, {}^2\mathbf{B}(t, r_{i+1/2}, s_{j+1/2}) &= \sum_{i,j} b^z_{i,j}(t) \frac{D_i^\alpha(r_{i+1/2}) D_j^\alpha(s_{j+1/2})}{|J(r_{i+1/2}, s_{j+1/2})|} \\ = f^\star\, {}^1\mathbf{H}_h^z(t, r_{i+1/2}, s_{j+1/2}) &= \sum_{i,j} h^z_{i+1/2,j+1/2}(t)\, {}^1\tilde{\mathbf{w}}^{\alpha,z}_{i+1/2,j+1/2}(r_{i+1/2}, s_{j+1/2}). \end{aligned}$$

6.3.2. *Test case in 2D with a change of variables and with perfect electric conductor boundary conditions*

We consider a physical domain which is a ring with perfect electric conductor boundary conditions. We have an exact solution where the electric field, a 1-form, is given by

$${}^1\mathbf{E} = -\cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J_{2\,k}(\omega\,r)d\theta + \cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J'_{2\,k}(\omega\,r)dr,$$

and the magnetic field, a 2- form, has the form,

$${}^2\mathbf{B} = \cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J_{2\,k}(\omega\,r)dr\,d\theta,$$

where $J_{2\,k}$ is the Bessel's function of the first kind with order $2\,k$ and $J'_{2\,k}$ her derivative. With the help of the Hodge star operator, we obtain the formula for electric displacement field:

$${}^1\mathbf{D} = -\cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J_{2\,k}(\omega\,r)dr + \cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J'_{2\,k}(\omega\,r)d\theta,$$

and for magnetizing field:

$${}^0\mathbf{H} = \cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J_{2\,k}(\omega\,r).$$

The intervals on which we have boundary conditions on perfect conductors are $(r,\theta) \in [\gamma_1, \gamma_2] \times [0, \frac{\pi}{2}]$ where γ_1, γ_2 are the first and second zero of Bessel's function of the first kind with order $2k$.
In this case the change of variables is nothing other than change of coordinates into polar coordinates.

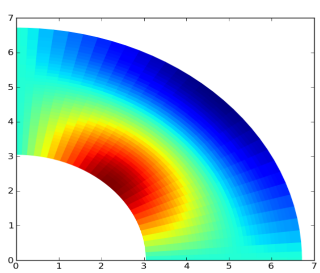


Figure 6: The first component of the electric displacement field Dx in physic domain with boundary conditions on perfect conductors.

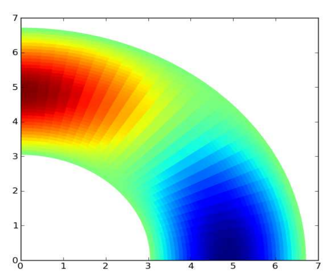


Figure 7: The second component of the electric displacement field Dy in physic domain with boundary conditions on perfect conductors.

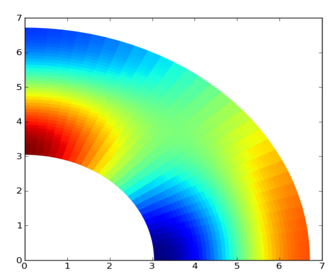


Figure 8: The magnetizing field H in physic domain with boundary conditions on perfect conductors.

We test our code with a time scheme of order 4 and we observe that the order of our scheme is given by spline's order as expected.

spline degree $\alpha = 1$		
Number of points	Errors L^2 for H	conv. order in H
10	0.00513786236267	
20	0.00238054129513	1.67535634073
40	0.000947723465878	1.82532724286
80	0.00035447325496	1.85426527326

spline degree $\alpha = 3$		
Number of points	Errors L^2 for H	conv. order in H
10	0.000245195436118	
20	1.91520985487e-05	3.67545675312
40	1.4651702633e-06	3.7021629499
80	9.27738247413e-08	3.9875224167

We are going up to time $T = 1$ for order 1 and $T = 2$ for order 3. These convergence orders are obtained with a time step $\Delta t = \min(\Delta_x, \Delta_y)/0.5$ when we use B-splines of degree 1 and $\Delta t = \min(\Delta_x, \Delta_y)/3.5$ when we use B-splines of degree 3.

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