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Discontinuous Galerkin methods for solving Helmholtz elastic wave equations for seismic imaging

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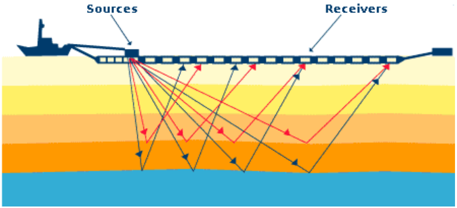
Depth Imaging Partnership DIP

TOTAL

July 23, 2014

Motivation

Examples of seismic applications



Motivation

Imaging method : the full wave inversion

- Iterative procedure using the wavefield in order to obtain quantitative high resolution images of the subsurface physical parameters

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- Harmonic-domain : imaging condition simple but huge computational cost

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Seismic imaging : time-domain or harmonic-domain ?

- Time-domain : imaging condition complicated but low computational cost
- Harmonic-domain : imaging condition simple but huge computational cost

Forward problem of the inversion process

- Elastic wave propagation in harmonic domain : Helmholtz equation
- Reduction of the size of the linear system

Motivation

Seismic imaging in heterogeneous complex media

- Complex topography
- High heterogeneities

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Use of unstructured meshes with FE methods

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DG method

- Flexible choice of interpolation orders ($p - adaptativity$)
- Very parallelizable method
- Increased computational cost as compared to classical FEM

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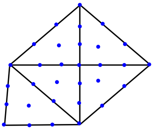
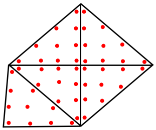


DOF of classical FEM



DOF of DGM

Motivation

Objective of this work

- Development of an hybridizable DG (HDG) method
- Comparison with a reference method : a standard nodal DG method

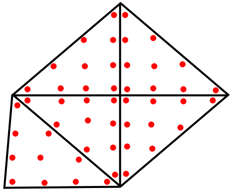


FIGURE : Degrees of freedom of DGM

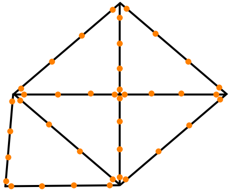


FIGURE : Degrees of freedom of HDGM

HDG methods

HDG methods

- B. Cockburn, J. Gopalakrishnan, R. Lazarov *Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems*, SIAM Journal on Numerical Analysis, Vol. 47 (2009)
- S. Lanteri, L. Li, R. Perrussel, *Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations*, COMPEL, Vol. 32 (2013) (time-harmonic domain)
- N.C. Nguyen, J. Peraire, B. Cockburn, *High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics*, J. of Comput. Physics, Vol. 230 (2011) (time domain for seismic applications)

Contents

2D Helmholtz elastic equations

First order formulation of Helmholtz wave equations

$\mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2$,

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla\cdot \underline{\underline{\sigma}}(\mathbf{x}) + f_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x})\ \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- $\mathbf{v}$: velocity vector
- $\underline{\underline{\sigma}}$: stress tensor
- $\underline{\underline{\varepsilon}}$: strain tensor

2D Helmholtz elastic equations

First order formulation of Helmholtz wave equations

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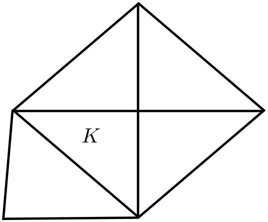
- ρ : mass density
- $\underline{\underline{C}}$: tensor of elasticity coefficients
- f_s : source term, $f_s \in L^2(\Omega)$

Contents

Notations and definitions

Notations

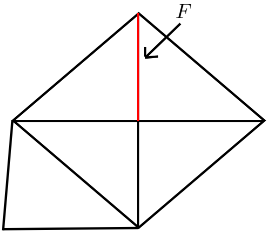
- $\mathcal{T}_h$ mesh of Ω composed of triangles K



Notations and definitions

Notations

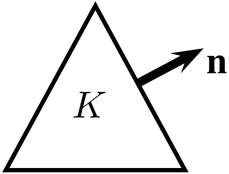
- $\mathcal{T}_h$ mesh of Ω composed of triangles K
- $\mathcal{F}_h$ set of all faces F of $\mathcal{T}_h$



Notations and definitions

Notations

- $\mathcal{T}_h$ mesh of Ω composed of triangles K
- $\mathcal{F}_h$ set of all faces F of $\mathcal{T}_h$
- $\mathbf{n}$ the normal outward vector of an element K



Notations and definitions

Approximations spaces

- $P_p(K)$ set of polynomials of degree at most p on K

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- $\mathbf{V}_h^p = \{\mathbf{v} \in \left(L^2(\Omega)\right)^2 : \quad \mathbf{v}|_K \in \mathbf{V}^p(K) = (P_p(K))^2, \forall K \in \mathcal{T}_h\}$

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- $\Sigma_h^p = \{\underline{\underline{\sigma}} \in \left(L^2(\Omega)\right)^3 : \quad \underline{\underline{\sigma}}|_K \in \Sigma^p(K) = (P_p(K))^3, \forall K \in \mathcal{T}_h\}$

Notations and definitions

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- $\mathbf{M}_h = \{\eta \in \left(L^2(\mathcal{F}_h)\right)^2 : \quad \eta|_F \in (P_p(F))^2, \forall F \in \mathcal{F}_h\}$

Notations and definitions

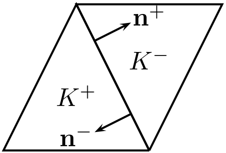
Definitions

- Jump $[\![\cdot]\!]$ of a vector $\mathbf{v}$ through F :

$$[\![\mathbf{v}]\!] = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^- = \mathbf{v}^+ \cdot \mathbf{n}^+ - \mathbf{v}^- \cdot \mathbf{n}^+$$

- Jump of a tensor $\underline{\underline{\sigma}}$ through F :

$$[\![\underline{\underline{\sigma}}]\!] = \underline{\underline{\sigma}}^+ \mathbf{n}^+ + \underline{\underline{\sigma}}^- \mathbf{n}^- = \underline{\underline{\sigma}}^+ \mathbf{n}^+ - \underline{\underline{\sigma}}^- \mathbf{n}^+$$



Contents

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} i\omega\rho\mathbf{v} - \nabla\cdot\underline{\sigma} &= 0 \\ i\omega\underline{\sigma} - \underline{\underline{C}}\,\underline{\varepsilon}(\mathbf{v}) &= 0 \end{cases}$$

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}} \right) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

$\widehat{\underline{\underline{\sigma}}}^K$ and $\widehat{\mathbf{v}}^K$ are numerical traces of $\underline{\underline{\sigma}}^K$ and $\mathbf{v}^K$ respectively on ∂K

HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{l} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}} \right) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array} \right.$$

We define :

$$\begin{array}{rcll} \widehat{\mathbf{v}}^F & = & \lambda^F, & \forall F \in \mathcal{F}_h, \\ \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} & = & \underline{\underline{\sigma}}^K \cdot \mathbf{n} - \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K} \right), & \text{on } \partial K \end{array}$$

where τ is the stabilization parameter ($\tau > 0$)

HDG formulation of the equations

Local HDG formulation

We replace $\widehat{\mathbf{v}}^K$ and $\left(\widehat{\underline{\underline{\sigma}}}^K \cdot \mathbf{n}\right)$ by their definitions into the local equations

$$\left\{\begin{array}{l} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \underline{\underline{\sigma}}^K \cdot \mathbf{n} \cdot \mathbf{w} \\ \displaystyle\qquad\qquad\qquad\qquad + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega\underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array}\right.$$

HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{l} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega\underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array} \right.$$

HDG formulation of the equations

Transmission condition

In order to determine λ^K, the continuity of the normal component of $\underline{\underline{\widehat{\sigma}}}^K$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F [\![\underline{\underline{\widehat{\sigma}}}^K \cdot \mathbf{n}]\!] \cdot \eta = 0$$

HDG formulation of the equations

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In order to determine λ^K, the continuity of the normal component of $\underline{\underline{\widehat{\sigma}}}^K$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F [\![\underline{\underline{\widehat{\sigma}}}^K \cdot \mathbf{n}]\!] \cdot \eta = 0$$

Replacing $\left(\underline{\underline{\widehat{\sigma}}}^K \cdot \mathbf{n}\right)$ and summing over all faces, the transmission condition becomes :

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \left(\underline{\underline{\sigma}}^K \cdot \mathbf{n}\right) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \eta = 0$$

HDG formulation of the equations

Global HDG formulation

$$
\begin{cases}
\displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \mathbf{w} = 0 \\
\displaystyle\int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^{\partial K} \cdot \underline{\underline{C}}_K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \\
\displaystyle\sum_{K \in \mathcal{T}_h} \int_{\partial K} \left(\underline{\underline{\sigma}}^K \cdot \mathbf{n}\right) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \eta = 0
\end{cases}
$$

Discretization of the HDG formulation

Local HDG formulation

$$\left\{ \begin{array}{l} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array} \right.$$

We define :

$$\underline{W}^K = \left(\underline{v_x}^K,\ \underline{v_z}^K,\ \underline{\sigma_{xx}}^K,\ \underline{\sigma_{zz}}^K,\ \underline{\sigma_{xz}}^K\right)^T$$

$$\underline{\Lambda} = \left(\underline{\Lambda}^{F_1},\ \underline{\Lambda}^{F_2}, ...,\ \underline{\Lambda}^{F_{n_f}}\right)^T, \text{ where } n_f = \text{card}(\mathcal{F}_h)$$

Discretization of the local HDG formulation

$$\mathbb{A}^K \underline{W}^K + \mathbb{C}^K \underline{\Lambda} = 0$$

Discretization of the HDG formulation

Transmission condition

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\underline{\underline{\sigma}}^K\cdot\mathbf{n}\right)\cdot\eta-\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbf{S}\left(\mathbf{v}^K-\lambda^{\partial K}\right)\cdot\eta=0$$

Discretization of the transmission condition

$$\sum_{K\in\mathcal{T}_h}\left[\mathbb{B}^K\underline{W}^K+\mathbb{L}^K\underline{\Lambda}\right]=0$$

Discretization of the HDG formulation

Transmission condition

$$\sum_{K \in \mathcal{T}_h} \left[\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda} \right] = 0$$

Local HDG scheme

$$\mathbb{A}^K \underline{W}^K + \mathbb{C}^K \underline{\Lambda} = 0$$

Discretization of the HDG formulation

Transmission condition

$$\sum_{K \in \mathcal{T}_h} \left[\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda} \right] = 0$$

Expression of $\underline{W}^K$ in terms of $\underline{\Lambda}$

$$\underline{W}^K = -(\mathbb{A}^K)^{-1} \mathbb{C}^K \underline{\Lambda}$$

Discretization of the HDG formulation

Transmission condition

$$\sum_{K \in \mathcal{T}_h} \left[\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda} \right] = 0$$

Expression of $\underline{W}^K$ in terms of $\underline{\Lambda}$

$$\underline{W}^K = -(\mathbb{A}^K)^{-1} \mathbb{C}^K \underline{\Lambda}$$

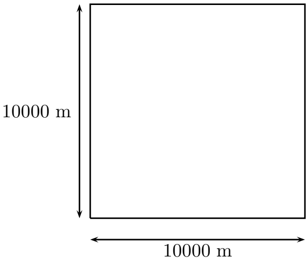
Global HDG system

$$\mathbb{K} \underline{\Lambda} = 0$$

with $\mathbb{K} = \sum_{K \in \mathcal{T}_h} \left[-\mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{C}^K + \mathbb{L}^K \right]$

Contents

Plane wave



Computational domain Ω setting

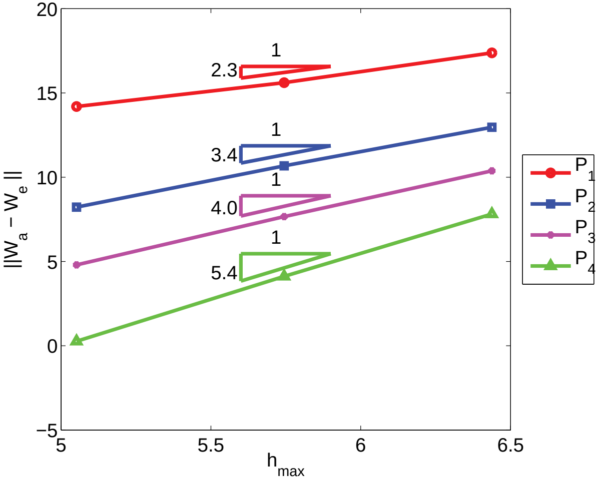
- Physical parameters :
  - $\rho = 2000 kg.m^{-3}$
  - $\lambda = 16 GPa$
  - $\mu = 8 GPa$
- Plane wave :

$$u = \nabla e^{i(k\cos\theta x + k\sin\theta y)}$$

where $k = \dfrac{\omega}{v_p}$

- $\theta = 0$
- Three meshes :
  - 3000 elements
  - 10000 elements
  - 45000 elements

Plane wave



Convergence order of the HDG scheme

Plane wave

Elements	Order	CPU Time (s)			Memory (MB)		
		HDG	UDG	IPDG	HDG	UDG	IPDG
3000	1	0.4			44		
10000	1	1.7			161		
45000	1	9.4			797		
3000	2	1.8			97		
10000	2	5.2			355		
45000	2	27.8			1746		
3000	3	5.8			170		
10000	3	12.7			624		
45000	3	67.8			3080		
3000	4	7.4			254		
10000	4	29.7			947		
45000	4	176.8			4653		

Plane wave

Elements	Order	CPU Time (s)			Memory (MB)		
		HDG	UDG	IPDG	HDG	UDG	IPDG
3000	1	0.4	1.5	1.4	44	288	58
10000	1	1.7	7.3	5.0	161	1076	221
45000	1	9.4	68.8	26.26	797	5492	1156
3000	2	1.8	6.1	5.5	97	804	215
10000	2	5.2	29.1	22.11	355	3097	852
45000	2	27.8	226.0	135.6	1746	15965	4454
3000	3	5.8	14.7	17.0	170	1656	598
10000	3	12.7	78.9	73.53	624	6600	2394
45000	3	67.8	646.7	492.0	3080	34597	12362
3000	4	7.4	28.7	42.6	254	2749	1324
10000	4	29.7	136.9	197.6	947	10098	5251
45000	4	176.8	1085.0	1409.4	4653	50297	27314

Plane wave

Elements	Order	CPU Time			Memory		
		HDG	UDG	IPDG	HDG	UDG	IPDG
3000	1	1	3.7	3.5	1	6.5	1.3
10000	1	1	4.3	2.9	1	6.7	1.4
45000	1	1	7.3	2.8	1	6.9	1.5
3000	2	1	3.4	3.1	1	8.3	2.2
10000	2	1	5.6	4.3	1	8.7	2.4
45000	2	1	8.1	4.9	1	9.1	2.6
3000	3	1	2.5	2.9	1	9.7	3.5
10000	3	1	6.2	5.8	1	10.6	3.8
45000	3	1	9.5	7.3	1	11.2	4.0
3000	4	1	3.9	5.6	1	10.8	5.2
10000	4	1	4.6	6.7	1	10.7	5.5
45000	4	1	6.1	8.0	1	10.8	5.9

Plane wave

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3000	2	1	3.4	3.1	1	8.3	2.2
10000	2	1	5.6	4.3	1	8.7	2.4
45000	2	1	8.1	4.9	1	9.1	2.6
3000	3	1	2.5	2.9	1	9.7	3.5
10000	3	1	6.2	5.8	1	10.6	3.8
45000	3	1	9.5	7.3	1	11.2	4.0
3000	4	1	3.9	5.6	1	10.8	5.2
10000	4	1	4.6	6.7	1	10.7	5.5
45000	4	1	6.1	8.0	1	10.8	5.9

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Conclusions-Perspectives

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- The HDG scheme has the correct convergence order $(p+1)$
- On a same mesh the HDG formulation is more competitive in terms of memory and computational time than the upwind flux DG formulation and the IPDG method

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Perspectives

- Develop 3D Upwind flux DG and HDG formulations for Helmholtz equations
- Adapt the program for parallel computing
- Solution strategy for the HDG linear system

Thank you

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Inria