EAGE EUROPEAN ASSOCIATION OF GEOSCIENTISTS & ENGINEERS
Workshop on High Performance Computing

Hybridizable Discontinuous Galerkin method for solving Helmholtz elastic wave equations

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[3] *TOTAL Exploration-Production*

Depth Imaging Partnership DIP

TOTAL

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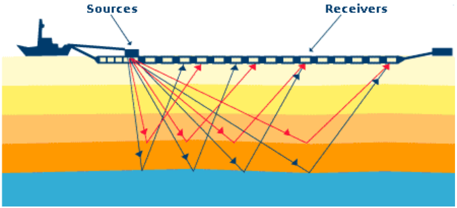
Motivation

Examples of seismic applications



Motivation

Imaging method : the full wave inversion

- Iterative procedure using the wavefield in order to obtain quantitative high resolution images of the subsurface physical parameters

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Seismic imaging : time-domain or harmonic-domain ?

- Time-domain : imaging condition complicated but low computational cost
- Harmonic-domain : imaging condition simple but huge computational cost

Motivation

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Seismic imaging : time-domain or harmonic-domain ?

- Time-domain : imaging condition complicated but low computational cost
- Harmonic-domain : imaging condition simple but huge computational cost

Forward problem of the inversion process

- Elastic wave propagation in harmonic domain : Helmholtz equation
- Reduction of the size of the linear system

Motivation

Seismic imaging in heterogeneous complex media

- Complex topography
- High heterogeneities

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DG method

- Flexible choice of interpolation orders ($p - adaptativity$)
- Highly parallelizable method
- Increased computational cost as compared to classical FEM

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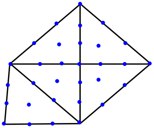
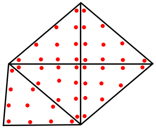


DOF of classical FEM



DOF of DGM

Motivation

Objective of this work

- Development of an hybridizable DG (HDG) method
- Comparison with a reference method : a standard nodal DG method

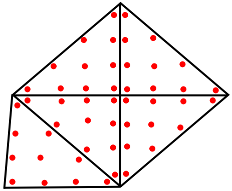


FIGURE : Degrees of freedom of DGM

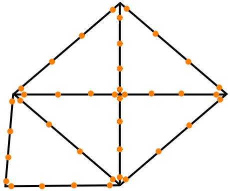


FIGURE : Degrees of freedom of HDGM

HDG methods

HDG methods

- B. Cockburn, J. Gopalakrishnan, R. Lazarov *Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems*, SIAM Journal on Numerical Analysis, Vol. 47 (2009)
- S. Lanteri, L. Li, R. Perrussel, *Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations*, COMPEL, Vol. 32 (2013) (time-harmonic domain)
- N.C. Nguyen, J. Peraire, B. Cockburn, *High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics*, J. of Comput. Physics, Vol. 230 (2011) (time domain for seismic applications)

Contents

2D Helmholtz elastic equations

First order formulation of Helmholtz wave equations

$\mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2$,

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla\cdot \underline{\underline{\sigma}}(\mathbf{x}) + f_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x})\ \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- Free surface condition : $\underline{\underline{\sigma}}\mathbf{n} = 0$ on Γ_l
- Absorbing boundary condition : $\underline{\underline{\sigma}}\mathbf{n} = v_p(\mathbf{v}\cdot\mathbf{n})\mathbf{n} + v_s(\mathbf{v}\cdot\mathbf{t})\mathbf{t}$ on Γ_a

- $\mathbf{v}$: velocity vector
- $\underline{\underline{\sigma}}$: stress tensor
- $\underline{\underline{\varepsilon}}$: strain tensor

2D Helmholtz elastic equations

First order formulation of Helmholtz wave equations

$\mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2$,

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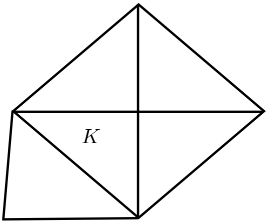
- ρ : mass density
- $\underline{\underline{C}}$: tensor of elasticity coefficients
- v_p : P-wave velocity
- v_s : S-wave velocity
- f_s : source term, $f_s \in L^2(\Omega)$

Contents

Notations and definitions

Notations

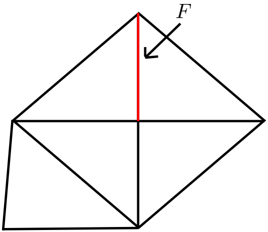
- $\mathcal{T}_h$ mesh of Ω composed of triangles K



Notations and definitions

Notations

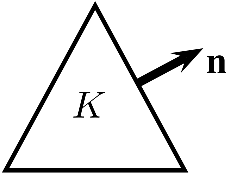
- $\mathcal{T}_h$ mesh of Ω composed of triangles K
- $\mathcal{F}_h$ set of all faces F of $\mathcal{T}_h$



Notations and definitions

Notations

- $\mathcal{T}_h$ mesh of Ω composed of triangles K
- $\mathcal{F}_h$ set of all faces F of $\mathcal{T}_h$
- $\mathbf{n}$ the normal outward vector of an element K



Notations and definitions

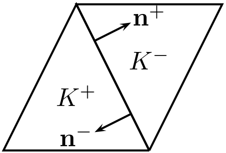
Definitions

- Jump $[\![\cdot]\!]$ of a vector $\mathbf{v}$ through F :

$$[\![\mathbf{v}]\!] = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^- = \mathbf{v}^+ \cdot \mathbf{n}^+ - \mathbf{v}^- \cdot \mathbf{n}^+$$

- Jump of a tensor $\underline{\underline{\sigma}}$ through F :

$$[\![\underline{\underline{\sigma}}]\!] = \underline{\underline{\sigma}}^+ \mathbf{n}^+ + \underline{\underline{\sigma}}^- \mathbf{n}^- = \underline{\underline{\sigma}}^+ \mathbf{n}^+ - \underline{\underline{\sigma}}^- \mathbf{n}^+$$



Contents

HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{rcl} i\omega\rho\mathbf{v} - \nabla\cdot\underline{\sigma} & = & 0 \\ i\omega\underline{\sigma} - \underline{\underline{C}}\,\underline{\varepsilon}(\mathbf{v}) & = & 0 \end{array} \right.$$

HDG formulation of the equations

Local HDG formulation

$$\left\{\begin{array}{l} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K\cdot\mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla\mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K}\cdot\mathbf{n}\cdot\mathbf{w} = 0 \\ \displaystyle\int_K i\omega\underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K\cdot\nabla\cdot\left(\underline{\underline{C}}^K\underline{\underline{\xi}}\right) - \int_{\partial K}\widehat{\mathbf{v}}^{\partial K}\cdot\underline{\underline{C}}^K\underline{\underline{\xi}}\cdot\mathbf{n} = 0 \end{array}\right.$$

$\widehat{\underline{\underline{\sigma}}}^K$ and $\widehat{\mathbf{v}}^K$ are numerical traces of $\underline{\underline{\sigma}}^K$ and $\mathbf{v}^K$ respectively on ∂K

HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{l} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}} \right) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array} \right.$$

We define :

$$\begin{array}{lcll} \widehat{\mathbf{v}}^F & = & \lambda^F, & \forall F \in \mathcal{F}_h, \\ \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} & = & \underline{\underline{\sigma}}^K \cdot \mathbf{n} - \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K} \right), & \text{on } \partial K \end{array}$$

where τ is the stabilization parameter ($\tau > 0$)

HDG formulation of the equations

Local HDG formulation

We replace $\widehat{\mathbf{v}}^K$ and $\left(\widehat{\underline{\underline{\sigma}}}^K \cdot \mathbf{n}\right)$ by their definitions into the local equations

$$\left\{\begin{array}{l} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \underline{\underline{\sigma}}^K \cdot \mathbf{n} \cdot \mathbf{w} \\ \qquad\qquad\qquad\qquad\displaystyle + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega\underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array}\right.$$

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} \int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

HDG formulation of the equations

Transmission condition

In order to determine λ^K, the continuity of the normal component of $\underline{\underline{\widehat{\sigma}}}^K$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F [\![\underline{\underline{\widehat{\sigma}}}^K \cdot \mathbf{n}]\!] \cdot \eta = 0$$

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Transmission condition

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$$\int_F [\![\underline{\underline{\widehat{\sigma}}}^K \cdot \mathbf{n}]\!] \cdot \eta = 0$$

Replacing $\left(\underline{\underline{\widehat{\sigma}}}^K \cdot \mathbf{n}\right)$ and summing over all faces, the transmission condition becomes :

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\underline{\underline{\sigma}}^K \cdot \mathbf{n}\right) \cdot \eta - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \eta = 0$$

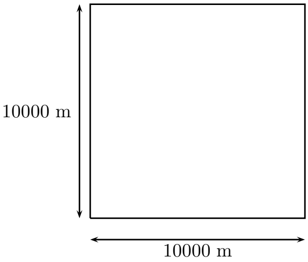
HDG formulation of the equations

Global HDG formulation

$$\begin{cases} \int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^{\partial K} \cdot \underline{\underline{C}}_K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \\ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left(\underline{\underline{\sigma}}^K \cdot \mathbf{n}\right) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^{\partial K}\right) \cdot \eta = 0 \end{cases}$$

Contents

Plane wave



Computational domain Ω setting

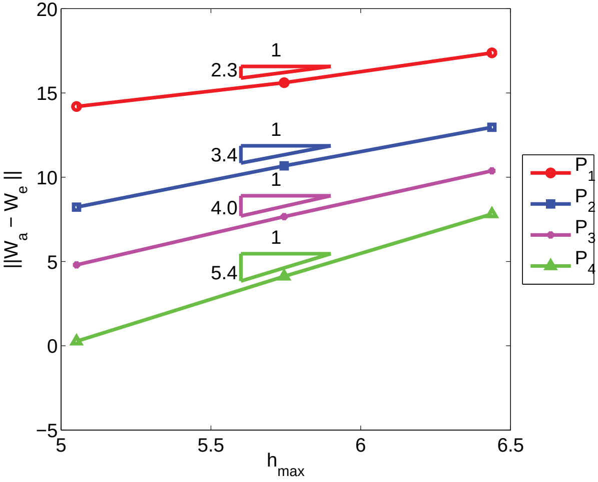
- Physical parameters :
  - $\rho = 2000 kg.m^{-3}$
  - $\lambda = 16 GPa$
  - $\mu = 8 GPa$
- Plane wave :

$$u = \nabla e^{i(k \cos\theta x + k \sin\theta y)}$$

where $k = \dfrac{\omega}{v_p}$

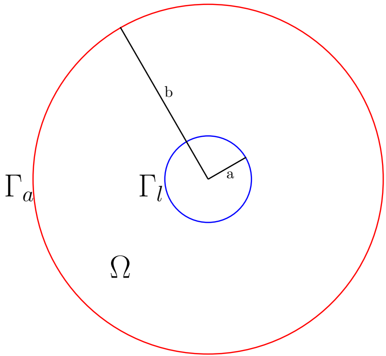
- $\theta = 0$
- Three meshes :
  - 3000 elements
  - 10000 elements
  - 45000 elements

Plane wave



Convergence order of the HDG scheme

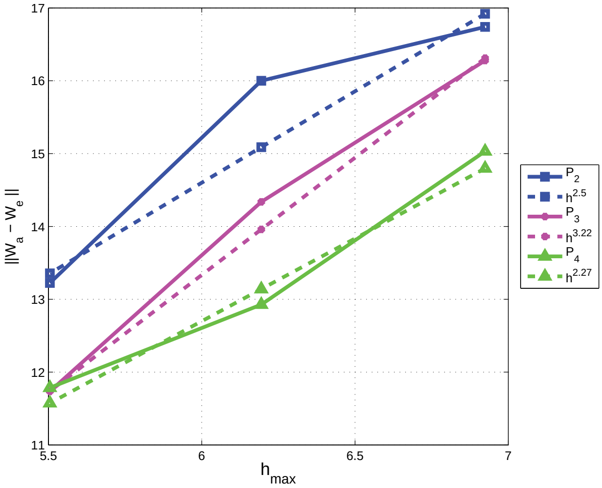
Disk-shaped scatterer problem



Computational domain Ω setting

- $a = 2000.0m$ and $b = 8000.0m$
- Physical parameters in Ω :
  - $\rho = 1kg.m^{-3}$
  - $\lambda = 8GPa$
  - $\mu = 4GPa$
- Γ_l free surface boundary : $\underline{\underline{\sigma}}\mathbf{n} = 0$
- Γ_a absorbing boundary : $\underline{\underline{\sigma}}\mathbf{n} = v_p(\mathbf{v} \cdot \mathbf{n})\mathbf{n} + v_s(\mathbf{v} \cdot \mathbf{t})\mathbf{t}$
- Three meshes :
  - 1200 elements
  - 5400 elements
  - 22000 elements

Disk-shaped scatterer problem



Convergence order of the HDG scheme

Disk-shaped scatterer problem

Elements	Order	CPU Time (s)			Memory (MB)		
		HDG	UDG	IPDG	HDG	UDG	IPDG
1200	2	0.7			32		
5100	2	3.0			161		
21000	2	14.0			728		
1200	3	1.7			57		
5100	3	7.6			283		
21000	3	34.8			1284		
1200	4	3.9			86		
5100	4	17.7			430		
21000	4	79.1			1953		

Disk-shaped scatterer problem

Elements	Order	CPU Time (s)			Memory (MB)		
		HDG	UDG	IPDG	HDG	UDG	IPDG
1200	2	0.7	2.6	2.4	32	269	70
5100	2	3.0	15.0	11.9	161	1360	369
21000	2	14.0	94.8	58.0	728	6578	1857
1200	3	1.7	5.4	6.8	57	525	190
5100	3	7.6	38.8	35.9	283	2921	1017
21000	3	34.8	252.0	197.8	1284	14131	5126
1200	4	3.9	10.5	15.7	86	895	428
5100	4	17.7	67.0	87.9	430	4537	2279
21000	4	79.1	452.8	520.7	1953	21186	11503

Disk-shaped scatterer problem

Elements	Order	CPU Time			Memory		
		HDG	UDG	IPDG	HDG	UDG	IPDG
1200	2	1	3.7	3.4	1	8.4	2.2
5100	2	1	5.0	4.0	1	8.4	2.3
21000	2	1	6.8	4.1	1	9.0	2.6
1200	3	1	3.1	4.0	1	9.2	3.3
5100	3	1	5.1	4.7	1	10.3	3.6
21000	3	1	7.2	5.7	1	11.0	4.0
1200	4	1	2.7	4.0	1	10.4	5.0
5100	4	1	3.8	5.0	1	10.5	5.3
21000	4	1	5.7	6.6	1	10.8	5.9

Disk-shaped scatterer problem

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		HDG	UDG	IPDG	HDG	UDG	IPDG
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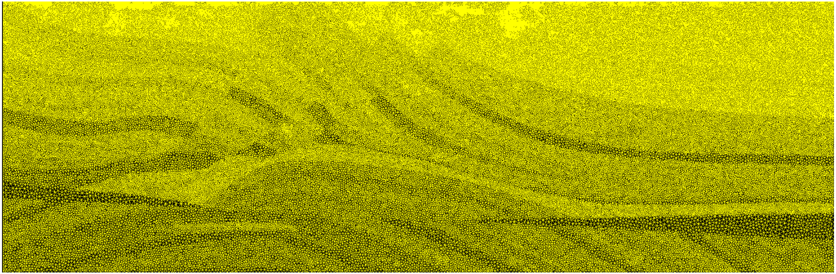
5100	3	1	5.1	3.8	1	10.3	3.6
21000	3	1	7.2	3.0	1	11.0	4.0
1200	4	1	2.7	1.9	1	10.4	5.0
5100	4	1	3.8	2.7	1	10.5	5.3
21000	4	1	5.7	5.4	1	10.8	5.9

Marmousi test-case



Computational domain Ω composed of 235000 triangles

Preliminary parallel results for the Marmousi test-case with the HDG-P1 scheme

	Max. CPU Time (s)	Max. Memory (MB)
sequential	64	4527
parallel (2 proc.)	48	2840
parallel (4 proc.)	30	1526

Contents

Conclusions-Perspectives

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- The HDG scheme has the correct convergence order $(p+1)$
- On a same mesh the HDG formulation is more competitive in terms of memory and computational time than the upwind flux DG formulation and the IPDG method

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Perspectives

- Performance analysis of the numerical code
- Develop 3D Upwind flux DG and HDG formulations for Helmholtz equations
- Solution strategy for the HDG linear system

Thank you !