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GEOMETRIC PROGRAMMING: AN ENGINEERING TOOL FOR COMPRESSOR DESIGN

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ABSTRACT

Geometric programming is a new optimization technique developed to solve nonlinear engineering design problems subject to either linear or nonlinear constraints. This paper outlines the procedure and illustrates its use with a series of examples. These examples include a compressor design, and a compressor transmission line problem. Geometric programming solutions not only yield optimal design parameters, but also generate certain invariant cost distributions unique to the geometric programming methodology.

THEORY

Geometric programming is a mathematical programming technique developed to solve nonlinear programming problems subject to linear or nonlinear constraints. The basic theory was originally developed by Richard Duffin and Clarence Zener [1] using the arithmetic - geometric mean inequality; hence the name geometric programming.

Define a nonlinear objective function in N variables and T_o terms of the following form:

MINIMIZE

$$Y(\bar{x}) = \sum_{t=1}^{T_o} \sigma_{ot} C_{ot} \prod_{n=1}^{N} x_n^{a_{otn}} \tag{1}$$

constrained by M inequality constraints given by,

$$g_m(\bar{x}) = \sum_{t=1}^{T_m} \sigma_{mt} C_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} \leq \sigma_m, \quad m=1,2,3,\cdots,M. \tag{2}$$

where:

$\sigma_{mt} = \pm 1$; $t=1,2,3,\cdots,T_m$; $m=0,1,3,\cdots,M$
$\sigma_m = \pm 1$; $m=1,2,3,\cdots,M$;
$C_{mt} > 0$; $t=1,2,3,\cdots,T_m$; $m=0,1,2,\cdots,M$
$x_n > 0$; $n=1,2,3,\cdots,N.$

This formulation will be designated as the Primal Geometric Program (PGP). It is conventional to refer to the value of $T-(N+1)$ as the degrees of difficulty in a geo-

metric program. Fundamental to the solution of nonlinear programming problems in the form of PGP is the Dual Geometric Program (DGP), given by the following:

MAXIMIZE

$$f(w_{mt}) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} \left(\frac{C_{mt} \, w_{mo}}{w_{mt}} \right)^{\sigma_{mt} w_{mt}} \right]^{\sigma} \tag{3}$$

subject to

$$\sigma = \sum_{t=1}^{T_o} \sigma_{ot} w_{ot} \tag{4}$$

and

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt} a_{mtn} w_{mt} = 0 \tag{5}$$

where,

$$w_{mo} = \sigma_m \sum_{t=1}^{T_m} \sigma_{mt} w_{mt} \geq 0, \quad m=1,2,3,\cdots,M$$
$$w_{oo} = 1 \tag{6}$$
$$\sigma = \pm 1$$

and it is understood that

$$\lim_{w_{mt} \to 0} \left[\frac{C_{mt} \, w_{mo}}{w_{mt}} \right]^{\sigma_{mt} w_{mt}} = 1$$

The variables w_{mt} are properly designated the geometric programming dual variables. The σ, σ_{ot} and σ_{mt} variables are introduced in order to absorb the signs of each term and constraint, and are called signum functions. The complete dual formulation is found in reference [2] and will not be repeated here.

In developing the dual problem, a dual constraint is generated by each primal variable, as is the normality condition given in (4). A dual variable is generated by each term in the primal problem. Hence when $T = N+1$ the dual constraints are sufficient to uniquely determine all dual variables. Once the dual variables of the DGP have been determined, then (3) can be evaluated since the signum functions and the cost coefficients are known from the problem formulation. Wilde and Beightler have shown that the following relationship holds at optimality:

$$f^*(w_{mt}) = Y^*(\bar{x}) \tag{7}$$

That is, at optimality the values of the primal and dual objective functions are equal. Using this result, one can obtain the values of the optimal primal variables from the set of dual variables through the following relations:

$$C_{ot} \prod_{m=1}^{N} X_m^{a_{otn}} = W_{ot} \sigma Y^*(\bar{x}) = W_{ot} \sigma f^*(\omega_{mt}) \quad (8)$$

and

$$C_{mt} \prod_{n=1}^{N} X_n^{a_{mtn}} = \frac{W_{mt}}{W_{mo}} \qquad \begin{array}{l} m = 1, 2, 3, \cdots, M \\ t = 1, 2, 3, \cdots, T_m \end{array} \quad (9)$$

Note that although equations (8) and (9) are nonlinear, they all possess only a single term. Hence, the equations are linear in a logarathmic transformation.

In order to avoid unnecessary confusion, a word is in order regarding notation. Note that all variables and constants are doubly subscripted except for the exponents of the primal variables, which contain three subscripts. Observe the following symbology:
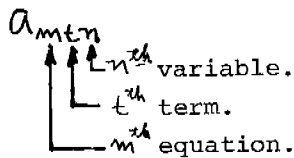
SUBSCRIPTS

$1^{\underline{st}}$ subscript: the equation

$2^{\underline{nd}}$ subscript: the term

$3^{\underline{rd}}$ subscript: the variable

For example; ω_{mt}

- t^{TH} term.
- m^{th} equation.
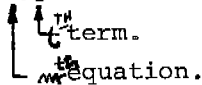
($m = 0$ represents the objective function;

$m > 0$ represents a constraint).

a_{mtn}

- n^{th} variable.
- t^{th} term.
- m^{th} equation.

Hence, the position of every variable within the problem formulation is easily determined through its subscripts. In order to illustrate these concepts, consider the following example [3].

EXAMPLE I: FILL-DIRT TRANSPORT

Suppose you are a fill-dirt contractor and

you have been hired to transport 400 cubic yards of dirt across a large river. In order to transport the material, a container must be constructed. The following information is available.

    a) Each round trip costs $0.10.

    b) The material for the container costs:
        1) bottom: 20.00 $/sq. yd.
        2) sides: 5.00 $/sq. yd.
        3) ends: 20.00 $/sq. yd.

Now, we wish to design a container which will minimize to total cost associated with the prescribed task. From the given data a nonlinear program can be formulated.

MINIMIZE:

$$Y(\bar{x}) = \frac{40}{X_1 X_2 X_3} + 20 X_1 X_2 + 10 X_1 X_3 + 40 X_2 X_3 \quad (10)$$

The DGP is given by the following:

MAXIMIZE:

$$f(\omega_{mt}) = \left[\frac{40}{\omega_{01}}\right]^{\omega_{01}} \left[\frac{20}{\omega_{02}}\right]^{\omega_{02}} \left[\frac{10}{\omega_{03}}\right]^{\omega_{03}} \left[\frac{40}{\omega_{04}}\right]^{\omega_{04}} \quad (11)$$

subject to:

$$\omega_{01} + \omega_{02} + \omega_{03} + \omega_{04} = 1$$
$$-\omega_{01} + \omega_{02} + \omega_{03} = 0$$
$$-\omega_{01} + \omega_{02} + \omega_{04} = 0$$
$$-\omega_{01} + \omega_{03} + \omega_{04} = 0$$

and $\quad \omega_{ot} \geq 0 \quad t = 1, 2, 3, 4.$

Note that $\quad \sigma = 1$
$$\sigma_{01} = \sigma_{02} = \sigma_{03} = \sigma_{04} = 1$$
$$\omega_{00} = 1$$

From the previous theory, the minimal cost is given by $Y^*(\bar{x}) = f^*(\omega_{mt})$ and $f(\omega_{mt})$ is found by maximizing $f(\omega_{mt})$ subject to the four linear equality constraints. However, for this particular problem the DGP possesses exactly four constraints with four unknowns. Hence, the optimal (unique) dual variables are given by the solution of these equations.

$$\omega_{01}^* = 2/5$$
$$\omega_{02}^* = 1/5$$
$$\omega_{03}^* = 1/5$$
$$\omega_{04}^* = 1/5$$

Note that what caused this phenomena is the fact that $T - (N+1) = 0$. In other words, we have zero degrees of difficulty. An interesting characteristic of geometric programming is that at optimality of the DGP, if all signum functions are positive, the ω_{ot}^* variables are actually the percent of optimal cost attributed to that particular term in the primal objective function. Hence, we know the following is true:

Construction Costs: 1/5 of the total cost is attributed to the bottom, sides, and ends of the container respectively.

Transportation Costs: 2/5 of the total cost is attributed to hauling the dirt.

An interesting thing has now occurred; we know the distribution of the project cost without knowing the project cost itself! The optimal cost is now recovered from equation (11).

$$f^{*}(\omega_{mt}) = \left(\frac{40}{2/5}\right)^{2/5}\left(\frac{20}{1/5}\right)^{1/5}\left(\frac{10}{1/5}\right)^{1/5}\left(\frac{40}{1/5}\right)^{1/5} = 100.$$

The optimal design variables are now easily recovered using equations (8) and (9):

$$40 x_1^{-1} x_2^{-1} x_3^{-1} = f^{*}(\omega_{mt})(2/5) = 40$$
$$20 x_1 x_2 = f^{*}(\omega_{mt})(1/5) = 20$$
$$10 x_1 x_3 = f^{*}(\omega_{mt})(1/5) = 10$$

Solving these equations, which are linear in the logarithms, we obtain:

$$x_1^{*} = 2.0 \; , \; x_2^{*} = 0.5 \; , \; x_3^{*} = 1.0$$

EXAMPLE II: FILL-DIRT REVISITED

Now suppose that a field engineer tells you that the cost per trip has doubled to $.20 per trip. What is the best operating policy now? A moments reflection on equations (4) and (5) yield the fact that the dual solution variables remain the same! That is;

$$\omega_{02}^{*} = \omega_{03}^{*} = \omega_{04}^{*} = 1/5 \; , \; \omega_{01}^{*} = 2/5.$$

The cost distributions at optimality are exactly the same. This occurs since the dual constraints from which the optimal solution is obtained depend only upon the exponents of the primal variables. Hence, provided the exponent structure remains the same, the optimal cost distribution is invarient with respect to changes in the technological coefficients. This characteristic is uniquely revealed through the Geometric Programming technique. Unfortunately, this characteristic is only true provided the problem is characterized by zero degrees of difficulty. Note that although the cost distribution remains unchanged, the optimal cost will change since the dual objective function does depend upon the cost coefficients. The new cost is readily obtained, and is given by:

$$f^{*}(\omega_{mt})_{NEW} = \frac{f^{*}(\omega_{mt})_{OLD}}{\left[\frac{40}{2/5}\right]^{2/5}}\left[\frac{80}{2/5}\right]^{2/5} = 132.$$

The new optimal design variables are calculated as before:

$$x_1^{*} = 2.300 \; , \; x_2^{*} = 0.575 \; , \; x_3^{*} = 1.150.$$

Note that although the optimal cost distribution remained unchanged, the design variables adjusted themselves to maintain the cost configuration. In simple terms, the container became larger due to the increased transportation cost.

EXAMPLE III: GAS TRANSMISSION SYSTEM

Consider a gas pipeline transmission system:
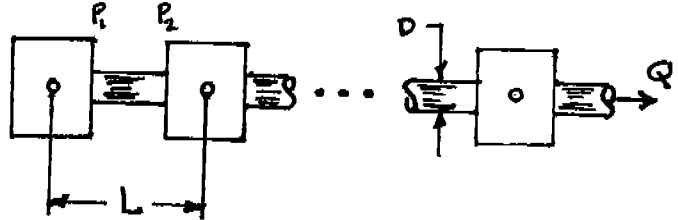
where compressor stations are placed L miles apart. Assume that the total annual cost of the transmission system and its operation is [4]:

$$C(D, P_1, L, r) = 7.84 D^2 P_1 + 450{,}000$$
$$+ 36{,}900 D + \frac{6.57 \cdot 10^6}{L} + \frac{7.72 \cdot 10^8}{L}(r^{.219} - 1) \quad (12)$$

where;

D = pipe inside diameter, inches.

P_1 = compressor discharge pressure, psia.

L = length between compressor stations, miles.

r = compression ratio = P_1/P_2

Furthermore, assume that the flowrate is:

$$Q = 3.39\left[\frac{(P_1^2 - P_2^2)D^5}{fL}\right]^{1/2} \quad (SCF/hr.) \quad (13)$$

with f = friction factor = $.008 D^{-1/3}$. (14)

Find the values for the design parameters P_1, L, and r which will deliver 100 million cubic feet of gas per day (4.17 SCF/hr) with minimum total cost.

Note that the problem as stated contains four design variables and one equality constraint. It will be convenient to use equation (13) to eliminate one of these variables. Using $Q = 4.17 \cdot 10^6$ SCF/hr. and the definition $r = P_1/P_2$; equation (13) can be restated;

$$P_1 = r\left[\frac{1.21 \cdot 10^{10} L \; D^{-5.33}}{(r^2 - 1)}\right]^{1/2} \quad (15)$$

Substituting this expression into (12), rearranging terms, and ignoring the constant $4.5 \cdot 10^6$ we obtain:

$$C(D, L, r) = 8.61 \cdot 10^5 L^{1/2} r D^{-2/3} (r^2-1)^{1/2}$$
$$+ 3.69 \cdot 10^4 D + 6.57 \cdot 10^6 L^{-1}$$
$$+ 7.72 \cdot 10^8 L^{-1} (r^{.219} - 1) \quad (16)$$

In order to represent equation (16) in the form of equation (1), define:

$$X_1 = L \quad , \quad X_2 = r \quad , \quad X_3 = D \; .$$

Hence, the problem statement becomes:

MINIMIZE: $Y(\bar{x}) = 8.61 \cdot 10^5 X_1^{1/2} X_2 X_3^{-2/3} (X_2-1)^{1/2}$
$$+ 3.69 \cdot 10^4 X_3 + 6.57 \cdot 10^6 X_1^{-1}$$
$$+ 7.72 \cdot 10^8 X_1^{-1} (X_2^{.219} - 1) \quad (17)$$

Note, that this objective function is not yet in the proper format, but by defining a new variable

$$X_4 \geq (X_2 - 1) \quad (18)$$

the equivalent problem statement is:

MINIMIZE: $Y(\bar{x}) = 8.61 \cdot 10^5 X_1^{1/2} X_2 X_3 X_4^{-1/2}$
$$+ 3.69 \cdot 10^4 X_3 + 7.72 \cdot 10^8 X_1^{-1} X_2^{.219} - 765.43 \cdot 10^6 X_1^{-1} \quad (19)$$

Subject to: $X_4 X_2^{-2} + X_2^{-2} \leq 1$ (20)

and $X_1, X_2, X_3, X_4 > 0$

The problem can now be solved using geometric programming. Note that one of the signum functions is negative σ_{04} and that there is one degree of difficulty, T − (N+1) = 6 − (4+1) = 1. Proceeding from equations (3) through (6) one obtains the DGP:

MAXIMIZE: $f(\omega_{mt}) = \left[\left(\dfrac{8.61 \cdot 10^5}{\omega_{01}} \right)^{\omega_{01}} \left(\dfrac{3.69 \cdot 10^4}{\omega_{02}} \right)^{\omega_{02}} \right.$
$$\left. \left(\dfrac{7.72 \cdot 10^8}{\omega_{03}} \right)^{\omega_{03}} \left(\dfrac{765.43 \cdot 10^6}{\omega_{04}} \right)^{-\omega_{04}} \left(\dfrac{\omega_{10}}{\omega_{11}} \right)^{\omega_{11}} \left(\dfrac{\omega_{10}}{\omega_{12}} \right)^{\omega_{12}} \right] \quad (21)$$

subject to: $\omega_{01} + \omega_{02} + \omega_{03} - \omega_{04} = 1$
$$.5\,\omega_{01} - \omega_{03} + \omega_{04} = 0$$
$$\omega_{01} + .219\,\omega_{03} - 2\omega_{11} - 2\omega_{12} = 0$$
$$-.67\,\omega_{01} + \omega_{02} = 0$$
$$-.5\,\omega_{01} + \omega_{11} = 0$$
$$\omega_{10} - \omega_{11} - \omega_{12} = 0$$
$$\omega_{mt} \geq 0 \quad , \quad m = 0, 1 \quad , \quad t = 1, 2, \cdots, T_m \, .$$

Note that there are six equality constraints and seven dual variables in the DGP. Hence, a true optimization exists. The DGP can be solved in any convenient fashion. In this case, an efficient solution technique would be to solve for any six dual variables in terms of a seventh and substitute these variables out of $f(\omega_{mt})$ to obtain an unconstrained objective function in only one variable. This problem is now easily solved using a one-dimensional

search scheme. Using this technique, the following dual solution was obtained.

$$\omega_{01}^* = 0.4552 \qquad \omega_{11}^* = 0.2276$$
$$\omega_{02}^* = 0.3094 \qquad \omega_{12}^* = 0.7519$$
$$\omega_{03}^* = 5.1520 \qquad \omega_{10}^* = 0.9795$$
$$\omega_{04}^* = 4.9200$$

Therefore, $f(\omega_{mt}^*) = 295.80 \cdot 10^4 = Y(\bar{x}^*)$.
Using equations (8) and (9) we obtain the optimal design variables:

$$X_1^* = 1.187 \, , \, X_2^* = 24.80 \, , \, X_3^* = 52.60 \, , \, X_4^* = 0.410.$$

And in terms of the original problem statement;

$$D = 24.8'' , \; L = 52.6\, min., \; r = 1.187 , \; P_1 = 284\, psi.$$

and

$$C^* = 3.408 \cdot 10^6 \; \$/yr.$$

Although this last example required some algebraic manipulations and a one-dimensional search, the geometric programming technique often yields the optimal solution to engineering design problems almost by inspection. A final example will serve to illustrate this advantage.

EXAMPLE IV: OPTIMAL STAGED COMPRESSOR

A classical problem in compressor design is that of finding the interstage pressures for an adiabatic reversible compression of an ideal gas. From thermodynamics, one seeks to minimize the energy consumption of an N stage system whose work is described by;

$$Y(P_2, P_3, \cdots, P_{N-1}) = \frac{RST}{\alpha} \left[\left(\frac{P_2}{P_1} \right)^\alpha + \left(\frac{P_3}{P_2} \right)^\alpha + \cdots + \left(\frac{P_N}{P_{N-1}} \right)^\alpha - N \right] \quad (22)$$

where:

$P_1 =$ inlet pressure.
$P_N =$ outlet pressure.
$R =$ gas constant.
$S =$ molar flow rate.
$T =$ inlet temperature.
$\alpha = (k-1)/k \, , \; k = (C_p / C_v).$

The PGP is:

MINIMIZE:

$$Y(\bar{x}) = C \left(\frac{x_1}{C_1} \right)^\alpha + C \left(\frac{x_2}{x_1} \right)^\alpha + \cdots + C \left(\frac{C_2}{x_{N-1}} \right)^\alpha \quad (23)$$

where:

$C_1, C_2 \, \text{\&} \, C$ are constants.

$x_m = P_m \, , \; m = 1, 2, 3, \cdots, N-1.$

The constraints of the DGP are obviously:

$$W_{01} + W_{02} + \cdots + W_{0N} = 1$$
$$\alpha W_{01} - \alpha W_{02} = 0$$
$$\alpha W_{02} - \alpha W_{03} = 0 \qquad (24)$$
$$\vdots$$
$$\alpha W_{0,N-1} - \alpha W_{0N} = 0$$

Note that there is exactly one more term than there are variables in the primal problem. Hence, the program possesses zero degrees of difficulty. From (24) we see that at optimality all dual variables are equal! This implies that all compressor stages contribute equally to the minimizing energy policy, and all compressor ratios must be equal. With a little experience in dealing with geometric programming, this result could be deduced by inspection of equation (22). The dual constraints were formulated to verify the final conclusion.

CLOSURE

In this paper we have attempted to illustrate the use of Geometric Programming in solving nonlinear engineering design problems. A series of examples are solved with emphasis placed on compressor related design formulations. The power of geometric programming lies in the fact that often a complex nonlinear engineering design problem can be solved through the solution to a set of linear equations. In addition, certain invarient cost distribution parameters can be obtained for some zero degree of difficulty problems, as a normal byproduct of the procedure. This technique can be of real value in the design of compressor systems.

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