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***Abstract*— In this communication, we discuss two estimation problems dealing with partial derivatives systems. Namely, estimating partial derivatives of a multivariate noisy signal and identifying parameters of partial differential equations. The multivariate noisy signal is expressed as a truncated Taylor expression in a small time interval. An algebraic method can be then used to estimate its partial derivatives in the operational domain. The same approach applies for the parameter identification problem. The main objects in the aforementioned algebraic method are particular differential operators called annihilators. Stafford's theorem can be applied to formalize the left ideal in the Weyl algebra formed by the annihilators. In the spatial domain, iterated integrals present in the estimates yield noise filtering.**

I. Introduction

Many problems in Engineering concern the estimation of partial derivatives of a noisy multivariate signal. For instance, they appear in the field of signal processing or control, or in economy issues as well.

Numerous methods were developed to tackle this numerical differentiation problem, the finite differences techniques being the most usually used.

The instability of possible solutions to these problems arise from the presence of noise due to the differentiation.

The use of algebraic tools for numerical differentiation in the multivariate case were addressed by Riachy et al. in [1], [2], [3]. Their inspiration comes from the algebraic approach initiated by M. Fliess and H. Sira-Ramírez [4] and from the solutions proposed by Mboup et al. in [5].

The algebraic method developed here is inspired by the parameter estimation methods elaborated in [6], [7], [8]. To illustrate this approach on a multi-dimensional parameter estimation problem, we examine a simple particular case of a partial differential equation. The example of the heat conduction on a thin rod is then discussed and treated by algebraic estimations. An earlier work in this direction, also based on algebraic techniques, can be found in [9].

In Section II, the two estimation problems are briefly described. The algebraic framework for the proposed solutions is given in Section III. More details on the solutions can be found in Section IV. Computer simulations to illustrate our estimates will be presented in the final version.

*This work was supported by Inria Lille - Nord Europe, France.

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II. Problem formulation

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero.

A. Derivatives estimation problem

Let $\mathbf{x} = (x_1, \dots, x_m)$ be an element in $\mathbb{K}^m$ ($m \in \mathbb{N}$). An m-tuple $N \in \mathbb{N}^m$ will be usually written as $N = (N_1, \dots, N_m)$. Let us consider the partial order $\preceq$ on $\mathbb{N}^m$ defined by $N \preceq M$ if $N_i \leq M_i$, for all $1 \leq i \leq m$.

In this paper, we wish to discuss the construction of closed-forms estimators for partial derivatives in the case of a multivariate signal $f(\mathbf{x})$ with $f : U \subset \mathbb{R}^m \to \mathbb{R}$ for U some neighborhood of 0. For a given $I = (i_1, \dots, i_m) \in \mathbb{N}^m$, we denote $|I| := i_1 + \dots + i_m$, $I! := i_1! \dots i_m!$, $\mathbf{x}^I = x_1^{i_1} \dots x_m^{i_m}$ and $\frac{\partial^I}{\partial \mathbf{x}^I} = \frac{\partial^{i_1}}{\partial x_1^{i_1}} \dots \frac{\partial^{i_m}}{\partial x_m^{i_m}}$.

In practical problems, the available signal is corrupted by a noise. Let us denote by $f_\mathbf{n}$ the noisy multivariate signal

$$f_\mathbf{n}(\mathbf{x}) = f(\mathbf{x}) + \boldsymbol{\varpi}(\mathbf{x}),$$

where $\boldsymbol{\varpi}(\mathbf{x})$ denotes an additive noise.

For the time being, we examine the function f only. Assume that f admits a Taylor series expansion at 0 and write:

$$f(\mathbf{x}) = \sum_{I \in \mathbb{N}^m} \frac{a_I}{I!} \mathbf{x}^I, \quad \text{where } a_I = \frac{\partial^I f}{\partial \mathbf{x}^I}(0).$$

For $N = (N_1, \dots, N_m) \in \mathbb{N}^m$, the truncated Taylor series f_N at order N is given by:

$$f_N(\mathbf{x}) = \sum_{I \preceq N} \frac{a_I}{I!} \mathbf{x}^I. \tag{1}$$

Recall that for a function $g : \mathbb{R}^m \to \mathbb{R}$, its multivariate Laplace transform is given by:

$$G(\mathbf{s}) := \mathcal{L}(g)(\mathbf{s}) = \int_{\mathbb{R}_+^m} g(\mathbf{t}) \, \mathrm{e}^{-\mathbf{s}\ {}^T\mathbf{t}} d\mathbf{t}, \tag{2}$$

where ${}^T\mathbf{t}$ denotes the transpose of $\mathbf{t} \in \mathbb{R}^m$. That implies, for instance:

$$\mathcal{L}\left(\frac{\mathbf{x}^I}{I!}\right) = \frac{1}{\mathbf{s}^{I+\mathbf{1}}}.$$

To realize $f_N(\mathbf{x})$ in the operational domain, we apply the Laplace transform (2) on the equation (1). It results:

$$F_N(\mathbf{s}) = \sum_{I \preceq N} \frac{a_I}{\mathbf{s}^{I+1}}, \tag{3}$$

where $\mathbf{s} = (s_1, \dots, s_m)$ denotes the Laplace variable and $\mathbf{s}^I = s_1^{i_1} \dots s_m^{i_m}$.

The idea presented here is to individually estimate each derivative a_J for $J \preceq N$. To formalize our procedure, we consider the following sets:

$$\Theta = \{a_I \mid I \preceq N\},\ \Theta_{\text{est}} = \{a_J\} \quad \text{and} \quad \Theta_{\overline{\text{est}}} = \Theta \setminus \Theta_{\text{est}}.$$

The definition of Θ, Θ_{est} and $\Theta_{\overline{\text{est}}}$ is clear: Θ contains all the parameters, Θ_{est} contains the parameters to be estimated and $\Theta_{\overline{\text{est}}}$ the remaining ones. The relation ($\mathcal{R}$) below follows from (3):

$$\mathcal{R} : P(\mathbf{s})F_N(\mathbf{s}) + Q(\mathbf{s}) + \overline{Q}(\mathbf{s}) = 0 \tag{4}$$

where

$$\begin{aligned} P(\mathbf{s}) &= \mathbf{s}^N,\ Q(\mathbf{s}) = -a_J \mathbf{s}^{N-J-1} \in \mathbb{K}_{\Theta_{\text{est}}}[\mathbf{s}, \frac{1}{\mathbf{s}}] \ \text{ and} \\ \overline{Q}(\mathbf{s}) &= -\sum_{I \preceq N, I \neq J} a_I \mathbf{s}^{N-I-1} \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right] \end{aligned} \tag{5}$$

By $\mathbb{K}_{\Theta_{\text{est}}}$ and $\mathbb{K}_{\Theta_{\overline{\text{est}}}}$, we denote respectively the algebraic extensions $\mathbb{K}_{\Theta_{\text{est}}} = \mathbb{K}(\Theta_{\text{est}})$ and $\mathbb{K}_{\Theta_{\overline{\text{est}}}} = \mathbb{K}(\Theta_{\overline{\text{est}}})$. Sharing the relation $\mathcal{R}$ into the polynomials P, Q and $\overline{Q}$ is justified by the fact that Q contains the coefficient to be estimated, while $\overline{Q}$ is formed by all the remaining terms. To obtain an equation containing only known terms and a_J, the polynomial $\overline{Q}$ must be eliminated. That will provide a formula for the estimate of a_J.

To annihilate $\overline{Q}$, some particular differential operators must be chosen to act on ($\mathcal{R}$). These operators are called *annihilators*. Such algebraic estimators for a_J will be constructed by using structural properties of the Weyl algebra.

Let us stress that if Π is an annihilator estimating a_J, the partial derivative of f at any other point $\mathbf{p} \in \mathbb{K}^m$ can be obtained by computing $\Pi\left(\mathcal{L}\left(f(\mathbf{x}+\mathbf{p})\right)\right)$.

For $\mathbf{x}, \tau \in \mathbb{K}^m$, we use the notation:

$$\int_{\mathbf{0}}^{\mathbf{x}} g(\tau) d\tau = \int_0^{x_1} \cdots \int_0^{x_m} g(\tau_1, \ldots, \tau_m)\, d\tau_1 \ldots d\tau_m.$$

Recall that for a multivariate function g and its Laplace transform G, the inverse Laplace transform satisfies

$$\mathcal{L}^{-1}\left(\frac{1}{\mathbf{s}^I}\frac{\partial^J G}{\partial \mathbf{s}^J}\right) = \frac{1}{(I-\mathbf{1})!}\int_{\mathbf{0}}^{\mathbf{x}} (\mathbf{x}-\tau)^{I-\mathbf{1}} (-\tau)^J g(\tau) d\tau. \tag{6}$$

Setting $v_{I,J}(\tau) = (\mathbf{x}-\tau)^I (-\tau)^J$, a shorter notation can be used:

$$\mathcal{L}^{-1}\left(\frac{1}{\mathbf{s}^I}\frac{\partial^J G}{\partial \mathbf{s}^J}\right) = \frac{1}{(I-\mathbf{1})!}\int_{\mathbf{0}}^{\mathbf{x}} v_{I-1,J}(\tau) g(\tau) d\tau \tag{7}$$

As we have pointed out, the resulting equation from the elimination of $\overline{Q}$ in ($\mathcal{R}$) contains the desired estimate of a_J and known terms. The return to the spatial domain is done by the action of the inverse Laplace transform on this equation.

B. Parameter estimation

A much alike method can be applied to identify parameters in a partial differential equation. To illustrate this assertion, we present the simple example of the one-dimensional heat equation. A similar algebraic approach was studied, for instance, in [9] for this problem and in [10] for the parameter identification of a linear model of the planar motion of a heavy rope. In both papers, uni-dimensional Laplace transforms are used and operational functions obtained as the solutions of an initial value problem. Contrariwise, here the equation is converted to the operational domain by using two-dimensional Laplace transform.

Consider the problem of the heat conduction in a thin rod of length 1. Let $w : (z,t) \mapsto w(z,t)$ be the function representing the temperature at position z at time t. The partial differential equation describing this problem is given by:

$$\frac{\partial^2}{\partial z^2} w(z,t) - \beta \frac{\partial}{\partial t} w(z,t) - \alpha w(z,t) = 0 \tag{8}$$

The initial temperature is assumed to be constant, say $w(z,0) = 30$. We denote it by $f : z \mapsto w(z,0)$. The boundaries are assumed to be perfectly isolated, so the heat may not flow at the boundaries. Therefore, $\frac{\partial}{\partial z} w(0,t) = \frac{\partial}{\partial z} w(1,t) = 0$. We assume also a Dirichlet boundary condition by setting $v : t \mapsto w(0,t) = 30\,\mathrm{e}^{-\frac{t}{4}}$.

The goal is to identify the parameters α and β. The algebraic method used in the previous subsection is applied here. The Laplace transform (2) is employed to realize the partial differential equation (8) as an algebraic equation in the Laplace variable $\mathbf{s} = (s_1, s_2)$:

$$\left(s_1^2 - \beta s_2 - \alpha\right) W(s_1, s_2) + \beta F(s_1) - s_1 V(s_2) = 0, \tag{9}$$

where $W(\mathbf{s})$, $F(s_1)$ and $V(s_2)$ denote the Laplace transforms of $w(z,t)$, $f(z)$ and $v(t)$, respectively. Since

$$F(s_1) = \frac{30}{s_1} \ \text{ and } \ V(s_2) = \mathcal{L}(v)(s_2) = \frac{120}{4s_2+1},$$

the equation (9) leads to:

$$\left(s_1^2 - \beta s_2 - \alpha\right) W(s_1, s_2) + 30\frac{\beta}{s_1} - 120\frac{s_1}{4\,s_2+1} = 0 \tag{10}$$

Here the set of parameters Θ_{est} to be estimated is

$$\Theta_{\text{est}} = \{\alpha, \beta\},$$

while $\Theta_{\overline{\text{est}}} = \emptyset$. Rewriting (10) in the form of a $\mathcal{R}$-relation (see (4)) gives:

$$\mathcal{R} : P(\mathbf{s})W(\mathbf{s}) + Q(\mathbf{s}) + \overline{Q}(\mathbf{s}) = 0 \tag{11}$$

where

$$P(\mathbf{s}) = (4s_2+1)\left(s_1^2 - \beta s_2 - \alpha\right) \ \text{ and} \tag{12}$$

$$Q(\mathbf{s}) = 30\left(4\frac{s_2}{s_1} + \frac{1}{s_1}\right)\beta \in \mathbb{K}_{\Theta_{\text{est}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right] \ \text{ and}$$

$$\overline{Q}(\mathbf{s}) = -120\,s_1 \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}[\mathbf{s}] \tag{13}$$

The problem of annihilating $\overline{Q}$ is simpler than in the previous subsection and it is tackled by finding suitable $\overline{Q}$-annihilators that will lead to a system in Θ_{est}. It is interesting to notice that in this case, the parameters to be estimated also appear in the coefficients of the polynomial P.

III. Annihilators via the Weyl algebra

We have indicated in the previous Section that our aim is to annihilate the polynomials $\overline{Q}$, see (5) and (12). That will be done by the action of annihilators, meaning certain differential operators (or polynomials in the variables $\frac{\partial^I}{\partial \mathbf{s}^I}$) with polynomial coefficients (or rational functions) in the variables $s_1, \ldots, s_m$. The Weyl algebra appears naturally in this context and its structural properties will be quite useful in the choice of the annihilators.

This algebraic viewpoint is inspired by the work of M. Fliess et al. [4], [11], [12]. Details about the algebraic notions defined in the sequel can be found in [13].

A. The Weyl Algebra

We recall below some basic definitions and properties of the Weyl algebra.

Definition 1: Let $m \in \mathbb{N} \setminus \{0\}$. The *Weyl algebra* $\mathsf{A}_m(\mathbb{K})$ (or A_m) is the $\mathbb{K}$-algebra with generators $p_1, q_1, \ldots, p_m, q_m$ and relations

$$[p_i, q_j] = \delta_{ij},\ [p_i, p_j] = [q_i, q_j] = 0, \forall\ 1 \le i, j \le m$$

where $[\cdot,\cdot]$ is the commutator defined by $[u, v] := uv - vu$, for all $u,\ v \in \mathsf{A}_m(\mathbb{K})$.

The Weyl algebra A_m can be realized as the algebra of polynomial differential operators on the polynomial ring $\mathbb{K}[\mathbf{s}]$ by setting:

$$p_i = \frac{\partial}{\partial s_i} \quad \text{and} \quad q_i = s_i \times \cdot\ , \forall\ 1 \le i \le m.$$

That implies that A_m can be written as $\mathsf{A}_m = \mathbb{K}[\mathbf{q}][\mathbf{p}] = \mathbb{K}[\mathbf{s}]\left[\frac{\partial}{\partial \mathbf{s}}\right]$. The algebra of differential operators $\mathsf{B}_m(\mathbb{K})$ (or B_m) on $\mathbb{K}[\mathbf{s}]$ with coefficients in the rational functions field $\mathbb{K}(\mathbf{s})$ is naturally related to $\mathsf{A}_m(\mathbb{K})$. In this case, we can write $\mathsf{B}_m := \mathbb{K}(\mathbf{q})[\mathbf{p}] = \mathbb{K}(\mathbf{s})\left[\dfrac{\partial}{\partial \mathbf{s}}\right]$.

A basis for A_m is given by $\left\{\mathbf{q}^I \mathbf{p}^J \mid I, J \in \mathbb{N}^m\right\}$ where $\mathbf{q} = q_1^{i_1} \ldots q_m^{i_m}$ and $\mathbf{p} = p_1^{i_1} \ldots p_m^{i_m}$. An operator $F \in \mathsf{A}_m$ can be written in a *canonical form*,

$$F = \sum_{I,J} \lambda_{IJ} \mathbf{q}^I \mathbf{p}^J \quad \text{with} \quad \lambda_{IJ} \in \mathbb{K}.$$

Similarly, an element $F \in \mathsf{B}_m$ can be written as

$$F = \sum_I g_I(\mathbf{s}) \frac{\partial^I}{\partial s^I}, \quad \text{where} \quad g_I(\mathbf{s}) \in \mathbb{K}(\mathbf{s}).$$

The *order* of F is defined as $\operatorname{ord}(F) = \max\{[I| \mid g_I(\mathbf{s}) \neq 0\}$. This definition holds for the Weyl algebra A_m as well, since $\mathsf{A}_m \subset \mathsf{B}_m$. Some useful properties of A_m and B_m are given by the following propositions:

Proposition 2: A_m is a domain. Moreover, A_m is simple and Noetherian.

These properties are shared by B_m. In addition, A_m is neither a principal right domain, nor a principal left domain. Nevertheless this is true for B_1:

Proposition 3: B_1 admits a left division algorithm, that is, if $F,\ G \in \mathsf{B}_1$, then there exists $Q,\ R \in \mathsf{B}_1$ such that $F = QG + R$ and $\operatorname{ord}(R) < \operatorname{ord}(G)$. Consequently, B_1 is a principal left domain.

Alas, this proposition does not hold for B_m for $m \ge 2$. But a celebrated theorem by T. Stafford (see [14]) provides an remarkable property on the number of generators of a left ideal in the Weyl algebra. Namely, Stafford proved that every left ideal of D ($D = \mathsf{A}_m$ or B_m) can be generated by two elements in D:

Theorem 4 (Stafford): Let $\mathfrak{a}$ be a left ideal of D generated by three elements F_1, F_2 and $F_3 \in D$. Then, there exist G_1 and $G_2 \in D$ such that

$$\mathfrak{a} = D(F_1 + G_1 F_3) + D(F_2 + G_2 F_3).$$

An effective implementation in `Maple`, named `Stafford`, of this important theorem can be found in the work of A. Quadrat and D. Robertz [15].

Remark 5: It is important to notice that the principality of B_1 was largely used in the initial works on algebraic methods applied to univariate numerical differentiation, such as [5] or parameter estimation in ordinary differential equations, see for instance [8]. In the multivariate case, the principality holds no longer, therefore the importance of Stafford's theorem.

To close this part, we remark a useful identity:

Remark 6: For arbitrary $N, M \in \mathbb{N}^r$, we have

$$\frac{\partial^N}{\partial \mathbf{s}^N} \frac{1}{\mathbf{s}^M} = \sum_{0 \preceq J \preceq N} \binom{N}{J} (-1)^{|N-J|} \frac{M^{\overline{N-J}}}{\mathbf{s}^{M+N-J}} \frac{\partial^J}{\partial \mathbf{s}^J},$$

where $\binom{N}{J} = \binom{n_1}{j_1} \ldots \binom{n_r}{j_r}$, $M^{\overline{N}} = m_1^{\overline{n_1}} \ldots m_r^{\overline{n_r}}$ and $m_i^{\overline{n_i}}$ denotes the rising factorial ($m_i^{\overline{n_i}} = m_i(m_i+1)\ldots(m_i+n_i-1)$).

B. Annihilators

From now on, we consider $m \ge 2$. In this subsection, we construct *annihilators*. These differential operators help to construct algebraic estimators, either of the partial derivatives, or else of parameters.

Definition 7: Let $R \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right]$. A *R-annihilator w.r.t.* B_m is an element of $\operatorname{Ann}_{\mathsf{B}_m}(R) = \{F \in \mathsf{B}_m \mid F(R) = 0\}$.

Let us remark that $\operatorname{Ann}_{\mathsf{B}_m}(R)$ is a left ideal of B_m. Therefore, by Stafford's theorem (Theorem 4), $\operatorname{Ann}_{\mathsf{B}_m}(R)$ is generated by two generators Π_1 and $\Pi_2 \in \mathsf{B}_m$:

$$\operatorname{Ann}_{\mathsf{B}_m}(R) = \mathsf{B}_m \Pi_1 + \mathsf{B}_m \Pi_2.$$

We call the annihilators Π_1 and Π_2 *minimal R-annihilators w.r.t.* B_m. Notice that $\operatorname{Ann}_{\mathsf{B}_m}(R)$ contains annihilators in finite integral form, *i.e.* operators with coefficients in $\mathbb{K}\left[\frac{1}{\mathbf{s}}\right]$.

Lemma 8: Consider $R(\mathbf{s}) = \alpha \mathbf{s}^N$, $N = (n_1, \ldots, n_m) \in \mathbb{Z}^m$ with $\alpha \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}$. A minimal R-annihilator is given by

$$s_i \frac{\partial}{\partial s_i} - n_i,\ \forall 1 \le i \le m.$$

Recall that the *degree* of a monomial $\mathbf{s}^I \in \mathbb{K}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right]$ is $|I|$. The *total degree* of a polynomial in $\mathbf{s}$ is the maximum degree of its monomials.

Remark 9: Consider $R \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right]$ with a monomial s^I of maximal degree. So R has total degree $|I|$. Let $i_k =$

$\max\{i_j \mid j=1,\ldots,m\}$. If $|I|>0$, then $\frac{\partial^{i_k+1}}{\partial s_k^{i_k+1}}$ is clearly an R-annihilator.

Now, recall that the polynomial to be annihilated is:

$$\overline{Q}(\mathbf{s}) \quad = \quad -\sum_{I\preceq N, I\neq J} a_I \mathbf{s}^{N-I-1} \in \mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s}, \frac{1}{\mathbf{s}}\right].$$

By the previous Remark, it results immediately:

Lemma 10: The differential operators $\frac{\partial^{n_k}}{\partial s_k^{n_k}}$ are $\overline{Q}$-annihilators, for all $1\leq k\leq m$.

In order to construct an alternative annihilator, an algorithm is sketched below:

Algorithm 11:

Input: a polynomial $R=\sum_{\substack{I\in\mathbb{Z}^m\\ \text{finite}}} b_I\mathbf{s}^I$ in $\mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s},\frac{1}{\mathbf{s}}\right]$ of total degree $d\in\mathbb{N}$

Output: an R-annihilator

1) set $\Pi = 1\in D$.
2) choose a monomial of degree d, say $b_J\mathbf{s}^J$ with $J=(j_1,\ldots,j_m)$ (so $|J|=d$).
3) choose $j_k=\min\{j_\ell>0 \mid \ell=1,\ldots,m\}$.
4) apply $\pi = s_k\frac{\partial}{\partial s_k} - j_k$ (see Lemma 8) on R.
5) (a) if $\pi(R)=0$, then return π and stop the algorithm.
   (b) if $\pi(R)\neq 0$, then set $\Pi=\pi\circ\Pi$ and return to step (2) with $R\leftarrow \pi(R)$.

Example 12: Consider $m=2$ and $R(s_1,s_2)=a_{00}s_1^2s_2+a_{01}s_1^2+a_{10}s_1s_2+a_{20}s_2\in\mathbb{K}_{\Theta_{\overline{\text{est}}}}[s_1,s_2]$. A R-annihilator constructed with the above algorithm is $\left(s_1\frac{\partial}{\partial s_1}-2\right)\circ\left(s_2\frac{\partial}{\partial s_2}-1\right)$.

The concept of an estimator must be defined in order to take into account the remaining terms in the relation $(\mathcal{R})$, after the action of an annihilator. Notice that the parameters to be estimated might appear in the set of coefficients of both polynomials Q and P, but they might be present exclusively in one of the two. Therefore a $\overline{Q}$-annihilator must not eliminate all terms in Θ_{est}, as formalized in the definition below:

Definition 13: An *estimator* $\pi\in\mathsf{B}$ is a $\overline{Q}$-annihilator satisfying

$$\operatorname{coeffs}(\pi(\mathcal{R}))\cap\mathbb{K}_{\Theta_{\text{est}}}\neq\emptyset,$$

where $\operatorname{coeffs}(R)$ denotes the set of coefficients of a polynomial $R\in\mathbb{K}_\Theta\left[\mathbf{s},\frac{1}{\mathbf{s}}\right]$.

It is implied by this definition that the criteria on the coefficients must be considered in the choice of annihilators in algorithm 11.

IV. Derivative estimation

For the sake of simplicity, the two-dimensional case only is presented in the sequel. In addition, to lighten the notation, the signal is written as $f(x,y)=\sum_{(i,j)\in\mathbb{N}^2}\frac{a_{ij}}{i!j!}x^iy^j$, where $a_{ij}=\frac{\partial^{i+j}f}{\partial x^i\partial y^j}(0)$. We write also B instead of B_2.

Assume that the parameter to be estimated is $a_{k\ell}$. For a truncated Taylor series at $N=(N_1,N_2)$, the polynomials appearing in the relation $(\mathcal{R})$ are (see (5)):

$$\begin{aligned}
P(s_1,s_2) &= s_1^{N_1}s_2^{N_2},\\
Q(s_1,s_2) &= -a_{k\ell}s_1^{N_1-k-1}s_2^{N_2-\ell-1}\in\mathbb{K}_{\Theta_{\text{est}}}\left[\mathbf{s},\frac{1}{\mathbf{s}}\right] \quad \text{and}\\
\overline{Q}(s_1,s_2) &= -\sum_{\substack{(i,j)\preceq N\\(i,j)\neq(k,\ell)}} a_{ij}s_1^{N_1-i-1}s_2^{N_2-j-1}\in\mathbb{K}_{\Theta_{\overline{\text{est}}}}\left[\mathbf{s},\frac{1}{\mathbf{s}}\right].
\end{aligned}$$

In what follows, we provide estimates for all derivatives a_{ij}, $0\leq i,j\leq 3$. Due to clear reasons of symmetry, it is enough to present the estimations of a_{00}, a_{10}, a_{20}, a_{30}, a_{11} and a_{21}.

For each $a_{k\ell}$ to be estimated, we use a truncation of the Taylor series at $N=(k,\ell)$. Notice that in this case, from (14) it follows:

$$Q(s_1,s_2)=-\frac{a_{k\ell}}{s_1s_2}.$$

The first step of the estimation is to determine minimal $\overline{Q}$-annihilators. To begin, Lemma 8 helps to find two $\overline{Q}$-annihilators $s_1\frac{\partial}{\partial s_1}+1$ and $s_2\frac{\partial}{\partial s_2}+1$. However, they are not estimators since they clearly eliminate Q as well.

We follow the algorithm 11 to determine a $\overline{Q}$-annihilator that is also a estimator.

The next table summarizes the various data about the coefficients $a_{k\ell}$:

coefficient	P	$\overline{Q}$	$\overline{Q}$ – annihilator
a_{00}	1	0	$\frac{1}{s_1s_2}$
a_{10}	s_1	$-\frac{a_{00}}{s_2}$	$\frac{1}{s_1^2s_2}\frac{\partial}{\partial s_1}$
a_{20}	s_1^2	$-\frac{a_{00}s_1}{s_2}-\frac{a_{10}}{s_2}$	$\frac{1}{s_1^3s_2}\frac{\partial^2}{\partial s_1^2}$
a_{30}	s_1^3	$-\frac{a_{00}s_1^2}{s_2}-\frac{a_{10}s_1}{s_2}-\frac{a_{20}}{s_2}$	$\frac{1}{s_1^4s_2}\frac{\partial^3}{\partial s_1^3}$
a_{11}	s_1s_2	$-a_{00}-\frac{a_{10}}{s_1}-\frac{a_{01}}{s_2}$	$\frac{1}{s_1^2s_2^2}\frac{\partial^2}{\partial s_1s_2}$
a_{21}	$s_1^2s_2$	$-a_{00}s_1-a_{10}-\frac{a_{01}s_1}{s_1}$ $-\frac{a_{11}}{s_2}-\frac{a_{20}}{s_1}$	$\frac{1}{s_1^3s_2^2}\frac{\partial^2}{\partial s_1s_2}\left(s_1\frac{\partial}{\partial s_1}-1\right)$

Let us remark that some annihilators are proposed in [1]. Using remark 6, it can be shown that the annihilators given above coincide with their minimal annihilators.

Remark 14: Another way to proceed would be to construct other two annihilators for each $\overline{Q}$. Then, to apply Stafford's theorem to obtain two minimal generators by using the package `Stafford`. The final step is to observe the criteria in Definition 13 in these generators to obtain an estimator.

Applying the inverse Laplace transform (6) provides the consequent estimates. Some summarized computations can be found below:

1) Estimating a_{00}. The action of the estimator provides the resulting expression in the operational domain:

$$\frac{F_N(s_1,s_2)}{s_1s_2}=\frac{a_{00}}{s_1^2s_2^2}.$$

After applying the inverse Laplace transform (6), we obtain the estimate:

$$a_{00}=\frac{1}{xy}\int_{\mathbf{0}}^{\mathbf{x}} f(\tau)d\tau$$

2) Estimating a_{21}. The action of the estimator provides the resulting expression in the operational domain: After applying the inverse Laplace transform (6), we obtain the estimate:

$$a_{21} = -\frac{360}{x^5 y^3} \int_{\mathbf{0}}^{\mathbf{x}} (v_{0,0,2,1} + v_{0,1,2,0} + 4v_{1,0,1,1} + 4v_{1,1,1,0} + v_{2,1,0,1} + v_{2,1,0,0})(\tau) f(\tau) d\tau$$

V. Parameter estimation

The polynomial $\overline{Q}$ to be annihilated in the equation (12) is $\overline{Q}(\mathbf{s}) = -s_1 \in \mathbb{K}_{\overline{\Theta_{\text{est}}}}[\mathbf{s}]$. Some natural $\overline{Q}$-annihilators are:

$$\pi_1 = \frac{\partial^2}{\partial s_1^2}, \ \pi_2 = \frac{\partial}{\partial s_2} \quad \text{and} \quad \pi_3 = \frac{\partial}{\partial s_2}\frac{\partial}{\partial s_1}. \qquad (14)$$

Although being unnecessary, the following Lemma can be checked by using `Stafford`:

Lemma 15: The two minimal $\overline{Q}$-annihilators are π_1 and π_2.

Applying π_1 on the relation (11) gives:

$$2W + 4s_1 \frac{\partial}{\partial s_1} W + \left(s_1^2 - \beta s_2 - \alpha\right) \frac{\partial^2}{\partial s_1^2} W + \frac{60}{s_1^3}\beta = 0$$

where $W = W(s_1, s_2)$. In order to apply the inverse Laplace transform (6), we divide the above equation by $s_1^3 s_2^2$. Using the notation $v_{I,J} = v_{I,J}(\tau,\eta) = (z-\tau)^{i_1}(t-\eta)^{i_2}(-\tau)^{j_1}(-\eta)^{j_2}$, for all $I = (i_1, i_2)$, $J = (j_1, j_2) \in \mathbb{N}^2$, we obtain in the spatial domain:

$$A_{11}\alpha + A_{12}\beta = B_1$$

where

$$\begin{aligned} A_{11} &= -\frac{1}{2}\int_{\mathbf{0}}^{(z,t)} v_{2,1,2,0} w(\tau,\eta) d\tau d\eta, \\ A_{12} &= \frac{1}{2} t z^5 - \frac{1}{2}\int_{\mathbf{0}}^{(z,t)} v_{2,0,2,0} w(\tau,\eta) d\tau d\eta \quad \text{and} \\ B_1 &= -\int_{\mathbf{0}}^{(z,t)} (v_{2,1,0,0} - 4v_{1,1,1,0} + v_{0,1,2,0}) w(\tau,\eta) d\tau d\eta. \end{aligned}$$

In the same manner, an expression resulting from the action of π_2 can be found:

$$\left(4s_1^2 - 4\alpha - (8s_2+1)\beta\right) W + (4s_2+1)\left(s_1^2 - \beta s_2 - \alpha\right)\frac{\partial}{\partial s_2} W + \frac{120}{s_1}\beta = 0.$$

After dividing the above equation by a suitable monomial in $\mathbf{s}$, we obtain $A_{21}\alpha + A_{22}\beta = B_2$ in the spatial domain. A system on $\overline{\Theta_{\text{est}}}$ results from the aforementioned actions of $\overline{Q}$-annihilators:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

with $A_{21} = -\int_{\mathbf{0}}^{(z,t)} \left(\frac{1}{4} v_{2,2,0,1} + v_{2,2,0,0} + 2v_{2,1,0,1}\right) w(\tau,\eta) d\tau d\eta$,

$$A_{22} = -\int_{\mathbf{0}}^{(z,t)} \left(\frac{1}{4} v_{2,2,0,0} + \frac{1}{2} v_{2,1,0,1} + 2v_{2,0,0,1} + 4v_{2,1,0,0}\right) w(\tau,\eta) d\tau d\eta + 10t^2 z^3$$

and

$$B_2 = -\int_{\mathbf{0}}^{(z,t)} \left(4v_{0,1,0,1} + 2v_{0,2,0,0} + \frac{1}{2} v_{0,2,0,1}\right) w(\tau,\eta) d\tau d\eta.$$

Solving the system provides the estimates of α and β.

VI. Conclusion

The goal of this communication was to examine two estimation problems under an algebraic viewpoint. The proposed algebraic method is inspired by the work of Fliess et al.

We have discussed the numerical derivation problem in the multivariate case. Basic properties of the Weyl algebra were applied to obtain the desired estimates.

A simple example of parameter estimation for the one-dimensional heat equation was also presented in order to demonstrate the algebraic method.

References

[1] S. Riachy, M. Mboup, and J.-P. Richard, "Multivariate numerical differentiation," *JCAM*, Aug. 2011. [Online]. Available: http://hal.inria.fr/inria-00637164

[2] S. Riachy, Y. Bachalany, M. Mboup, and J.-P. Richard, "An algebraic method for multi-dimensional derivative estimation," in *MED'08, 16th IEEE Mediterranean Conference on Control and Automation*. Ajaccio, Corsica, France: IEEE, 2008. [Online]. Available: http://hal.inria.fr/inria-00275461

[3] ——, "Différenciation numérique multivariable I : estimateurs algébriques et structure," in *Sixième Conférence Internationale Francophone d'Automatique Nancy, France, 2-4 juin 2010*, Nancy, France, 2010. [Online]. Available: http://hal.inria.fr/inria-00463786

[4] M. Fliess and H. Sira-Ramírez, "An algebraic framework for linear identification," *ESAIM Control Optim. Calc. Variat.*, vol. 9, pp. 151–168, 2003.

[5] M. Mboup, C. Join, and M. Fliess, "Numerical differentiation with annihilators in noisy environement," *Numerical Algorithms*, vol. 50, pp. 439–467, 2009.

[6] R. Ushirobira, W. Perruquetti, M. Mboup, and M. Fliess, "Estimation algébrique des paramètres intrinsèques d'un signal sinusoïdal biaisé en environnement bruité," in *Proc. Gretsi*, Bordeaux, France, Sept. 2011.

[7] ——, "Algebraic parameter estimation of a biased sinusoidal waveform signal from noisy data," in *Sysid 2012, 16th IFAC Symposium on System Identification*, Brussels, Belgique, 2012.

[8] ——, "Algebraic parameter estimation of a multi-sinusoidal waveform signal from noisy data," in *European Control Conference*, Zurich, Suisse, Apr. 2013. [Online]. Available: http://hal.inria.fr/hal-00819048

[9] J. Rudolph and F. Woittennek, "An algebraic approach to parameter identification in linear infinite dimensional systems," in *Proc. 16th Mediterranean Conference on Control and Automation*, june 2008, pp. 332–337.

[10] N. Gehring, T. Knüppel, J. Rudolph, and F. Woittennek, "Algebraic identification of heavy rope parameters," in *Proc. 16th IFAC Symposium on System Identification*, July 2012, pp. 161–166.

[11] M. Fliess, M. Mboup, H. Mounier, and H. Sira-Ramírez, "Questioning some paradigms of signal processing via concrete examples," in *in Algebraic Methods in Flatness, Signal Processing and State Estimation*, G. S.-N. H. Sira-Ramírez, Ed. Editiorial Lagares, 2003, pp. 1–21.

[12] M. Mboup, "Parameter estimation for signals described by differential equations," *Applicable Analysis*, vol. 88, pp. 29–52, 2009.

[13] J. McConnell and J. Robson, *Noncommutative Noetherian Rings*, A. M. Soc., Ed. Hermann, 2000.

[14] J. T. Stafford, "Module structure of Weyl algebras," *J. London Math. Soc.*, vol. 18, no. 3, pp. 429–442, 1978.

[15] A. Quadrat and D. Robertz, "Computation of bases of free modules over the weyl algebras," *J. Symbolic Comput.*, vol. 42, pp. 1113–1141, 2007.