Stable Kneser Graphs are almost all not weakly Hom-Idempotent

Pablo Torres, Mario Valencia-Pabon

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Pablo Torres † Mario Valencia-Pabon‡

Abstract

A graph G is said to be *hom-idempotent* if there is an homomorphism from G^2 to G, and *weakly hom-idempotent* if for some $n \geq 1$ there is a homomorphism from G^{n+1} to G^n. Larose et al. [*Eur. J. Comb. 19:867-881, 1998*] proved that Kneser graphs $KG(n,k)$ are not weakly hom-idempotent for $n \geq 2k+1$, $k \geq 2$. We show that 2-stable Kneser graphs $KG(n,k)_{2-\text{stab}}$ are not weakly hom-idempotent, for $n \geq 2k+2$, $k \geq 2$. Moreover, for $s, k \geq 2$, we prove that s-stable Kneser graphs $KG(ks+1,k)_{s-\text{stab}}$ are circulant graphs and so hom-idempotent graphs.

Keywords: Cartesian product of graphs, Stable Kneser graphs, Cayley graphs, Hom-idempotent graphs.

1 Introduction

Let $[n]$ denote the set $\{1,\ldots,n\}$. For positive integers $n \geq 2k$, the Kneser graph $KG(n,k)$ has as vertices the k-subsets of $[n]$ and two vertices are connected by an edge if they have empty intersection. In a famous paper, Lovász [8] showed that its chromatic number $\chi(K(n,k))$ is equal to $n-2k+2$. After this result, Schrijver [10] proved that the chromatic number remains the same when we consider the subgraph $KG(n,k)_{2-\text{stab}}$ of $KG(n,k)$ obtained by restricting the vertex set to the k-subsets that are 2-*stable*, that is, that do not contain two consecutive elements of $[n]$ (where 1 and n are considered also to be consecutive). Schrijver [10] also proved that the 2-stable Kneser graphs are *vertex critical* (or χ-*critical*), i.e. the chromatic number of any proper subgraph of $KG(n,k)_{2-\text{stab}}$ is strictly less than $n-2k+2$; for this reason, the 2-stable Kneser graphs are also known as the Schrijver graphs. After these general advances, a lot of work has been done concerning properties of Kneser graphs and stable Kneser graphs (see [2, 3, 7, 1, 9] and references therein). For example, it is well known that for $n \geq 2k+1$ the automorphism group of the Kneser graph $KG(n,k)$ is the symmetric group induced by the permutation action on $[n]$; see [4] for a textbook account. More recently, Braun [1] showed that the automorphism group of the 2-stable Kneser graphs $KG(n,k)_{2-\text{stab}}$ is the dihedral group of order $2n$.

The *cartesian product* $G \square H$ of two graphs G and H has vertex set $V(G) \times V(H)$, two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism.

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An *homomorphism* from a graph G into a graph H, denoted by $G \to H$, is an edge-preserving map from $V(G)$ to $V(H)$. If H is a subgraph of G and $\phi : G \to H$ has the property that $\phi(u) = u$ for every vertex u of H, then ϕ is called a *retraction* and H is called a *retract* of G. If $\phi : G \to H$ is a bijection and ϕ^{-1} is also a homomorphism from H to G, then ϕ is an *isomorphism* and we write $G \simeq H$. In particular, if $G = H$ then ϕ is an *automorphism* if and only if it is bijective. Two graphs G and H are *homomorphically equivalent*, denoted by $G \leftrightarrow H$, if $G \to H$ and $H \to G$. A graph G is called a *core* if it has no proper retracts, i.e., any homomorphism $\phi : G \to G$ is an automorphism of G. It is well known that any finite graph G is homomorphically equivalent to at least one core $G^\bullet$, as can be seen by selecting $G^\bullet$ as a retract of G with a minimum number of vertices. In this way, $G^\bullet$ is uniquely determined up to isomorphism, and it makes sense to think of it as *the core* of G. It is widely known that Kneser graphs are core. Moreover, it is not difficult to deduce that any χ-critical graph is a core. Therefore, any 2-stable Kneser graph is also a core, because they are χ-critical graphs [10].

Let A be a group and S a subset of A that is closed under inverses and does not contain the identity. The *Cayley gragh* $Cay(A, S)$ is the graph whose vertex set is A, two vertices u, v being joined by an edge if $u^{-1}v \in S$. Cayley graphs of cyclic groups are often called *circulants*.

A graph G is said *vertex-transitive* if its automorphism group $\mathrm{AUT}(G)$ acts transitively on its vertex-set. It's well known that Cayley graphs and Kneser graphs are vertex-transitive. However, 2-stable Kneser graphs are not vertex-transitive in general. For example, no automorphism in $KG(6, 2)_{2-\mathrm{stab}}$ sends $\{1, 3\}$ to $\{1, 4\}$, since $\mathrm{AUT}(KG(6, 2)_{2-\mathrm{stab}})$ is the dihedral group of order 12 acting on the set $\{1, 2, \ldots, 6\}$.

We write G^n for the n-fold cartesian product of a graph G. A graph G is said *hom-idempotent* if there is a homomorphism from G^2 to G, and *weakly hom-idempotent* if for some $n \geq 1$ there is a homomorphism from G^{n+1} to G^n. Larose et al. [7] showed that the Kneser graphs are not weakly hom-idempotent. However, the technique used by Larose et al. [7] cannot be extended directly to the 2-stable Kneser graphs.

A subset $S \subseteq [n]$ is *s-stable* if any two of its elements are at least "at distance s apart" on the n-cycle, that is, if $s \leq |i - j| \leq n - s$ for distinct $i, j \in S$. For $s, k \geq 2$ and $n \geq ks$, the s-stable Kneser graph $KG(n, k)_{s-\mathrm{stab}}$ is the subgraph of $KG(n, k)$ obtained by restricting the vertex set of $KG(n, k)$ to the s-stable k-subsets of $[n]$.

In this paper, we show that almost all 2-stable Kneser graphs are not weakly hom-idempotent. Moreover, for $s, k \geq 2$, we show that s-stable Kneser graphs $KG(ks + 1, k)_{s-\mathrm{stab}}$ are circulant graphs and so hom-idempotent graphs. In the sequel, we will use the term *modulo* $[n]$ to denote arithmetic operations on the set $[n]$ where n represents the 0.

2 2-stable Kneser graphs

As we have mentioned in the previous section, Braun [1] showed that the automorphism group of the 2-stable Kneser graph $KG(n, k)_{2-\mathrm{stab}}$ is the dihedral group D_{2n} of order $2n$. We denote the elements of D_{2n} as follows (arithmetic operations are taken modulo $[n]$):

- *Rotations*: Let σ^0 be the identity permutation on $[n]$ and, for $1 \leq i \leq n-1$, let $\sigma^i = \sigma^{i-1} \circ \sigma^1$, where σ^1 is the circular permutation $(1, 2, \ldots, n-1, n)$.
- *Reflexions*:

– Case n odd. For $1 \leq i \leq n$, let ρ_i be the permutation formed by the product of the transpositions $(i+1, i-1)(i+2, i-2) \ldots (i+\frac{n-1}{2}, i-\frac{n-1}{2})$, where i is a fix point.

– Case n even. For $1 \leq i \leq \frac{n}{2}$, we have two types of reflexions: let ρ_i be the permutation formed by the product of the transpositions $(i+1, i-1)(i+2, i-2) \ldots (i+\frac{n}{2}-1, i-\frac{n}{2}+1)$, where i and $i+\frac{n}{2}$ are fix points; and let δ_i be the permutation formed by the product of transpositions $(i, i-1)(i+1, i-2) \ldots (i+\frac{n}{2}-1, i-\frac{n}{2})$ without fix point.

An *automorphism* ϕ of a graph G is called a *shift* of G if $\{u, \phi(u)\} \in E(G)$ for each $u \in V(G)$. In other words, a shift of G maps every vertex to one of its neighbors [7].

Lemma 1. *Let $n \geq 2k+2$. Then, the only two shifts of the 2-stable Kneser graph $KG(n,k)_{2-stab}$ are the rotations σ^1 and σ^{n-1}.*

Proof. It is very easy to deduce that the circular permutations $\sigma^1 = (1, 2, 3, \ldots, n-1, n)$ and $\sigma^{n-1} = (\sigma^1)^{-1} = (n, n-1, n-2, \ldots, 2, 1)$ are both shifts of the graph $KG(n,k)_{2-\mathrm{stab}}$. In order to prove that they are the only two shifts of $KG(n,k)_{2-\mathrm{stab}}$, we will proceed by cases. The arithmetic operations are taken modulo $[n]$.

- *Rotations.* Clearly, the identity permutation σ^0 is not a shift. Now, we claim that for each $i \in \{2, 3, \ldots, n-2\}$, there exists a vertex v_i in $KG(n,k)_{2-\mathrm{stab}}$ such that $\{1, i+1\} \subseteq v_i$. In fact, vertex v_i can be computed as follows:
  - If $2 \leq i \leq 2k-1$ then, set $v_i = \{1, 1+2.1, 1+2.2, \ldots, 1+2.(k-1)\}$ if i is even, otherwise set $v_i = \{1, 2+2.1, 2+2.2, \ldots, 2+2.(k-1)\}$.
  - If $2k \leq i \leq n-2$ then, set $v_i = \{1, 1+2.1, 1+2.2, \ldots, 1+2.(k-2), i+1\}$.

  Now, for each $2 \leq i \leq n-2$, we know that $\sigma^i(1) = 1+i$ and therefore, $1+i \in \sigma^i(v_i)$ which implies that $\{v_i, \sigma^i(v_i)\}$ is not an edge of $KG(n,k)_{2-\mathrm{stab}}$. Thus, for $2 \leq i \leq n-2$, σ^i is not a shift of $KG(n,k)_{2-\mathrm{stab}}$.

- *Reflexions.* We consider two cases:
  - Case n odd. For each $1 \leq i \leq n$, let v_i be a vertex in $KG(n,k)_{2-\mathrm{stab}}$ such that $i \in v_i$. Trivially, such vertex v_i always exists. Now, we know that i is a fix point under the permutation ρ_i and thus, $i \in \rho_i(v_i)$ which implies that $\{v_i, \rho_i(v_i)\}$ is not an edge of $KG(n,k)_{2-\mathrm{stab}}$. Thus, for $1 \leq i \leq n$, ρ_i is not a shift of $KG(n,k)_{2-\mathrm{stab}}$.
  - Case n even. Analogous to the previous case, we can show that ρ_i is not a shift of $KG(n,k)_{2-\mathrm{stab}}$, for $1 \leq i \leq \frac{n}{2}$. Now, for each $1 \leq i \leq \frac{n}{2}$, let $v_i = \{i+1, i+1+2.1, i+1+2.2, \ldots, i+1+2.(k-2), i-2\}$. Clearly, v_i is a 2-stable set, since $i+1+2.(k-2)$ and $i-2$ are not consecutive integers modulo $[n]$. So, v_i is a vertex of $KG(n,k)_{2-\mathrm{stab}}$ such that $\{i+1, i-2\} \subseteq v_i$. However, $\{i+1, i-2\} \subseteq \delta_i(v_i)$ which implies that $\{v_i, \delta_i(v_i)\}$ is not an edge of $KG(n,k)_{2-\mathrm{stab}}$. Thus, for $1 \leq i \leq \frac{n}{2}$, δ_i is not a shift of $KG(n,k)_{2-\mathrm{stab}}$.

□

Larose et al. [7] showed the following useful results:

Proposition 1 (Proposition 2.3 in [7]). *A graph G is hom-idempotent if and only if $G \leftrightarrow Cay(Aut(G^\bullet), S_{G^\bullet})$, where $Aut(G^\bullet)$ denotes the automorphism group of the core $G^\bullet$ of G and $S_{G^\bullet}$ denotes the set of all shifts of $G^\bullet$.*

Theorem 1 (Theorem 5.1 in [7]). *Let G be a χ-critical graph. Then G is weakly hom-idempotent if and only if it is hom-idempotent.*

Proposition 2. *Let $n \geq 2k+2$ and let G denotes the graph $KG(n,k)_{2-stab}$. Then, $G \not\to$ Cay($Aut(G), S_G$), where S_G are the shifts of G.*

Proof. We know that the automorphism group of the graph $KG(n,k)_{2-\text{stab}}$ is the dihedral group D_{2n} on $[n]$. Moreover, by Lemma 1, we known that the only two shifts of $KG(n,k)_{2-\text{stab}}$ are the circular permutations $\sigma^1 = (1,2,\ldots,n-1,n)$ and its inverse permutation $\sigma^{n-1} = (\sigma^1)^{-1} = (n,n-1,\ldots,2,1)$. Therefore, the Cayley graph $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ is a 2-regular graph, that is, each vertex (i.e. each automorphism in D_{2n}) has exactly two neighbors. In fact, notice that the vertex identity σ^0 has as neighbors the vertices σ^1 and σ^{n-1}, and as it's well known, any Cayley graph is vertex-transitive, and thus, any vertex in $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ has exactly two neighbors. So, $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ is a disjoint union of cycles, which implies that $2 \leq \chi(\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})) \leq 3$. Moreover, we known that $\chi(KG(n,k)_{2-\text{stab}}) = n-2k+2 \geq 4$, because $n \geq 2k+2$. Therefore $KG(n,k)_{2-\text{stab}} \not\to \text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$. □

As mentioned in the previous section, we know that any 2-stable Kneser graph is a core. Therefore, by Propositions 1 and 2, and by Theorem 1, we have the following result.

Theorem 2. *For any $n \geq 2k+2$, the 2-stable Kneser graphs $KG(n,k)_{2-stab}$ are not weakly hom-idempotent.*

3 s-stable Kneser graphs $KG(ks+1,k)_{s-\textbf{stab}}$

Let $\overline{G}$ denote the complement graph of the graph G, i.e. $\overline{G}$ has the same vertex set of G and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Let p be a positive integer. The pth power of a graph G, that we denoted by $G^{(p)}$, is the graph having the same vertex set as G and where two vertices are adjacent in $G^{(p)}$ if the distance between them in G is at most equal to p, where the distance of two vertices in a graph G is the number of edges on the shortest path connecting them.

Let $n \geq 2k$ be positive integers. The Cayley graphs $\text{Cay}(\mathbb{Z}_n, \{k, k+1, \ldots, n-k\})$, that we denoted by $G(n,k)$, are known as *circular graphs* [11, 6], where $\mathbb{Z}_n$ denote the cyclic group of order n. It is well known that the Kneser graph $KG(n,k)$ contains an induced subgraph isomorphic to $G(n,k)$. In fact, let $C(n,k)$ be the subgraph of $KG(n,k)$ obtained by restricting the vertex set of $KG(n,k)$ to the shifts modulo $[n]$ of the k-subset $\{1,2,\ldots,k\}$, that is, $\{1,2,\ldots,k\}, \{2,3,\ldots,k+1\}, \ldots, \{n,1,2,\ldots,k-1\}$. Define $\phi : G(n,k) \to C(n,k)$ by putting $\phi(u) = \{u+1, u+2, \ldots, u+k\}$ where the arithmetic operations are taken modulo $[n]$. Clearly, ϕ is a graph isomorphism. Notice also that the graph $G(n,k)$ is isomorphic to the graph $\overline{C_n^{(k-1)}}$, i.e. the complement graph of the $(k-1)$th power of a cycle C_n. Vince [11] has shown that $\chi(G(n,k)) = \lceil \frac{n}{k} \rceil$.

In the remaining of this section, we will always assume w.l.o.g. that any vertex $v = \{v_1, v_2, \ldots, v_k\}$ of the s-stable Kneser graph $KG(ks+1,k)_{s-\text{stab}}$ is such that $v_1 < v_2 < \ldots < v_k$, where $s, k \geq 2$. For $i \in [k-1]$, let $l_i(v) = v_{i+1} - v_i$ and $l_k(v) = v_1 + (ks+1) - v_k$. If C is the cycle on $ks+1$ points labeled by integers $1, 2, \ldots, ks+1$ in the clockwise direction and $v = \{v_1, v_2, \ldots, v_k\}$ is a vertex of the s-stable Kneser graph $KG(ks+1,k)_{s-\text{stab}}$, then $l_i(v)$ gives the distance in the clockwise direction between v_i and v_{i+1} in C.

Lemma 2. *Let $s, k \geq 2$ and let $v = \{v_1, v_2, \ldots, v_k\}$ be a vertex of $KG(ks+1,k)_{s-stab}$. Then, $l_i(v) \in \{s, s+1\}$ for all $i \in [k]$. Moreover, there exists exactly one $i' \in [k]$ such that $l_{i'}(v) = s+1$.*

Proof. By definition, $l_i(v) \geq s$ for any $i \in [k]$. The result follows from the fact that $\sum_{i=1}^{k} l_i(v) = ks+1$. □

Lemma 3. *Let $s, k \geq 2$. The number of vertices of the graph $KG(ks+1,k)_{s-stab}$ is equal to $ks+1$.*

Proof. Again, let C be the cycle on $ks+1$ points labeled by integers $1, 2, \ldots, ks+1$ in the clockwise direction. From Lemma 2, we have that each vertex of $KG(ks+1,k)_{s-\text{stab}}$ is uniquely determined by a clockwise circular interval of length $s+1$ in C. Trivially there exist $ks+1$ distinct clockwise circular intervals of length $s+1$ in C and the lemma holds. □

Proposition 3. *Let $s, k \geq 2$. Then, $G(ks+1,k) \simeq KG(ks+1,k)_{s-stab}$.*

Proof. Let C be a cycle on $ks+1$ points. We assume that the vertices of $G(ks+1,k)$ are disposed over C in clockwise increasing order from 0 to ks. By Lemmas 2 and 3, we are able to define the application $\phi : G(ks+1,k) \to KG(ks+1,k)_{s-\text{stab}}$ as follows: let u be a vertex of $G(ks+1,k)$ such that $u = jk+i$, where $0 \leq j \leq s-1$ and $0 \leq i \leq k-1$. Then, $\phi(u) = \{u_1, \ldots, u_k\}$ where,

$$u_r = \begin{cases} j+1+(r-1)s, & \text{if } 1 \leq r \leq k-i \\ j+2+(r-1)s, & \text{if } k-i+1 \leq r \leq k. \end{cases}$$

Finally, define $\phi(ks) = \{s+1, 2s+1, \ldots, ks+1\}$. It is not difficult to prove that ϕ is a bijective function. It remains to show that ϕ is indeed a graph isomorphism. Let u, v be two vertices in $C(ks+1,k)$. In the sequel, we assume that $v > u$. In fact, if $u > v$ we can always swap u and v. Let $u = jk+i$, where $0 \leq j \leq s-1$ and $0 \leq i \leq k-1$. Let $t = v-u$, where $1 \leq t \leq k-1$. By construction, $\phi(v) \setminus \{v_p : k-i-t+1 \leq p \leq k-i\} \subset \phi(u)$, where the arithmetic operations are taken modulo $[k]$. As $t \leq k-1$ then, we must have that $\phi(u) \cap \phi(v) \neq \emptyset$. Analogously, let $k(s-1)+2 \leq v-u \leq ks$ and let $t = ks+1-(v-u)$. Then, $\phi(u) \setminus \{u_p : k-i-t+1 \leq p \leq k-i\} \subset \phi(v)$, where the arithmetic operations are taken modulo $[k]$. Again, as $t \leq k-1$ then, we must have that $\phi(u) \cap \phi(v) \neq \emptyset$. Notice that, by the previous argument, if $v-u = k$ or $v-u = k(s-1)+1$ then, $\phi(u) \cap \phi(v) = \emptyset$. Therefore, let $t = v-u$, where $k+1 \leq t \leq k(s-1)$ and assume that there exists $r \in [k]$ such that $u_r \in \phi(u) \cap \phi(v)$. We consider the following cases:

- Case $1 \leq r \leq k-i$. As $u = jk+i$ then, $v = jk+i+xk+y$, where $0 \leq j \leq s-1$, $0 \leq i \leq k-1$, $1 \leq x \leq s-1$, and $0 \leq i+y \leq k-1$. If $u_r = v_{r'}$, with $1 \leq r' \leq k-(i+y)$ then, $u_r = j+1+(r-1)s = j+x+1+(r'-1)s = v_{r'}$ and thus $x = (r-r')s$. As $x \geq 1$ then, $r-r' > 0$ and thus $x \geq s$ which is a contradiction. If $k-(i+y)+1 \leq r' \leq k$ then, $j+1+(r-1)s = j+x+2+(r'-1)s$ which implies that $x = (r-r')s-1$. However, $r \leq k-i$ and $r' \geq k-i-y+1$ and so, $x \leq (y-1)s-1$. Moreover, as $x \geq 1$ then, $y \geq 2$ and thus $x = s-1$, but it implies that $t \geq k(s-1)+2$ which is a contradiction.

- Case $k-i+1 \leq r \leq k$. As the previous case, $v = (j+x)k+(i+y)$, with $1 \leq x \leq s-1$ and $0 \leq i+y \leq k-1$. If $u_r = j+2+(r-1)s = j+x+1+(r'-1)s = v_{r'}$, with $1 \leq r' \leq k-(i+y)$ then, $x = (r-r')s+1$ which implies that $x \geq (y+1)s+1 \geq s$, that is a contradiction. If $k-(i+y)+1 \leq r' \leq k$ then, $j+2+(r-1)s = j+x+2+(r'-1)s$. Then, $x = (r-r')s$ and as $x \geq 1$ then $r-r' > 0$, but it implies that $x \geq s$ which is again a contradiction.

Therefore, vertices u, v in $C(ks+1, k)$ are adjacent if and only if vertices $\phi(u), \phi(v)$ in $KG(ks+1, k)_{s-\text{stab}}$ are adjacent. □

Notice that if $n \geq 2r+3$ the automorphism group of the r^{th} power of a n-cycle, $C_n^{(r)}$, is the dihedral group D_{2n} (see Remark 1) and thus, $\text{AUT}(\overline{C_n^{(r)}})$ is also D_{2n}.

Remark 1. *For $r \geq 1$ and $n \geq 2r+3$ the automorphism group of $C_n^{(r)}$ is the dihedral group D_{2n}.*

In fact, it is not hard to see that D_{2n} is a subgroup of $\text{AUT}(C_n^{(r)})$. Let $\alpha \in \text{AUT}(C_n^{(r)})$ and let $N[i]$ denote the closed neighborhood of a vertex i in $C_n^{(r)}$. Assume that α sends two consecutive vertices i, j in the cycle to non consecutive vertices i', j' in the cycle, respectively. Observe that $|N[i] \cap N[j]| = 2r$, but $|N[i'] \cap N[j']| \leq 2r-1$ since there exists at least two vertices in $N[i'] \setminus N[j']$. Therefore, any automorphism of $C_n^{(r)}$ sends consecutive vertices to consecutive vertices. From this fact it is easy to see that any automorphism of $C_n^{(r)}$ belongs to D_{2n}.

Therefore, by Proposition 3, the automorphism group of the graph $KG(ks+1, k)_{s-\text{stab}}$ is the dihedral group $D_{2(ks+1)}$. However, the problem to compute the automorphism group of the graph $KG(n, k)_{s-\text{stab}}$ for $k \geq 2$, $s > 2$, and $n > ks+1$ is still open.

Another direct consequence of Proposition 3 is that $\chi(KG(ks+1, k)_{s-\text{stab}}) = s+1$. This fact has been found also by Meunier (see Proposition 1 in [9]).

Let Cay(A, S) be a Cayley graph. If $a^{-1}Sa = S$ for all $a \in A$, then $Cay(A, S)$ is called a *normal Cayley graph*.

Lemma 4 ([5])**.** *Any normal Cayley graph is hom-idempotent.*

Note that all Cayley graphs on abelian groups are normal, and thus hom-idempotents. In particular, the *circulant* graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). Therefore, by Proposition 3 and Lemma 4, we have the following result.

Theorem 3. *Let $s, k \geq 2$. Then, $KG(ks+1, k)_{s-stab}$ is hom-idempotent.*

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