The Erdös–Hajnal conjecture for paths and antipaths

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The Erdős-Hajnal Conjecture for Paths and Antipaths

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Abstract

We prove that for every k, there exists $c_k > 0$ such that every graph G on n vertices with no induced path P_k or its complement $\overline{P_k}$ contains a clique or a stable set of size n^{c_k}.

Keywords: Erdős-Hajnal, path, antipath, Ramsey

An *n-graph* is a graph on n vertices. For every vertex x, $N(x)$ denotes the neighborhood of x, that is the set of vertices y such that xy is an edge. The degree $\deg(x)$ is the size of $N(x)$. In this note, we only consider classes of graphs that are closed under induced subgraphs. Moreover a class $\mathcal{C}$ is *strict* if it does not contain all graphs. It is said to have the *(weak) Erdős-Hajnal property* if there exists some $c > 0$ such that every graph of $\mathcal{C}$ contains a clique or a stable set of size n^c where n is the size of G. The Erdős-Hajnal conjecture [8] asserts that every strict class of graphs has the Erdős-Hajnal property; see [3] for a survey. This fascinating question is open even for graphs not inducing a cycle of length five. When excluding a single graph H, Alon, Pach and Solymosi showed in [2] that it suffices to consider *prime* H, namely graphs without nontrivial modules (a *module* is a subset V' of vertices such that for every $x, y \in V'$, $N(x) \setminus V' = N(y) \setminus V'$). A natural approach is then to study classes of graphs with intermediate difficulty, hoping to get a proof scheme which could be extended. A natural prime candidate to forbid is certainly the path. Unfortunately, even excluding the path on five vertices seems already hard. Chudnovsky and Zwols studied the class $\mathcal{C}_k$ of graphs not inducing the path P_k on k vertices or its complement $\overline{P_k}$. They proved the Erdős-Hajnal property for P_5 and $\overline{P_6}$-free graphs [7]. This was extended for P_5 and $\overline{P_7}$-free graphs by Chudnovsky and Seymour [6]. Moreover structural results have been provided for $\mathcal{C}_5$ [4, 5]. We show in this note that for every fixed k, the class $\mathcal{C}_k$ has the Erdős-Hajnal property. An n-graph is an *ε-stable set* if it has at most $\varepsilon\binom{n}{2}$ edges. The complement of an ε-stable set is an *ε-clique*. Fox and Sudakov [11] proved the following:

Theorem 1 ([11]). *For every positive integer k and every $\varepsilon \in (0, 1/2)$, there exists $\delta > 0$ such that every n-graph G satisfies one of the following:*

- *G induces all graphs on k vertices.*
- *G contains an ε-stable set of size at least δn.*
- *G contains an ε-clique of size at least δn.*

Note that a stronger result was previously showed by Rödl [14] using Szemerédi's regularity lemma, but Fox and Sudakov's proof provides a much better quantitative estimate ($\delta = 2^{-ck(\log 1/\varepsilon)^2}$ for some constant c). They further conjecture that a polynomial estimate should hold, which would imply the Erdős-Hajnal conjecture.

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In a graph G, a *biclique of size* t is a (not necessarily induced) complete bipartite subgraph (X, Y) such that both $|X|, |Y| \geq t$. Observe that it does not require any condition inside X or inside Y. Erdős, Hajnal and Pach proved in [9] that for every strict class $\mathcal{C}$, there exists some $c > 0$ such that for every n-graph G in $\mathcal{C}$, G or its complement $\overline{G}$ contains a biclique of size n^c. This "half" version of the conjecture was improved to a "three quarter" version by Fox and Sudakov [10], where they show the existence of a polynomial size stable set or biclique. Following the notations of [12], a class $\mathcal{C}$ of graphs has the *strong Erdős-Hajnal property* if there exists a constant c such that for every n-graph G in $\mathcal{C}$, G or $\overline{G}$ contains a biclique of size cn. It was proved that having the strong Erdős-Hajnal property implies having the (weak) Erdős-Hajnal property:

Theorem 2 ([1, 12])**.** *If $\mathcal{C}$ is a class of graphs having the strong Erdős-Hajnal property, then $\mathcal{C}$ has the weak Erdős-Hajnal property.*

Proof. (sketch) Let c be the constant of the strong Erdős-Hajnal property, meaning that for every n-graph G in $\mathcal{C}$, G or $\overline{G}$ contains a biclique of size cn. Let $c' > 0$ be such that $c^{c'} \geq 1/2$. We prove by induction that every n-graph G in $\mathcal{C}$ induces a P_4-free graph of size $n^{c'}$. By our hypothesis on $\mathcal{C}$, there exists, say, a biclique (X, Y) of size cn in G. Applying the induction hypothesis inside both X and Y, we form a P_4-free graph on $2(cn)^{c'} \geq n^{c'}$ vertices. The Erdős-Hajnal property of $\mathcal{C}$ follows from the fact that every P_4-free $n^{c'}$-graph has a clique or a stable set of size at least $n^{c'/2}$. □

We now prove our main result. The key lemma is an adaptation of Gyárfás' proof of the χ-boundedness of P_k-free graphs, see [13].

Lemma 3. *For every $k \geq 2$, there exists $\varepsilon_k > 0$ and c_k (with $0 < c_k \leq 1/2$) such that every connected n-graph G with $n \geq 2$ satisfies one of the following:*

- *There exists a vertex of degree more than $\varepsilon_k n$.*
- *For every vertex v, G contains an induced P_k starting at v.*
- *The complement $\overline{G}$ of G contains a biclique of size $c_k n$.*

Proof. We proceed by induction on k. For $k = 2$, since G is connected, every vertex is the endpoint of an edge (that is, a P_2). Thus we can arbitrarily define $\varepsilon_2 = c_2 = 1/2$.

If $k > 2$, let $\varepsilon_k = \frac{\varepsilon_{k-1}}{(2+\varepsilon_{k-1})}$ and $c_k = \frac{c_{k-1}(1-\varepsilon_k)}{2}$. Let us assume that the first item is false. We will show that the second or the third item is true. Let v_1 be any vertex and $S = V(G) \setminus (N(v_1) \cup \{v_1\})$. The size s of S is at least $(1 - \varepsilon_k)n - 1$. If S have only *small* connected components, meaning of size at most $s/2$, then one can divide the connected components into two parts with at least $(s + 1)/4$ vertices each, and no edges between both parts. This gives in $\overline{G}$ a biclique of size $(s + 1)/4 \geq \frac{(1-\varepsilon_k)n}{4}$, thus of size at least $c_k n$ since $c_k \leq \frac{1-\varepsilon_k}{4}$. Otherwise, S has a *giant* connected component S', meaning of size s' more than $s/2$. Let v_2 be a vertex adjacent both to v_1 and to some vertex in S'. Observe that v_2 exists since G is connected. Consider now the graph G_2 induced by $S' \cup \{v_2\}$. The maximum degree in G_2 is still at most $\varepsilon_k n = \varepsilon_{k-1}(1-\varepsilon_k)n/2 \leq \varepsilon_{k-1}(s'+1)$. By the induction hypothesis, either the second or the third item is true for G_2 with parameter $k - 1$. The second item gives an induced P_{k-1} in G_2 starting at v_2, thus an induced P_k in G starting at v_1. The third item gives a biclique of size $c_{k-1}|G_2|$ in $\overline{G_2}$. Since $|G_2| = s' + 1 \geq \frac{1-\varepsilon_k}{2}n$, this gives a biclique of size at least $\frac{c_{k-1}(1-\varepsilon_k)}{2}n = c_k n$ and concludes the proof. □

Theorem 4. *For every $k \geq 2$, $\mathcal{C}_k$ has the strong Erdős-Hajnal property. Thus, by Theorem 2, the class $\mathcal{C}_k$ has the (weak) Erdős-Hajnal property.*

Proof. Let ε_k be as defined in Lemma 3 and $\varepsilon = \varepsilon_k/8 > 0$. By Theorem 1, there exists $\delta > 0$ such that every graph G not inducing P_k or $\overline{P_k}$ does contain an ε-stable set or an ε-clique of size at least δn. Free to consider the complement of G, we can assume that G contains an ε-stable set S_0 of size δn. We start by deleting in S_0 all the vertices with degree in S_0 at least $2\varepsilon s_0$ where s_0 is the size of S_0. Since the average degree in S_0

is at most εs_0, we do not delete more than half of the vertices. We call S the remaining subgraph which is a 4ε-stable set of size $s \geq \delta n/2$ with maximum degree less than $4\varepsilon s$.

Let G_S be the graph induced by S. Our goal is to find a constant c such that $\overline{G_S}$ have a biclique of size cs, which gives a biclique in $\overline{G}$ of size at least $c\delta n/2$ and concludes the proof. Assume first that G_S only has *small* connected components, meaning of size less than $s/2$. Then one can partition the connected components of G_S in order to get a biclique in $\overline{G_S}$ of size $s/4$. Otherwise, G_S has a connected component S' of size $s' \geq s/2$. The degree of every vertex in S' is at most $8\varepsilon s' = \varepsilon_k s'$, and S' does not contain any induced P_k since G does not. By Lemma 3, there exists a biclique of size $c_k s' \geq c_k s/2$ in the complement of the graph induced by S', thus in $\overline{G_S}$. □

[1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić and M. Sharir, Crossing patterns of semi-algebraic sets. *Journal of Combinatorial Theory Series A*, 111(2):310–326, 2005.

[2] N. Alon, J. Pach, and J. Solymosi, Ramsey-type theorems with forbidden subgraphs. *Combinatorica*, 21:155–170, 2001.

[3] M. Chudnovsky, The Erdős-Hajnal Conjecture - A Survey. *Journal of Graph Theory*, 75:178–190, 2014.

[4] M. Chudnovsky and P. Maceli, Simplicial vertices in graphs with no induced four-edge path or four-edge antipath, and the H_6 conjecture. *To appear.*

[5] M. Chudnovsky, P. Maceli and I. Penev, Excluding four-edge path and their complements. *Submitted for publication.*

[6] M. Chudnovsky and P. Seymour, Excluding paths and antipaths. *To appear.*

[7] M. Chudnovsky and Y. Zwols, Large cliques or stable sets in graphs with no four-edge path and no five-edge path in the complement. *Journal of Graph Theory*, 70:449–472, 2012.

[8] P. Erdős and A. Hajnal, Ramsey-type theorems. *Discrete Applied Mathematics*, 25:37–52, 1989.

[9] P. Erdős, A. Hajnal and J. Pach, Ramsey-type theorem for bipartite graphs. *Geombinatorics*, 10:64–68, 2000.

[10] J. Fox and B. Sudakov, Density theorems for bipartite graphs and related Ramsey-type results. *Combinatorica*, 29:153–196, 2009.

[11] J. Fox and B. Sudakov, Induced Ramsey-type theorems. *Advances in Mathematics*, 219:1771–1800, 2008.

[12] J. Fox and J. Pach, Erdős-Hajnal-type results on intersection patterns of geometric objects. *Horizon of Combinatorics* (G.O.H. Katona et al., eds.), Bolyai Society Studies in Mathematics, Springer, 79–103, 2008.

[13] A. Gyárfás, Problems from the world surrounding perfect graphs. *Zastos. Mat.*, 413–441, 1987.

[14] V. Rödl, On universality of graphs with uniformly distributed edges. *Discrete Mathematics*, 59:125–134, 1986.