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# The $\mathbb{Q}$-curve Construction for Endomorphism-Accelerated Elliptic Curves

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**Abstract.** We give a detailed account of the use of $\mathbb{Q}$-curve reductions to construct elliptic curves over $\mathbb{F}_{p^2}$ with efficiently computable endomorphisms, which can be used to accelerate elliptic curve-based cryptosystems in the same way as Gallant–Lambert–Vanstone (GLV) and Galbraith–Lin–Scott (GLS) endomorphisms. Like GLS (which is a degenerate case of our construction), we offer the advantage over GLV of selecting from a much wider range of curves, and thus finding secure group orders when $p$ is fixed for efficient implementation. Unlike GLS, we also offer the possibility of constructing twist-secure curves. We construct several one-parameter families of elliptic curves over $\mathbb{F}_{p^2}$ equipped with efficient endomorphisms for every $p > 3$, and exhibit examples of twist-secure curves over $\mathbb{F}_{p^2}$ for the efficient Mersenne prime $p = 2^{127} - 1$.

**Keywords:** Elliptic curve cryptography; endomorphism; exponentiation; GLS; GLV; $\mathbb{Q}$-curves; scalar decomposition; scalar multiplication

*This is an extended treatment of the curves and techniques introduced in [44], including two more families of curves, more efficient scalar decompositions, and more detail on exceptional CM curves and 4-dimensional decompositions.*

## 1 Introduction

Let $\mathcal{E}$ be an elliptic curve over a finite field $\mathbb{F}_q$, and let $\mathcal{G} \subseteq \mathcal{E}(\mathbb{F}_q)$ be a cyclic subgroup of prime order $N$. When implementing cryptographic protocols in $\mathcal{G}$, the fundamental operation is *scalar multiplication* (or *exponentiation*):

$$\text{Given } P \text{ in } \mathcal{G} \text{ and } m \text{ in } \mathbb{Z}, \text{ compute } [m]P := \underbrace{P \oplus \cdots \oplus P}_{m \text{ times}}.$$

The literature on general scalar multiplication algorithms is vast, and we will not explore it in detail here (see [15, §2.8,§11.2] and [8, Chapter 9] for introductions to exponentiation and multiexponentiation algorithms). For our purposes, it suffices to note that the dominant factor in scalar multiplication time using conventional algorithms is the bitlength $\lceil \log_2 |m| \rceil$ of $m$. As a basic

example, we may compute $[m]P$ using a variant of the classic binary method, which requires at most $\lceil \log_2 |m| \rceil$ doublings and (in the worst case) about as many addings in $\mathcal{G}$ (in general, $\log_2 |m| \sim \log_2 N \sim \log_2 q$).

Suppose $\mathcal{E}$ is equipped with an efficient $\mathbb{F}_q$-endomorphism $\psi$. By *efficient*, we mean that we can compute the image $\psi(P)$ of any point $P$ in $\mathcal{E}(\mathbb{F}_q)$ for the cost of $O(1)$ operations in $\mathbb{F}_q$. In practice, we want this to cost no more than a few doublings in $\mathcal{E}(\mathbb{F}_q)$.

Assume $\psi(\mathcal{G}) \subseteq \mathcal{G}$, or equivalently, that $\psi$ restricts to an endomorphism of $\mathcal{G}$.[1] Now $\mathcal{G}$ is a finite cyclic group, isomorphic to $\mathbb{Z}/N\mathbb{Z}$, and every endomorphism of $\mathbb{Z}/N\mathbb{Z}$ is just an integer multiplication modulo $N$. Hence, $\psi$ acts on $\mathcal{G}$ as multiplication by some integer eigenvalue $\lambda_\psi$: that is,

$$\psi|_{\mathcal{G}} = [\lambda_\psi]_{\mathcal{G}} \qquad \text{for some } -N/2 < \lambda_\psi \leq N/2 \ .$$

The eigenvalue $\lambda_\psi$ is a root of the characteristic polynomial of $\psi$ in $\mathbb{Z}/N\mathbb{Z}$.

Returning to the problem of scalar multiplication: we want to compute $[m]P$. Rewriting $m$ as

$$m = a + b\lambda_\psi \pmod{N}$$

for some $a$ and $b$, we can compute $[m]P$ using the relation

$$[m]P = [a]P \oplus [b\lambda_\psi]P = [a]P \oplus [b]\psi(P)$$

and a 2-dimensional multiexponentation such as Straus's algorithm [47], which has a loop length of $\log_2 \|(a,b)\|_\infty$: that is, $\log_2 \|(a,b)\|_\infty$ doubles and at most as many adds[2] (recall $\|(a,b)\|_\infty = \max(|a|,|b|)$). If $|\lambda_\psi|$ is not too small, then we can easily find $(a,b)$ such that $\log_2 \|(a,b)\|_\infty$ is roughly $\frac{1}{2}\log_2 N$ (we remove the "If" and the "roughly" for our $\psi$ in §4.)

The endomorphism therefore lets us replace conventional $\log_2 N$-bit scalar multiplications with $(\frac{1}{2}\log_2 N)$-bit multiexponentiations. In basic binary methods this means halving the loop length, cutting the number of doublings in half.

Of course, in practice we are not halving the execution time. The precise speedup depends on a variety of factors, including the choice of exponentiation and multiexponentiation algorithms, the cost of computing $\psi$, and the cost of doublings and addings in terms of bit operations—to say nothing of the cryptographic protocol, which may prohibit some other conventional speedups. For example: in [16], Galbraith, Lin, and Scott report experiments where cryptographic operations on GLS curves required between 70% and 83% of the time

[1] The assumption is satisfied, almost by default, in the context of classical discrete log-based cryptosystems. If $\psi(\mathcal{G}) \not\subseteq \mathcal{G}$, then $\mathcal{E}[N](\mathbb{F}_q) = \mathcal{G} + \psi(\mathcal{G}) \cong (\mathbb{Z}/N\mathbb{Z})^2$, so $N^2 \mid \#\mathcal{E}(\mathbb{F}_q)$ and $N \mid q-1$; such $\mathcal{E}$ are cryptographically inefficient, and discrete logs in $\mathcal{G}$ are vulnerable to the Menezes–Okamoto–Vanstone and Frey–Rück reductions [33, 14]. However, the assumption should be verified carefully in the context of pairing-based cryptography, where $\mathcal{G}$ and $\psi$ with $\psi(\mathcal{G}) \not\subseteq \mathcal{G}$ arise naturally.

[2] Straus's algorithm serves as a simple and convenient reference example here: like all multiexponentiation algorithms, its running time depends essentially on $\log_2 \|(a,b)\|_\infty$. It is not the fastest multiexponentiation, nor is it uniform or constant-time; as such, it is not recommended for real-world implementations.

required for the previous best practice curves—with the variation depending on the architecture, the underlying curve arithmetic, and the protocol in question.

To put this technique into practice, we need a source of cryptographic elliptic curves equipped with efficient endomorphisms. In the large characteristic case[3], there are two archetypal constructions:

1. The classic *Gallant–Lambert–Vanstone* (GLV) construction [17] uses elliptic curves with explicit complex multiplication (CM) by quadratic orders with tiny discriminants. These curves can be found by reducing CM curves over number fields modulo suitable primes.
2. The more recent *Galbraith–Lin–Scott* (GLS) construction [16]. Here, curves over $\mathbb{F}_p$ are viewed over $\mathbb{F}_{p^2}$; the $p$-power sub-Frobenius induces an extremely efficient endomorphism on the quadratic twist (which can have prime order).

GLV and GLS have been combined to give higher-dimensional variants for elliptic curves ([29], [50]), and extended to hyperelliptic curves ([4], [27], [41], [48]).

*New endomorphisms from $\mathbb{Q}$-curves.* This work develops a new source of elliptic curves over $\mathbb{F}_{p^2}$ with efficient endomorphisms: reductions of quadratic $\mathbb{Q}$-curves.

**Definition 1.** *A* quadratic $\mathbb{Q}$-curve of degree $d$ *is an elliptic curve* $\mathcal{E}$ without *CM, defined over a quadratic number field* $K$*, such that there exists an isogeny of degree* $d$ *from* $\mathcal{E}$ *to its Galois conjugate* ${}^{\sigma}\mathcal{E}$ *(formed by applying* $\sigma$ *to the coefficients of the defining equation of* $\mathcal{E}$*), where* $\langle\sigma\rangle = \mathrm{Gal}(K/\mathbb{Q})$.

$\mathbb{Q}$-curves are well-established objects of interest in number theory, where they have formed a natural setting for generalizations of the Modularity Theorem. We recommend Ellenberg [12] for an excellent introduction to the theory.

Our application of quadratic $\mathbb{Q}$-curves is somewhat more prosaic: given a $d$-isogeny $\widetilde{\mathcal{E}} \to {}^{\sigma}\widetilde{\mathcal{E}}$ over a quadratic field, we reduce modulo an inert prime $p$ to obtain an isogeny $\mathcal{E} \to {}^{\sigma}\mathcal{E}$ over $\mathbb{F}_{p^2}$. We then exploit the fact that the $p$-power Frobenius isogeny maps ${}^{\sigma}\mathcal{E}$ back onto $\mathcal{E}$; composing with the reduced $d$-isogeny, we obtain an endomorphism of $\mathcal{E}$ of degree $dp$. For efficiency, $d$ must be small; happily, for small values of $d$, Hasegawa has written down universal one-parameter families of $\mathbb{Q}$-curves [20]. We thus obtain one-parameter families of elliptic curves over $\mathbb{F}_{p^2}$ equipped with efficient non-integer endomorphisms.[4]

For concrete examples, we consider $d = 2, 3, 5$, and 7 in §5, §6, §7, and §8, respectively. We define our curves in short Weierstrass form for maximum generality and flexibility, but we also give isomorphisms to Montgomery, twisted Edwards, and Doche–Icart–Kohel models where appropriate.

[3] We are primarily interested in the large characteristic case, where $q = p$ or $p^2$, so we do not discuss $\tau$-adic or Frobenius expansion-style techniques (see [42], [28, §5]).

[4] While our curves are defined over an extension field, the extension degree is only 2, so Weil descent attacks offer no advantage when solving DLP instances (cf. [16, §9]).

*Comparison with GLV.* Like GLV, our method involves reducing curves defined over number fields to obtain curves over finite fields with explicit CM. However, we emphasise a profound difference: in our method, the curves over number fields generally do not have CM themselves.

GLV curves are necessarily isolated examples—and the really useful examples are extremely limited in number (cf. [29, App. A]). The scarcity of GLV curves is their Achilles' heel: as noted in [16], if $p$ is fixed then there is no guarantee that there will exist a GLV curve with prime (or almost-prime) order over $\mathbb{F}_p$. While we discuss this phenomenon further in §9, it is instructive to consider the example discussed in [16, §1]: the most efficient GLV curves have CM discriminants $-3$ and $-4$. If we are working at the 128-bit security level, then taking $p = 2^{255} - 19$ allows particularly fast arithmetic in $\mathbb{F}_p$. But the largest prime factor of the order of a curve over $\mathbb{F}_p$ with CM discriminant $-4$ (resp. $-3$) has 239 (resp. 230) bits: using these curves wastes 9 (resp. 13) potential bits of security. In fact, we are lucky with $-3$ and $-4$: for all of the other discriminants offering endomorphisms of degree at most 3, we can do no better than a 95-bit prime factor, which represents a catastrophic 80-bit loss of relative security.

In contrast, our construction yields true families of curves, covering $\sim p$ isomorphism classes over $\mathbb{F}_{p^2}$ for any choice of $p$. This gives us a vastly higher probability of finding secure curves over practically important fields.

*Comparison with GLS.* Like GLS, we construct curves over $\mathbb{F}_{p^2}$ equipped with an inseparable endomorphism. And like GLS, each of our families offer around $p$ distinct isomorphism classes of curves, making it easy to find secure group orders when $p$ is fixed.

But unlike GLS, our curves have $j$-invariants in $\mathbb{F}_{p^2}$ and not $\mathbb{F}_p$: they are not isomorphic to, or twists of, subfield curves. This allows us to find twist-secure curves: that is, curves with secure orders whose twists also have secure orders.

Twist-security is crucial in many elliptic curve-based protocols (including [23], [24], [5], [31], and [7]); it is particularly important in modern $x$-coordinate-only ECDH implementations, such as Bernstein's `Curve25519` software [2] (which anticipates the kind of fault attack detailed in [13]). GLS curves cannot be used in any of these constructions.

As we will see in §3, our construction degenerates to GLS when $d = 1$. Our construction is therefore a sort of generalized GLS—though it is not the higher-degree generalization anticipated by Galbraith, Lin, and Scott themselves, which composes a $p$-power sub-Frobenius endomorphism with a non-rational separable isogeny and its dual isogeny (cf. [16, Theorem 1]).

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## 2 Notation and Elementary Constructions

Throughout, we work over fields of characteristic not 2 or 3. Let

$$\mathcal{E} : y^2 = x^3 + Ax + B$$

be an elliptic curve over such a field $K$. (Recall that for an equation in this form to define an elliptic curve, the *discriminant* $-16(4A^3+27B^2)$ must be nonzero.)

*Galois conjugates.* For every automorphism $\sigma$ of $\overline{K}$, we have a conjugate curve

$$^{\sigma}\mathcal{E} : y^2 = x^3 + (^{\sigma}A)x + (^{\sigma}B) \ .$$

If $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ is an isogeny, then we obtain a conjugate isogeny $^{\sigma}\phi : {}^{\sigma}\mathcal{E}_1 \to {}^{\sigma}\mathcal{E}_2$ by applying $\sigma$ to the defining equations of $\phi$, $\mathcal{E}_1$, and $\mathcal{E}_2$. We write $(p)$ for the $p$-th powering automorphism of $\overline{\mathbb{F}}_p$. We note that $(p)$ is trivial to compute on $\mathbb{F}_{p^2} = \mathbb{F}_p(\sqrt{\Delta})$, since $^{(p)}(a + b\sqrt{\Delta}) = a - b\sqrt{\Delta}$ for all $a$ and $b$ in $\mathbb{F}_p$.

*Quadratic twists and twisted endomorphisms.* For every nonzero $\lambda$ in $\overline{K}$, the quadratic *twist* of $\mathcal{E}$ by $\lambda$ is the curve over $K(\lambda^2)$ defined by

$$\mathcal{E}^{\lambda} : y^2 = x^3 + \lambda^4 Ax + \lambda^6 B \ .$$

The twisting isomorphism $\delta(\lambda) : \mathcal{E} \to \mathcal{E}^{\lambda}$ is defined over $K(\lambda)$ by

$$\delta(\lambda) : (x, y) \longmapsto (\lambda^2 x, \lambda^3 y) \ .$$

Observe that $\delta(\lambda_1)\delta(\lambda_2) = \delta(\lambda_1\lambda_2)$ for any $\lambda_1, \lambda_2$; and $\delta(-\lambda) = -\delta(\lambda)$. For every $K$-endomorphism $\psi$ of $\mathcal{E}$, there is a twisted $K(\lambda^2)$-endomorphism

$$\psi^{\lambda} := \delta(\lambda)\psi\delta(\lambda^{-1}) \ \text{ in } \ \mathrm{End}(\mathcal{E}^{\lambda}) \ .$$

It is important to distinguish conjugates (marked by left-superscripts) from twists (marked by right-superscripts); note that $^{\sigma}(\mathcal{E}^{\lambda}) = (^{\sigma}\mathcal{E})^{^{\sigma}\lambda}$ for all $\lambda$ and $\sigma$.

*Quadratic twists over finite fields.* If $\mu$ is a nonsquare in $K = \mathbb{F}_q$, then $\mathcal{E}^{\sqrt{\mu}}$ is a quadratic twist of $\mathcal{E}$. But $\mathcal{E}^{\sqrt{\mu_1}}$ and $\mathcal{E}^{\sqrt{\mu_2}}$ are $\mathbb{F}_q$-isomorphic for all nonsquares $\mu_1$ and $\mu_2$ in $\mathbb{F}_q$ (the isomorphism $\delta(\sqrt{\mu_1/\mu_2})$ is defined over $\mathbb{F}_q$ because $\mu_1/\mu_2$ must be a square); so, up to $\mathbb{F}_q$-isomorphism, it makes sense to speak of *the* quadratic twist. We let $\mathcal{E}'$ denote the quadratic twist when the choice of nonsquare is not important. Similarly, if $\psi$ is an $\mathbb{F}_q$-endomorphism of $\mathcal{E}$, then $\psi'$ denotes the corresponding twisted $\mathbb{F}_q$-endomorphism of $\mathcal{E}'$.

*Traces and cardinalities.* If $K = \mathbb{F}_q$, then $\pi_{\mathcal{E}}$ denotes the $q$-power Frobenius endomorphism of $\mathcal{E}$. The characteristic polynomial of $\pi_{\mathcal{E}}$ has the form

$$\chi_{\mathcal{E}}(T) = T^2 - t_{\mathcal{E}}T + q;$$

the *trace* $t_{\mathcal{E}}$ satisfies the Hasse bound $|t_{\mathcal{E}}| \le 2\sqrt{q}$. Recall $\#\mathcal{E}(\mathbb{F}_q) = q + 1 - t_{\mathcal{E}}$ and $t_{\mathcal{E}'} = -t_{\mathcal{E}}$, so

$$\#\mathcal{E}(\mathbb{F}_q) + \#\mathcal{E}'(\mathbb{F}_q) = 2(q+1) \ . \tag{1}$$

*Explicit isogenies.* Let $\mathcal{S} \subset \mathcal{E}$ be a finite subgroup defined over $K$; Vélu's formulæ ([26, §2.4], [49]) compute the explicit (normalized) quotient isogeny

$$\phi : \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{S} : y^2 = x^3 + A_{\mathcal{S}} x + B_{\mathcal{S}} \ ,$$

mapping $(x, y)$ to $(\phi_x(x), y\phi'_x(x))$ for some $\phi_x$ in $K(x)$. We will need explicit formulæ for the cases $\#\mathcal{S} = 2, 3, 5$, and 7. If $\mathcal{S} = \{0, (\alpha, 0)\}$ has order 2, then

$$A_{\mathcal{S}} = -4A - 15\alpha^2 \ , \quad B_{\mathcal{S}} = B - 7\alpha(3\alpha^2 + A) \ , \quad \text{and} \quad \phi_x(x) = x + \tfrac{3\alpha^2 + A}{x - \alpha} \ . \tag{2}$$

If $\mathcal{S}$ has odd order $d = 2e + 1$, then it is defined by a kernel polynomial $F(x) = \sum_{i=0}^{e} f_i x^{e-i}$ (so $F(x(P)) = 0$ if and only if $P$ is in $\mathcal{S} \setminus \{0\}$); and then

$$A_{\mathcal{S}} = (1 - 10e)A - 30(f_1/f_0)^2 + 60 f_2/f_0 \ , \tag{3}$$

$$B_{\mathcal{S}} = (1 - 28e)B + 28 f_1/f_0 + 70(f_1/f_0)^3 - 210 f_1 f_2/f_0^2 + 210 f_3/f_0 \ , \tag{4}$$

and

$$\phi_x(x) = (2e + 1)x + 2\tfrac{f_1}{f_0} - 4(x^3 + Ax + B)\big(\tfrac{F'(x)}{F(x)}\big)' - 2(3x^2 + A)\tfrac{F'(x)}{F(x)} \ . \tag{5}$$

*Legendre symbols.* The Legendre symbol $\big(n/p\big)$ is defined to be 1 if $n$ is a square mod $p$, $-1$ if $n$ is not a square mod $p$, and 0 if $p$ divides $n$.

*Reduced lattice bases.* We work exclusively with the infinity norm $\| \cdot \|_\infty$ in this article (recall $\|(a, b)\|_\infty := \max(|a|, |b|)$). An ordered basis $[\mathbf{e}_1, \mathbf{e}_2]$ of a lattice in $\mathbb{Z}^2$ is *reduced* if

$$\|\mathbf{e}_1\|_\infty \le \|\mathbf{e}_2\|_\infty \le \|\mathbf{e}_1 - \mathbf{e}_2\|_\infty \le \|\mathbf{e}_1 + \mathbf{e}_2\|_\infty \ ; \tag{6}$$

a reduced basis has minimal length with respect to $\| \cdot \|_\infty$ (see [22]).

## 3 Quadratic $\mathbb{Q}$-curves and their Reductions

Suppose $\widetilde{\mathcal{E}}/\mathbb{Q}(\sqrt{\Delta})$ is a quadratic $\mathbb{Q}$-curve of prime degree $d$ (as in Definition 1), where $\Delta$ is a discriminant prime to $d$, and let $\widetilde{\phi} : \widetilde{\mathcal{E}} \to {}^{\sigma}\widetilde{\mathcal{E}}$ be the corresponding $d$-isogeny (where $\sigma$ is the conjugation of $\mathbb{Q}(\sqrt{\Delta})$ over $\mathbb{Q}$). In general, $\widetilde{\phi}$ is only defined over a quadratic extension $\mathbb{Q}(\sqrt{\Delta}, \gamma)$ of $\mathbb{Q}(\sqrt{\Delta})$ (cf. [18, Prop. 3.1]), but we can always reduce to the case where $\gamma = \sqrt{\pm d}$ (see [18, remark p. 385]). Indeed, the $\mathbb{Q}$-curves of degree $d$ that we treat below all have $\gamma = \sqrt{-d}$; so to simplify matters, from now on we will

*Assume $\widetilde{\phi}$ is defined over $\mathbb{Q}(\sqrt{\Delta}, \sqrt{-d})$.*

Let $p$ be a prime inert in $\mathbb{Q}(\sqrt{\Delta})$ (equivalently, $\Delta$ is not a square in $\mathbb{F}_p$), of good reduction for $\widetilde{\mathcal{E}}$ and prime to $d$. If $\mathcal{O}$ is the ring of integers of $\mathbb{Q}(\sqrt{\Delta})$, then

$$\mathbb{F}_{p^2} = \mathcal{O}/(p) = \mathbb{F}_p(\sqrt{\Delta}) \ .$$

Looking at the Galois groups of our fields, we have a series of injections

$$\langle (p) \rangle = \mathrm{Gal}(\mathbb{F}_p(\sqrt{\Delta})/\mathbb{F}_p) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q}) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\sqrt{\Delta}, \sqrt{-d})/\mathbb{Q}) \ .$$

The image of $(p)$ in $\mathrm{Gal}(\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q})$ is $\sigma$, because $p$ is inert in $\mathbb{Q}(\sqrt{\Delta})$. We extend $\sigma$ to the automorphism of $\mathbb{Q}(\sqrt{\Delta}, \sqrt{-d})$ that is the image of $(p)$: that is,

$$^{\sigma}\big(\alpha + \beta\sqrt{\Delta} + \gamma\sqrt{-d} + \delta\sqrt{-d\Delta}\big) = \alpha - \beta\sqrt{\Delta} + \big(-d/p\big)\,\big(\gamma\sqrt{-d} - \delta\sqrt{-d\Delta}\big) \quad (7)$$

for all $\alpha, \beta, \gamma$, and $\delta \in \mathbb{Q}$.

Now let $\mathcal{E}/\mathbb{F}_{p^2}$ be the reduction modulo $p$ of $\widetilde{\mathcal{E}}$. The curve $^{\sigma}\widetilde{\mathcal{E}}$ reduces to $^{(p)}\mathcal{E}$, while the $d$-isogeny $\widetilde{\phi} : \widetilde{\mathcal{E}} \to {}^{\sigma}\widetilde{\mathcal{E}}$ reduces to a $d$-isogeny $\phi : \mathcal{E} \to {}^{(p)}\mathcal{E}$ over $\mathbb{F}_{p^2}$.

Applying $\sigma$ to $\widetilde{\phi}$, we obtain a second $d$-isogeny $^{\sigma}\widetilde{\phi} : {}^{\sigma}\widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}$ travelling in the opposite direction, which reduces mod $p$ to a conjugate isogeny $^{(p)}\phi : {}^{(p)}\mathcal{E} \to \mathcal{E}$ defined over $\mathbb{F}_{p^2}$. Composing $^{\sigma}\widetilde{\phi}$ with $\widetilde{\phi}$ yields endomorphisms $^{\sigma}\widetilde{\phi} \circ \widetilde{\phi}$ of $\widetilde{\mathcal{E}}$ and $\widetilde{\phi} \circ {}^{\sigma}\widetilde{\phi}$ of $^{\sigma}\widetilde{\mathcal{E}}$, each of degree $d^2$. But (by definition) $\widetilde{\mathcal{E}}$ and $^{\sigma}\widetilde{\mathcal{E}}$ do not have CM, so all of their endomorphisms are integer multiplications; and since the only integer multiplications of degree $d^2$ are $[d]$ and $[-d]$, we conclude that

$$^{\sigma}\widetilde{\phi} \circ \widetilde{\phi} = [\epsilon_p d]_{\widetilde{\mathcal{E}}} \quad \text{and} \quad \widetilde{\phi} \circ {}^{\sigma}\widetilde{\phi} = [\epsilon_p d]_{^{\sigma}\widetilde{\mathcal{E}}} \ , \quad \text{where} \quad \epsilon_p \in \{\pm 1\} \ .$$

Technically, $^{\sigma}\widetilde{\phi}$ and $^{(p)}\phi$ are—*up to sign*—the dual isogenies of $\widetilde{\phi}$ and $\phi$, respectively. The sign $\epsilon_p$ depends on $p$: if $\tau$ is the extension of $\sigma$ to $\mathbb{Q}(\sqrt{\Delta}, \sqrt{-d})$ that is *not* the image of $(p)$, then $^{\tau}\widetilde{\phi} \circ \widetilde{\phi} = [-\epsilon_p d]_{\widetilde{\mathcal{E}}}$. Reducing modulo $p$, we see that

$$^{(p)}\phi \circ \phi = [\epsilon_p d]_{\mathcal{E}} \quad \text{and} \quad \phi \circ {}^{(p)}\phi = [\epsilon_p d]_{^{(p)}\mathcal{E}} \ .$$

The map $(x, y) \mapsto (x^p, y^p)$ defines $p$-isogenies

$$\pi_p : {}^{(p)}\mathcal{E} \longrightarrow \mathcal{E} \quad \text{and} \quad {}^{(p)}\pi_p : \mathcal{E} \longrightarrow {}^{(p)}\mathcal{E} \ .$$

Observe that $^{(p)}\pi_p \circ \pi_p = \pi_{\mathcal{E}}$ and $\pi_p \circ {}^{(p)}\pi_p = \pi_{^{(p)}\mathcal{E}}$. Composing $\pi_p$ with $\phi$ yields a degree-$pd$ endomorphism

$$\psi := \pi_p \circ \phi \in \mathrm{End}(\mathcal{E}) \ .$$

We also obtain an $\mathbb{F}_{p^2}$-endomorphism $\psi'$ on the quadratic twist $\mathcal{E}'$.

If $d$ is very small, then $\psi$ and $\psi'$ are efficient because $\phi$ is defined by polynomials of degree about $d$, and $\pi_p$ acts as a simple conjugation on coordinates in $\mathbb{F}_{p^2}$ (as in Eq. (7)). In this article we concentrate on prime $d \leq 7$.

**Theorem 1.** *The endomorphisms $\psi$ and $\psi'$ satisfy*

$$\psi^2 = [\epsilon_p d]\pi_{\mathcal{E}} \qquad \textit{and} \qquad (\psi')^2 = [-\epsilon_p d]\pi_{\mathcal{E}'} \ ,$$

*respectively. There exists an integer $r$ satisfying*[5]

$$dr^2 = 2p + \epsilon_p t_{\mathcal{E}} \quad (8)$$

[5] We warn the reader that the integer $r$ here corresponds to $\epsilon_p r$ in [44].

*such that*

$$[r]\psi = [p] + \epsilon_p \pi_{\mathcal{E}} \qquad \textit{and} \qquad [r]\psi' = [p] - \epsilon_p \pi_{\mathcal{E}'} \ ; \tag{9}$$

*the characteristic polynomial of both $\psi$ and $\psi'$ is*

$$P_\psi(T) = P_{\psi'}(T) = T^2 - rdT + dp \ .$$

*Proof.* Clearly $\pi_p \circ \phi = {}^{(p)}\phi \circ {}^{(p)}\pi_p$, so

$$\psi^2 = \pi_p \phi \pi_p \phi = \pi_p \phi^{(p)} \phi^{(p)} \pi_p = \pi_p [\epsilon_p d]^{(p)} \pi_p = [\epsilon_p d] \pi_p {}^{(p)} \pi_p = [\epsilon_p d] \pi_{\mathcal{E}} \ .$$

Similarly, choosing a nonsquare $\mu$ in $\mathbb{F}_{p^2}$, so $\mathcal{E}' = \mathcal{E}^{\sqrt{\mu}}$ and $\psi' = \psi^{\sqrt{\mu}}$, we find

$$(\psi')^2 = \delta(\mu^{\frac{1}{2}})\psi^2 \delta(\mu^{-\frac{1}{2}}) = \delta(\mu^{\frac{1}{2}(1-p^2)})[\epsilon_p d]\pi_{\mathcal{E}'} = \delta(-1)[\epsilon_p d]\pi_{\mathcal{E}'} = [-\epsilon_p d]\pi_{\mathcal{E}'} \ .$$

The degree of $\psi$ (and $\psi'$) is $dp$, so both have a characteristic polynomial in the form $P_\psi(T) = T^2 - aT + dp$ for some integer $a$. Hence,

$$[a]\psi = \psi^2 + [dp] = [\epsilon_p d]\pi + [dp] \ . \tag{10}$$

Squaring both sides, replacing $\psi^2$ with $[\epsilon_p d]\pi$, and then factoring out $[\epsilon_p d]\pi$, we find $a^2 = 2dp + d\epsilon_p t_{\mathcal{E}}$. It follows that $d \mid a^2$; but $d$ is squarefree, so $a = dr$ for some integer $r$, whence Eq. (8) and the characteristic polynomial. Putting $a = dr$ in Eq. (10) yields $[r]\psi = [p] + \epsilon_p \pi$; a similar argument for $\psi'$, using $\psi'^2 = [-\epsilon_p d]\pi_{\mathcal{E}'}$, yields $[r]\psi' = [p] - \epsilon_p \pi$, completing Eq. (9). □

**Corollary 1.** *The curves $\mathcal{E}$ and $\mathcal{E}'$ are ordinary if and only if $r \neq 0$; and then*

$$|r| = [\mathbb{Z}[\psi] : \mathbb{Z}[\pi_{\mathcal{E}}]] = [\mathbb{Z}[\psi'] : \mathbb{Z}[\pi_{\mathcal{E}'}]] \ .$$

*Proof.* The curves $\mathcal{E}$ and $\mathcal{E}'$ are ordinary (not supersingular) if and only if $p \nmid t_{\mathcal{E}}$, if and only if $p \nmid r$ (using Eq. (8) and $p \nmid d$). But the Hasse bound gives $|t_{\mathcal{E}}| \leq 2p$, so $|r| \leq 2\sqrt{p/d}$; the only $r$ in this interval divisible by $p$ is 0, proving the first claim. If $\mathcal{E}$ is ordinary, then $\mathbb{Z}[\pi_{\mathcal{E}}]$ and $\mathbb{Z}[\psi]$ are quadratic imaginary orders of discriminant $d^2r^2 - 4dp$ and ${t_{\mathcal{E}}}^2 - 4p^2 = r^2(d^2r^2 - 4dp)$, respectively, so $|r|$ is the conductor of $\mathbb{Z}[\pi'_{\mathcal{E}}]$ in $\mathbb{Z}[\psi]$. The same holds for $\psi'$ and $\pi_{\mathcal{E}'}$ on $\mathcal{E}'$. □

Equation (8) relates $r$ to the orders of $\mathcal{E}$ and $\mathcal{E}'$: we find

$$\#\mathcal{E}(\mathbb{F}_{p^2}) = (p + \epsilon_p)^2 - \epsilon_p d r^2 \quad \text{and} \quad \#\mathcal{E}'(\mathbb{F}_{p^2}) = (p - \epsilon_p)^2 + \epsilon_p d r^2 \ . \tag{11}$$

If $\mathcal{E}$ is supersingular, so $r = 0$, then [51, Theorem 1.1] yields a stronger statement: $\mathcal{E}(\mathbb{F}_{p^2}) \cong (\mathbb{Z}/(p + \epsilon_p)\mathbb{Z})^2$ and $\mathcal{E}'(\mathbb{F}_{p^2}) \cong (\mathbb{Z}/(p - \epsilon_p)\mathbb{Z})^2$.

**Corollary 2.** *Suppose $\mathcal{E}$ is ordinary. If $\mathcal{G} \subseteq \mathcal{E}(\mathbb{F}_{p^2})$ (resp. $\mathcal{G}' \subseteq \mathcal{E}'(\mathbb{F}_{p^2})$) is a cyclic subgroup such that $\psi(\mathcal{G}) \subseteq \mathcal{G}$ (resp. $\psi'(\mathcal{G}') \subseteq \mathcal{G}'$), then the eigenvalue of $\psi$ on $\mathcal{G}$ (resp. $\psi'$ on $\mathcal{G}'$) is*

$$\lambda_\psi \equiv \frac{p + \epsilon_p}{r} \pmod{\#\mathcal{G}} \quad \textit{and} \quad \lambda_{\psi'} \equiv \frac{p - \epsilon_p}{r} \pmod{\#\mathcal{G}'} \ .$$

*Proof.* Theorem 1 states that $[r]\psi = [p] + \epsilon_p \pi$ in $\mathrm{End}(\mathcal{E})$, so $\ker([p] + \epsilon_p \pi)$ contains $\mathcal{E}[r]$, and hence $[p] + \epsilon_p \pi$ is divisible by $r$ in $\mathrm{End}(\mathcal{E})$: indeed, the quotient is $\psi$. The result follows on restricting to $\mathcal{G}$; the argument for $\psi'$ and $\mathcal{G}'$ is the same. □

*$\mathbb{Q}$-curves: Where from, and why?* Now we just need a source of quadratic $\mathbb{Q}$-curves of small degree. Elkies [11] shows that all $\mathbb{Q}$-curves correspond to rational points on certain modular curves:[6] Let $X^*(d)$ be the quotient of the modular curve $X_0(d)$ by all of its Atkin–Lehner involutions. If $e$ is a point in $X^*(d)(\mathbb{Q})$ and $E$ is a preimage of $e$ in $X_0(d)(\mathbb{Q}(\sqrt{\Delta})) \setminus X_0(d)(\mathbb{Q})$ for some $\Delta$, then $E$ parametrizes (up to $\overline{\mathbb{Q}}$-isomorphism) a $d$-isogeny $\widetilde{\phi} : \widetilde{\mathcal{E}} \to {}^{\sigma}\widetilde{\mathcal{E}}$ over $\mathbb{Q}(\sqrt{\Delta})$, where $\sigma$ is the involution of $\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q}$.

Luckily enough, for very small $d$, both $X_0(d)$ and $X^*(d)$ have genus zero—so not only can we get plenty of rational points on $X^*(d)$, we can get a whole one-parameter family of $\mathbb{Q}$-curves of degree $d$. Hasegawa gives explicit universal curves for $d = 2$, 3, and 7 in [20, Theorem 2.2]: for each squarefree integer $\Delta \neq 1$, every $\mathbb{Q}$-curve of degree $d = 2, 3, 7$ over $\mathbb{Q}(\sqrt{\Delta})$ is $\overline{\mathbb{Q}}$-isomorphic to a rational specialization of one of these families.

Crucially, Hasegawa's families for $d = 2, 3$, and 7 are defined for *any* squarefree $\Delta$; so we are free to start by fixing $p$, before choosing a $\Delta$ to suit. Indeed, the particular choice of $\Delta$ is theoretically irrelevant—since all quadratic extensions of $\mathbb{F}_p$ are isomorphic—so we may choose any practically convenient value for $\Delta$, such as one permitting faster arithmetic in $\mathbb{F}_{p^2} = \mathbb{F}_p(\sqrt{\Delta})$.

Of course, one might ask why it is necessary to use this characteristic-zero theory when we could simply search for curves over $\mathbb{F}_{p^2}$ with a $d$-isogeny to their Galois conjugate. But for low degrees $d$ where $X_0(d)$ has genus zero, every such curve arises as the reduction mod $p$ of a $\mathbb{Q}$-curve over $\mathbb{Q}(\sqrt{\Delta})$ where $\Delta$ is a nonsquare mod $p$. Indeed, the curves over $\mathbb{F}_{p^2}$ with $d$-isogenies to their Galois conjugates correspond (up to isomorphism) to points in $X^*(d)(\mathbb{F}_p)$ with preimages in $X_0(d)(\mathbb{F}_{p^2}) \setminus X_0(d)(\mathbb{F}_p)$. But $X_0(d)$ is isomorphic to $\mathbb{P}^1$ over the ground field in the cases we consider, so every such point lifts trivially to a point in $X^*(d)(\mathbb{Q})$ with preimages in $X_0(d)(\mathbb{Q}(\sqrt{\Delta})) \setminus X_0(d)(\mathbb{Q})$: that is, to a $\mathbb{Q}$-curve.

We therefore lose no candidate curves over $\mathbb{F}_{p^2}$ by reducing $\mathbb{Q}$-curves mod $p$ instead of working directly over $\mathbb{F}_{p^2}$. What we gain by working with $\mathbb{Q}$-curves is some simplicity[7] and universality in our proofs, to say nothing of the wealth of mathematical literature to be raided for examples and theorems. For example: instead of deriving and proving defining equations for these families over $\mathbb{F}_{p^2}$, we can just conveniently borrow Hasegawa's universal $\mathbb{Q}$-curve equations.

*GLS as the degenerate case $d = 1$.* Suppose $\widetilde{\phi} : \widetilde{\mathcal{E}} \to {}^{\sigma}\widetilde{\mathcal{E}}$ is an isogeny of degree $d = 1$: that is, an isomorphism. Then $j(\widetilde{\mathcal{E}}) = j({}^{\sigma}\widetilde{\mathcal{E}}) = {}^{\sigma}j(\widetilde{\mathcal{E}})$, so $j(\widetilde{\mathcal{E}})$ is in $\mathbb{Q}$, and $\widetilde{\mathcal{E}}$ is $\mathbb{Q}(\sqrt{\Delta})$-isomorphic to a curve defined over $\mathbb{Q}$. Suppose, then, that

[6] The reader unfamiliar with modular curves can get away with the following here: the modular curve $X_0(d)$ parametrizes isomorphism classes of (cyclic) $d$-isogenies. If $d$ is prime—which is the only case we need here—then there is a unique Atkin–Lehner involution $\omega$ on $X_0(d)$: its action corresponds to exchanging isogenies $\phi$ with their duals $\phi^\dagger$. The quotient $X_0(d) \to X^*(d) := X_0(d)/\langle\omega\rangle$ is a double cover, mapping (the isomorphism class of) an isogeny $\phi$ to the pair (of isomorphism classes) $\{\phi, \phi^\dagger\}$.

[7] Especially since $\mathbb{Q}$-curves (by definition) have no non-integer endomorphisms, while every elliptic curve over $\mathbb{F}_{p^2}$ has complex multiplication.

$\widetilde{\mathcal{E}}$ is defined over $\mathbb{Q}$ and base-extended to $\mathbb{Q}(\sqrt{\Delta})$: then $\widetilde{\mathcal{E}} = {}^{\sigma}\widetilde{\mathcal{E}}$, and we can apply our construction taking $\widetilde{\phi} : \widetilde{\mathcal{E}} \to {}^{\sigma}\widetilde{\mathcal{E}}$ to be the identity map. Reducing modulo an inert $p$, we have $\psi = \pi_p$ and $\psi^2 = \pi_p^2 = \pi_{\mathcal{E}}$, so $\psi$ has eigenvalue $\pm 1$ on cryptographic subgroups of $\mathcal{E}(\mathbb{F}_{p^2})$: clearly, $\psi$ is of no use to us for scalar decompositions. However, the twisted endomorphism $\psi'$ on $\mathcal{E}'$ satisfies $(\psi')^2 = -\pi_{\mathcal{E}'}$, so the eigenvalue of $\psi'$ on cryptographic subgroups is a square root of $-1$, which is large enough to yield good scalar decompositions. We have recovered the GLS endomorphism (cf. [16, Theorem 2]).

While $\mathcal{E}'(\mathbb{F}_{p^2})$ may have prime order, $\mathcal{E}(\mathbb{F}_{p^2})$ cannot: the fixed points of $\pi_p$ form a subgroup of order $p+1-t_0$, where $t_0^2 - 2p = t_{\mathcal{E}}$ (and the complementary subgroup has order $p+1+t_0$). Hence, the largest prime divisor of $\#\mathcal{E}(\mathbb{F}_{p^2})$ can be no larger than $O(p)$; the curve $\mathcal{E}'$ can therefore never be twist-secure.

## 4 Short Scalar Decompositions

Before moving on to concrete families and examples, we will show that the endomorphisms developed in §3 yield short scalar decompositions.

Suppose $\mathcal{G} \cong \mathbb{Z}/N\mathbb{Z}$ is a cyclic subgroup of $\mathcal{E}(\mathbb{F}_{p^2})$ such that $\psi(\mathcal{G}) \subseteq \mathcal{G}$. Corollary 2 shows that $\psi$ acts as the eigenvalue $\lambda_\psi \equiv (p+\epsilon_p)/r \pmod{N}$ (a square root of $\epsilon_p d$) on $\mathcal{G}$. Given an integer $m$, we want to compute a decomposition

$$m = a + b\lambda_\psi \pmod{N}$$

so as to efficiently compute $[m]P = [a]P \oplus [b]\psi(P)$ for $P$ in $\mathcal{G}$. The decomposition is not unique: far from it. The set of all decompositions $(a,b)$ of $m$ is the lattice coset $(m,0) + \mathcal{L}$, where

$$\mathcal{L} := \langle (N,0), (-\lambda_\psi, 1) \rangle \subset \mathbb{Z}^2$$

is the lattice of decompositions of 0: that is, of integer pairs $(a,b)$ such that $a + b\lambda_\psi \equiv 0 \pmod{N}$.

We want to find a decomposition of $m$ where $a$ and $b$ have minimal bitlength: that is, where $\lceil \log_2 \|(a,b)\|_\infty \rceil$ is as small as possible. The following algorithm[8] computes an optimal decomposition of $m$ given a reduced basis of $\mathcal{L}$.

**Algorithm 1** *Given a reduced basis $[\mathbf{b}_1, \mathbf{b}_2]$ for $\mathcal{L}$, computes a decomposition of minimal bitlength (and bitlength at most $\lceil \log_2 \|\mathbf{b}_2\|_\infty \rceil$) for any given integer $m$.*

**Input** *An integer $m$ and a reduced basis $[\mathbf{b}_1, \mathbf{b}_2]$ for $\mathcal{L} = \langle (N,0), (-\lambda_\psi, 1) \rangle$.*
**Output** *A pair of integers $(a,b)$ such that $m \equiv a + b\lambda_\psi \pmod{N}$*
**Step 1** *Let $\alpha := mb_{22}/N$ and $\beta := -mb_{12}/N$.*
**Step 2** *Let $\mathbf{c}$ be the shortest of the four vectors $\lfloor \alpha \rfloor \mathbf{b}_1 + \lfloor \beta \rfloor \mathbf{b}_2$, $\lfloor \alpha \rfloor \mathbf{b}_1 + \lceil \beta \rceil \mathbf{b}_2$, $\lceil \alpha \rceil \mathbf{b}_1 + \lfloor \beta \rfloor \mathbf{b}_2$, and $\lceil \alpha \rceil \mathbf{b}_1 + \lceil \beta \rceil \mathbf{b}_2$.*

[8] Algorithm 1 differs from the standard technique (cf. [17, §4]), based on Babai rounding [1], in Step 2. Instead of choosing $\mathbf{c}$ to be the shortest of the four vectors, Babai rounding approximates it by selecting $\mathbf{c}' = \lfloor \alpha \rceil \mathbf{e}_1 + \lfloor \beta \rceil \mathbf{e}_2$ (this is the correct choice for most $m$). In terms of bitlength, this means an excess of one bit in the worst case.

**Step 3** *Return* $(a, b) := (m, 0) - \mathbf{c}$.

*Proof.* It is easily checked that $(\alpha, \beta)$ is the unique solution in $\mathbb{Q}^2$ to the linear system $\alpha\mathbf{b}_1 + \beta\mathbf{b}_2 = (m, 0)$ (here we use the fact that $N = \det \mathcal{L}$). Then $\mathbf{c}$ is the closest vector to $(m, 0)$ in $\mathcal{L}$ by Theorem 19 of [22], so $\|(a, b)\|_\infty$ is minimal over all decompositions of $m$. For the bound on $\|(a, b)\|_\infty$, set $\mathbf{c}' := \lfloor\alpha\rceil\mathbf{b}_1 + \lfloor\beta\rceil\mathbf{b}_2$; then $\|(m, 0) - \mathbf{c}\|_\infty \leq \|(m, 0) - \mathbf{c}'\|_\infty$. The triangle inequality and $|x - \lfloor x\rceil| \leq 1/2$ for all $x$ in $\mathbb{Q}$ imply $\|(m, 0) - \mathbf{c}'\|_\infty \leq \max(\|\mathbf{b}_1\|_\infty, \|\mathbf{b}_2\|_\infty) = \|\mathbf{b}_2\|_\infty$. □

It remains to precompute a reduced basis for $\mathcal{L}$. If $|\lambda_\psi|$ is not unusually small, then there exists a reduced basis of size $O(\sqrt{N})$.[9] Traditionally, we would compute it using the Gauss reduction or Euclidean algorithms (cf. [22], [17, §4] and [15, §17.1.1]), but in our case lattice reduction algorithms are unnecessary: following the approach outlined in [43], we can immediately write down a reduced basis for a large sublattice of $\mathcal{L}$, which coincides with $\mathcal{L}$ when $\mathcal{G} = \mathcal{E}(\mathbb{F}_{p^2})$. If $\mathcal{G}$ has a small cofactor in $\mathcal{E}(\mathbb{F}_{p^2})$, then we can easily modify the sublattice basis to give a proper reduced basis for $\mathcal{L}$ (as we will do in Lemmas 2, 3, and 4 below).

**Lemma 1.** *The vectors* $\mathbf{e}_1 = (p + \epsilon_p, -r)$ *and* $\mathbf{e}_2 = (-\epsilon_p dr, p + \epsilon_p)$ *generate a sublattice* $\mathcal{L}_0 \subseteq \mathcal{L}$ *of index* $[\mathcal{L} : \mathcal{L}_0] = \#\mathcal{E}(\mathbb{F}_{p^2})/N$. *In particular, if* $\#\mathcal{E}(\mathbb{F}_{p^2}) = N$, *then* $\mathcal{L} = \mathcal{L}_0$. *For large* $p$,

- *if* $\epsilon_p = -1$, *then* $[\mathbf{e}_1, \mathbf{e}_2]$ *is reduced;*
- *if* $\epsilon_p = 1$, *then* $[\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1]$ *(if* $r > 0$*) or* $[\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1]$ *(if* $r < 0$*) is reduced.*

*In either case, the bitlength of the reduced basis is* $\lceil\log_2(p + \epsilon_p)\rceil$.

*Proof.* Corollary (2) implies $r\lambda_\psi \equiv p + \epsilon_p$ and $r\epsilon_p d \equiv (p + \epsilon_p)\lambda_\psi \pmod{N}$, so $\mathbf{e}_1$ and $\mathbf{e}_2$ are in $\mathcal{L}$; they are linearly independent, so they generate a sublattice. The determinant is $(p + \epsilon_p)^2 - \epsilon_p dr^2$, which is $\#\mathcal{E}(\mathbb{F}_{p^2})$ by Eq. (11). Recall that $r$ is in $O(\sqrt{p})$ and $d$ is very small, so $\|\mathbf{e}_1\|_\infty = \|\mathbf{e}_2\|_\infty = p + \epsilon_p$. The bases satisfy Inequality (6), and hence are reduced. □

## 5 Endomorphisms from Quadratic $\mathbb{Q}$-curves of Degree 2

Let $\Delta$ be a squarefree integer. Hasegawa defines a one-parameter family

$$\widetilde{\mathcal{E}}_{2,\Delta,s} : y^2 = x^3 + 2(C_{2,\Delta}(s) - 24)x - 8(C_{2,\Delta}(s) - 16)$$

of $\mathbb{Q}$-curves of degree 2 over $\mathbb{Q}(\sqrt{\Delta})$ in [20, Theorem 2.2], where

$$C_{2,\Delta}(s) := 9(1 + s\sqrt{\Delta})$$

and $s$ is a free parameter taking values in $\mathbb{Q}$. Observe that ${}^\sigma\widetilde{\mathcal{E}}_{2,\Delta,s} = \widetilde{\mathcal{E}}_{2,\Delta,-s}$.

[9] General bounds on the constant hidden by the $O(\cdot)$ appear in [41], but they are far from tight for inseparable endomorphisms. Lemma 1 gives much better results for our endomorphisms in cryptographic contexts.

To realise the $\mathbb{Q}$-curve structure, observe that $\widetilde{\mathcal{E}}_{2,\Delta,s}$ has a rational 2-torsion point $(4,0)$. We compute the normalized quotient isogeny $\widetilde{\mathcal{E}}_{2,\Delta,s} \to \widetilde{\mathcal{E}}_{2,\Delta,s}/\langle(4,0)\rangle$ using Eq. (2); but then we observe that $\widetilde{\mathcal{E}}_{2,\Delta,s}/\langle(4,0)\rangle = ({}^{\sigma}\widetilde{\mathcal{E}}_{2,\Delta,s})^{\sqrt{-2}}$, so composing the quotient with the twisting isomorphism $\delta(1/\sqrt{-2})$ yields a 2-isogeny

$$\widetilde{\phi}_{2,\Delta,s} : \widetilde{\mathcal{E}}_{2,\Delta,s} \longrightarrow {}^{\sigma}\widetilde{\mathcal{E}}_{2,\Delta,s}$$

defined by the rational map

$$\widetilde{\phi}_{2,\Delta,t} : (x,y) \longmapsto \left(\frac{-x}{2} - \frac{C_{2,\Delta}(s)}{x-4}, \frac{y}{\sqrt{-2}}\left(\frac{-1}{2} + \frac{C_{2,\Delta}(s)}{(x-4)^2}\right)\right) .$$

(The arbitrary choice of one of the two square roots of $-2$ results in an arbitrary sign on $\widetilde{\phi}_{2,\Delta,s}$.) Conjugating and composing again, we find that

$${}^{\sigma}\widetilde{\phi}_{2,\Delta,s} \circ \widetilde{\phi}_{2,\Delta,s} = [\epsilon 2]_{\widetilde{\mathcal{E}}_{2,\Delta,s}} \qquad \text{where } \epsilon = \begin{cases} -1 & \text{if } {}^{\sigma}\sqrt{-2} = \sqrt{-2} \\ +1 & \text{if } {}^{\sigma}\sqrt{-2} = -\sqrt{-2} \end{cases} \tag{12}$$

—and similarly, $\widetilde{\phi}_{2,\Delta,s} \circ {}^{\sigma}\widetilde{\phi}_{2,\Delta,s} = [\epsilon 2]_{{}^{\sigma}\widetilde{\mathcal{E}}_{2,\Delta,s}}$.

The discriminant of the family $\widetilde{\mathcal{E}}_{2,\Delta,s}$ is $2^9 \cdot C_{2,\Delta}(s)^2 \cdot {}^{\sigma}C_{2,\Delta}(s)$, and

$$j(\widetilde{\mathcal{E}}_{2,\Delta,s}) = \frac{-12^3(C_{2,\Delta}(s)-24)^3}{C_{2,\Delta}(s)^2 \cdot {}^{\sigma}C_{2,\Delta}(s)}$$

(letting $s \to \infty$,[10] we find $j(\widetilde{\mathcal{E}}_{2,\Delta,\infty}) = 1728$). We see that $\widetilde{\mathcal{E}}_{2,\Delta,s}$ reduces modulo any inert $p > 3$ to give a family of elliptic curves over $\mathbb{F}_{p^2}$, and then every value of $s$ in $\mathbb{F}_p$ yields an elliptic curve over $\mathbb{F}_{p^2}$. (Proposition 1 below shows that at most two of these curves are isomorphic.)

**Theorem 2.** *Let $p > 3$ be prime, fix a nonsquare $\Delta$ modulo $p$, so $\mathbb{F}_{p^2} = \mathbb{F}_p(\sqrt{\Delta})$, and let $\mathcal{E}_{2,\Delta,s}$ and $\phi_{2,\Delta,s}$ be the reductions modulo $p$ of $\widetilde{\mathcal{E}}_{2,\Delta,s}$ and $\widetilde{\phi}_{2,\Delta,s}$.*

*For each $s$ in $\mathbb{F}_p$, the curve $\mathcal{E}_{2,\Delta,s}/\mathbb{F}_{p^2}$ has an efficient $\mathbb{F}_{p^2}$-endomorphism*

$$\psi_{2,\Delta,s} := \pi_p \circ \phi_{2,\Delta,s}$$

*of degree $2p$ such that $\psi_{2,\Delta,s}^2 = [\epsilon_p 2]\pi_{\mathcal{E}_{2,\Delta,s}}$ and $(\psi'_{2,\Delta,s})^2 = [-\epsilon_p 2]\pi_{\mathcal{E}'_{2,\Delta,s}}$, where*

$$\epsilon_p := -\left(-2/p\right) = \begin{cases} -1 & \text{if } p \equiv 1,3 \pmod 8 , \\ +1 & \text{if } p \equiv 5,7 \pmod 8 . \end{cases}$$

[10] "Letting $s \to \infty$" has a proper technical meaning here (and also in §6, §8, and §9)—even over finite fields. The parameter $s$ is defined by Hasegawa's choice (following Fricke) of a rational parametrization of the modular curve $X_0(2)$: that is, a birational map between $\mathbb{P}^1$ and $X_0(2)$. Under this parametrization, the point at infinity on $\mathbb{P}^1$ corresponds to the isomorphism class of the 2-isogeny (in fact, the endomorphism) $1+\iota$, where $\iota$ is an automorphism of order 4 of the curve with $j$-invariant 1728; this reflects what we find when we put $s = \infty$ in the formula for $j(\widetilde{\mathcal{E}}_{2,\Delta,s})$.

*There exists an integer $r$ satisfying $2r^2 = 2p + \epsilon_p t_{\mathcal{E}_{2,\Delta,s}}$ such that*

$$[r]\psi_{2,\Delta,s} = [p] + \epsilon_p \pi_{\mathcal{E}_{2,\Delta,s}} \qquad \textit{and} \qquad [r]\psi'_{2,\Delta,s} = [p] - \epsilon_p \pi_{\mathcal{E}'_{2,\Delta,s}} \ ;$$

*the characteristic polynomial of $\psi_{2,\Delta,s}$ and $\psi'_{2,\Delta,s}$ is $P_{2,\Delta,s}(T) = T^2 - 2rT + 2p$.*

*In particular, if $\mathcal{E}_{2,\Delta,s}$ is ordinary and $\mathcal{G} \subseteq \mathcal{E}_{2,\Delta,s}(\mathbb{F}_{p^2})$ is a cyclic subgroup of order $N$ such that $\psi_{2,\Delta,s}(\mathcal{G}) \subseteq \mathcal{G}$, then the eigenvalue of $\psi_{2,\Delta,s}$ on $\mathcal{G}$ is*

$$\lambda_{2,\Delta,s} \equiv (p + \epsilon_p)/r \equiv \pm\sqrt{\epsilon_p 2} \pmod{N} \ .$$

*Proof.* Apply Theorem 1 and Corollary 2 to $\widetilde{\phi}_{2,\Delta,s}$ using Eq. (12). □

**Proposition 1.** *If $p > 7$, then $\#\{j(\mathcal{E}_{2,\Delta,s}) : s \in \mathbb{F}_p\} = p$ if $-7$ is a square in $\mathbb{F}_p$, and $p - 1$ otherwise.*

*Proof.* Suppose $j(\mathcal{E}_{2,\Delta,s_1}) = j(\mathcal{E}_{2,\Delta,s_2})$, with $s_1 \neq s_2$. Equating the $j$-invariants symbolically, we must have $F_0(s_1, s_2) = 2\sqrt{\Delta}F_1(s_1, s_2)$, where the polynomials $F_0(T_1, T_2) = (\Delta T_1 T_2 + 1)(81\Delta T_1 T_2 - 175) + 49\Delta(T_1 + T_2)^2$ and $F_1(T_1, T_2) = (T_1 + T_2)(63\Delta T_1 T_2 - 65)$ have coefficients in $\mathbb{F}_p$. If $s_1$ and $s_2$ are in $\mathbb{F}_p$, then $F_0(s_1, s_2) = F_1(s_1, s_2) = 0$; this happens if and only if $s_2 = -s_1$ and either $s_i\sqrt{\Delta} = \pm 1$ (which is impossible) or $\pm\frac{5}{9}\sqrt{-7}$, whence the result. □

Both $\mathcal{E}_{2,\Delta,s}(\mathbb{F}_{p^2})$ and $\mathcal{E}'_{2,\Delta,s}(\mathbb{F}_{p^2})$ contain points of order 2: they generate the kernels of $\psi_{2,\Delta,s}$ and $\psi'_{2,\Delta,s}$. If $\#\mathcal{E}_{2,\Delta,s}(\mathbb{F}_{p^2}) = 2^k N$ and $\#\mathcal{E}'_{2,\Delta,s}(\mathbb{F}_{p^2}) = 2^{k'} N'$ with $N$ and $N'$ odd, then Eq. (1) modulo 8 implies that either $k = k' = 1$, or $k = 2$ and $k' \geq 3$, or $k \geq 3$ and $k' = 2$. Equation (11) modulo 3 implies that if $p \equiv 2 \pmod 3$ then either $\mathcal{E}_{2,\Delta,s}(\mathbb{F}_{p^2})$ or $\mathcal{E}'_{2,\Delta,s}(\mathbb{F}_{p^2})$ contains a point of order 3.

*Optimal decompositions.* In view of the Pohlig–Hellman–Silver reduction [35] and the rational 2-torsion point on $\mathcal{E}_{2,\Delta,s}$, the "optimal" situation for discrete-log based cryptosystems is when $\mathcal{E}_{2,\Delta,s}(\mathbb{F}_{p^2}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathcal{G}$ with $\#\mathcal{G}$ prime (though for faster arithmetic, we may want a cofactor of 4 instead of 2; we consider this later). Lemma 2 constructs an optimal basis for the GLV lattice $\mathcal{L}$ in this case. We can use this basis in Algorithm 1 to decompose scalar multiplications in $\mathcal{G}$ as $[m]P = [a]P \oplus [b]\psi_{2,\Delta,s}(P)$ where $a$ and $b$ have at most $\lceil \log_2 p \rceil$ bits.

**Lemma 2.** *Suppose $\mathcal{E}_{2,\Delta,s}(\mathbb{F}_{p^2}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ with $N$ odd, and let $\mathcal{L} = \langle (N, 0), (-\lambda_{2,\Delta,s}, 1) \rangle$. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be defined as in Lemma 1. For large $p$,*

- *if $\epsilon_p r \geq 0$, then $[-\mathbf{e}_2/2, \mathbf{e}_1 + \mathbf{e}_2/2]$ is a reduced basis for $\mathcal{L}$;*
- *if $\epsilon_p r < 0$, then $[-\mathbf{e}_2/2, \mathbf{e}_1 - \mathbf{e}_2/2]$ is a reduced basis for $\mathcal{L}$.*

*In either case, the bitlength of the reduced basis is $\lceil \log_2(p + \epsilon_p - |r|) \rceil$.*

*Proof.* Let $\mathcal{L}_0 := \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$ with $\mathbf{e}_1 := (p + \epsilon_p, -r)$ and $\mathbf{e}_2 := (-\epsilon_p 2r, p + \epsilon_p)$ as in Lemma 1; then $[\mathcal{L} : \mathcal{L}_0] = 2$, so exactly one of $\mathbf{e}_1/2$, $\mathbf{e}_2/2$, or $(\mathbf{e}_1 + \mathbf{e}_2)/2$ is in $\mathcal{L}$. Equation (11) shows that $2N = (p + \epsilon_p)^2 - 2\epsilon_p r^2$ with $|r| < \sqrt{2p}$; since $N$ is odd, $r$ must also be odd, so $\mathbf{e}_2/2$ is in $\mathbb{Z}^2$ but $\mathbf{e}_1/2$ and $(\mathbf{e}_1 + \mathbf{e}_2)/2$ are

not—and hence they cannot be in $\mathcal{L}$, either. We conclude that $\mathcal{L} = \langle \mathbf{e}_1, \mathbf{e}_2/2 \rangle$. Inequality (6) is satisfied by $\mathbf{b}_1 := -\mathbf{e}_2/2$ and $\mathbf{b}_2 := \mathbf{e}_1 \pm \mathbf{e}_2/2$ (with the sign chosen according to whether $\epsilon_p r$ is positive or negative), so $[\mathbf{b}_1, \mathbf{b}_2]$ is a reduced basis for $\mathcal{L}$. The longer of the vectors is $\mathbf{b}_2$, and $\|\mathbf{b}_2\|_\infty = p + \epsilon_p - |r|$. □

*Example 1.* Let $p = 2^{127} - 1$ and $\Delta = -1$. Taking $s = 28106$ in the family $\mathcal{E}_{2,-1,s}/\mathbb{F}_p(\sqrt{\Delta})$ yields a twist-secure curve at the 128-bit security level: we have $\epsilon_p = 1$ and $t_{\mathcal{E}_{2,-1,28106}} = -272082382382015736940757543628153813996$, so

$$\#\mathcal{E}_{2,-1,28106}(\mathbb{F}_{p^2}) = p^2 + 1 - t_{\mathcal{E}_{2,-1,28106}} = 2 \cdot N \quad \text{and}$$
$$\#\mathcal{E}'_{2,-1,28106}(\mathbb{F}_{p^2}) = p^2 + 1 + t_{\mathcal{E}_{2,-1,28106}} = 2 \cdot N'$$

where $N$ and $N'$ are 253-bit primes.[11] Algorithm 1 and Lemma 2 transform 253-bit scalar multiplications in $\mathcal{E}_{2,-1,28106}(\mathbb{F}_{p^2})[N]$ into 128-bit multiexponentations. This value of $s$ is the "smallest" (counting upwards from 1) yielding a curve-twist pair such that both curve orders are twice a prime. The curve coefficients, being linear in $s$, are relatively small; but while small coefficients are important in optimized implementations, here this is no more than a happy coincidence—we did not explicitly search for an example with convenient coefficients.

*Montgomery models.* The curve $\mathcal{E}_{2,\Delta,s}$ has a Montgomery model over $\mathbb{F}_{p^2}$ if and only if $2C_{2,\Delta}(s)$ is a square in $\mathbb{F}_{p^2}$ by [34, Proposition 1]—or equivalently, if $1 + s\sqrt{\Delta}$ is a square in $\mathbb{F}_{p^2}$ (since 2 is always a square in $\mathbb{F}_{p^2}$). Setting

$$B^{\mathrm{M}}_{2,\Delta}(s) := (2C_{2,\Delta}(s))^{1/2} \qquad \text{and} \qquad A^{\mathrm{M}}_{2,\Delta}(s) := 12/B^{\mathrm{M}}_{2,\Delta}(s) \ ,$$

the birational map $(x, y) \mapsto (X/Z, Y/Z) = \big((x-4)/B^{\mathrm{M}}_{2,\Delta}(s), y/B^{\mathrm{M}}_{2,\Delta}(s)^2\big)$ takes us from $\mathcal{E}_{2,\Delta,s}$ to the projective Montgomery model

$$\mathcal{E}^{\mathrm{M}}_{2,\Delta,s} : B^{\mathrm{M}}_{2,\Delta}(s)Y^2Z = X\big(X^2 + A^{\mathrm{M}}_{2,\Delta}(s)XZ + Z^2\big)$$

(we may replace the term $B^{\mathrm{M}}_{2,\Delta}(s)Y^2Z$ in the defining equation with a conveniently small multiple of $Y^2Z$, if desired, by scaling the $Y$ coordinate).

Montgomery models offer a particularly efficient arithmetic using only the $X$ and $Z$ coordinates [32]. The induced endomorphism on the $(X : Z)$-line is

$$\psi^{\mathrm{M}}_{2,\Delta,s} : (X : Z) \longmapsto \big(X^{2p} + A^{\mathrm{M}}_{2,\Delta}(s)^p X^p Z^p + Z^{2p} : -2A^{\mathrm{M}}_{2,\Delta}(s)^{p-1} X^p Z^p\big) \ ;$$

an implementation of fast scalar multiplication using $\psi^{\mathrm{M}}_{2,\Delta,s}$ is detailed in [9].

*Twisted Edwards models.* Every Montgomery model corresponds to a twisted Edwards model, and vice versa (cf. [3] and [21]). Indeed, $\mathcal{E}_{2,\Delta,s}$ is isomorphic to the twisted Edwards model

$$\mathcal{E}^{\mathrm{TE}}_{2,\Delta,s} : \big(12 + 2B^{\mathrm{M}}_{2,\Delta}(s)\big)x_1^2 + x_2^2 = 1 + \big(12 - 2B^{\mathrm{M}}_{2,\Delta}(s)\big)x_1^2x_2^2$$

via $(x, y) \mapsto (x_1, x_2) = \big((x-4)/y, (x - 4 - B^{\mathrm{M}}_{2,\Delta}(s))/(x - 4 + B^{\mathrm{M}}_{2,\Delta}(s))\big)$. Composing with $\psi_{2,\Delta,s}$ yields an endomorphism $\psi^{\mathrm{TE}}_{2,\Delta,s}$ of $\mathcal{E}^{\mathrm{TE}}_{2,\Delta,s}$.

[11] We computed the traces for all of our examples using a new specialized variant of the SEA algorithm [38] under development with François Morain and Charlotte Scribot, implemented in NTL [39].

*Optimal decompositions for cofactor 4.* Every curve with a twisted Edwards or Montgomery model has order divisible by 4; indeed, $C_{2,\Delta}(s)$ is a square in $\mathbb{F}_{p^2}$ (so $\mathcal{E}^{\mathrm{M}}_{2,\Delta,s}$ and $\mathcal{E}^{\mathrm{TE}}_{2,\Delta,s}$ are defined over $\mathbb{F}_{p^2}$) if and only if $\mathcal{E}_{2,\Delta,s}$ has full rational 2-torsion. The optimal situation for discrete log-based cryptography on these curves is therefore when $\mathcal{E}_{2,\Delta,s}(\mathbb{F}_{p^2}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathcal{G}$ with $\#\mathcal{G}$ prime. Lemma 3 gives a reduced basis for the GLV lattice in this case, which we can use in Algorithm 1 to decompose scalar multiplications in $\mathcal{G}$ as $[m]P = [a]P \oplus [b]\psi_{2,\Delta,s}(P)$ where $a$ and $b$ have at most $\lceil \log_2 p \rceil - 1$ bits.

**Lemma 3.** *Suppose $\mathcal{E}_{2,\Delta,s}(\mathbb{F}_{p^2}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/N\mathbb{Z}$ with $N$ odd, and let $\mathcal{L} = \langle (N,0), (-\lambda_{2,\Delta,s}, 1) \rangle$. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be defined as in Lemma 1. For large $p$,*

- *if $\epsilon_p = 1$ and $r \geq 0$, then $[(\mathbf{e}_1 + \mathbf{e}_2)/2, \mathbf{e}_2/2]$ is a reduced basis for $\mathcal{L}$;*
- *if $\epsilon_p = 1$ and $r < 0$, then $[(\mathbf{e}_1 - \mathbf{e}_2)/2, -\mathbf{e}_2/2]$ is a reduced basis for $\mathcal{L}$;*
- *if $\epsilon_p = -1$ and $r \geq 0$, then $[\mathbf{e}_1/2, \mathbf{e}_2/2]$ is a reduced basis for $\mathcal{L}$;*
- *otherwise, if $\epsilon_p = -1$ and $r < 0$, then $[\mathbf{e}_1/2, -\mathbf{e}_2/2]$ is a reduced basis for $\mathcal{L}$.*

*In each case, the bitlength of the reduced basis is $\lceil \log_2(p + \epsilon_p) \rceil - 1$.*

*Proof.* The sublattice $\mathcal{L}_0 = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$ has index 4 in $\mathcal{L}$. Equation (11) implies $(p + \epsilon_p)^2 \equiv 2\epsilon_p r^2 \pmod 4$, and since $p + \epsilon_p$ is even, $r$ must be even as well; so we can replace the relation $\epsilon_p r \lambda_{2,\Delta,s} \equiv p + \epsilon_p$ with $\epsilon_p (r/2) \lambda_{2,\Delta,s} \equiv (p + \epsilon_p)/2 \pmod N$. Hence, both $\mathbf{e}_1/2$ and $\mathbf{e}_2/2$ are in $\mathcal{L}$; so they must form a basis for $\mathcal{L}$. The listed bases are combinations of $\mathbf{e}_1/2$ and $\mathbf{e}_2/2$ satisfying Ineq. (6), and are therefore reduced. The longest vector in each basis has length $(p + \epsilon_p)/2$. □

*Doche–Icart–Kohel models.* Doubling-oriented Doche–Icart–Kohel models are defined by equations of the form $y^2 = x(x^2 + Dx + 16D)$. These curves have a rational 2-isogeny $\phi$ with kernel $\langle (0,0) \rangle$, and both $\phi$ and its dual $\phi^\dagger$ are in a special form that allows marginally faster doubling using the factorization $[2] = \phi^\dagger \phi$ (see [10, §3.1] for details). Our curves $\mathcal{E}_{2,\Delta,s}$ come equipped with a rational 2-isogeny, so it is natural to try putting them in Doche–Icart–Kohel form. The same 2-isogeny plays two rôles in this situation: as a factor of our endomorphism for scalar decomposition, and as a factor of the doubling map for Doche–Icart–Kohel arithmetic. We emphasize that these two applications are distinct and complementary, and their benefits are cumulative. We have an isomorphism from $\mathcal{E}_{2,\Delta,s}$ to the Doche–Icart–Kohel model

$$\mathcal{E}^{\mathrm{DIK}}_{2,\Delta,s} : v^2 = u\big(u^2 + \tfrac{1152}{C_{2,\Delta}(s)} u + 16 \cdot \tfrac{1152}{C_{2,\Delta}(s)}\big) \ ,$$

defined by $(x,y) \mapsto (u,v) = (\alpha(x-4), \alpha^{3/2} y)$ where $\alpha = 96/C_{2,\Delta}(s)$; if $C_{2,\Delta}(s)$ is not a square in $\mathbb{F}_{p^2}$, then $\mathcal{E}^{\mathrm{DIK}}_{2,\Delta,s}$ is $\mathbb{F}_{p^2}$-isomorphic to $\mathcal{E}'_{2,\Delta,s}$.

## 6 Endomorphisms from Quadratic $\mathbb{Q}$-curves of Degree 3

Let $\Delta$ be a squarefree integer. Hasegawa defines a one-parameter family

$$\widetilde{\mathcal{E}}_{3,\Delta,s} : y^2 = x^3 - 3(2C_{3,\Delta}(s) + 1)x + \big(C_{3,\Delta}(s)^2 + 10C_{3,\Delta}(s) - 2\big)$$

of $\mathbb{Q}$-curves of degree 3 over $\mathbb{Q}(\sqrt{\Delta})$ in [20, Theorem 2.2], where

$$C_{3,\Delta}(s) := 2(1 + s\sqrt{\Delta})$$

and $s$ is a free parameter taking values in $\mathbb{Q}$. Observe that ${}^{\sigma}\widetilde{\mathcal{E}}_{3,\Delta,s} = \widetilde{\mathcal{E}}_{3,\Delta,-s}$.

To realize the degree-3 $\mathbb{Q}$-curve structure, note that $x - 3$ defines an order-3 subgroup $\mathcal{S} = \{0, (3, \pm{}^{\sigma}C_{3,\Delta}(s))\}$ of $\widetilde{\mathcal{E}}_{3,\Delta,s}(\mathbb{Q}(\sqrt{\Delta}))$. Computing the normalized quotient isogeny $\widetilde{\mathcal{E}}_{3,\Delta,s} \to \widetilde{\mathcal{E}}_{3,\Delta,s}/\mathcal{S}$ using Eqs. (3), (4), and (5), we observe that $\widetilde{\mathcal{E}}_{3,\Delta,s}/\mathcal{S} = ({}^{\sigma}\widetilde{\mathcal{E}}_{3,\Delta,s})^{\sqrt{-3}}$; so composing the quotient with $\delta(1/\sqrt{-3})$ yields an explicit 3-isogeny $\widetilde{\phi}_{3,\Delta,s} : \widetilde{\mathcal{E}}_{3,\Delta,s} \to {}^{\sigma}\widetilde{\mathcal{E}}_{3,\Delta,s}$ defined by the rational map

$$\widetilde{\phi}_{3,\Delta,s} : (x, y) \longmapsto \Big( (\widetilde{\phi}_{3,\Delta,s})_x(x), \frac{y}{\sqrt{-3}} \frac{d(\widetilde{\phi}_{3,\Delta,s})_x}{dx}(x) \Big)$$

where

$$(\widetilde{\phi}_{3,\Delta,s})_x(x) = -\frac{1}{3}\Big( x + \frac{12 \cdot {}^{\sigma}C_{3,\Delta}(s)}{x-3} + \frac{4 \cdot {}^{\sigma}C_{3,\Delta}(s)^2}{(x-3)^2} \Big) .$$

Conjugating and composing again, we see that

$${}^{\sigma}\widetilde{\phi}_{3,\Delta,s} \circ \widetilde{\phi}_{3,\Delta,s} = \epsilon[3]_{\widetilde{\mathcal{E}}_{3,\Delta,s}} \quad \text{where} \quad \epsilon = \begin{cases} -1 & \text{if } {}^{\sigma}\sqrt{-3} = \sqrt{-3} \\ +1 & \text{if } {}^{\sigma}\sqrt{-3} = -\sqrt{-3} \end{cases} \tag{13}$$

(and similarly, $\widetilde{\phi}_{3,\Delta,s} \circ {}^{\sigma}\widetilde{\phi}_{3,\Delta,s} = \epsilon[3]_{{}^{\sigma}\widetilde{\mathcal{E}}_{3,\Delta,s}}$).

This family has discriminant $2^4 \cdot 3^3 \cdot C_{3,\Delta}(s) \cdot {}^{\sigma}C_{3,\Delta}(s)^3$ and $j$-invariant

$$j(\widetilde{\mathcal{E}}_{3,\Delta,s}) = \frac{2^8 \cdot 3^3 \cdot (2C_{3,\Delta}(s) + 1)^3}{C_{3,\Delta}(s) \cdot {}^{\sigma}C_{3,\Delta}(s)^3}$$

(letting $s \to \infty$, we see that $j(\widetilde{\mathcal{E}}_{3,\Delta,\infty}) = 0$). Hence, $\widetilde{\mathcal{E}}_{3,\Delta,s}$ reduces modulo any inert $p > 3$ to give a family of elliptic curves over $\mathbb{F}_{p^2}$, and then every value of $s$ in $\mathbb{F}_p$ yields an elliptic curve over $\mathbb{F}_{p^2}$. A calculation similar to Proposition 1 shows that we get at least $p - 8$ non-isomorphic curves in this way.

**Theorem 3.** *Let $p > 3$ be prime, fix a nonsquare $\Delta$ modulo $p$, so $\mathbb{F}_{p^2} = \mathbb{F}_p(\sqrt{\Delta})$, and let $\mathcal{E}_{3,\Delta,s}$ and $\phi_{3,\Delta,s}$ be the reductions modulo $p$ of $\widetilde{\mathcal{E}}_{3,\Delta,s}$ and $\widetilde{\phi}_{3,\Delta,s}$.*

*For each $s$ in $\mathbb{F}_p$, the curve $\mathcal{E}_{3,\Delta,s}/\mathbb{F}_{p^2}$ has an efficient $\mathbb{F}_{p^2}$-endomorphism*

$$\psi_{3,\Delta,s} := \pi_p \circ \phi_{3,\Delta,s}$$

*of degree $3p$, such that $\psi_{3,\Delta,s}^2 = [\epsilon_p 3]\pi_{\mathcal{E}_{3,\Delta,s}}$ and $(\psi'_{3,\Delta,s})^2 = [-\epsilon_p 3]\pi_{\mathcal{E}'_{3,\Delta,s}}$, where*

$$\epsilon_p := -\left(-3/p\right) = \begin{cases} +1 & \text{if } p \equiv 2 \pmod 3 , \\ -1 & \text{if } p \equiv 1 \pmod 3 . \end{cases}$$

*There exists an integer $r$ satisfying $3r^2 = 2p + \epsilon_p t_{\mathcal{E}_{3,\Delta,s}}$ such that*

$$[r]\psi_{3,\Delta,s} = [p] + \epsilon_p \pi_{\mathcal{E}_{3,\Delta,s}} \qquad \text{and} \qquad [r]\psi'_{3,\Delta,s} = [p] - \epsilon_p \pi_{\mathcal{E}_{3,\Delta,s}} \ ;$$

*the characteristic polynomial of $\psi_{3,\Delta,s}$ and $\psi'_{3,\Delta,s}$ is $P_{3,\Delta,s}(T) = T^2 - 3rT + 3p$.*

*In particular, if $\mathcal{E}_{3,\Delta,s}$ is ordinary and $\mathcal{G} \subseteq \mathcal{E}_{3,\Delta,s}(\mathbb{F}_{p^2})$ is a cyclic subgroup of order $N$ such that $\psi_{3,\Delta,s}(\mathcal{G}) \subseteq \mathcal{G}$, then the eigenvalue of $\psi_{3,\Delta,s}$ on $\mathcal{G}$ is*

$$\lambda_{3,\Delta,s} \equiv (p + \epsilon_p)/r \equiv \pm\sqrt{\epsilon_p 3} \pmod{N} .$$

*Proof.* Follows from Theorem 1 and Corollary 2 using Eq. (13). □

*Optimal decompositions.* The kernel of $\psi_{3,\Delta,s}$ is generated by the rational points $(3, \pm^\sigma C_{3,\Delta}(s))$, so $\#\mathcal{E}_{3,\Delta,s}(\mathbb{F}_{p^2})$ is always divisible by 3. However, the nontrivial points in the kernel of the twisted endomorphism $\psi'_{3,\Delta,s}$ are *not* defined over $\mathbb{F}_{p^2}$ (they are conjugates), so it is possible for $\mathcal{E}'_{3,\Delta,s}(\mathbb{F}_{p^2})$ to have prime order.

From the point of view of the Pohlig–Hellman–Silver reduction, the "most secure" curves in $\mathcal{E}_{3,\Delta,s}/\mathbb{F}_{p^2}$ have $\mathcal{E}_{3,\Delta,s}(\mathbb{F}_{p^2}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathcal{G}$, with $\mathcal{G}$ of prime order. Lemma 4 gives an optimal basis for the GLV lattice $\mathcal{L}$ in this case (for a prime-order twist, the basis of Lemma 1 is already optimal).

**Lemma 4.** *Suppose $\mathcal{E}_{3,\Delta,s}(\mathbb{F}_{p^2}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ with $N$ prime to 3, and let $\mathcal{L} = \langle (N, 0), (-\lambda_{3,\Delta,s}, 1) \rangle$. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be defined as in Lemma 1. For large $p$,*

- *if $\epsilon_p r \geq 0$, then $[\mathbf{e}_2/3, \mathbf{e}_1 + 2\mathbf{e}_2/3]$ is a reduced basis of $\mathcal{L}$;*
- *if $\epsilon_p r < 0$, then $[\mathbf{e}_2/3, \mathbf{e}_1 - 2\mathbf{e}_2/3]$ is a reduced basis of $\mathcal{L}$.*

*In either case, the bitlength of the reduced basis is $\lceil \log_2(p + \epsilon_p - 2|r|) \rceil$.*

*Proof.* The proof is essentially the same as for Lemma 2, with 3 in place of 2. The sublattice $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$ has index 3 in $\mathcal{L}$, so exactly one of $\frac{1}{3}\mathbf{e}_1$, $\frac{1}{3}\mathbf{e}_2$, $\frac{1}{3}(\mathbf{e}_1 + \mathbf{e}_2)$, and $\frac{1}{3}(\mathbf{e}_1 - \mathbf{e}_2)$ is in $\mathcal{L}$. Equation (11) gives $3N = (p + \epsilon_p)^2 - 3\epsilon_p r^2$; but $p \equiv -\epsilon_p \pmod{3}$, so $\frac{1}{3}\mathbf{e}_2$ is in $\mathbb{Z}^2$. On the other hand, $3 \nmid r$ (since otherwise $3 \mid N$), so neither $\frac{1}{3}\mathbf{e}_1$ nor $\frac{1}{3}(\mathbf{e}_1 \pm \mathbf{e}_2)$ is in $\mathbb{Z}^2$. Hence $\langle \mathbf{e}_1, \frac{1}{3}\mathbf{e}_2 \rangle$ is the only lattice in $\mathbb{Z}^2$ containing $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$ with index 3, so $\mathcal{L} = \langle \mathbf{e}_1, \frac{1}{3}\mathbf{e}_2 \rangle$. The vectors $\frac{1}{3}\mathbf{e}_2$ and $\mathbf{e}_1 \pm 2\mathbf{e}_2/3$ satisfy Ineq. (6), so they form a reduced basis for $\mathcal{L}$; the longest of their components is $p + \epsilon_p - 2|r|$. □

*Example 2.* Let $p = 2^{127} - 1$; then $\Delta = -1$ is a nonsquare in $\mathbb{F}_p$. The parameter value $s = 10400$ yields a twist-secure curve at the 128-bit security level: $\#\mathcal{E}_{3,-1,10400}(\mathbb{F}_{p^2}) = 3 \cdot N$ and $\#\mathcal{E}'_{3,-1,10400}(\mathbb{F}_{p^2}) = N'$, where $N$ and $N'$ are 253- and 254-bit primes, respectively. As in Example 1, this is the smallest value of $s$ yielding a curve-twist pair with orders in this form. Any scalar multiplication in $\mathcal{E}_{3,-1,10400}(\mathbb{F}_{p^2})[N]$ or $\mathcal{E}'_{3,-1,10400}(\mathbb{F}_{p^2})[N']$ can be computed via a 127-bit multiexponentiation using Algorithm 1 with the basis of Lemma 4. (We warn the reader that here, one of the curve coefficients is quadratic in $s$; so small values of $s$ may not yield particularly convenient coefficients for serious implementations.)

*Doche–Icart–Kohel models.* We can exploit the 3-isogeny on $\mathcal{E}_{3,\Delta,s}$ for faster tripling (cf. [10, §3.2]): $\mathcal{E}_{3,\Delta,s}$ is isomorphic to the tripling-oriented Doche–Icart-Kohel model

$$\mathcal{E}^{\mathrm{DIK}}_{3,\Delta,s} : v^2 = u^3 + 3 \cdot \frac{9}{C_{3,\Delta}(s)^p}(u + 1)^2$$

via $(x, y) \mapsto (u, v) = \big(\alpha(x - 3), \alpha^{3/2} y\big)$ where $\alpha := 3C_{3,\Delta}(s)^{-p}$. This is an $\mathbb{F}_{p^2}$-isomorphism if $C_{3,\Delta}(s)$ is a square in $\mathbb{F}_{p^2}$; otherwise, $\mathcal{E}^{\mathrm{DIK}}_{3,\Delta,s} \cong_{\mathbb{F}_{p^2}} \mathcal{E}'_{3,\Delta,s}$.

## 7 Endomorphisms from Quadratic $\mathbb{Q}$-curves of Degree 5

For $d = 5$, Hasegawa notes that it is impossible to give a universal $\mathbb{Q}$-curve for arbitrary squarefree $\Delta$: there exists a quadratic $\mathbb{Q}$-curve of degree 5 over $\mathbb{Q}(\sqrt{\Delta})$ if and only if $\left(5/p_i\right) = 1$ for every prime $p_i \neq 5$ dividing $\Delta$ (see [20, Proposition 2.3]). This restricts our choice of $\Delta$ for a given $p$.

The special case $\Delta = -1$ is particularly interesting: by the above, there exists a family of $\mathbb{Q}$-curves of degree 5 over $\mathbb{Q}(\sqrt{-1})$, and every prime $p \equiv 3 \pmod 4$ is inert in $\mathbb{Q}(\sqrt{-1})$. We work this case out in detail below. The remaining case $p \equiv 1 \pmod 4$ is a straightforward exercise: given a fixed prime $p > 5$, we choose a squarefree $\Delta$ meeting the condition above, then apply [20, Theorem 2.4] to derive a family of degree-5 $\mathbb{Q}$-curves over $\mathbb{Q}(\sqrt{\Delta})$ amenable to the construction of §3.[12]

Of course, compared with $d = 2$ and 3, endomorphisms with separable degree $d = 5$ are intrinsically slower. The chief interest of this family is that unlike with $d = 2$ and 3, here neither the generic curve nor its twist have rational torsion points, so it is possible for reductions and their twists to both have prime order.

Let $\widetilde{\mathcal{E}}_{5,-1,s}$ be the family of elliptic curves over $\mathbb{Q}(\sqrt{-1})$ defined by

$$\widetilde{\mathcal{E}}_{5,-1,s} : y^2 = x^3 + A_{5,-1}(s)x + B_{5,-1}(s)$$

where

$$\begin{aligned} A_{5,-1}(s) &:= -27s(11s-2)\left(3(6s^2+6s-1) - 20s(s-1)\sqrt{-1}\right) \ , \\ B_{5,-1}(s) &:= 54s^2(11s-2)^2\left((13s^2+59s-9) - 2(s-1)(20s+9)\sqrt{-1}\right) \ , \end{aligned}$$

and $s$ is a free parameter taking values in $\mathbb{Q}$.

The family $\widetilde{\mathcal{E}}_{5,-1,s}$ is a family of $\mathbb{Q}$-curves of degree 5: the polynomial

$$(1+2\sqrt{-1})(x - 3s(11s-2)(2-\sqrt{-1}))^2 + 81s(11s-2)(1+s\sqrt{-1})^2$$

defines the kernel $\mathcal{S}$ of a 5-isogeny $\widetilde{\phi}_{5,-1,s} : \widetilde{\mathcal{E}}_{5,-1,s} \to {}^{\sigma}\widetilde{\mathcal{E}}_{5,-1,s}$ over $\mathbb{Q}(\sqrt{-1})$, which is the composition of the normalized quotient $\widetilde{\mathcal{E}}_{5,-1,s} \to \widetilde{\mathcal{E}}_{5,-1,s}/\mathcal{S}$ (as in Eqs. (3), (4), and (5)) with the twisting isomorphism $\delta(5/(1+2\sqrt{-1}))$. Conjugating and composing again, we find

$${}^{\sigma}\widetilde{\phi}_{5,\Delta,s} \circ \widetilde{\phi}_{5,\Delta,s} = [5]_{\mathcal{E}_{5,\Delta,s}} \qquad \text{and} \qquad \widetilde{\phi}_{5,\Delta,s} \circ {}^{\sigma}\widetilde{\phi}_{5,\Delta,s} = [5]_{{}^{\sigma}\mathcal{E}_{5,\Delta,s}} \ . \tag{14}$$

The family has discriminant $-2^6 3^{12} s^3 (11s-2)^3 (1+s^2)(1+s\sqrt{-1})^4$, and

$$j(\widetilde{\mathcal{E}}_{5,-1,s}) = \frac{-64\left(3(6s^2+6s-1) - 20(s^2-s)\sqrt{-1}\right)^3}{(1+s^2)(1+s\sqrt{-1})^4} \ .$$

Hence, $\widetilde{\mathcal{E}}_{5,-1,s}$ is an elliptic curve for all $s$ in $\mathbb{Q}\setminus\{0, 2/11\}$, and these $\widetilde{\mathcal{E}}_{5,-1,s}$ have good reduction at any $p > 5$ inert in $\mathbb{Q}(\sqrt{-1})$. The analogue of Proposition 1 shows that we get at least $p - 25$ non-isomorphic curves in this way.

[12] In [44], the author suggested that for any $p \equiv 1 \pmod 4$ one could use $\Delta = -11$ with Hasegawa's parameters in [20, Table 6] in the construction of §3. This is incorrect: by Dirichlet's theorem, half of the $p \equiv 1 \pmod 4$ are not inert in $\mathbb{Q}(\sqrt{-11})$.

**Theorem 4.** *Let $p$ be a prime congruent to 3 modulo 4, so $\mathbb{F}_{p^2} = \mathbb{F}_p(\sqrt{-1})$, and let $\mathcal{E}_{5,-1,s}$ and $\phi_{5,-1,s}$ be the reductions mod $p$ of $\widetilde{\mathcal{E}}_{5,-1,s}$ and $\widetilde{\phi}_{5,-1,s}$.*

*For each $s \neq 0$ or $2/11$ in $\mathbb{F}_p$, the curve $\mathcal{E}_{5,-1,s}/\mathbb{F}_{p^2}$ has an efficient $\mathbb{F}_{p^2}$-endomorphism*

$$\psi_{5,-1,s} := \pi_p \circ \phi_{5,-1,s}$$

*of degree $5p$ such that $\psi_{5,-1,s}^2 = [5]\pi_{\mathcal{E}_{5,-1,s}}$ and $(\psi'_{5,-1,s})^2 = [-5]\pi_{\mathcal{E}'_{5,-1,s}}$. There exists an integer $r$ satisfying $5r^2 = 2p + t_{\mathcal{E}_{5,-1,s}}$ such that*

$$[r]\psi_{5,-1,s} = [p] + \pi_{\mathcal{E}_{5,-1,s}} \qquad \text{and} \qquad [r]\psi'_{5,-1,s} = [p] - \pi_{\mathcal{E}_{5,-1,s}} \ ;$$

*the characteristic polynomial of $\psi_{5,-1,s}$ and $\psi'_{5,-1,s}$ is $P_{5,-1,s}(T) = T^2 - 5rT + 5p$.*

*In particular, if $\mathcal{E}_{5,-1,s}$ is ordinary and $\mathcal{G} \subseteq \mathcal{E}_{5,-1,s}(\mathbb{F}_{p^2})$ is a cyclic subgroup of order $N$ such that $\psi_{5,-1,s}(\mathcal{G}) \subseteq \mathcal{G}$, then the eigenvalue of $\psi_{5,-1,s}$ on $\mathcal{G}$ is*

$$\lambda_{5,-1,s} \equiv (p+1)/r \equiv \pm\sqrt{5} \pmod{N} \ .$$

*Proof.* Follows from Theorem 1 and Corollary 2 using Eq. (14). □

Reductions of curves in $\widetilde{\mathcal{E}}_{5,-1,s}$ may have prime order, and so can their twists. In this situation, Algorithm 1 with the basis of Lemma 1 computes optimal scalar decompositions for $\psi_{5,-1,s}$ (of bitlength at most $\lceil \log_2(p+1) \rceil$).

*Example 3.* Let $p = 2^{127} - 1$ and $\Delta = -1$. Taking $s = 7930$ in the degree-5 family yields a twist-secure curve at the 128-bit security level: the trace of $\mathcal{E}_{5,-1,7930}$ is

$$t_{\mathcal{E}_{2,-1,28106}} = 160084314926568661653252069280514036151 \ ,$$

so $\#\mathcal{E}_{5,-1,7930}(\mathbb{F}_{p^2})$ and $\#\mathcal{E}'_{5,-1,7930}(\mathbb{F}_{p^2})$ are *both* 254-bit primes. We transform 254-bit scalar multiplications in $\mathcal{E}_{5,-1,7930}(\mathbb{F}_{p^2})$ into a 127-bit multiexponentiations using Algorithm 1 with the basis of Lemma 1. As in Examples 1 and 2, this is the smallest value of $s$ yielding a curve-twist pair with both curves of prime order. (Here the curve coefficients are quartic and sextic in $s$, so the smallness of $s$ has little effect on the convenience of the coefficients for implementations—however, as we remarked above, this family is essentially of theoretical interest.)

## 8 Endomorphisms from Quadratic $\mathbb{Q}$-curves of Degree 7

For completeness, we include a family of $\mathbb{Q}$-curves of degree 7. These curves are less interesting for practical applications, since the higher degree renders the endomorphism intrinsically slower than the curves with $d = 2, 3$, and 5.

Let $\Delta$ be a squarefree integer. Hasegawa defines a one-parameter family

$$\widetilde{\mathcal{E}}_{7,\Delta,s} : y^2 = x^3 + A_{7,\Delta}(s)x + B_{7,\Delta}(s)$$

of $\mathbb{Q}$-curves of degree 7 over $\mathbb{Q}(\sqrt{\Delta})$ in [20, Theorem 2.2], where

$$A_{7,\Delta}(s) = -3C_{7,\Delta}(s)(85 + 96s\sqrt{\Delta} + 15s^2\Delta) \ ,$$
$$B_{7,\Delta}(s) = 14C_{7,\Delta}(s)\big(9(3s^4\Delta^2 + 130s^2\Delta + 171) + 16(9s^2\Delta + 163)s\sqrt{\Delta}\big) \ ,$$
$$C_{7,\Delta}(s) = 7(27 + s^2\Delta) \ ,$$

and $s$ is a free parameter taking values in $\mathbb{Q}$. Observe that ${}^\sigma\widetilde{\mathcal{E}}_{7,\Delta,s} = \widetilde{\mathcal{E}}_{7,\Delta,-s}$.

The family $\widetilde{\mathcal{E}}_{7,\Delta,s}$ is a family of quadratic $\mathbb{Q}$-curves of degree 7. More explicitly: $\widetilde{\mathcal{E}}_{7,\Delta,s}$ has a subgroup $\mathcal{S}$ of order 7 defined by the kernel polynomial

$$(x-C_{7,\Delta}(s))^3-4^2(1-s\sqrt{\Delta})^2C_{7,\Delta}(s)\big[3(x-C_{7,\Delta}(s))+4(1-s\sqrt{\Delta})(27+s\sqrt{\Delta})\big]\ .$$

While $\mathcal{S}$ is defined over $\mathbb{Q}(\sqrt{\Delta})$, none of its nontrivial points are. Computing the normalized quotient $\widetilde{\mathcal{E}}_{7,\Delta,s} \to \widetilde{\mathcal{E}}_{7,\Delta,s}/\mathcal{S}$ (using Eqs. (3), (4), and (5)) and composing with the twisting isomorphism $\delta(1/\sqrt{-7})$ yields an explicit 7-isogeny $\widetilde{\phi}_{7,\Delta,s} : \widetilde{\mathcal{E}}_{7,\Delta,s} \to \widetilde{\mathcal{E}}_{7,\Delta,s}$. Conjugating and composing again, we see that

$$ {}^\sigma\widetilde{\phi}_{7,\Delta,s} \circ \widetilde{\phi}_{7,\Delta,s} = [\epsilon 7]_{\mathcal{E}_{7,\Delta,s}} \quad \text{where } \epsilon = \begin{cases} -1 & \text{if } {}^\sigma\sqrt{-7} = \sqrt{-7} \\ +1 & \text{if } {}^\sigma\sqrt{-7} = -\sqrt{-7} \end{cases} \tag{15}$$

(and similarly, $\widetilde{\phi}_{7,\Delta,s} \circ {}^\sigma\widetilde{\phi}_{7,\Delta,s} = [\epsilon 7]_{{}^\sigma\mathcal{E}_{7,\Delta,s}}$).

The discriminant of $\widetilde{\mathcal{E}}_{7,\Delta,s}$ is $2^{12}\cdot 3^6\cdot 7\cdot C_{7,\Delta}(s)^2(1-s^2\Delta)(1-s\sqrt{\Delta})^6$, and

$$j\big(\widetilde{\mathcal{E}}_{7,\Delta,s}\big) = \frac{(27+s^2\Delta)\big(85+96s\sqrt{\Delta}+15s^2\Delta\big)^3}{(1-s^2\Delta)(1-s\sqrt{\Delta})^6}$$

(letting $s\to\infty$, we find $j(\widetilde{\mathcal{E}}_{7,\Delta,\infty}) = -3375$); so $\widetilde{\mathcal{E}}_{7,\Delta,s}$ reduces modulo any inert $p>7$ to give a family of elliptic curves $\mathcal{E}_{7,\Delta,s}/\mathbb{F}_{p^2}$, and then any value of $s$ in $\mathbb{F}_p$ such that $s^2\neq -27/\Delta$ yields an elliptic curve over $\mathbb{F}_{p^2}$. A calculation similar to Proposition 1 shows that we get at least $p-48$ non-isomorphic curves in this way, when $p$ is sufficiently large.

**Theorem 5.** *Let $p>7$ be prime, fix a nonsquare $\Delta$ modulo $p$, so $\mathbb{F}_{p^2}=\mathbb{F}_p(\sqrt{\Delta})$, and let $\mathcal{E}_{7,\Delta,s}$ and $\phi_{7,\Delta,s}$ be the reductions modulo $p$ of $\widetilde{\mathcal{E}}_{7,\Delta,s}$ and $\widetilde{\phi}_{7,\Delta,s}$.*

*For each $s$ in $\mathbb{F}_p$ such that $s^2\neq -27/\Delta$, the curve $\mathcal{E}_{7,\Delta,s}/\mathbb{F}_{p^2}$ has an efficient $\mathbb{F}_{p^2}$-endomorphism $\psi_{7,\Delta,s} := \pi_p\circ\phi_{7,\Delta,s}$ of degree $7p$ satisfying*

$$\psi_{7,\Delta,s}^2 = [\epsilon_p 7]\pi_{\mathcal{E}_{7,\Delta,s}} \quad \textit{and} \quad (\psi'_{3,\Delta,s})^2 = [-\epsilon_p 7]\pi_{\mathcal{E}'_{7,\Delta,s}}$$

*where*

$$\epsilon_p := -\big(-7/p\big) = \begin{cases} +1 & \textit{if } p\equiv 3,5,6 \pmod 7\ , \\ -1 & \textit{if } p\equiv 1,2,4 \pmod 7\ . \end{cases}$$

*There exists an integer $r$ satisfying $7r^2 = 2p+\epsilon_p t_{\mathcal{E}_{7,\Delta,s}}$ such that*

$$[r]\psi_{7,\Delta,s} = [p]+\epsilon_p\pi_{\mathcal{E}_{7,\Delta,s}} \qquad \textit{and} \qquad [r]\psi'_{7,\Delta,s} = [p]-\epsilon_p\pi_{\mathcal{E}_{7,\Delta,s}}\ ;$$

*the characteristic polynomial of $\psi_{7,\Delta,s}$ and $\psi'_{7,\Delta,s}$ is $P_{7,\Delta,s}(T) = T^2-7rT+7p$.*

*In particular, if $\mathcal{E}_{7,\Delta,s}$ is ordinary and $\mathcal{G}\subseteq\mathcal{E}_{7,\Delta,s}(\mathbb{F}_{p^2})$ is a cyclic subgroup of order $N$ such that $\psi_{7,\Delta,s}(\mathcal{G})\subseteq\mathcal{G}$, then the eigenvalue of $\psi_{7,\Delta,s}$ on $\mathcal{G}$ is*

$$\lambda_{7,\Delta,s} \equiv (p+\epsilon_p)/r \equiv \pm\sqrt{\epsilon_p 7} \pmod N\ .$$

*Proof.* Follows from Theorem 1 and Corollary 2 using Eq. (15). □

## 9 Exceptional CM and 4-dimensional Decompositions

By definition, $\mathbb{Q}$-curves do not have CM. However, if $\widetilde{\mathcal{E}}_s$ is a family of $\mathbb{Q}$-curves then some isolated curves in $\widetilde{\mathcal{E}}_s$ may have CM. These exceptional curves are of interest for 4-dimensional scalar decompositions: they form a natural generalization of the GLV+GLS curves described by Longa and Sica [29].

Briefly: if $\widetilde{\mathcal{E}}/\mathbb{Q}(\sqrt{\Delta})$ has CM by an order of small discriminant, then we can compute an explicit endomorphism $\tilde{\rho}$ of $\widetilde{\mathcal{E}}$ of small degree (using Stark's algorithm [46], say), which then yields an efficient endomorphism $\rho$ on the reduction $\mathcal{E}$ of $\widetilde{\mathcal{E}}$ modulo $p$, exactly as in the GLV construction. If $\widetilde{\mathcal{E}}$ is $d$-isogenous to ${}^{\sigma}\widetilde{\mathcal{E}}$ and $p$ is inert in $\mathbb{Q}(\sqrt{\Delta})$, then $\mathcal{E}$ also has the degree-$dp$ endomorphism $\psi$ constructed in §3. The endomorphisms $[1], \rho, \psi$, and $\rho\psi$ may then be used as a basis for the 4-dimensional decomposition techniques elaborated in [29].

*Practical limitations of 4-dimensional decompositions.* "$\mathbb{Q}$-curves with CM" inherit the chief drawback of the GLV construction: as noted in §1, we cannot hope to find secure (and twist-secure) curves when $p$ is fixed. This scarcity of secure curves is easily explained: reductions of CM endomorphisms (including GLV endomorphisms) are *separable*, and efficient separable endomorphisms have extremely small degree, so that their (dense) defining polynomials can be evaluated quickly.[13] But the degree of an endomorphism is the norm of the corresponding CM-order element; and to have non-integers of very small norm, the CM-order must have a tiny discriminant. Up to twists, the number of elliptic curves with CM discriminant $-D$ is the class number $h(-D)$ (which is asymptotically in $O(\sqrt{D})$). The six orders containing endomorphisms of degree $\leq 3$ have class number 1, and hence only one corresponding $j$-invariant. For $-D = -4$, corresponding to $j = 1728$, there are two or four $\mathbb{F}_{p^2}$-isomorphism classes; for $-D = -3$, corresponding to $j = 0$, we have two or six; and otherwise we have only two. In particular, there are at most 18 pairwise non-isomorphic curves over $\mathbb{F}_{p^2}$ with a nontrivial endomorphism of degree at most 3.

Over a fixed finite field, the probability that any of these curves will have a secure group order, let alone be twist-secure, is very low: roughly speaking, we expect to try $O(\log^2 p)$ random curves over $\mathbb{F}_{p^2}$ before finding a twist-secure one (see [40], for example, for more accurate heuristics). In practice, then, we cannot use these curves when $p$ is fixed for efficiency. Higher-dimensional scalar decomposition speedups therefore come at the cost of suboptimal field arithmetic: we pay for shorter loop lengths with comparatively slower field (and hence group) operations, to say nothing of a more complicated multiexponentiation algorithm.

We must therefore choose between 4-dimensional decompositions and faster underlying field arithmetic. Here we have chosen the latter, so we do not treat CM curves in depth. However, we enumerate the exceptional CM curves in our families in Theorem 6, to provide a convenient source of curves for readers interested in exploring and implementing 4-dimensional techniques.

[13] By dense, we mean that these polynomials have many nonzero terms; the cost of their evaluation therefore depends linearly on the degree.

*Exceptional CM curves.* Any one-dimensional family of $\mathbb{Q}$-curves has only finitely many exceptional CM curves, up to isomorphism, and it is easy to compute them.

**Theorem 6.** *The exceptional CM curves in the families $\widetilde{\mathcal{E}}_{2,\Delta,s}$, $\widetilde{\mathcal{E}}_{3,\Delta,s}$, $\widetilde{\mathcal{E}}_{5,-1,s}$, and $\widetilde{\mathcal{E}}_{7,\Delta,s}$ are as follows. (In each table, if $s\sqrt{\Delta}$ takes the given value then $\widetilde{\mathcal{E}}_{d,\Delta,s}$ has CM by the order of discriminant $-D_0f^2$, where $-D_0$ is the fundamental discriminant and $f$ is the conductor.)*

1. *The following table lists the CM fibres in $\widetilde{\mathcal{E}}_{2,\Delta,s}$ (completing Quer's list [36, §5], where $s\sqrt{\Delta} = 0$, $\pm\frac{5}{9}\sqrt{-7}$, and $\infty$ are missing).*

| $s\sqrt{\Delta}$ | $-D_0f^2$ | $s\sqrt{\Delta}$ | $-D_0f^2$ | $s\sqrt{\Delta}$ | $-D_0f^2$ | $s\sqrt{\Delta}$ | $-D_0f^2$ |
|---|---|---|---|---|---|---|---|
| $\infty$ | $-4\cdot 1^2$ | $\pm\frac{5}{9}\sqrt{-7}$ | $-7\cdot 1^2$ | $\pm\frac{1}{2}\sqrt{5}$ | $-20\cdot 1^2$ | $\pm\frac{5}{18}\sqrt{13}$ | $-52\cdot 1^2$ |
| $\pm\frac{7}{12}\sqrt{3}$ | $-4\cdot 3^2$ | $0$ | $-8\cdot 1^2$ | $\pm\frac{2}{3}\sqrt{2}$ | $-24\cdot 1^2$ | $\pm\frac{70}{99}\sqrt{2}$ | $-88\cdot 1^2$ |
| $\pm\frac{161}{360}\sqrt{5}$ | $-4\cdot 5^2$ | $\pm\frac{20}{49}\sqrt{6}$ | $-8\cdot 3^2$ | $\pm\frac{4}{9}\sqrt{5}$ | $-40\cdot 1^2$ | $\pm\frac{145}{882}\sqrt{37}$ | $-148\cdot 1^2$ |
| | | | | | | $\pm\frac{1820}{9801}\sqrt{29}$ | $-232\cdot 1^2$ |

2. *The following table lists the CM fibres in $\widetilde{\mathcal{E}}_{3,\Delta,s}$ (completing Quer's list [37, §6], where $s\sqrt{\Delta} = 0$, $\pm\frac{1}{4}\sqrt{-11}$, $\pm\frac{5}{2}\sqrt{-2}$, and $\infty$ are missing).*

| $s\sqrt{\Delta}$ | $-D_0f^2$ | $s\sqrt{\Delta}$ | $-D_0f^2$ | $s\sqrt{\Delta}$ | $-D_0f^2$ |
|---|---|---|---|---|---|
| $\infty$ | $-3\cdot 1^2$ | $\pm\frac{5}{2}\sqrt{-2}$ | $-8\cdot 1^2$ | $\pm\frac{1}{2}\sqrt{2}$ | $-24\cdot 1^2$ |
| $0$ | $-3\cdot 2^2$ | $\pm\frac{1}{4}\sqrt{-11}$ | $-11\cdot 1^2$ | $\pm\frac{1}{4}\sqrt{17}$ | $-51\cdot 1^2$ |
| $\pm\frac{5}{9}\sqrt{3}$ | $-3\cdot 4^2$ | $\pm\sqrt{5}$ | $-15\cdot 1^2$ | $\pm\frac{5}{32}\sqrt{41}$ | $-123\cdot 1^2$ |
| $\pm\frac{9}{20}\sqrt{5}$ | $-3\cdot 5^2$ | $\pm\frac{11}{25}\sqrt{5}$ | $-15\cdot 2^2$ | $\pm\frac{53}{500}\sqrt{89}$ | $-267\cdot 1^2$ |
| $\pm\frac{55}{252}\sqrt{21}$ | $-3\cdot 7^2$ | | | | |

3. *The only CM fibres in $\widetilde{\mathcal{E}}_{5,-1,s}$ are $\widetilde{\mathcal{E}}_{5,-1,1}$ (defined over $\mathbb{Q}$) and $\widetilde{\mathcal{E}}_{5,-1,-9/13}$; both have $j$-invariant $66^3$ and CM by the order of discriminant $-4\cdot 2^2$.*
4. *The following table lists the CM fibres in $\widetilde{\mathcal{E}}_{7,\Delta,s}$.*

| $s\sqrt{\Delta}$ | $-D_0f^2$ | $s\sqrt{\Delta}$ | $-D_0f^2$ |
|---|---|---|---|
| $\infty$ | $-7\cdot 1^2$ | $\pm\sqrt{5}$ | $-35\cdot 1^2$ |
| $0$ | $-7\cdot 2^2$ | $\pm\frac{1}{3}\sqrt{13}$ | $-91\cdot 1^2$ |
| $\pm\frac{1}{3}\sqrt{7}$ | $-7\cdot 4^2$ | $\pm\frac{5}{39}\sqrt{61}$ | $-427\cdot 1^2$ |

*Proof.* Suppose $\widetilde{\mathcal{E}}/\mathbb{Q}(\sqrt{\Delta})$ is isogenous to ${}^\sigma\widetilde{\mathcal{E}}$. If $\widetilde{\mathcal{E}}$ has CM by the order of discriminant $-D_0f^2$, then so does ${}^\sigma\widetilde{\mathcal{E}}$; hence, both $j(\widetilde{\mathcal{E}})$ and ${}^\sigma j(\widetilde{\mathcal{E}}) = j({}^\sigma\widetilde{\mathcal{E}})$ are roots of the Hilbert class polynomial $H_{-D_0f^2}$. But $H_{-D_0f^2}$ is irreducible over $\mathbb{Q}$, so either $H_{-D_0f^2}(T) = T - j(\widetilde{\mathcal{E}})$ with $j(\widetilde{\mathcal{E}}) = j({}^\sigma\widetilde{\mathcal{E}})$ in $\mathbb{Q}$, or $H_{-D_0f^2}(T) = (T - j(\widetilde{\mathcal{E}}))(T - j({}^\sigma\widetilde{\mathcal{E}}))$. Tables 1 and 2 list every quadratic imaginary discriminant $-D_0f^2$ such that $\deg H_{-D_0f^2}$ (which is the class number $h(-D_0f^2)$) is 1 or 2, along with the associated $j$-invariants: these can be found in the Echidna database [25], or (re)computed using Magma ([6], [30]) or Sage [45]. To find the exceptional CM curves in each of our families, we solve for rational $s$ and squarefree $\Delta$ such that the $j$-invariant of the family appears in Table 1 or 2. $\square$

Theorem 6 gives a simple alternative construction for some of the curves investigated by Guillevic and Ionica in [19]: the curves $E_{1,c}$ and $E_{2,c}$ of [19, §2] are $\mathcal{E}_{2,\Delta,s}^{\sqrt{3}}$ with $c = s\sqrt{\Delta}$ and $\mathcal{E}_{3,\Delta,s}$ with $c = -2s\sqrt{\Delta}$, respectively. The 255-bit curve of [19, Ex. 1] is a twist of $\mathcal{E}_{2,5,4/9}$ by $\sqrt{3}$. This curve is not twist-secure.

**Table 1.** The thirteen quadratic imaginary discriminants $-D_0 f^2$ of class number 1, together with the $j$-invariants of the elliptic curves over $\overline{\mathbb{Q}}$ with CM by the quadratic order of each discriminant.

| $-D_0 f^2$ | $j$-invariant | $-D_0 f^2$ | $j$-invariant | $-D_0 f^2$ | $j$-invariant | $-D_0 f^2$ | $j$-invariant |
|---|---|---|---|---|---|---|---|
| $-3 \cdot 1^2$ | $0$ | $-4 \cdot 1^2$ | $12^3$ | $-8 \cdot 1^2$ | $20^3$ | $-43 \cdot 1^2$ | $-960^3$ |
| $-3 \cdot 2^2$ | $2 \cdot 30^3$ | $-4 \cdot 2^2$ | $66^3$ | $-11 \cdot 1^2$ | $-2^{15}$ | $-67 \cdot 1^2$ | $-5280^3$ |
| $-3 \cdot 3^2$ | $-3 \cdot 20^3$ | $-7 \cdot 1^2$ | $-15^3$ | $-19 \cdot 1^2$ | $-96^3$ | $-163 \cdot 1^2$ | $-640320^3$ |
| | | $-7 \cdot 2^2$ | $255^3$ | | | | |

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**Table 2.** The twenty-nine quadratic imaginary discriminants $-D_0 f^2$ of class number 2, together with the $j$-invariants of the elliptic curves over $\overline{\mathbb{Q}}$ with CM by the quadratic order of each discriminant.

| $-D_0 f^2$ | $j$-invariants |
|---|---|
| $-3 \cdot 4^2$ | $12 \cdot 15^3(35010 \pm 20213\sqrt{3})$ |
| $-3 \cdot 5^2$ | $-96^3(369830 \pm 165393\sqrt{5})$ |
| $-3 \cdot 7^2$ | $-3 \cdot 480^3(52518123 \pm 11460394\sqrt{21})$ |
| $-4 \cdot 3^2$ | $3 \cdot 4^3(399849 \pm 230888\sqrt{3})$ |
| $-4 \cdot 4^2$ | $2 \cdot 3^3(761354780 \pm 538359129\sqrt{2})$ |
| $-4 \cdot 5^2$ | $12^3(12740595841 \pm 5697769392\sqrt{5})$ |
| $-7 \cdot 4^2$ | $15^3(40728492440 \pm 15393923181\sqrt{7})$ |
| $-8 \cdot 2^2$ | $10^3(26125 \pm 18473\sqrt{2})$ |
| $-8 \cdot 3^2$ | $20^3(23604673 \pm 9636536\sqrt{6})$ |
| $-11 \cdot 3^2$ | $-44 \cdot 16^3(104359189 \pm 18166603\sqrt{33})$ |
| $-15 \cdot 1^2$ | $-5 \cdot 3^3(1415 \pm 637\sqrt{5})/2$ |
| $-15 \cdot 2^2$ | $5 \cdot 3^3(274207975 \pm 122629507\sqrt{5})/2$ |
| $-20 \cdot 1^2$ | $5 \cdot 4^3(1975 \pm 884\sqrt{5})$ |
| $-24 \cdot 1^2$ | $12^3(1399 \pm 988\sqrt{2})$ |
| $-35 \cdot 1^2$ | $-5 \cdot 32^3(360 \pm 161\sqrt{5})$ |
| $-40 \cdot 1^2$ | $5 \cdot 12^3(24635 \pm 11016\sqrt{5})$ |
| $-51 \cdot 1^2$ | $-4 \cdot 48^3(6263 \pm 1519\sqrt{17})$ |
| $-52 \cdot 1^2$ | $60^3(15965 \pm 4428\sqrt{13})$ |
| $-88 \cdot 1^2$ | $60^3(14571395 \pm 10303524\sqrt{2})$ |
| $-91 \cdot 1^2$ | $-96^3(5854330 \pm 1623699\sqrt{13})$ |
| $-115 \cdot 1^2$ | $-5 \cdot 96^3(48360710 \pm 21627567\sqrt{5})$ |
| $-123 \cdot 1^2$ | $-480^3(6122264 \pm 956137\sqrt{41})$ |
| $-148 \cdot 1^2$ | $60^3(91805981021 \pm 15092810460\sqrt{37})$ |
| $-187 \cdot 1^2$ | $-68 \cdot 240^3(2417649815 \pm 586366209\sqrt{17})$ |
| $-232 \cdot 1^2$ | $60^3(1399837865393267 \pm 259943365786104\sqrt{29})$ |
| $-235 \cdot 1^2$ | $-5 \cdot 1056^3(69903946375 \pm 31261995198\sqrt{5})$ |
| $-267 \cdot 1^2$ | $-4 \cdot 240^3(177979346192125 \pm 18865772964857\sqrt{89})$ |
| $-403 \cdot 1^2$ | $-480^3(11089461214325319155 \pm 3075663155809161078\sqrt{13})$ |
| $-427 \cdot 1^2$ | $-5280^3(53028779614147702 \pm 6789639488444631\sqrt{61})$ |

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