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We discuss a general form of the exchange energy for a homogeneous system of interacting electrons in two spatial dimensions, which is particularly suited in the presence of a generic spin-orbit interaction. The theory is best formulated in terms of a generalized fractional electronic polarization. Remarkably, we find that a net generalized polarization does not necessarily translate into an increase in the magnitude of the exchange energy, a fact that, in turn, favors unpolarized states. Our results account qualitatively for the findings of recent experimental investigations.

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INTRODUCTION

In most cases, as a first step in assessing the role of the electronic interactions, an analysis of the role played by the exchange energy is illuminating and most often very useful. As we show here, the exchange energy acquires particularly interesting and unusual properties in two dimensions in the presence of spin orbit of the Rashba type. In this case, the chiral nature of the single-particle states involved competes with standard requirements that are commonly identified with the exchange interaction.

The interplay of spin-orbit coupling and Coulomb interaction is unexplored, except for the initial and limited work of Refs. 1 and 2, where some quasiparticle properties were investigated. Most of the work is relevant in the high-density limit in which only weak corrections to the standard behavior were identified. Systematic studies of the mean-field phase diagram of this interesting system are provided elsewhere.[3–5]

Alongside the general theoretical relevance of such systematic studies, a timely practical application of such an analysis can be found in the study of spin-polarization states in two-dimensional hole systems in a GaAs heterostructure.[6]

The expression for the exchange energy per particle in a homogeneous electron liquid is well known.[7] In particular, for the two-dimensional case, which is of relevance in the present discussion, the formula is given by

$$\mathcal{E}_{x,0}(r_s,p) = -\frac{4\sqrt{2}}{3\pi}\frac{(1+p)^{3/2}+(1-p)^{3/2}}{r_s}, \tag{1}$$

where we measure energy in Rydbergs and the significance of the subindex 0 will be presently made clear.

In Eq. (1), the parameter p is defined as follows:

$$p = \frac{n_+ - n_-}{n}, \tag{2}$$

where n is the particle number density and $n_\pm$ refers to the particle density corresponding to the two spin subbands. $r_s = 1/\sqrt{\pi a_B^2 n}$ is the usual density parameter in two dimensions. It is important to note that Eq. (1) is obtained under the assumption that the spins are quantized along a common arbitrary quantization axis $\hat{z}$. Therefore, when, as in this particular case, the label $\pm$ refers to the orientation $\uparrow\downarrow$ of the spin, p assumes the meaning of a fractional spin polarization. It is easy to see that the exchange energy (1) monotonically attains its maximum magnitude for $p=1$, and therefore favors the polarization of the system. In fact, within the Hartree-Fock approximation, a transition to a fully spin polarized state occurs for $r_s = \frac{3\pi}{8(2-\sqrt{2})} \simeq 2.01$.[7]

As we will presently show, this scenario changes in interesting ways in the presence of the spin-orbit interaction. In the quasi-two-dimensional electronic systems present in a high-quality semiconducting heterostructure, the latter can come in different guises ranging from Rashba and Dresselhaus couplings to terms induced by the application of an external magnetic field.[8] The following is a systematic study of the effects of spin-orbit interactions of various types on the exchange energy and the polarization properties of a clean two-dimensional system. The theory can be elegantly formulated in a compact way by introducing a generic form of spin-orbit coupling, something that allows us to draw quite general conclusions.

The paper is organized as follows: Sec. I introduces our model Hamiltonian including a spin-orbit coupling of generic form. Section II describes the solution of the corresponding noninteracting problem in terms of the generalized polarization. In Sec. III, which represents the main part of the paper, the role of the exchange energy is studied in detail. In particular, it is demonstrated there that the presence of spin-orbit coupling leads to a qualitatively different relation between this quantity and the generalized polarization. Section IV contains a discussion of recent experiments on the polarization of holes in GaAs heterostructures in the light of these findings. Finally, some general remarks and conclusions are provided in the last section.

I. MODEL HAMILTONIAN

We consider the following generic model two-dimensional electronic system described by the Hamiltonian[9]

$$\hat{H}_n = \sum_i \hat{H}_{0,n}^{(i)} + \frac{1}{2}\sum_{i\neq j}\frac{e^2}{|\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j|}, \tag{3}$$

where the single-particle terms are of the form

$$\hat{H}_{0,n} = \frac{\hat{\mathbf{p}}^2}{2m} + i\gamma \frac{(\hat{p}_-)^n \hat{\sigma}_+ - (\hat{p}_+)^n \hat{\sigma}_-}{2}, \tag{4}$$

with the motion taking place in the x-y plane. In Eq. (4), we have defined $\hat{p}_\pm = \hat{p}_x \pm i\hat{p}_y$ and $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$. Here, n is an integer number assuming values from 0 to 3. In view of its structure, we will refer to the second term in Eq. (4) as a generalized spin-orbit coupling.

In the simplest case, $n=0$ corresponds to the familiar Zeeman coupling with $\gamma = \frac{g\mu_B B}{2}$ for which the standard result [Eq. (1)] applies. More complicated $n=0$ expressions are contemplated in the Appendix.

For $n>0$, Eq. (4) describes different types of bona fide spin-orbit interactions. In particular, for $n=1$, we obtain a form equivalent to the Rashba[10,11] or Dresselhaus[12] spin-orbit Hamiltonians, appropriate for two-dimensional conduction electrons in III-V semiconductors. Furthermore, the cases $n=2$ and $n=3$ are appropriate for holes in III-V semiconductors such as GaAs. Explicit expressions are provided in the Appendix.

From a physical point of view, the $n=2$ term arises in the presence of an external in-plane magnetic field, the coefficient γ being proportional to the value B of such field. For hole carriers in GaAs heterostructures, this term is the dominating low-field effect in the high-symmetry growth directions [001] and [111], for which the Zeeman term can be safely assumed to be approximately vanishing.[8]

Finally, the $n=3$ term corresponds to a Rashba spin-orbit coupling for holes, the value of γ being in this case approximately proportional to the value of the average electric field of the confining potential.[13]

II. NONINTERACTING ELECTRONS

The eigenstates of Eq. (4) are plane waves, a fact that allows one to write the spin-orbit term in the form $-\gamma(\hbar k)^n \boldsymbol{\sigma}\cdot\hat{s}_{\mathbf{k}}$, where $\hat{s}_{\mathbf{k}}$ is defined as follows:

$$\hat{s}_{\mathbf{k}} = -\sin(n\phi_{\mathbf{k}})\hat{x} + \cos(n\phi_{\mathbf{k}})\hat{y}. \tag{5}$$

Here, $\phi_{\mathbf{k}} = \arctan(k_y/k_x)$ is the polar angle spanned by $\mathbf{k}$. The unit vector $\hat{s}_{\mathbf{k}}$ determines the direction of the quantization axis for the particular value of the wave vector $\mathbf{k}$. The two possible spin orientations immediately give the eigenstates

$$\varphi_{\mathbf{k},\pm}(\mathbf{r}) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{2L^2}} \begin{pmatrix} \pm 1 \\ ie^{in\phi_{\mathbf{k}}} \end{pmatrix}, \tag{6}$$

with energies[14]

$$\epsilon_{\mathbf{k},\pm} = \frac{\hbar^2 k^2}{2m} \mp \gamma(\hbar k)^n. \tag{7}$$

Note that in view of the structure of the spinors, the integer n can be seen as a spin direction winding number.

It is not difficult to show[3] that a class of determinantal many-body states with compact momentum space occupation, which are homogeneous and isotropic in the plane of motion, is uniquely determined by the areal density n and the fractional polarization p defined as in Eq. (2).[5,15] It must be immediately noted here that in the general case, p should not be interpreted as a spin polarization for this is, in general, vanishing in these states. p does merely determines the two Fermi vectors $k_\pm$ of the so-called spin-split subbands via the relations

$$k_\pm = \sqrt{2\pi n(1\pm p)}, \tag{8}$$

which uniquely determine the occupation numbers $n_{\mathbf{k}\pm}$ of the subbands.

It is readily found that the noninteracting energy per particle (in Rydbergs) of such Slater determinants is given by

$$\mathcal{E}_n^{(0)}(r_s,p) = \frac{1+p^2}{r_s^2} - \bar{\gamma}\frac{2^{n/2}}{r_s^n}\frac{(1+p)^{1+n/2} - (1-p)^{1+n/2}}{1+n/2}, \tag{9}$$

where we have defined the dimensionless coupling

$$\bar{\gamma} = \frac{m^{n-1}e^{2(n-2)}}{\hbar^{n-2}}\gamma. \tag{10}$$

The first term in Eq. (9) is the contribution of the kinetic energy, while the second corresponds to the spin-orbit interaction.

In this case, minimization of the total energy with respect to the value of p, $\frac{\partial \mathcal{E}_n^{(0)}}{\partial p} = 0$, leads to the (finite) ground-state value of the equilibrium generalized polarization $p_{min}^{(0)}$. Interestingly, the result only depends on the dimensionless quantity

$$g = 2^{n/2}\bar{\gamma}r_s^{2-n}, \tag{11}$$

so that it can be compactly expressed as follows:

$$p_{min}^{(0)} = \begin{cases} g & \text{for } n=0 \\ g\sqrt{1-\frac{g^2}{4}} & \text{for } n=1 \\ g & \text{for } n=2 \\ g\sqrt{\frac{-3g^4+6g^2-2+2(1-2g^2)^{3/2}}{g^6}} & \text{for } n=3. \end{cases} \tag{12}$$

For small coupling, this quantity behaves as

$$p_{min}^{(0)} \simeq g + \frac{n(n-2)}{8}g^3 + O(g^5). \tag{13}$$

One can then immediately notice that since for $n=1$ we have $g=\sqrt{2}\bar{\gamma}r_s$, in this case, the high-density regime is equivalent to a vanishing spin orbit. The opposite is obtained for $n=3$, since $g=2\sqrt{2}\bar{\gamma}/r_s$. For quadratic spin orbit $g=2\bar{\gamma}$, and the fractional generalized polarization p is independent of the density.

It is also useful to define here the depopulation coupling strength $\bar{\gamma}_d^{(0)}$ as the particular value of $\bar{\gamma}$ for which, at a given

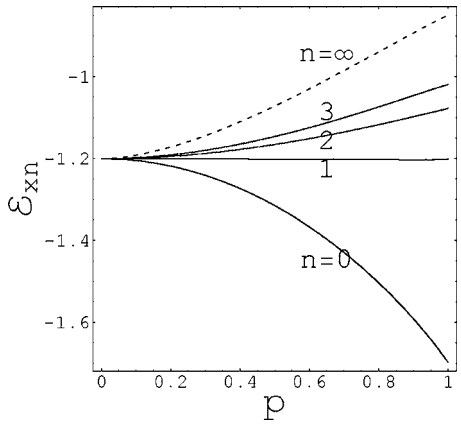

FIG. 1. Plot of the exchange energy per particle $\mathcal{E}_{xn}(r_s,p)$ (in Ry units) as a function of p. Here, $r_s=1$. The different values of n are noted, and the limiting curve $n=\infty$ is also displayed (dashed line).

density, the upper band empties and $p_{min}^{(0)}=1$. This is readily obtained from Eq. (12) to be given by

$$\bar{\gamma}_d^{(0)} = \begin{cases} \dfrac{1}{r_s^2} & \text{for } n=0 \\ \dfrac{r_s^{n-2}}{2^{n-1}} & \text{for } n=1,2,3. \end{cases} \tag{14}$$

III. EFFECTS OF EXCHANGE

We turn next to the effect of exchange. If translational invariance is not broken,[5] a single Slater determinant can be constructed with plane-wave states characterized by generic orientation of the spin-quantization axis $\hat{s}_{\mathbf{k}}$ and occupation numbers $n_{\mathbf{k}\pm}$. The total exchange energy can be written in the following elegant general form:[3,4]

$$E_x = -\frac{1}{2L^2}\sum_{\mathbf{k},\mathbf{k}';\mu,\mu'=\pm} v_{\mathbf{k}-\mathbf{k}'}\frac{1+\mu\mu'\hat{s}_{\mathbf{k}}\cdot\hat{s}_{\mathbf{k}'}}{2}n_{\mathbf{k}\mu}n_{\mathbf{k}'\mu'}, \tag{15}$$

which represents a functional of $n_{\mathbf{k}\pm}$ and $\hat{s}_{\mathbf{k}}$. Equation (15) immediately reduces to the familiar textbook result when $\hat{s}_{\mathbf{k}}=\hat{z}$. In this case, only states with parallel spin contribute.

Making use of Eq. (5) in Eq. (15) leads to the following result for the exchange energy per particle (in Rydbergs):

$$\mathcal{E}_{x,n}(r_s,p) = \mathcal{E}_{x,0}(r_s,p) + \delta\mathcal{E}_{x,n}(r_s,p), \tag{16}$$

where the correction to Eq. (1) can be obtained from the following quadrature:

$$\delta\mathcal{E}_{x,n}(r_s,p) = \frac{\sqrt{2}}{4\pi r_s}\int_{\sqrt{1-p}}^{\sqrt{1+p}} x\,dx\int_{\sqrt{1-p}}^{\sqrt{1+p}} y\,dy \times\int_0^{2\pi}\frac{1-\cos n\theta}{\sqrt{x^2+y^2-2xy\cos\theta}}d\theta. \tag{17}$$

The resulting exchange energy is plotted in Fig. 1 as a function of p. We notice that for $p=0$, the exchange energy is independent of the spin-quantization axis orientations $\hat{s}_{\mathbf{k}}$. This can be understood by realizing that the corresponding many-body state can be constructed by repeated application of $b_{\mathbf{k}+}^{\dagger}b_{\mathbf{k}-}^{\dagger}$, an operator that creates a spin singlet and is therefore independent of the spin-quantization direction. As a consequence, the $\hat{s}_{\mathbf{k}}$ dependence of all the physical quantities (e.g., the exchange and the spin-orbit energies) stems only from the existence of regions of momentum space where $n_{\mathbf{k}+}\neq n_{\mathbf{k}-}$. It is also important to remark that only for $n=0$ is the magnitude of the exchange energy maximum for $p=1$. For the other cases, the minimum occurs at $p=p_1^*\simeq 0.915$ (although this is not obvious from Fig. 1) for $n=1$ and at $p=0$ for $n\geqslant 2$.

The different behaviors of the exchange energy in the various cases lead to dissimilar results.

For $n=0$, the fact that the minimum of $\mathcal{E}_{x,0}$ occurs at $p=1$ leads to an enhancement of p_{min}, i.e., to the familiar enhancement of the spin polarization. The opposite is true for $n\geqslant 2$ since in these cases the minimum of $\mathcal{E}_{x,0}$ occurs at $p=0$.

For completeness, we remark that in the limiting case of very large winding number n, the result can be obtained simply neglecting the $\cos n\theta$ contribution in Eq. (17). For comparison, the corresponding curve is shown as a dashed line in Fig. 1. Note that in this case, the magnitude of the exchange energy is minimum for $p=1$, the value being given by

$$\lim_{n\to\infty}\mathcal{E}_{x,n}(r_s,1) = -\frac{8}{3\pi r_s}. \tag{18}$$

The situation for the $n=1$ case is more complex, although in all cases the exchange only leads to a very small deviation from $p_{min}^{(0)}$. Specifically, p_{min} is slightly enhanced for $p_{min}^{(0)}<p_1^*$ and slightly diminished for $p_{min}^{(0)}>p_1^*$, being unrenormalized for $p_{min}^{(0)}=p_1^*$.

A similar argument leads one to conclude that the critical value $\bar{\gamma}_d$ for which the upper spin band empties (at fixed r_s) decreases from its noninteracting value for $n=0$ $(g=1)$, while it does increase in the other cases.

Studying the limit of small p is of particular interest, since it corresponds to a determination of the generalized susceptibility. In this case, a direct inspection of the integral of Eq. (17) leads to the following asymptotic formula:

$$\mathcal{E}_{x,n}(r_s,p) \simeq -\frac{8\sqrt{2}}{3\pi r_s} - \frac{C_n}{r_s}p^2, \tag{19}$$

where we have defined the quantity

$$C_n = \frac{\sqrt{2}}{\pi}\sum_{m=0}^{n}\frac{1}{1-2m}. \tag{20}$$

The resulting value for p_{min} is then given by

$$p_{min} \simeq \frac{g}{1-C_n r_s}. \tag{21}$$

Equation (21) simply expresses the fact that in this limit, the effect of the interactions is to renormalize the noninteracting result $p_{min}^{(0)}\simeq g$ via the denominator $(1-C_n r_s)^{-1}$. Inter-

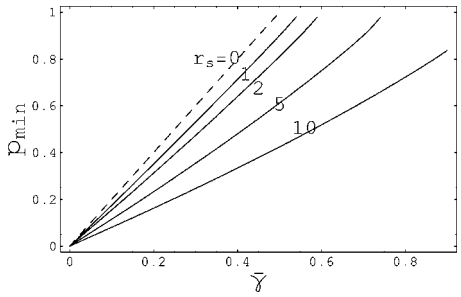

FIG. 2. Plot of the fractional generalized polarization p_{min} as function of the parameter $\bar{\gamma}$ for different values of r_s. Here, $n=2$. The increase of the depopulation value $\bar{\gamma}_d$ with r_s is manifested.

estingly, the latter corresponds to an enhancement only for $n=0$.

In particular, for $n=0$, we recover the well-known Hartree-Fock differential instability occurring at $r_s=\frac{\pi}{\sqrt{2}}$.[7] On the other hand, for $n=1$, we have $C_1=0$, the whole renormalization effect being solely associated with correlation effect. Finally, for $n\geqslant 2$, C_n is (ever increasingly) negative leading to a perhaps iconoclastic exchange driven quenching of the generalized polarization.

Expression (21) is valid in the limit of $\bar{\gamma}\to 0$ or, for $n>1$, when r_s is large. The generic case when p_{min} is not small must be obtained numerically. As an example, we show in Fig. 2 the value of p_{min} as a function of the dimensionless coupling strength $\bar{\gamma}$ for different values of the density parameter r_s in the case of $n=2$. We notice that for low densities, the depopulation value $\bar{\gamma}_d$, for which $p_{min}=1$, is considerably increased by exchange. This is in stark contrast with the familiar $n=0$ case. In particular, for the $n=2$ case of Fig. 2, we have:

$$\bar{\gamma}_d=\frac{1}{2}+\left(\frac{1}{12}-\frac{1}{9\pi}\right)r_s, \tag{22}$$

where the second term represents the increase of the depopulation field due to exchange effects. In this particular case, the correction is linear in the density parameter r_s.

IV. APPLICATION TO SPIN POLARIZED HOLE SYSTEMS

Spin-polarization experiments have recently been performed on GaAs two-dimensional hole systems with growth direction along [113] and [100].[6] In these studies, the magnitude of the in-plane depopulation field B_d, when only one band is occupied, is surmised from the measured longitudinal magnetoresistance. While a small suppression of B_d with respect to its noninteracting value B_d^0 is observed in the [113] case, basically no suppression is observed for the [100] growth direction. This must be contrasted with the fact that, as discussed in Ref. 6, in the absence of spin-orbit interaction and at the densities under consideration, the expected ratio for electrons is of order $B_d^0/B_d\sim 10$.

The experiments of Ref. 6 are carried out at somewhat low densities ($r_s\simeq 10-15$), a regime in which a simple Hartree-Fock treatment, as well as somewhat more sophisticated analytical treatments designed to approximately include correlation effects, generally fails to provide reliable quantitative results. On the other hand, Monte Carlo analysis of the system being to date nonexistent, it is reasonable to expect that many of the qualitative features established by a study of the exchange energy will prove sufficiently robust to justify such a preliminary discussion.

In these quantum wells, the introduction of a magnetic field induces, in the effective Hamiltonian of two-dimensional holes, the spin dependent terms given in Eqs. (A3), (A4), and (A7) of the Appendix. All these terms can be reduced to some of the generic forms contemplated by our model Hamiltonian [Eq. (4)]. The corresponding coupling strength γ for each of these terms can then be extracted, as, for instance, explicitly done in Eqs. (A5), (A6), and (A8). One can then make use of the definitions (10) and (11), alongside suitable numerical parameters, to determine the relevant dimensionless coupling strength.

For our numerical estimates, we use values appropriate for the case of Ref. 6. In particular, $W=200$ Å for the width of the quantum well, $m\simeq 0.2m_0$ for the effective mass, $\epsilon=12.4$ for the background dielectric constant, and $n=3\times 10^{10}$ cm^{-2} for the hole density (corresponding to $r_s\simeq 10$).

We first consider the [113] growth direction. In this case, the various physical properties are anisotropic in the plane, the principal directions being given by $x\equiv[1\bar{1}0]$ and $y\equiv[33\bar{2}]$. For the density under consideration, the depopulation field is approximately given by $B_d\simeq 10$ T along x and $B_d\simeq 5$ T along y. This gives the following results for various dimensionless couplings. For Eq. (A3), we have

$$|g_{21}^x|=0.19 \quad \text{and} \quad |g_{21}^y|=0.09. \tag{23}$$

For Eq. (A4),

$$|g_{03}^x|=0.23 \quad \text{and} \quad |g_{03}^y|=0.03. \tag{24}$$

Finally, for Eq. (A7),

$$|g_{01}^x|=0.14 \quad \text{and} \quad |g_{01}^y|=0.16. \tag{25}$$

We recall here that Eq. (A3) is a term of type $n=2$, while Eqs. (A4) and (A7) are both of type $n=0$. We note that the quadratic spin orbit, although in general smaller, has a strength which is comparable to that of the terms of the Zeeman type.

For the [100] growth direction, the Zeeman term [Eq. (A7)] is vanishing, and the depopulation magnetic field is approximately $B_d\simeq 10$ T. Therefore, we obtain that the quadratic spin orbit and the Zeeman term cubic in B have comparable strength: $|g_{03}|\simeq|g_{21}|\simeq 0.2$.

These estimates suggest that the apparent quenching of the many-body enhancement of the spin susceptibility is due to the presence of a large $n=2$ spin-orbit coupling. Moreover, the presence of a sizable $n=3$ Rashba spin-orbit term would also result in a reduction of the generalized polarization. This is consistent with the experimental finding that the suppression is most noticeable for [100] quantum wells, for which the quadratic spin orbit is comparatively stronger.

From our theory, one can moreover surmise that a larger many-body enhancement of the susceptibility is expected in

the limit of very low densities when, due to the $\propto k^2$ and $\propto k^3$ dependences, the $n=2$ and $n=3$ terms become less relevant. The enhancement should also be more noticeable in the limit of a very narrow well. In the particular case of the [113] growth direction, this happens because the linear Zeeman term [Eq. (A7)] can, in principle, become dominant. For the [100] growth direction, on the other hand, the term (A7) vanishes while the cubic Zeeman term [Eq. (A4)] is largest, this in spite of the $\sim W^4$ proportionality of the coupling strength. The reason is that the depopulation field is large in a very narrow well. This is obtained since $B_d \sim 1/W^{4/3}$ while the magnitude of the quadratic spin orbit [Eq. (A3)] behaves like $\sim W^{2/3}$, thereby losing its relevance.

It should be kept in mind that, being based on a perturbative treatment and not taking in account orbital effects, these conclusions should be taken at best as qualitative.[16]

DISCUSSION AND CONCLUSIONS

The main conclusion of our analysis is that in the presence of quadratic and cubic spin-orbit interactions, the magnitude of the exchange energy decreases with increasing generalized polarization. This results in a quenched value of p and in a corresponding increase of the value of the depopulation coupling $\bar{\gamma}_d$. By the same token, the corresponding generalized susceptibility is also quenched. This interesting phenomenon stems from the in-plane rotation of the spin-quantization axis induced by the spin-orbit coupling proper ($n=1,2,3$) and from the universal structure of Eq. (15), which only depends on the momentum space occupation (i.e., the generalized polarization p) and the spin orientations $\hat{s}_{\mathbf{k}}$. As one can verify by making use of the very same equation, for a given value of p, the only difference between the various spin-orbit interactions stems from the different form acquired by $\hat{s}_{\mathbf{k}}$. Since for larger n the spins are "less parallel," it is clear that the magnitude of the exchange energy will decrease for larger n.

Being based on a study of the exchange energy only, our theory can be expected to be strictly valid in the high-density limit. The reason is that it is in this regime that the exchange energy represents the first interaction correction to the noninteracting result, with correlation effects becoming comparatively smaller as r_s decreases. On the other hand, at not too low densities, the physics of the exchange is still expected to give qualitatively reasonable results. This conclusion appears to be corroborated by the apparent observed reduction of the many-body enhancement of the spin susceptibility in dilute hole systems in which, beside the familiar Zeeman term, the magnetic field also induces large quadratic spin-orbit interactions.

It is not difficult to prove that, when the spin-orbit coupling terms described by Eq. (4) are present in isolation, the many-body states, as parametrized by r_s and p, are also self-consistent solutions of the Hartree-Fock equations.[3,4] On the other hand this is not the case when multiple concomitant terms are present. In such situations, the circular symmetry is broken, and the interacting problem is considerably more complicated. Such situations must be treated case by case. In particular, the spin-quantization directions $\hat{s}_{\mathbf{k}}$ must be determined self-consistently.[17] This problem has nontrivial solutions in the case of broken symmetry states, like, for instance, in the case of ferromagnetic phases.[3,4]

As a final remark we stress is that since the generalized fractional polarization does not correspond directly to an actual magnetization, strictly speaking, one cannot draw direct conclusions about the enhancement of the spin-spin response from measurements of the depopulation field B_d. In general, the spin susceptibility is enhanced by the exchange, in a way similar to the usual case without spin orbit.[4] The bare spin-spin susceptibility involves the response to a pure $n=0$ perturbation, which, from an experimental point of view, is not straightforward to realize for the case of hole systems of Ref. 6. In fact, as we have argued, the external magnetic field induces also a change of the $n=2$ spin-orbit coupling.

A detailed study of the linear spin-spin response and of the phase diagram, within the framework of the Hartree-Fock approximation and in the presence of spin-orbit coupling, will be the subject of future publications. Interestingly, the transition to a ferromagnetic state occurs at densities that are, in general, *larger* than in the absence of spin-orbit interaction.[3,4]

APPENDIX

We obtain and discuss here the spin dependent contributions to the effective Hamiltonian appropriate to the highest two-dimensional heavy-hole subband in the presence of an in-plane magnetic field $\mathbf{B}=B_x\hat{x}+B_y\hat{y}$ for a typical III-V semiconductor quantum well. The specific numerical value of the parameters will be chosen to be appropriate to the case of GaAs.

We begin by approximately describing the motion of the holes in the bulk through the standard Luttinger Hamiltonian,[18] which in spherical approximation takes the form

$$\hat{H}_h = -\frac{1}{2m_0}\left[\left(\gamma_1 + \frac{5}{2}\tilde{\gamma}\right)\hat{\mathbf{p}}^2 - 2\tilde{\gamma}(\hat{\mathbf{J}}\cdot\hat{\mathbf{p}})^2\right], \tag{A1}$$

where $\hat{J}_i$ are 4×4 spin-3/2 matrices and, for the moment, cubic corrections have been neglected. For GaAs, $\gamma_1=6.85$ and $\tilde{\gamma}=(\gamma_2+\gamma_3)/2=2.5$. Within this context, the effect of the magnetic field can be described by introducing the Zeeman Hamiltonian $\hat{H}_Z=-2\kappa\mu_B\mathbf{B}\cdot\hat{\mathbf{J}}$, where for GaAs $\kappa=1.2$, a convenient choice of the vector potential being provided by $\mathbf{A}=zB_y\hat{x}-zB_x\hat{y}$.

As a specific model case, we consider here the confinement associated with an infinite rectangular well of width W. The corresponding effective Hamiltonian for holes in the highest two-dimensional subband can be written as

$$\hat{H}_0 = \mathcal{E}_0(\hat{\mathbf{p}}) + \delta\hat{H}_{21} + \delta\hat{H}_{03}, \tag{A2}$$

where $\mathcal{E}_0(\mathbf{p})$ is the subband energy dispersion for $\mathbf{B}=0$, and $\delta\hat{H}_{21}$ and $\delta\hat{H}_{03}$ are spin dependent terms associated with the external magnetic field with subindices indicating the value of the integer n and the power of their dependence on the magnetic field. Their explicit form can be found by making

use of perturbation theory[8] in **B** and in the wave vector **k**. For the first term, we obtain

$$\delta\hat{H}_{21} = \frac{a\mu_B W^2}{\pi^2\hbar^2}\frac{B_+p_+^2\hat{\sigma}_- + B_-p_-^2\hat{\sigma}_+}{2}, \tag{A3}$$

where $B_\pm = B_x \pm iB_y$ and the explicit form of the numerical coefficient is given by $a = \frac{1024\tilde{\gamma}^2}{9\pi^2(3\gamma_1+10\tilde{\gamma})} - \frac{3\kappa}{2}$. For the second term, we find

$$\delta\hat{H}_{03} = b\mu_B^3\left(\frac{m_0W^2}{\pi^2\hbar^2}\right)^2\frac{B_+^3\hat{\sigma}_- + B_-^3\hat{\sigma}_+}{2}, \tag{A4}$$

where $b = \frac{\kappa(\pi^2-6)}{2} - \frac{27\tilde{\gamma}^2}{8(2\gamma_1+5\tilde{\gamma})}$. We should remark that the present results do differ from the ones one would infer from the corresponding formulas appearing in Ref. 8. For GaAs, we have $a \simeq -0.2$ and $b \simeq 1.5$.

With a suitable spin rotation, these contributions can be both transformed to the form of generic spin-orbit defined in our model Hamiltonian [Eq. (4)]. In particular, $\delta\hat{H}_{21}$ and $\delta\hat{H}_{03}$ are of the type $n=2$ and $n=0$, respectively. Because of the isotropy implied by the spherical approximation, the value of the corresponding coupling strength γ extracted by comparison to Eq. (4) only depends on the magnitude of the magnetic field B and is immediately found to be

$$\gamma_{21} = \frac{a\mu_B W^2 B}{\pi^2\hbar^2} \tag{A5}$$

and

$$\gamma_{03} = b\mu_B^3\left(\frac{W^2m_0}{\pi^2\hbar^2}\right)^2 B^3, \tag{A6}$$

for $\delta\hat{H}_{21}$ and $\delta\hat{H}_{03}$, respectively.

For a given growth direction, these results can be extended beyond the spherical approximation to include the appropriate cubic anisotropy as, for instance, done in Ref. 8. Following this procedure, one then obtains in Eq. (A2) an additional anisotropic linear Zeeman term. By choosing coordinates along the principal axes, this can be generally expressed as

$$\delta\hat{H}_{01} = \frac{\mu_B}{2}(g_xB_x\hat{\sigma}_x + g_yB_y\hat{\sigma}_y). \tag{A7}$$

As it turns out, this term vanishes (even beyond the spherical approximation) for the high-symmetry growth directions [100] and [111]. However, in the case of the [113] growth direction, Eq. (A7) is nonvanishing and the principal axes x and y are along the $[1\bar{1}0]$ and $[33\bar{2}]$, respectively. A perturbative estimate of the suitable Landé g factors for the case of an infinite rectangular well gives[8] $g_x \simeq -0.17$ and $g_y \simeq 0.41$. The coupling strength γ, as defined in Eq. (4), of this $n=0$ term depends on the direction of the magnetic field, and is given by

$$\gamma_{01}^i = \frac{g_i\mu_B}{2}B, \tag{A8}$$

for the particular case of an external field of magnitude B along one of the two principal axes ($i \equiv x$ or y).

Finally, we mention that including a transverse electric field, one can develop a perturbation theory in $\mathcal{E}_z$ and **k** to obtain a term corresponding to the $n=3$ spin-orbit coupling appearing in Eq. (4). In this case, the explicit form of the coefficient reads

$$\gamma = \frac{512e\tilde{\gamma}^2W^4}{9\hbar^3\pi^6(\gamma_1-2\tilde{\gamma})(3\gamma_1+10\tilde{\gamma})}\mathcal{E}_z. \tag{A9}$$

This again differs from the corresponding expression surmised from Ref. 8.

[1] G.-H. Chen and M. E. Raikh, Phys. Rev. B **60**, 4826 (1999).
[2] D. S. Saraga and D. Loss, Phys. Rev. B **72**, 195319 (2005).
[3] G. F. Giuliani and S. Chesi, *Proceedings of Highlights in the Quantum Theory of Condensed Matter* (Edizioni della Normale, Pisa, 2005).
[4] S. Chesi and G. F. Giuliani (unpublished).
[5] Spatially inhomogeneous solutions have been investigated in G. Simion and G. F. Giuliani (unpublished).
[6] R. Winkler, E. Tutuc, S. J. Papadakis, S. Melinte, M. Shayegan, D. Wasserman, and S. A. Lyon, Phys. Rev. B **72**, 195321 (2005).
[7] G. F. Giuliani and G. Vignale, *Quantum Theory of the Electron Liquid* (Cambridge University Press, Cambridge, 2005).
[8] R. Winkler, *Spin-Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems* (Springer, Berlin, 2003).
[9] In this expression, standard terms associated with the neutralizing background have been omitted for simplicity (Ref. 7).
[10] Y. A. Bychkov and E. Rashba, JETP Lett. **39**, 78 (1984).
[11] Y. A. Bychkov and E. Rashba, J. Phys. C **17**, 6039 (1984).
[12] G. Dresselhaus, Phys. Rev. **100**, 580 (1955).
[13] This is the case when the heavy-hole–light-hole subband splitting is independent of the field: see, however, the discussion in B. Habib, E. Tutuc, S. Melinte, M. Shayegan, D. Wasserman, S. A. Lyon, and R. Winkler, Appl. Phys. Lett. **85**, 3151 (2005).
[14] The form of the spin-orbit coupling we consider is obviously not valid at large k.
[15] We will assume that for the $n=1$ Rashba case, both chiral bands are occupied. When this is not the case (as it can, in principle, occur at very low densities or large spin-orbit coupling), one can tackle the problem by introducing the device of generalized chirality as done in Ref. 3.
[16] As pointed out in Ref. 6, a potentially relevant phenomenon we have neglected is that of the effective-mass enhancement associated with the magnetic field.
[17] This also happens in the presence of terms equivalent to the same n by different spin rotations, as for example in the simultaneous presence of Rashba and Dresselhaus spin orbits.
[18] J. M. Luttinger, Phys. Rev. **102**, 1030 (1956).