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Oscillating *p*-branes

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Coset methods are used to construct the action describing the dynamics associated with the spontaneous breaking of the Poincaré symmetries of D dimensional space-time due to the embedding of a p-brane with codimension $N = D - p - 1$. The resulting world volume action is an $ISO(1, p + N)$ invariant generalization of the Nambu-Goto action in $d = p + 1$ dimensional space-time. Analogous results are obtained for an anti-de Sitter (AdS) p-brane with codimension N embedded in D dimensional AdS space, yielding an $SO(2, p + N)$ invariant version of the Nambu-Goto action in $d = p + 1$ dimensional space-time. Attention is focused on a supersymmetric extension of the $D = 6$ Minkowski space case with an embedded $p = 3$ brane; a particular realization of which is provided by a non-BPS vortex. Here both the Nambu-Goto-Akulov-Volkov action and its dual tensor form are presented.

I. BRANE OSCILLATIONS IN MINKOWSKI SPACE

The nonlinear realization of spontaneously broken space-time symmetries provides a systematic method of obtaining dynamical properties of a lower dimensional probe brane whose presence is responsible for the symmetry breakdown. In the case that the bulk is D dimensional Minkowski space, the p-brane defect breaks the isometry group from $ISO(1, D - 1)$ down to that of the $d = (1 + p)$ dimensional world volume and its $N = (D - d)$ dimensional complement, $ISO(1, p) \times SO(N)$. The long wavelength modes of the p-brane are described by the Nambu-Goldstone bosons associated with the collective coordinate translations transverse to the brane into the N dimensional covolume. Indeed, the Nambu-Goto action governing the zero mode fields' dynamics is readily obtained in a model independent way by nonlinearly realizing the broken symmetries on the Nambu-Goldstone fields [1,2]. The symmetry generators of $ISO(1, D - 1)$ are the D dimensional translations $P^{\mathcal{M}}$ and Lorentz transformations $M^{\mathcal{MN}}$ where $\mathcal{M}, \mathcal{N} = 0, 1, \ldots, p + N$. For the case of a p-brane with codimension N, the d dimensional world volume Poincaré generators, P^μ for space-time translations and $M^{\mu\nu}$ for Lorentz rotations, with $\mu, \nu = 0, 1, \ldots, p$, form an unbroken subgroup $ISO(1, p)$ of the D dimensional Poincaré group $ISO(1, D - 1)$. In addition, the rotations in the N dimensional covolume with generators $T^{ij} = M^{p+i, p+j}$, where $i, j = 1, 2, \ldots N$, form an unbroken subgroup $SO(N)$. The broken symmetry charges are the generators of translations transverse to the brane, denoted by $Z_i = -P^{p+i}$, and the generators of the broken bulk Lorentz

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transformations, denoted by $K_i^\mu = 2M^{p+i, \mu}$. The d dimensional Lorentz scalars Z_i are in the fundamental (vector) representation of the unbroken $SO(N)$ subgroup. Likewise, the d dimensional Lorentz vector K_i^μ is also a $SO(N)$ vector. The explicit form of the charge algebra is given in [3] and can be found as the Minkowski limit of the anti-de Sitter (AdS) algebra of Eqs. (2.1), (2.2), (2.3), and (2.4).

The D dimensional Poincaré group of transformations can be realized by group elements acting on the $ISO(1, D - 1)/SO(1, p) \times SO(N)$ coset element Ω which is constructed from the P^μ, Z_i, K_i^μ charges as

$$\Omega(x) \equiv e^{ix^\mu P_\mu} e^{i\phi^i(x)Z_i} e^{iv_i^\mu(x)K_\mu^i}. \quad (1.1)$$

Here x^μ are the d dimensional world volume space-time coordinates of the p-brane in the static gauge, while $\phi^i(x)$ and $v_i^\mu(x)$ are the collective coordinate Nambu-Goldstone bosons associated with the broken D dimensional Poincaré symmetries and correspond to the massless excitation modes of the brane. Left multiplication of the coset elements Ω by an $ISO(1, D - 1)$ group element g, which is specified by global transformation parameters ϵ^μ, z^i, b_i^μ, $\lambda^{\mu\nu}$, θ^{ij}, so that

$$g(x) = e^{i\epsilon^\mu P_\mu} e^{iz^i Z_i} e^{ib_i^\mu K_\mu^i} e^{(i/2)\lambda^{\mu\nu} M_{\mu\nu}} e^{(i/2)\theta^{ij} T_{ij}} \quad (1.2)$$

results in transformations of the space-time coordinates and the Nambu-Goldstone fields according to the general form [1]

$$g(x)\Omega(x) = \Omega'(x')h(x). \quad (1.3)$$

The transformed coset element, $\Omega'(x')$, is a function of the transformed world volume coordinates and the total variations of the fields

$$\Omega'(x') = e^{ix'^\mu P_\mu} e^{i\phi'^i(x')Z_i} e^{iv_i'^\mu(x')K_\mu^i}, \quad (1.4)$$

while $h(x)$ is a field dependent element of the stability group $SO(1, p) \times SO(N)$:

$$h(x) = e^{(i/2)\alpha_{\mu\nu}(x)M^{\mu\nu}} e^{(i/2)\beta_{ij}(x)T^{ij}}. \qquad (1.5)$$

Exploiting the algebra of the $ISO(1, D-1)$ charges, along with extensive use of the Baker-Campbell-Hausdorff formulas, the $ISO(1, D-1)$ transformations are obtained for infinitesimal transformation parameters as

$$x'^{\mu} = (\eta^{\mu\nu} - \lambda^{\mu\nu})x_{\nu} + \epsilon^{\mu} + 2b_i^{\mu}\phi^i(x)$$

$$\phi_i'(x') = (\delta_{ij} + \theta_{ij})\phi^j(x) + z_i + 2b_i^{\mu}x_{\mu}$$

$$v_i'^{\mu}(x') = (\eta^{\mu\nu} - \lambda^{\mu\nu})(\delta_{ij} + \theta_{ij})v_{\nu j}(x)$$
$$- b_j^{\nu}M_{\nu k}^{j\rho}(x)(\coth\sqrt{M(x)})_{\rho i}^{k\mu}$$

$$\alpha^{\mu\nu}(x) = \lambda^{\mu\nu} - 4b_j^{\lambda}[M^{-1/2}(x)]_{\lambda k}^{j\rho}\{(\tanh\sqrt{M(x)/4})_{\rho i}^{k\mu}v_i^{\nu}(x)$$
$$- (\tanh\sqrt{M(x)/4})_{\rho i}^{k\nu}v_i^{\mu}(x)\}$$

$$\beta_{ij}(x) = \theta^{ij} - 4b_l^{\lambda}[M^{-1/2}(x)]_{\lambda k}^{l\rho}\{(\tanh\sqrt{M(x)/4})_{\rho j}^{k\mu}v_{\mu i}(x)$$
$$- (\tanh\sqrt{M(x)/4})_{\rho i}^{k\mu}v_{\mu j}(x)\}, \qquad (1.6)$$

where the matrix M is defined as

$$M_{ij}^{\mu\nu} = 4(\eta^{\mu\nu}U_{ij} - 2v_j^{\mu}v_i^{\nu} + W^{\mu\nu}\delta_{ij}), \qquad (1.7)$$

with

$$U_{ij} = v_i^{\mu}\eta_{\mu\nu}v_j^{\nu} \qquad W^{\mu\nu} = v_i^{\mu}\delta^{ij}v_j^{\nu} \qquad (1.8)$$

and $\eta_{\mu\nu}$ is the d dimensional Minkowski space metric tensor with signature $(+1, -1, \ldots, -1)$. In the above, the space-time indices are raised, lowered, and contracted using $\eta_{\mu\nu}$ while the Kronecker δ_{ij} is similarly used for the $SO(N)$ indices. Both Nambu-Goldstone fields ϕ^i and v_i^{μ} transform inhomogeneously under the broken translations Z^i and broken local Lorentz transformations K_i^{μ}.

The nonlinearly realized $ISO(1, D-1)$ transformations induce a coordinate and field dependent general coordinate transformation of the world volume space-time coordinates. From the x^{μ} coordinate transformation given above, the general coordinate Einstein transformation for the world volume space-time coordinate differentials is given by

$$dx'^{\mu} = dx^{\nu}G_{\nu}^{\mu}(x), \qquad (1.9)$$

with $G_{\nu}^{\mu}(x) = \partial x'^{\mu}/\partial x^{\nu}$. The $ISO(1, D-1)$ invariant interval can be formed using the metric tensor $g_{\mu\nu}(x)$ so that $ds^2 = dx^{\mu}g_{\mu\nu}(x)dx^{\nu} = ds'^2 = dx'^{\mu}g'_{\mu\nu}(x')dx'^{\nu}$ where the metric tensor transforms as

$$g'_{\mu\nu}(x') = G_{\mu}^{-1\rho}(x)g_{\rho\sigma}(x)G_{\nu}^{-1\sigma}(x). \qquad (1.10)$$

The form of the vielbein (and hence the metric tensor) as well as the $ISO(1, D-1)$ covariant derivatives of the Nambu-Goldstone boson fields and the spin and $SO(N)$ connections can be extracted from the Maurer-Cartan one-form, $\Omega^{-1}d\Omega$, which can be expanded in terms of the generators as

$$\omega(x) = \Omega^{-1}d\Omega$$
$$= i(\omega^m P_m + \omega_{Zi}Z^i + \omega_{Ki}^m K_{im} + \tfrac{1}{2}\omega_M^{mn}M_{mn}$$
$$+ \tfrac{1}{2}\omega_T^{ij}T_{ij}). \qquad (1.11)$$

Here Latin indices $m, n = 0, 1, \ldots, p$ are used to distinguish tangent space induced local Lorentz transformation properties from world volume Einstein transformation properties which are denoted using Greek indices. In what follows, Latin indices are raised and lowered with use of the Minkowski metric tensors, η^{mn} and η_{mn}, while Greek indices are raised and lowered with use of the induced space-time metric tensors, $g^{\mu\nu}$ and $g_{\mu\nu}$.

Using the transformation property of the coset element, Eq. (1.3), the Maurer-Cartan one-form transforms according to

$$\omega'(x') = h(x)\omega(x)h^{-1}(x) + h(x)dh^{-1}(x), \qquad (1.12)$$

with $h(x) = e^{(i/2)[\alpha^{mn}(x)M_{mn}+\beta^{ij}(x)T_{ij}]}$ as given in Eqs. (1.5) and (1.6). Expanding in terms of the D dimensional Poincaré charges, the individual one-forms transform according to their local Lorentz and $SO(N)$ nature so that

$$\omega'^m(x') = \omega^n(x)\Lambda_n{}^m(\alpha(x))$$

$$\omega'_{Zi}(x') = R_{ij}(\beta(x))\omega_{Zj}(x)$$

$$\omega'^m_{Ki}(x') = R_{ij}(\beta(x))\omega^n_{Kj}(x)\Lambda_n{}^m(\alpha(x)) \qquad (1.13)$$

$$\omega'^{mn}_M(x') = \omega^{rs}_M\Lambda_r{}^m(\alpha(x))\Lambda_s{}^n(\alpha(x)) - d\alpha^{mn}(x)$$

$$\omega'_{Tij}(x') = R_{ik}(\beta(x))R_{jl}(\beta(x))\omega_{Tkl}(x) - d\beta_{ij}(x).$$

For infinitesimal transformations, the local Lorentz transformations are $\Lambda_n{}^m(\alpha(x)) = \delta_n{}^m + \alpha_n{}^m(x)$ and the local $SO(N)$ transformations are $R_{ij}(\beta(x)) = \delta_{ij} + \beta_{ij}(x)$.

Using the Feynman formula for the variation of an exponential operator in conjunction with the Baker-Campbell-Hausdorff formulas, the individual world volume one-forms appearing in the above decomposition of the covariant Maurer-Cartan one-form are secured as

$$\omega^m = dx^\mu e_\mu{}^m = dx^\mu\{\delta_\mu{}^m + v_{\mu j}(U^{-1/2}(\cosh\sqrt{4U} - 1)U^{-1/2})_{ji}v_i^m - \partial_\mu\phi_j(U^{-1/2}\sinh\sqrt{4U})_{ji}v_i^m\}$$

$$\omega_{Zi} = dx^\mu\omega_{Zi\mu} = dx^\mu(\cosh\sqrt{4U})_{ij}\{\partial_\mu\phi_j - (U^{-1/2}\tanh\sqrt{4U})_{jk}v_{k\mu}\}$$

$$\omega_{Ki}^m = dx^\mu\omega_{Ki\mu}^m = dx^\mu(\partial_\mu v_j^n)(\sinh\sqrt{M}M^{-1/2})_{ni}^{jm} \tag{1.14}$$

$$\omega_M^{mn} = dx^\mu\omega_{M\mu}^{mn} = -4dx^\mu(\partial_\mu v_i^r)\{[(\cosh\sqrt{M} - 1)M^{-1}]_{rj}^{im}v_j^n - [(\cosh\sqrt{M} - 1)M^{-1}]_{rj}^{in}v_j^m\}$$

$$\omega_{Tij} = dx^\mu\omega_{Tij\mu} = -4dx^\mu(\partial_\mu v_l^m)\{[(\cosh\sqrt{M} - 1)M^{-1}]_{mj}^{ln}v_{ni} - [(\cosh\sqrt{M} - 1)M^{-1}]_{mi}^{ln}v_{nj}\}.$$

The covariant coordinate differential ω^m is related to the world volume coordinate differential dx^μ by the vielbein, $e_\mu{}^m$, as $\omega^m = dx^\mu e_\mu{}^m$.

The vielbein transforms as $e_\mu'^m(x') = G_\mu^{-1\nu}(x)e_\nu{}^n(x)\Lambda_n{}^m(\alpha(x))$. Since the Jacobian of the $x^\mu \to x'^\mu$ transformation is simply

$$d^d x' = d^d x\, \det G, \tag{1.15}$$

it follows that $d^d x' \det e'(x') = d^d x \det e(x)$ since $\det\Lambda = 1$. Thus an $ISO(1, D - 1)$ invariant action [4] for the Nambu-Goto fields is constructed as

$$\Gamma = -\sigma\int d^d x\, \det e(x), \tag{1.16}$$

with σ the brane tension.

As seen from the expression for the vielbein, the action does not contain any derivatives of the vector field v_μ^i. Its equation of motion implies the covariant constraint of the inverse Higgs mechanism [5], $\omega_{Zi} = 0$. This allows v_μ^i to be expressed in terms of $\partial_\mu\phi^i$:

$$\partial_\mu\phi^i = (U^{-1/2}\tanh\sqrt{4U})_{ij}v_\mu^j. \tag{1.17}$$

Applying this to the vielbein gives

$$e_\mu{}^m = \delta_\mu{}^m + \partial_\mu\phi^i G_{ij}\partial^m\phi^j, \tag{1.18}$$

where

$$G_{ij} \equiv H_{ik}^{-1/2}[\sqrt{(1 - H)} - 1]_{kl}H_{lj}^{-1/2} \tag{1.19}$$

is an $SO(N)$ symmetric matrix with

$$H_{ij} \equiv \partial_\mu\phi_i\partial^\mu\phi_j. \tag{1.20}$$

This in turn yields the Nambu-Goto form of the $ISO(1, D - 1)$ invariant action for the Nambu-Goldstone fields ϕ^i as

$$\Gamma = -\sigma\int d^d x\, \det e$$
$$= -\sigma\int d^d x\sqrt{\det(\delta^{ij} - \partial_\mu\phi^i\partial^\mu\phi^j)}. \tag{1.21}$$

The form of this action can also be secured by considering the $ISO(1, D - 1)$ invariant interval in D dimensional Minkowski space with coordinates $X^{\mathcal{M}} = (x^\mu, \phi^i(x))$ which reads

$$ds^2 = dX^{\mathcal{M}}\eta_{\mathcal{MN}}dX^{\mathcal{N}}$$
$$= dx^\mu\eta_{\mu\nu}dx^\nu - d\phi^i(x)\delta_{ij}d\phi^j(x), \tag{1.22}$$

where the static gauge coordinate oscillations of the p-brane into the covolume are identified with the N Nambu-Goldstone fields $\phi^i(x)$. This induces a metric $g_{\mu\nu}$ on the world volume as

$$ds^2 = dx^\mu[\eta_{\mu\nu} - \partial_\mu\phi^i\delta_{ij}\partial_\nu\phi^j]dx^\nu = dx^\mu g_{\mu\nu}dx^\nu. \tag{1.23}$$

Consequently, the $ISO(1, D - 1)$ invariant action is again given as in Eq. (1.21) takes the form

$$\Gamma = -\sigma\int d^d x\sqrt{(-1)^p\det g}$$
$$= -\sigma\int d^d x\sqrt{\det(\delta^{ij} - \partial_\mu\phi^i\partial^\mu\phi^j)}. \tag{1.24}$$

The induced Nambu-Goto vielbein $e_\mu{}^m$ is just the one given in (1.18) which produces the metric in Eq. (1.23), $g_{\mu\nu} = e_\mu{}^m\eta_{mn}e_\nu{}^n$.

II. PROBE BRANES IN ADS SPACE

Embedding an AdS_d probe brane in AdS_D space breaks the $SO(2, D - 1)$ symmetry of the AdS_D bulk space down to the $SO(2, d - 1)$ isometry group of the AdS_d world volume and its complement: $SO(2, p) \times SO(N)$. The $SO(2, D - 1)$ isometries of the AdS_D bulk space are generated by the D dimensional (pseudo)translation operators, denoted by $P^{\mathcal{M}}$, and the Lorentz transformations, $M^{\mathcal{MN}}$, $\mathcal{M}, \mathcal{N} = 0, 1, \ldots, p + N$, collectively satisfying the algebra [6]

$$[M^{\mathcal{MN}}, M^{\mathcal{RS}}] = -i(\eta^{\mathcal{MR}}M^{\mathcal{NS}} - \eta^{\mathcal{MS}}M^{\mathcal{NR}} + \eta^{\mathcal{NS}}M^{\mathcal{MR}} - \eta^{\mathcal{NR}}M^{\mathcal{MS}})$$
$$[M^{\mathcal{MN}}, P^{\mathcal{L}}] = i(P^{\mathcal{M}}\eta^{\mathcal{NL}} - P^{\mathcal{N}}\eta^{\mathcal{ML}}) \tag{2.1}$$
$$[P^{\mathcal{M}}, P^{\mathcal{N}}] = -im^2M^{\mathcal{MN}}.$$

Here $m^2 > 0$ is a constant inverse squared length scale characterizing the AdS bulk space as well as the AdS world volume. The Minkowski space case with symmetry breaking pattern $ISO(1, D - 1) \to ISO(1, d - 1)$ is secured as the $m^2 = 0$ limit. For an AdS p-brane with codimension N, the d dimensional world volume symmetry generators, P^μ

for (pseudo)translations and $M^{\mu\nu}$ for Lorentz rotations, with $\mu, \nu = 0, 1, \ldots, p$, form an unbroken subgroup $SO(2, p)$ of the D dimensional AdS$_D$ symmetry group $SO(2, D-1)$. The specific case with one codimension, $N = 1$, was discussed in [6]. The N dimensional covolume rotation generators, $T^{ij} = M^{p+i,p+j}$, where $i, j = 1, 2, \ldots N$, form an unbroken subgroup $SO(N)$. The broken symmetry charges are the generators of (pseudo)translations transverse to the brane, denoted by $Z_i = -P^{p+i}$, and the generators of the broken bulk Lorentz transformations, denoted by $K_i^{\mu} = 2M^{p+i,\mu}$. Both the d dimensional Lorentz scalars Z_i and the d dimensional Lorentz vector K_i^{μ} transform according to the fundamental (vector) representation of the unbroken $SO(N)$ subgroup. The unbroken generators of the $SO(2, D-1)$ algebra [cf. Eq. (2.1)] satisfy

$$
\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho} \\
&\quad - \eta^{\nu\rho}M^{\mu\sigma}) \\
[M^{\mu\nu}, P^{\lambda}] &= i(P^{\mu}\eta^{\nu\lambda} - P^{\nu}\eta^{\mu\lambda}) \\
[P^{\mu}, P^{\nu}] &= -im^2 M^{\mu\nu} \\
[T^{ij}, T^{kl}] &= i(\delta^{ik}T^{jl} - \delta^{il}T^{jk} + \delta^{jl}T^{ik} - \delta^{jk}T^{il}) \\
[P^{\mu}, T^{ij}] &= 0 = [M^{\mu\nu}, T^{ij}],
\end{aligned}
\tag{2.2}
$$

while the broken generators carry representations of the unbroken symmetries and rotate the unbroken into broken charges as

$$
\begin{aligned}
[M^{\mu\nu}, Z_i] &= 0, \qquad [M^{\mu\nu}, K_i^{\lambda}] = i(K_i^{\mu}\eta^{\nu\lambda} - K_i^{\nu}\eta^{\mu\lambda}) \\
[T^{ij}, Z_k] &= -i(Z_i\delta^{jk} - Z_j\delta^{ik}) \\
[T^{ij}, K^{\mu k}] &= -i(K^{\mu i}\delta^{jk} - K^{\mu j}\delta^{ik}) \\
[P^{\mu}, Z_i] &= -\frac{i}{2}m^2 K_i^{\mu}, \qquad [K_i^{\mu}, P^{\nu}] = -2i\eta^{\mu\nu}Z_i.
\end{aligned}
\tag{2.3}
$$

Finally, the commutation relations between broken generators take the form

$$
\begin{aligned}
[Z_i, Z_j] &= -im^2 T^{ij} \\
[K_i^{\mu}, K_j^{\nu}] &= 4i(\delta_{ij}M^{\mu\nu} - \eta^{\mu\nu}T^{ij}) \\
[K_i^{\mu}, Z_j] &= -2i\delta_{ij}P^{\mu}.
\end{aligned}
\tag{2.4}
$$

These results are instrumental in deriving the $SO(2, D-1)$ invariant action describing the motion of the AdS$_d$ probe brane in AdS$_D$ space which will be obtained using three different constructions. For a purely group theoretic derivation of D-brane actions in AdS$_3$, see Ref. [7]. In Sec. (2.1) the coset method is used, while in (2.2) an embedding method is employed. Finally in Sec. (2.3) the action is obtained through a hybrid of these two methods.

A. Coset method

The invariant action of the Nambu-Goldstone fields along with their $SO(2, D-1)$ transformations and those

of the coordinates can be obtained using coset methods. Parametrizing a coset element as

$$
\Omega(x) \equiv e^{ix^{\mu}P_{\mu}} e^{i\phi^i(x)Z_i} e^{iv_i^{\mu}(x)K_{\mu}^i}
\tag{2.5}
$$

and expanding the Maurer-Cartan one-form as in Eq. (1.11), the veilbein $\omega^m = dx^{\mu} e_{\mu}^{\ m}$ and the $SO(2, D-1)$ covariant derivative of ϕ^i, $\omega_Z^i = dx^{\mu}\nabla_{\mu}\phi^i$, are extracted as

$$
\begin{aligned}
e_{\mu}^{\ m} &= \cosh\sqrt{m^2\phi^2}\,\bar{e}_{\mu}^{\ n}\{\delta_n^{\ m} + v_{in}[U^{-1/2}(\cosh\sqrt{4U} - 1) \\
&\quad \times U^{-1/2}]_{ij}v_j^m - \mathcal{D}_m\phi^i(U^{-1/2}\sinh\sqrt{4U})_{ij}v_j^m\} \\
\nabla_{\mu}\phi^i &= -\cosh\sqrt{m^2\phi^2}\,\bar{e}_{\mu}^{\ n}\{(U^{-1/2}\sinh\sqrt{4U})_{ij}v_{nj} \\
&\quad + \mathcal{D}_n\phi^j(\cosh\sqrt{4U})_{ji}\}.
\end{aligned}
\tag{2.6}
$$

Here $\bar{e}_{\mu}^{\ m}$ is the AdS$_d$ background vielbein [6] defined as

$$
\bar{e}_{\mu}^{\ m}(x) = \frac{\sin\sqrt{m^2 x^2}}{\sqrt{m^2 x^2}}P_{T\mu}^{\ m}(x) + P_{L\mu}^{\ m}(x),
\tag{2.7}
$$

and the transverse and longitudinal projectors for x^{μ} are

$$
P_{T\mu\nu}(x) = \eta_{\mu\nu} - \frac{x_{\mu}x_{\nu}}{x^2}, \qquad P_{L\mu\nu}(x) = \frac{x_{\mu}x_{\nu}}{x^2}.
\tag{2.8}
$$

The partially covariant derivative $\mathcal{D}_m\phi^i$ is defined as

$$
\mathcal{D}_m\phi^i = e^{-A(\phi)}\bar{e}_j^{\ i}\bar{e}_m^{-1\mu}\partial_{\mu}\phi^j,
\tag{2.9}
$$

where $A(\phi) = \ln\cosh\sqrt{m^2\phi^2}$ is the warp factor with $\phi^2 = \phi^i\phi^i$. In addition, $\bar{e}_i^{\ j}$, which is identified in Sec. (2.3) as the Euclidean AdS$_N$ background covolume vielbein, takes the form

$$
\bar{e}_{ij} = \frac{\sinh\sqrt{m^2\phi^2}}{\sqrt{m^2\phi^2}}P_{Tij}(\phi) + P_{Lij}(\phi),
\tag{2.10}
$$

where the transverse and longitudinal projectors for ϕ^i are

$$
P_{Tij}(\phi) = \delta_{ij} - \frac{\phi_i\phi_j}{\phi^2}, \qquad P_{Lij}(\phi) = \frac{\phi_i\phi_j}{\phi^2}.
\tag{2.11}
$$

An $SO(2, D-1)$ invariant action is then constructed as

$$
\Gamma = -\sigma \int d^d x \det e.
\tag{2.12}
$$

Since this action is independent of v_m^i space-time derivatives, the v_m^i field equation takes the form of the constraint $\nabla\phi^i = 0$ and allows the nonpropagating v_m^i degrees of freedom to be eliminated in terms of the partially covariant derivative of ϕ^i as

$$
\mathcal{D}_m\phi^i = (U^{-1/2}\tanh\sqrt{4U})_{ij}v_m^j.
\tag{2.13}
$$

Using Eq. (2.13) in the expression for the vielbein $e_{\mu}^{\ m}$, cf. Eq. (2.6), then gives the factorized form of the vielbein

$$e_\mu{}^m(x) = e^{A(\phi)}\bar{e}_\mu{}^n(x)N_n{}^m(x)$$

$$= \cosh\sqrt{m^2\phi^2}\,\bar{e}_\mu{}^n[\delta_n{}^m + \mathcal{D}_n\phi^i(x)G_{ij}\mathcal{D}^m\phi^j(x)], \tag{2.14}$$

where the Nambu-Goto vielbein, $N_n{}^m$, is given by

$$N_n{}^m(x) = \delta_n{}^m + \mathcal{D}_n\phi^i(x)G_{ij}\mathcal{D}^m\phi^j(x), \tag{2.15}$$

and, as before, the warped $SO(N)$ matrix G_{ij} is

$$G_{ij} = H_{ik}^{-1/2}[\sqrt{(1-H)} - 1]_{kl}(H^{-1/2})_{lj}, \tag{2.16}$$

but now with H_{ij} defined in terms of the partially covariant derivatives as

$$H_{ij} = \mathcal{D}_r\phi^i(x)\eta^{rs}\mathcal{D}_s\phi^j(x). \tag{2.17}$$

Hence, the $SO(2, D-1)$ invariant action for the AdS_d p-brane takes the form

$$\Gamma = -\sigma \int d^dx \det\bar{e} \cosh^d\sqrt{(m^2\phi^2)}$$

$$\times \sqrt{\det[\delta^{ij} - \mathcal{D}_m\phi^i\eta^{mn}\mathcal{D}_n\phi^j]}. \tag{2.18}$$

B. Embedding method

Alternatively, the embedding of the probe brane in the AdS_D space can be implemented by picking specific coordinates with which to describe the AdS_d and covolume subspaces. The AdS_D space can be simply described as the $SO(2, D-1)$ invariant hyperboloidal hypersurface,

$$\frac{1}{m^2} = X_0^2 - X_1^2 - X_2^2 - \cdots - X_{D-1}^2 + X_D^2$$

$$= X^{\mathcal{M}}\hat{\eta}_{\mathcal{MN}}X^{\mathcal{N}}, \tag{2.19}$$

embedded in a $(D+1)$-dimensional pseudo-Euclidean space defined with invariant interval

$$ds^2 = dX^{\mathcal{M}}\hat{\eta}_{\mathcal{MN}}dX^{\mathcal{N}} \tag{2.20}$$

characterized by the metric tensor $\hat{\eta}_{\mathcal{MN}}$ of signature $(+1, -1, -1, \ldots, -1, +1)$, where here $\mathcal{M}, \mathcal{N} = 0, 1, \ldots, D-1, D$ and $X^{\mathcal{M}}$ are the pseudo-Euclidean space homogeneous coordinates.

AdS_D space containing an AdS_d p-brane, which is a d dimensional world volume with an AdS metric embedded as a hypersurface at the covolume coordinates $X^{p+i} = 0$ for $i = 1, 2, \ldots, N$, can be described by the coordinates $X^{\mathcal{M}} = (X^\mu, X^{p+i}, X^D)$ with isotropic AdS_d coordinates X^μ and covolume direction cosine coordinates X^{p+i}:

$$X^\mu = a(x^2)x^\mu \cosh\sqrt{(m^2\phi^2)}: \quad \mu = 0, 1, \ldots, p$$

$$X^{p+i} = \frac{1}{m}\frac{\phi^i}{\sqrt{\phi^2}}\sinh\sqrt{(m^2\phi^2)}: \quad i = 1, 2, \ldots, N \tag{2.21}$$

$$X^D = \frac{1}{m}b(x^2)\cosh\sqrt{(m^2\phi^2)}.$$

Here x^μ are the intrinsic coordinates of the AdS_d world volume and the ϕ^i are the covolume coordinates. To satisfy the equation (2.19) of the AdS_D hyperbola, $a(x^2)$ and $b(x^2)$ are related as

$$1 = m^2x^2a^2(x^2) + b^2(x^2). \tag{2.22}$$

Hence the $SO(2, D-1)$ invariant interval, Eq. (2.20), becomes

$$ds^2 = e^{2A(r)}d\bar{s}^2 - dr^2, \tag{2.23}$$

where the warp factor is $A(\phi) = \ln\cosh\sqrt{(m^2\phi^2)}$ and $d\bar{s}^2 = dx^\mu\bar{g}_{\mu\nu}(x)dx^\nu$ is the AdS_d invariant interval with $\bar{g}_{\mu\nu}$ the AdS_d metric tensor to be given below. The interval on the covolume is found to be

$$dr^2 = d\phi^i\bar{g}_{ij}d\phi^j, \tag{2.24}$$

where the metric $\bar{g}_{ij}$ is found to be

$$\bar{g}_{ij} = \bar{e}_i{}^k\delta_{kl}\bar{e}_j{}^l = \frac{\sinh^2\sqrt{(m^2\phi^2)}}{m^2\phi^2}P_{Tij}(\phi) + P_{Lij}(\phi), \tag{2.25}$$

with the transverse and longitudinal projectors for ϕ_i defined as in Eq. (2.11).

The AdS_d subspace has the isotropic coordinates x^μ of an $SO(2, p)$ invariant hyperboloid, $\frac{1}{m^2} = X_0^2 - X_1^2 - \cdots - X_p^2 + X_D^2$, embedded at $\phi^i = 0 = X^{p+i}$. This subsurface maintains the coordinate relation equation (2.22). This in turn leads to a form for the AdS_d metric tensor given by

$$\bar{g}_{\mu\nu}(x) = a^2(x^2)P_{T\mu\nu}(x) + \left[\left(a(x^2) + 2x^2\frac{da(x^2)}{dx^2}\right)^2\right.$$

$$\left. + 4\frac{x^2}{m^2}\left(\frac{db(x^2)}{dx^2}\right)^2\right]P_{L\mu\nu}(x)$$

$$= a^2(x^2)P_{T\mu\nu}(x) + \frac{(a(x^2) + 2x^2\frac{da(x^2)}{dx^2})^2}{(1 - m^2x^2a^2(x^2))}P_{L\mu\nu}. \tag{2.26}$$

With the specific choice of parameters

$$a(x^2) = \frac{\sin\sqrt{m^2x^2}}{\sqrt{m^2x^2}}, \qquad b(x^2) = \cos\sqrt{m^2x^2}, \tag{2.27}$$

the AdS_d background vielbein, Eq. (2.7), and metric are obtained in the same coordinate system as implied by the coset method in Sec. (2.1). The induced metric on the probe brane's world volume, $g_{\mu\nu}$, with $SO(2, D-1)$ isometries

follows from

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu = dx^\mu [e^{2A(\phi)} \bar{g}_{\mu\nu} - \partial_\mu \phi^i \bar{g}_{ij} \partial_\nu \phi^j] dx^\nu, \tag{2.28}$$

and the $SO(2, D-1)$ invariant action is therefore given by

$$\Gamma = -\sigma \int d^d x \sqrt{(-1)^p \det g}, \tag{2.29}$$

in agreement with Eq. (2.18).

C. Hybrid method

The $SO(2, D-1)$ isometry group of AdS$_D$ space contains as subgroups $SO(2, p)$, generated by $M^{\mu\nu}$ and P^μ, and $SO(1, N)$, generated by T^{ij} and Z^i, which are the isometry groups of an AdS$_d$ subspace and its Euclidean AdS$_N$ covolume, respectively. Below, the coset method is used to obtain the background metric and invariant interval in each of these two subspaces. With the introduction of a warp factor, the two intervals are combined to construct an $SO(2, D-1)$ invariant interval. The embedding method is then employed to obtain the $SO(2, D)$ invariant action for the AdS$_d$ probe brane.

The background vielbein and spin connection of the AdS$_d$ world volume can be found by considering the Maurer-Cartan one-form made from just the (pseudo)translation $SO(2, p)/SO(1, p)$ coset element

$$\bar{\Omega} = e^{ix^m P_m}. \tag{2.30}$$

The AdS$_d$ covariant coordinate differential $\bar{\omega}^m$ and spin connection $\bar{\omega}_M^{mn}$, are obtained from the AdS$_d$ coordinate one-form,

$$\bar{\omega} = \bar{\Omega}^{-1} d\bar{\Omega} = i[\bar{\omega}^m P_m + \bar{\omega}_M^{mn} M_{mn}], \tag{2.31}$$

where

$$\bar{\omega}^m = \frac{\sin\sqrt{m^2 x^2}}{\sqrt{m^2 x^2}} P_T^{mn}(x) dx_n + P_L^{mn}(x) dx_n$$
$$\bar{\omega}_M^{mn} = [\cos\sqrt{m^2 x^2} - 1] \frac{(x^m dx^n - x^n dx^m)}{x^2}, \tag{2.32}$$

with the transverse and longitudinal projectors for x^μ defined as in Eq. (2.8). The differential $\bar{\omega}^m$ is related to the x^μ world volume coordinate differential via the AdS$_d$ background vielbein $\bar{e}_\mu{}^m(x)$ as

$$\bar{\omega}^m = dx^\mu \bar{e}_\mu{}^m(x). \tag{2.33}$$

Using Eq. (2.32) along with $d = dx^\mu \partial_\mu^x$, the vielbein presented in Eq. (2.7) is obtained.

Likewise, the vielbein and hence metric for the covolume Euclidean AdS$_N$ space can be obtained by considering the Maurer-Cartan one-form made from just the broken (pseudo)translation $SO(1, N)/SO(N)$ coset element, ϕ^i being the covolume coordinates,

$$\bar{\bar{\Omega}} = e^{i\phi^i Z_i}. \tag{2.34}$$

The covolume covariant coordinate differential, $\bar{\omega}^i$, is obtained from the one-form

$$\bar{\bar{\omega}} = \bar{\bar{\Omega}}^{-1} d\bar{\bar{\Omega}} = i[\bar{\omega}^i Z_i + \tfrac{1}{2} \bar{\omega}_T^{ij} T_{ij}], \tag{2.35}$$

where

$$\bar{\omega}^i = d\phi^j \bar{e}_j{}^i$$
$$\bar{\omega}_T^{ij} = [\cosh\sqrt{m^2 \phi^2} - 1] \frac{(\phi^i d\phi^j - \phi^j d\phi^i)}{\phi^2}, \tag{2.36}$$

with the covolume vielbein $\bar{e}_j{}^i$ given in Eq. (2.10).

In order to obtain the $SO(2, D-1)$ invariant action for the probe p-brane most directly, consider the $SO(2, D-1)$ invariant interval in D dimensional AdS space with coordinates $X^{\mathcal{M}} = (x^\mu, \phi^i)$

$$ds^2 = dX^{\mathcal{M}} \bar{g}_{\mathcal{MN}} dX^{\mathcal{N}} = e^{2A(\phi)} d\bar{s}^2 - dr^2, \tag{2.37}$$

where the warp factor $A(\phi) = \ln \cosh\sqrt{(m^2 \phi^2)}$ is introduced and $d\bar{s}^2 = dx^\mu \bar{g}_{\mu\nu}(x) dx^\nu$ is the AdS$_d$ invariant interval with $\bar{g}_{\mu\nu}$ the AdS$_d$ world volume metric tensor, $\bar{g}_{\mu\nu} = \bar{e}_\mu{}^m \eta_{mn} \bar{e}_\nu{}^n$. Similarly, $dr^2 = d\phi^i \bar{g}_{ij} d\phi^j$ is the invariant interval for the Euclidean AdS$_N$ covolume with metric given by $\bar{g}_{ij} = \bar{e}_i{}^k \delta_{kl} \bar{e}_j{}^l$.

The static gauge coordinate motions of the p-brane into the covolume are identified with N Nambu-Goldstone fields $\phi^i(x)$. The induced metric on the probe brane's world volume, $g_{\mu\nu}$, with $SO(2, D-1)$ isometries follows from

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu = dx^\mu [e^{2A(\phi)} \bar{g}_{\mu\nu} - \partial_\mu \phi^i \bar{g}_{ij} \partial_\nu \phi^j] dx^\nu, \tag{2.38}$$

and the $SO(2, D-1)$ invariant action is therefore given by

$$\Gamma = -\sigma \int d^d x \sqrt{(-1)^p \det g}. \tag{2.39}$$

The induced metric, Eq. (2.38), can be factorized into the product of warped background vielbein, $e^A \bar{e}_\mu{}^m$, and the Nambu-Goto metric, n_{mn}, for the Nambu-Goldstone fields

$$g_{\mu\nu} = (e^{A(\phi)} \bar{e}_\mu{}^m) n_{mn} (e^{A(\phi)} \bar{e}_\nu{}^n), \tag{2.40}$$

where

$$n_{mn} = \eta_{mn} - \mathcal{D}_m \phi^i(x) \delta_{ij} \mathcal{D}_n \phi^j(x), \tag{2.41}$$

with the partially covariant derivative $\mathcal{D}_m \phi^i = \mathcal{D}_m^{ij} \phi^j$ defined as in Eq. (2.9). With these definitions, the $SO(2, D-1)$ invariant action (2.39) becomes

$$\Gamma = -\sigma \int d^d x \det \bar{e} \, e^{dA(\phi)} \sqrt{(-1)^p \det n}. \tag{2.42}$$

The determinant of the Nambu-Goto metric in turn can be expressed in terms of the determinant of a $SO(N)$ matrix

$$\det n = (-1)^p \det[\delta^{ij} - \mathcal{D}_m \phi^i \eta^{mn} \mathcal{D}_n \phi^j]. \tag{2.43}$$

This yields the AdS$_d$ probe p-brane $SO(2, D-1)$ invariant action in the form

$$\Gamma = -\sigma \int d^d x \det\bar{e} \cosh^d\sqrt{(m^2\phi^2)}$$
$$\times \sqrt{\det[\delta^{ij} - \mathcal{D}_m\phi^i \eta^{mn} \mathcal{D}_n\phi^j]}, \quad (2.44)$$

as obtained in Eqs. (2.18) and (2.29).

III. A NON-BPS VORTEX IN $D = 6$, $\mathcal{N} = (1, 0)$ SUPERSPACE

The formation of a non-Bogomol'nyi-Prasad-Sommerfield (non-BPS) saturated vortex in six-dimensional superspace completely breaks the supersymmetry (SUSY) as well as two of the space translation symmetries. The oscillations of the vortex into superspace are described by two Nambu-Goldstone bosons and four Goldstinos. The case of a BPS vortex with partial supersymmetry breaking in the Abelian Higgs model was discussed in [8], while coset methods were used to construct effective actions describing a non-BPS $p = 2$ brane embedded in $D = 4$, $N = 1$ superspace in [9] and a non-BPS $p = 3$ brane embedded in $D = 5$, $N = 1$ superspace in [10]. In general, $D = 6$, $\mathcal{N} = (1, 1)$ superspace has eight supersymmetries with the associated eight component complex (Dirac) spinor supersymmetry charges $\mathcal{Q}_a$; $a, b = 1, \dots, 8$ and $\bar{\mathcal{Q}} \equiv \mathcal{Q}^\dagger \Gamma^0$. The nonvanishing (anti)commutation relations are

$$\{\mathcal{Q}_a, \bar{\mathcal{Q}}^b\} = 2\Gamma_b^{\mathcal{M}b} P_{\mathcal{M}},$$
$$[M^{\mathcal{M}\mathcal{N}}, \mathcal{Q}_a] = -\tfrac{1}{2}\Gamma_a^{\mathcal{M}\mathcal{N}b}\mathcal{Q}_b \quad (3.1)$$
$$[M^{\mathcal{M}\mathcal{N}}, \bar{\mathcal{Q}}^b] = \tfrac{1}{2}\bar{\mathcal{Q}}^a \Gamma_a^{\mathcal{M}\mathcal{N}b}.$$

The number of complex supersymmetry generators can be reduced from eight to four using left and right Weyl projectors defined as

$$\mathcal{P}_L \equiv \tfrac{1}{2}(1 - \Gamma_*), \qquad \mathcal{P}_R \equiv \tfrac{1}{2}(1 + \Gamma_*), \quad (3.2)$$

with $\Gamma_* = -\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5$. The resulting $(1, 0)$ $D = 6$

supersymmetry algebra is obtained by constraining the supersymmetry generators to satisfy $\mathcal{Q}_L = \mathcal{P}_L \mathcal{Q}$ and hence $\mathcal{P}_R \mathcal{Q}_L = 0$ so that the supersymmetry charges obey the algebra

$$\{\mathcal{Q}_{La}, \bar{\mathcal{Q}}_L^b\} = 2\mathcal{P}_{La}{}^c\Gamma_c^{\mathcal{M}b} P_{\mathcal{M}}$$
$$[M^{\mathcal{M}\mathcal{N}}, \mathcal{Q}_{La}] = -\tfrac{1}{2}\Gamma_a^{\mathcal{M}\mathcal{N}b}\mathcal{Q}_{Lb}$$
$$[M^{\mathcal{M}\mathcal{N}}, \bar{\mathcal{Q}}_L^b] = \tfrac{1}{2}\bar{\mathcal{Q}}_L^a\Gamma_a^{\mathcal{M}\mathcal{N}b}, \quad (3.3)$$

while remaining nontrivial commutators form the Poincaré algebra

$$[M^{\mathcal{M}\mathcal{N}}, M^{\mathcal{R}\mathcal{S}}] = -i(\eta^{\mathcal{M}\mathcal{R}}M^{\mathcal{N}\mathcal{S}} - \eta^{\mathcal{M}\mathcal{S}}M^{\mathcal{N}\mathcal{R}}$$
$$+ \eta^{\mathcal{N}\mathcal{S}}M^{\mathcal{M}\mathcal{R}} - \eta^{\mathcal{N}\mathcal{R}}M^{\mathcal{M}\mathcal{S}}) \quad (3.4)$$
$$[M^{\mathcal{M}\mathcal{N}}, P^{\mathcal{L}}] = i(P^{\mathcal{M}}\eta^{\mathcal{N}\mathcal{L}} - P^{\mathcal{N}}\eta^{\mathcal{M}\mathcal{L}}).$$

With the formation of the non-BPS vortex in this $D = 6$, $\mathcal{N} = (1, 0)$ superspace, the resulting world volume is just $d = 4$ Minkowski space with a bosonic codimension $N = 2$. Thus the isometry group of the target superspace is broken by the vortex to the Poincaré group of $d = 4$ space-time which is $ISO(1, 3)$. As such, all symmetry charges can be expressed in terms of their $SO(1, 3)$ Lorentz group and $SO(2)$ covolume rotation group representation content. The bosonic charges are given as in the previous sections now with $i = 1, 2$ for the $SO(2)$ covolume rotation indices so that the $T^{ij} = \tfrac{1}{2}\epsilon^{ij}T$ obey the algebra of Eqs. (2.1), (2.2), (2.3), and (2.4) with $m^2 = 0$. The supersymmetry charges form two complex $d = 4$ Weyl spinors denoted by Q_α, S_α and their complex conjugates $\bar{Q}_{\dot\alpha}$ and $\bar{S}_{\dot\alpha}$. The left-handed Weyl projection of the $D = 6$ spinor of supersymmetry charges, $\mathcal{Q}_{La}$, contains Q_α and $\bar{S}^{\dot\alpha}$, while the conjugate spinor $\bar{\mathcal{Q}}_L^a$ contains the charges S^α and $\bar{Q}_{\dot\alpha}$. The $D = 6$ supersymmetry algebra is then given by the centrally extended $d = 4$, $N = 2$ SUSY algebra with nonvanishing commutators

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu, \qquad [M^{\mu\nu}, Q_\alpha] = -\tfrac{1}{2}\sigma^{\mu\nu\beta}_\alpha Q_\beta,$$
$$\{S_\alpha, \bar{S}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu, \qquad [M^{\mu\nu}, S_\alpha] = -\tfrac{1}{2}\sigma^{\mu\nu\beta}_\alpha S_\beta,$$
$$\{Q_\alpha, S^\beta\} = -2iZ\delta_\alpha{}^\beta, \qquad [M^{\mu\nu}, \bar{Q}^{\dot\alpha}] = \bar{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta},$$
$$\{\bar{Q}_{\dot\alpha}, \bar{S}^{\dot\beta}\} = 2i\bar{Z}\delta_{\dot\alpha}{}^{\dot\beta}, \qquad [M^{\mu\nu}, \bar{S}^{\dot\alpha}] = \bar{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\bar{S}^{\dot\beta},$$

$$[T, Q_\alpha] = -Q_\alpha, \qquad [\bar{K}^\mu, Q_\alpha] = -2\sigma^\mu_{\alpha\dot\alpha}\bar{S}^{\dot\alpha},$$
$$[T, S_\alpha] = -S_\alpha, \qquad [K^\mu, \bar{Q}^{\dot\alpha}] = 2\bar{\sigma}^{\mu\dot\alpha\alpha}S_\alpha,$$
$$[T, \bar{Q}_{\dot\alpha}] = \bar{Q}_{\dot\alpha}, \qquad [\bar{K}^\mu, S_\alpha] = -2\sigma^\mu_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha},$$
$$[T, \bar{S}_{\dot\alpha}] = \bar{S}_{\dot\alpha}, \qquad [K^\mu, \bar{S}^{\dot\alpha}] = 2\bar{\sigma}^{\mu\dot\alpha\alpha}Q_\alpha,$$

$$(3.5)$$

where we have defined $Z = Z^1 - iZ^2$, $\bar{Z} = Z^1 + iZ^2$, $K^\mu = K^{1\mu} - iK^{2\mu}$, $\bar{K}^\mu = K^{1\mu} + iK^{2\mu}$.

Denoting the $D = 6$, $\mathcal{N} = (1, 0)$ super-Poincaré group by G and the $d = 4$ Poincaré stability group $ISO(1, 3)$ by H, all symmetry transformations can be realized by group elements in G acting on the G/H coset element Ω formed from the P^μ, Z_i, K_i^μ, Q_α, $\bar{Q}^{\dot\alpha}$, S_α, $\bar{S}^{\dot\alpha}$ charges as

$$\Omega(x) \equiv e^{ix^\mu P_\mu} e^{i\phi^i(x)Z_i}$$
$$\times e^{i[\theta(x)^\alpha Q_\alpha + \bar{\theta}_{\dot\alpha}(x)\bar{Q}^{\dot\alpha} + \lambda(x)^\alpha S_\alpha + \bar{\lambda}_{\dot\alpha}(x)\bar{S}^{\dot\alpha}]} e^{iv_i^\mu(x)K_\mu^i}, \quad (3.6)$$

where x^μ are the four-dimensional Minkowski space world volume coordinates, $\phi^i(x)$ and $v_i^\mu(x)$ are the collective coordinate Nambu-Goldstone bosons associated with the broken D dimensional Poincaré symmetries, and $\theta_\alpha(x)$, $\bar{\theta}^{\dot\alpha}(x)$, $\lambda_\alpha(x)$, $\bar{\lambda}^{\dot\alpha}(x)$ are the Goldstinos associated with the broken supersymmetries. Collectively, these fields describe the motion of the vortex into the superspace covolume. Using the Maurer-Cartan one-forms, the covariant coordinate differentials and the covariant derivatives of all of the Nambu-Goldstone fields can be extracted.

Expanding the Maurer-Cartan one-form in terms of the generators as

$$\omega(x) = \Omega^{-1} d\Omega$$

$$= i[\omega^m P_m + \omega_{Zi} Z^i + \omega_Q^\alpha Q_\alpha + \omega_{\bar{Q}\dot\alpha} \bar{Q}^{\dot\alpha} + \omega_S^\alpha S_\alpha$$

$$+ \omega_{\bar{S}\dot\alpha} \bar{S}^{\dot\alpha} + \omega_{Ki}^m K_{im} + \tfrac{1}{2}\omega_M^{mn} M_{mn} + \tfrac{1}{2}\omega_T^{ij} T_{ij}]$$

$$(3.7)$$

yields the vierbein and the covariant derivative of the Nambu-Goldstone bosons ϕ^i as

$$\omega^m = dx^\mu e_\mu{}^m = dX^n N_n{}^m$$

$$\omega_{Zi} = dx^\mu \nabla_\mu \phi^i$$

$$= dX^m[\nabla_m \Phi^j (\cosh\sqrt{4U})_{ji} - v_m^j (U^{-1/2}\sinh\sqrt{4U})_{ji}].$$

$$(3.8)$$

Here $dX^m = dx^\mu \hat{e}_\mu{}^m$ is the Akulov-Volkov coordinate differential with $\hat{e}_\mu{}^m$ the Akulov-Volkov vierbein

$$\hat{e}_\mu{}^m = \delta_\mu{}^m - i[\theta\sigma^m \overleftrightarrow{\partial}_\mu \bar\theta + \lambda\sigma^m \overleftrightarrow{\partial}_\mu \bar\lambda] \qquad (3.9)$$

and $\mathcal{D}_m = \hat{e}_m^{-1\mu}\partial_\mu$ the SUSY covariant derivative. The Nambu-Goto vierbein $N_n{}^m$ is obtained as

$$N_n{}^m = \delta_n{}^m + v_m^i[U^{-1/2}(\cosh\sqrt{4U} - 1)U^{-1/2}]_{ij}v^{jm}$$

$$- \nabla_m\Phi^i[U^{-1/2}\sinh\sqrt{4U}]_{ij}v^{jm}, \qquad (3.10)$$

where the Nambu-Goldstone boson differentials $d\Phi^i = dX^m\nabla_m\Phi^i$ are[1]

$$d\Phi^1 = d\phi^1 + [(\lambda d\theta - \theta d\lambda) + (\bar\lambda d\bar\theta - \bar\theta d\bar\lambda)]$$

$$d\Phi^2 = d\phi^2 - i[(\lambda d\theta - \theta d\lambda) - (\bar\lambda d\bar\theta - \bar\theta d\bar\lambda)]. \qquad (3.11)$$

The invariant action describing the motion of the 3-brane vortex into the superspace covolume is given by the invariant synthesis of the Akulov-Volkov and Nambu-Goto actions as

$$\Gamma = -\sigma\int d^4x \det e = -\sigma\int d^4x \det\hat{e} \det N, \qquad (3.12)$$

with σ the brane tension. Since the action is independent of v_μ^i derivatives, this field is nondynamical and it can be eliminated using its field equation, or equivalently the inverse Higgs mechanism. Doing so yields the covariant constraint $\omega_{Zi} = 0$ which allows v_μ^i to be eliminated in terms of $\nabla_m\Phi^i$ as

$$\nabla_m\Phi^i = [U^{-1/2}\tanh\sqrt{4U}]_{ij}v_m^j. \qquad (3.13)$$

Substituting this into the Nambu-Goto vierbein eliminates v_μ^i from $N_n{}^m$ giving

$$N_n{}^m = \delta_n{}^m + \nabla_n\Phi^i G_{ij}\nabla^m\Phi^j, \qquad (3.14)$$

where now the $SO(2)$ matrix G_{ij} [recall Eq. (1.19)] is given in terms of the SUSY covariant derivatives of Φ as

$$G_{ij} = H_{ik}^{-1/2}[\sqrt{(1-H)} - 1]_{kl}H_{lj}^{-1/2}, \qquad (3.15)$$

with H_{ij} [recall Eq. (1.20)] expressed as

$$H_{ij} \equiv \nabla_m\Phi_i\nabla^m\Phi_j. \qquad (3.16)$$

Hence the determinant of the Nambu-Goto vierbein becomes

$$\det N = \det\sqrt{(1-H)} = \det\sqrt{(\delta_{ij} - \nabla^m\Phi_i\eta_{mn}\nabla^n\Phi_j)}, \qquad (3.17)$$

which after exploiting the 2×2 nature of the H matrices takes the form

$$\det N = \sqrt{(1 - \text{Tr}[H] + \det H)}$$

$$= \sqrt{(1 - \text{Tr}[\nabla^m\Phi_i\nabla_m\Phi_j] + \det\nabla^m\Phi_i\nabla_m\Phi_j)}. \qquad (3.18)$$

Thus the final form of the Akulov-Volkov-Nambu-Goto invariant action describing the covolume oscillations of a non-BPS vortex embedded in $D = 6$, $\mathcal{N} = (1,0)$ superspace is secured as

$$\Gamma = -\sigma\int d^4x \det\hat{e} \det\sqrt{(\delta_{ij} - \nabla^m\Phi_i\eta_{mn}\nabla^n\Phi_j)} = -\sigma\int d^4x \det\hat{e}\sqrt{(1 - \text{Tr}[\nabla^m\Phi_i\nabla_m\Phi_j] + \det\nabla^m\Phi_i\nabla_m\Phi_j)}$$

$$= -\sigma\int d^4x \det\hat{e}\sqrt{(1 - \nabla^m\Phi_i\nabla_m\Phi_i + \tfrac{1}{2}\epsilon_{ij}\epsilon_{kl}\nabla^m\Phi_i\nabla_m\Phi_k\nabla^n\Phi_j\nabla_n\Phi_l)}. \qquad (3.19)$$

The Akulov-Volkov vierbein $\hat{e}_\mu{}^m$ is given by Eq. (3.9) and the SUSY covariant derivatives of the Nambu-Goldstone boson fields are given by the differentials $d\Phi^i = dX^m\nabla_m\Phi^i$, where

$$\nabla_m\Phi^1 = \mathcal{D}_m\phi^1 + \lambda\overleftrightarrow{\mathcal{D}}_m\theta + \bar\lambda\overleftrightarrow{\mathcal{D}}_m\bar\theta$$

$$\nabla_m\Phi^2 = \mathcal{D}_m\phi^2 - i\lambda\overleftrightarrow{\mathcal{D}}_m\theta + i\bar\lambda\overleftrightarrow{\mathcal{D}}_m\bar\theta. \qquad (3.20)$$

The Nambu-Goldstone bosons can equivalently be described by tensor gauge fields [10–13]. The dual action which involves the replacement of one or both of the scalar

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[1]The analogous expression in the case of a codimension 1, $p = 3$ brane embedded in $D = 5$, $\mathcal{N} = 1$ superspace given in Ref. [10] should be amended to a similar form replacing $\partial_\mu(\theta\lambda - \bar\theta\bar\lambda)$ with $(\theta\partial_\mu\lambda - \bar\theta\partial_\mu\bar\lambda)$.

fields by tensor fields can be constructed using Lagrange multipliers. For the mixed scalar-tensor action, a Lagrange multiplier denoted by L^m is introduced and used to define the vector $l_m = \nabla_m \Phi^2$ so that the action (3.19) (with Φ^1 relabeled as φ) can then be written as

$$\Gamma = -\sigma \int d^4x \det\hat{e}$$
$$\times \{\sqrt{1 - \nabla^\mu\varphi\nabla_\mu\varphi - l^2 + l^2(\nabla\varphi)^2 - (l^\mu\nabla_\mu\varphi)^2}$$
$$+ L^\mu(l_\mu - \nabla_\mu\Phi^2)\}. \tag{3.21}$$

The Φ^2 equation of motion yields the identity

$$\partial_\mu(\det\hat{e}\,\hat{e}_m^{-1\mu}L^m) = 0 \tag{3.22}$$

whose solution is

$$F^\mu = \det\hat{e}\,\hat{e}_m^{-1\mu}L^m = \epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}, \tag{3.23}$$

where $B_{\mu\nu}$ is the dual tensor gauge field. Recalling that

$$\nabla_m\Phi^2 = \hat{e}_m^{-1\mu}\{\partial_\mu\phi^2 - i\lambda\overleftrightarrow{\partial}_\mu\theta + i\bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta}\}, \tag{3.24}$$

Φ^2 can be eliminated from the action by an integration by parts to yield

$$\Gamma = -\sigma \int d^4x \det\hat{e}$$
$$\times \{\sqrt{1 - \nabla^\mu\varphi\nabla_\mu\varphi - l^2 + l^2(\nabla\varphi)^2 - (l^\mu\nabla_\mu\varphi)^2}$$
$$+ L^\mu l_\mu + iL^m\hat{e}_m^{-1\mu}(\lambda\overleftrightarrow{\partial}_\mu\theta - \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta})\}. \tag{3.25}$$

The l^μ equation of motion

$$L^a = \frac{[l^m(1 - (\nabla\varphi)^2 + (l\cdot\nabla\varphi)\nabla^m\varphi]}{\sqrt{(1 - (\nabla\varphi)^2 - l^2 + l^2(\nabla\varphi)^2 - (l^\mu\nabla_\mu\varphi)^2)}} \tag{3.26}$$

can then be used to eliminate the l^μ field to obtain the mixed scalar field φ and tensor field $L^m = (1/\det\hat{e})\hat{e}_\mu{}^m\epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}$ dual action [13]

$$\Gamma = -\sigma \int d^4x \det\hat{e}$$
$$\times \{\sqrt{1 - \nabla^\mu\varphi\nabla_\mu\varphi + L^m L_m + (L^m\nabla_m\varphi)^2}$$
$$+ iL^m\hat{e}_m^{-1\mu}(\lambda\overleftrightarrow{\partial}_\mu\theta - \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta})\}. \tag{3.27}$$

Finally, the completely dual tensor gauge field action can be obtained by introducing another Lagrange multiplier K^m to define the vector field $k_m = \nabla_m\varphi = \hat{e}_m^{-1\mu}[\partial_\mu\phi^1 + (\lambda\overleftrightarrow{\partial}_\mu\theta + \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta})]$. With these substitutions the action (3.27) can be written as

$$\Gamma = -\sigma \int d^4x \det\hat{e}\{\sqrt{1 - k^m k_m + L^m L_m - (L^m k_m)^2}$$
$$+ iL^m\hat{e}_m^{-1\mu}(\lambda\overleftrightarrow{\partial}_\mu\theta - \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta}) + K^m(k_m - \nabla_m\varphi)\}. \tag{3.28}$$

As previously, the φ field equation $\partial_\mu[\det\hat{e}\,\hat{e}_m^{-1\mu}K^m] = 0$ is solved in terms of the second tensor gauge field giving

$$G^\mu = \det\hat{e}\,\hat{e}_m^{-1\mu}K^m = \epsilon^{\mu\nu\rho\sigma}\partial_\nu C_{\rho\sigma}. \tag{3.29}$$

The scalar field φ can then be eliminated using integration by parts to obtain the action

$$\Gamma = -\sigma \int d^4x \det\hat{e}\{\sqrt{1 - k^m k_m + L^m L_m - (L^m k_m)^2}$$
$$+ K^m k_m + iL^m\hat{e}_m^{-1\mu}(\lambda\overleftrightarrow{\partial}_\mu\theta - \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta})$$
$$- K^m\hat{e}_m^{-1\mu}(\lambda\overleftrightarrow{\partial}_\mu\theta + \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta})\}. \tag{3.30}$$

Lastly, use of the k_m equation of motion

$$K^m = \frac{k^m + (L^n k_n)L^m}{\sqrt{1 - k^m k_m + L^m L_m - (L^m k_m)^2}} \tag{3.31}$$

allows the complete dual form of the action to be secured as

$$\Gamma = -\sigma \int d^4x \det\hat{e}\{\sqrt{(1 + L^2)(1 + K^2) - (L\cdot K)^2}$$
$$+ iL^m\hat{e}_m^{-1\mu}(\lambda\overleftrightarrow{\partial}_\mu\theta - \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta})$$
$$- K^m\hat{e}_m^{-1\mu}(\lambda\overleftrightarrow{\partial}_\mu\theta + \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta})\}. \tag{3.32}$$

The $SO(2)$ invariance of the action can be made more manifest by introducing the notation

$$V_1^m = K^m, \qquad V_2^m = L^m$$
$$J_{1m} = \hat{e}_m^{-1\mu}(\lambda\overleftrightarrow{\partial}_\mu\theta + \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta}) \tag{3.33}$$
$$J_{2m} = \hat{e}_m^{-1\mu}(\lambda\overleftrightarrow{\partial}_\mu\theta - \bar{\lambda}\overleftrightarrow{\partial}_\mu\bar{\theta})$$

in terms of which the action reads

$$\Gamma = -\sigma \int d^4x \det\hat{e}\{\det\sqrt{(1 + V_i^m V_{jm})} - V_i^m J_{mi}\}$$
$$= -\sigma \int d^4x \det\hat{e}$$
$$\times \left\{\sqrt{1 + V_i^m V_{im} + \frac{1}{2}\epsilon_{ij}\epsilon_{kl}(V_i^m V_{km})(V_j^n V_{ln})} - V_i^m J_{mi}\right\}. \tag{3.34}$$

Using the Akulov-Volkov vierbein to convert tangent space indices to world volume indices and vice versa so that, for example, $J_i^\mu = \hat{e}_m^{-1\mu}J_i^m$, and introducing the Akulov-Volkov metric $\hat{g}_{\mu\nu} = \hat{e}_\mu{}^m\eta_{mn}\hat{e}_\nu{}^n$ and the world volume tensor density fields $F_i^\mu = \det\hat{e}\,\hat{e}_m^{-1\mu}V_i^m$ so that $F_{i\mu} = (1/\det\hat{e})\hat{g}_{\mu\nu}F_i^\nu$, the action takes the final form:

$$\Gamma = -\sigma \int d^4x \{ \sqrt{-\det \hat{g}} \sqrt{1 + F_{i\mu}\hat{g}^{\mu\nu}F_{i\nu} + \det(F_{i\mu}\hat{g}^{\mu\nu}F_{j\nu})}$$
$$- F_{i\mu}J_i^\mu \}$$
$$= -\sigma \int d^4x \{ \sqrt{-\det \hat{g}} \sqrt{\det(1 + F_{i\mu}\hat{g}^{\mu\nu}F_{j\nu})} - F_{i\mu}J_i^\mu \}$$
$$= -\sigma \int d^4x \{ \sqrt{-\det[\hat{g}_{\mu\nu} + F_{i\mu}F_{i\nu}]} - F_{i\mu}J_i^\mu \}.$$

$$(3.35)$$

If, in addition to translation symmetries, the presence of the vortex leads to the spontaneous breakdown of various global internal symmetries, then the effective world volume theory also contains the associated Nambu-Goldstone scalar fields, while the dual theory is a function of the corresponding non-Abelian tensor gauge fields [12]. A vortex of this type occurs in supersymmetric $U(N_C)$ gauge theory with $N_F = N_C$ fundamental hypermultiplets in $D = 4$ [14] and also in its $D = 6$ [15] dimensional generalization.

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