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To cite this version:

Arnaud Debussche, Sylvain de Moor, Martina Hofmanova. A Regularity Result for Quasilinear Stochastic Partial Differential Equations of Parabolic Type. SIAM Journal on Mathematical Analysis, Society for Industrial and Applied Mathematics, 2015, 47 (2), pp.1590-1614. 10.1137/130950549 . hal-00935892v2

HAL Id: hal-00935892

https://hal.archives-ouvertes.fr/hal-00935892v2

Submitted on 26 May 2015

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A REGULARITY RESULT FOR QUASILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

ARNAUD DEBUSSCHE, SYLVAIN DE MOOR, AND MARTINA HOFMANOVÁ

Abstract. We consider a non degenerate quasilinear parabolic stochastic partial differential equation with a uniformly elliptic diffusion matrix. It is driven by a nonlinear noise. We study regularity properties of its weak solution satisfying classical *a priori* estimates. In particular, we determine conditions on coefficients and initial data under which the weak solution is Hölder continuous in time and possesses spatial regularity. Our proof is based on an efficient method of increasing regularity: the solution is rewritten as the sum of two processes, one solves a linear parabolic SPDE with the same noise term as the original model problem whereas the other solves a linear parabolic PDE with random coefficients. This way, the required regularity can be achieved by repeatedly making use of known techniques for stochastic convolutions and deterministic PDEs.

1. Introduction

In this paper, we are interested in the regularity of weak solutions of non degenerate quasilinear parabolic stochastic partial differential equation driven by a multiplicative noise. Let $D \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $T > 0$ and set $D_T = (0,T) \times D$, $S_T = (0,T] \times \partial D$. We study the following problem

(1.1)
$$\begin{cases} \mathrm{d}u = \operatorname{div}(B(u))\,\mathrm{d}t + \operatorname{div}(A(u)\nabla u)\,\mathrm{d}t + F(u)\,\mathrm{d}t + H(u)\,\mathrm{d}W, & \text{in } D_T, \\ u = 0, & \text{on } S_T, \\ u(0) = u_0, & \text{in } D. \end{cases}$$

where W is a cylindrical Wiener process on some Hilbert space K and H is a mapping with values in the space of the γ-radonifying operators from K to certain Sobolev spaces. The diffusion matrix A is assumed to be smooth and uniformly elliptic and the initial condition u_0 is random in general. The precise description of the problem setting will be given in the next section.

It is a well known fact in the field of PDEs and SPDEs that many equations do not, in general, have classical or strong solutions and can be solved only in some weaker sense. Unlike deterministic problems, in the case of stochastic equations we can only ask whether the solution is smooth in the

Date: February 25, 2015.

space variable since the time regularity is limited by the regularity of the stochastic integral. Thus, the aim of the present work is to determine conditions on coefficients and initial data under which there exists a spatially smooth solution to (1.1).

Such a regularity result is fundamental and interesting by itself. Equations of the form (1.1) appear in many sciences. Regularity of solutions is an important property when one wants to study qualitative behaviour. It is also a preliminary step when studying numerical approximations and implies strong uniqueness for (1.1). Our original motivation is that such models arise as limits of random kinetic equations (see [12]).

The issue of existence of a classical solution to deterministic parabolic problems is well understood, among the main references stands the extensive book [19] which is mainly concerned with the solvability of initial-boundary value problems and the Cauchy problem to the basic linear and quasilinear second order PDEs of parabolic type. Special attention is paid to the connection between the smoothness of solutions and the smoothness of known data entering into the problem (initial condition and coefficients), nevertheless, due to the technical complexity of the proofs a direct generalization to the stochastic case is not obvious.

In the case of linear parabolic problems, let us mention the classical Schauder theory (see e.g. [20]) that provides *a priori* estimates relating the norms of solutions of initial- boundary value problems, namely the parabolic Hölder norms, to the norms of the known quantities in the problems. These results are usually employed in order to deal with quasilinear equations: the application of the Schauder fixed point theorem leads to the existence of a smooth solution under suitable hypotheses on the coefficients. In our proof, we make use of the Schauder theory as well, yet in an entirely different approach.

Regularity of parabolic problems in the stochastic setting was also studied in several works. In the previous work of the third author [18], semilinear parabolic SPDEs (i.e. the diffusion matrix A being independent of the solution) were studied and a regularity result established by using semigroup arguments (see also [25], [26]). In [13], [14], a maximum principle is obtained for an SPDE similar to (1.1) but with a more general diffusion H, it may depend on the gradient of u. In [16], existence and uniqueness of strong solutions to SPDEs with drift given by the subdifferential of a quasi-convex function is proved. Hölder continuity of solutions to nonlinear parabolic systems under suitable structure conditions was proved in [5] by energy methods. Quasilinear stochastic porous media equations are studied in [3], [4] and specific techniques for these equations are used. In comparison to this work, the quasilinear case considered in the present paper is more delicate and different techniques need to be applied.

The transposition of the deterministic method exposed in [19] seems to be quite difficult. Fortunately, we may use a trick to avoid this. We use a very simple idea: a weak solution to (1.1) that satisfies *a priori* estimates

is decomposed into two parts $u = y + z$ where z is a solution to a linear parabolic SPDE with the same noise term as (1.1) and y solves a linear parabolic PDE with random coefficients. As a consequence, the problem of regularity of u is reduced to showing regularity of z and regularity of y which can be handled by known techniques for stochastic convolutions and deterministic PDEs. It is rather surprising that this classical idea used to treat semilinear equations can be applied also for quasilinear problems.

Let us explain this method more precisely. As the main difficulties come from the second order and stochastic terms, for simplicity of the introduction we assume $B = F = 0$. Let u be a weak solution to

$$\begin{cases} \mathrm{d}u = \operatorname{div}(A(u)\nabla u)\,\mathrm{d}t + H(u)\,\mathrm{d}W, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

and let z be a solution to

$$\begin{cases} \mathrm{d}z = \Delta z\,\mathrm{d}t + H(u)\,\mathrm{d}W, \\ z(0) = 0. \end{cases}$$

Then z is given by the stochastic convolution with the semigroup generated by the Laplacian, denoted by $(S(t))_{t\geq 0}$, i.e.

$$z(t) = \int_0^t S(t-s)H(u)\,\mathrm{d}W(s)$$

and regularization properties are known. Setting $y = u - z$ it follows immediately that y solves

$$\begin{cases} \partial_t y = \operatorname{div}(A(u)\nabla y) + \operatorname{div}((A(u) - \mathrm{I})\nabla z), \\ y(0) = u_0, \end{cases} \tag{1.3}$$

which is a (pathwise) deterministic linear parabolic PDE. According to *a priori* estimates for (1.2), it holds for all $p \in [2, \infty)$ and $u_0 \in L^p(\Omega \times D)$:

$$u \in L^p(\Omega; L^\infty(0,T; L^p(D))) \cap L^p(\Omega; L^2(0,T; W^{1,2}(D))),$$

and making use of the factorization method it is possible to show that z possesses enough regularity so that ∇z is a function with good integrability properties. Now, a classical result for deterministic linear parabolic PDEs with discontinuous coefficients (see [19]) yields Hölder continuity of y (in time and space) and consequently also Hölder continuity of u itself. Having this in hand, the regularity of z can be increased to a level where the Schauder theory for linear parabolic PDEs with Hölder continuous coefficients applies to (1.3) (see [20]) and higher regularity of y is obtained. Repeating this approach then allows us to conclude that u is λ-Hölder continuous in time for all $\lambda < 1/2$ and possesses as much regularity in space as allowed by the regularity of the coefficients and the initial data. In this article, in order to avoid lengthy proofs and notations, we restrict to spatial regularity less than 5. Our method extends to higher regularity, see Remark 2.8.

The paper is organized as follows. In Section 2, we introduce the basic setting and state our regularity results, Theorem 2.6, Theorem 2.7. Section 3 gives preliminary results concerning the stochastic convolution and the smoothness of the solutions of linear parabolic equations. These are adapted from [19] and [20], we have to explicit the dependance of the constant with respect to some data and to treat low time regularity. The remainder of the paper is devoted to the proof of Theorem 2.6 and Theorem 2.7 that is divided into several parts. In Section 4, we establish our first regularity result, Theorem 2.6, that gives some Hölder continuity in time and space of a weak solution to (1.1). The regularity is then improved in the final Section 5 and Theorem 2.7 is proved.

2. Notations, hypotheses and the main result

2.1. **Notations.** In this paper, we adopt the following conventions. For $r \in [1,\infty]$, the Lebesgue spaces $L^r(D)$ are denoted by L^r and the corresponding norm by $\|\cdot\|_r$. In order to measure higher regularity of functions we make use of the Bessel potential spaces $H^{a,r}(D)$, $a \in \mathbb{R}$ and $r \in (1,\infty)$.

In order to motivate the use of these spaces let us recall their basic properties (for a thorough exposition we refer the reader to the books of Triebel [23], [24]). In the case of $\mathbb{R}^N$ the Bessel potential spaces are defined in terms of Fourier transform of tempered distributions: let $a \in \mathbb{R}$, $r \in (1,\infty)$ then

$$H^{a,r}(\mathbb{R}^N) = \left\{ f \in \mathcal{S}'(\mathbb{R}^N);\ \|f\|_{H^{a,r}} := \left\| \mathcal{F}^{-1}(1+|\xi|^2)^{a/2}\mathcal{F}f \right\|_{L^r} < \infty \right\}$$

and they belong to the Triebel-Lizorkin scale $F^a_{r,s}(\mathbb{R}^N)$ in the sense that $H^{a,r}(\mathbb{R}^N) = F^a_{r,2}(\mathbb{R}^N)$. The Bessel potential spaces $H^{a,r}(\mathbb{R}^N)$ behave well under the complex interpolation, i.e. for $a_0, a_1 \in \mathbb{R}$ and $r_0, r_1 \in (1,\infty)$ it holds that

$$[H^{a_0,r_0}(\mathbb{R}^N), H^{a_1,r_1}(\mathbb{R}^N)]_\theta = H^{a,r}(\mathbb{R}^N), \tag{2.1}$$

where $\theta \in (0,1)$ and $a = (1-\theta)a_0 + \theta a_1$, $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, which makes them more suitable for studying regularity for linear elliptic and parabolic problems. Indeed, under the assumption of bounded imaginary powers of a positive operator $\mathcal{A}$ on a Banach space X, the domains of fractional powers of $\mathcal{A}$ are given by the complex interpolation as well: let $0 \le \alpha < \beta < \infty$, $\theta \in (0,1)$ then

$$[D(\mathcal{A}^\alpha), D(\mathcal{A}^\beta)]_\theta = D(\mathcal{A}^{(1-\theta)\alpha+\theta\beta}).$$

Furthermore, the expression (2.1) and the obvious identity $H^{m,r}(D) = W^{m,r}(D)$ for $m \in \mathbb{N}_0$ and $r \ge 1$ suggest how the spaces $H^{a,r}(D)$ may be defined for a general domain D: if $a \ge 0$ and $m \in \mathbb{N}$ such that $a \le m < a+1$ then we define

$$H^{a,r}(D) := [W^{m,r}(D), L^r(D)]_{(m-a)/m}.$$

If D is sufficiently regular then $H^{a,r}(D)$ coincides with the space of restrictions to D of functions in $H^{a,r}(\mathbb{R}^N)$ and the Sobolev embedding theorem holds true. The spaces $H^{a,r}_0(D)$, $a \ge 0$, $r \in (1,\infty)$, are then defined as the

closure of $C_c^\infty(D)$ in $H^{a,r}(D)$. Note, that $H_0^{a,r}(D) = H^{a,r}(D)$ if $a \leq 1/r$ and $H_0^{a,r}(D)$ is strictly contained in $H^{a,r}$ if $a > 1/r$. Besides, an interpolation result similar to (2.1) holds for these spaces as well

$$[H_0^{a_0,r_0}(D), H_0^{a_1,r_1}(D)]_\theta = H_0^{a,r}(D).$$

For notational simplicity, we denote the norm of $H^{a,r}$ by $\|\cdot\|_{a,r}$.

Remark 2.1. The spaces $H^{a,r}(D)$ are generally different from the Sobolev-Slobodeckij spaces $W^{a,r}(D)$ which belong to the Besov scale $B^a_{r,s}(D)$ in the sense that $W^{a,r}(D) = B^a_{r,r}(D)$ if $a > 0$, $a \notin \mathbb{N}$. Nevertheless, we have the following two relations which link the two scales of function spaces together

$$W^{a,r}(D) = H^{a,r}(D) \quad \text{if} \quad a \in \mathbb{N}_0,\, r \in [1,\infty) \quad \text{or} \quad a \geq 0,\, r = 2,$$

and

$$H^{a+\varepsilon,r}(D) \hookrightarrow W^{a,r}(D) \hookrightarrow H^{a-\varepsilon,r}(D) \qquad a \in \mathbb{R},\, r \in (1,\infty),\, \varepsilon > 0.$$

Below we use the Laplace operator with Dirichlet boundary conditions, denoted by Δ_D. Considered as an operator on L^r, its domain is $H_0^{2,r}$ and it is the infinitesimal generator of an analytic semigroup denoted by $S = (S(t))_{t\geq 0}$. Moreover, it follows from the above considerations that the domains of its fractional powers coincide with the Bessel potential spaces, that is $D((-\Delta_D)^\alpha) = H_0^{2\alpha,r}$, $\alpha \geq 0$. Therefore, one can build a fractional power scale (or a Sobolev tower, see [2], [15]) generated by $(L^r, -\Delta_D)$ to get

$$\big[\big(H_0^{2\alpha,r}, -\Delta_{D,2\alpha,r}\big);\, \alpha \geq 0\big], \tag{2.2}$$

where $-\Delta_{D,2\alpha,r}$ is the $H_0^{2\alpha,r}$-realization of $-\Delta_D$. Having this in hand, an important result [2, Theorem V.2.1.3] describes the behavior of the semigroup S in this scale. More precisely, the operator $\Delta_{D,2\alpha,r}$ generates an analytic semigroup $S_{2\alpha,r}$ on $H_0^{2\alpha,r}$ which is naturally obtained from S by restriction, i.e. $S_{2\alpha,r}(t)$ is the $H_0^{2\alpha,r}$-realization of $S(t)$, $t \geq 0$, and we have the following regularization property: for any $\delta > 0$ and $t > 0$, $S_{2\alpha,r}(t)$ maps $H_0^{2\alpha,r}$ into $H_0^{2\alpha+\delta,r}$ with

$$\big\|S_{2\alpha,r}(t)\big\|_{\mathcal{L}(H_0^{2\alpha,r},H_0^{2\alpha+\delta,r})} \leq \frac{C}{t^{\delta/2}}. \tag{2.3}$$

For notational simplicity of the sequel we do not directly specify the spaces where the operators Δ_D and $S(t)$, $t \geq 0$, are acting since this is always clear from the context.

Another important scale of function spaces which is used throughout the paper are the Hölder spaces. In particular, if X and Y are two Banach spaces and $\alpha \in (0,1)$, $C^\alpha(X;Y)$ denotes the space of bounded Hölder continuous functions with values in Y equipped with the norm

$$\|f\|_{C^\alpha(X;Y)} = \sup_{x\in X} \|f(x)\|_Y + \sup_{x,x'\in X, x\neq x'} \frac{\|f(x)-f(x')\|_Y}{\|x-x'\|_X^\alpha}.$$

In the sequel, we consider the spaces $C^\alpha(\overline{D}) = C^\alpha(\overline{D};\mathbb{R})$, $C^\alpha([0,T];Y)$ where $Y = H^{a,r}$ or $Y = C^\beta(\overline{D})$ and $C^\alpha([0,T]\times\overline{D}) = C^\alpha([0,T]\times\overline{D};\mathbb{R})$... Besides, we employ Hölder spaces with different regularity in time and space, i.e. $C^{\alpha,\beta}([0,T]\times\overline{D})$. For $\alpha\in(0,1/2)$, $\beta\in(0,1)$, they are equipped with the norm

$$\|f\|_{C^{\alpha,\beta}} = \sup_{(t,x)}|f(t,x)| + \sup_{(t,x)\neq(s,y)}\frac{|f(t,x)-f(s,y)|}{\max\{|t-s|^\alpha,|x-y|^\beta\}}.$$

For larger indices $\alpha+l/2$, $\beta+k$ with $\alpha\in(0,1/2)$, $\beta\in(0,1)$ and l,k non negative integers, the norm is defined by

$$\|f\|_{C^{\alpha+l/2,\beta+k}} = \sum_{2r\le l,|\gamma|\le k,2r+|\gamma|\le\max\{l,k\}}\|\partial_t^r\partial^\gamma f\|_{C^{\alpha,\beta}}.$$

We have denoted by ∂_t the partial derivative with respect to the time variable t and, for a multi-index $\gamma=(\gamma_1,\dots,\gamma_N)$, $\partial^\gamma = \partial_1^{\gamma_1}\dots\partial_N^{\gamma_N}$ where ∂_i is the partial derivative with respect to x_i. For $\alpha=\beta/2$, $k=l$, these are the classical Hölder spaces used to measure the regularity of solutions of parabolic problems. Here, we need these slightly more general spaces. In this work, we always use $\alpha=\beta/2$. This is a very natural choice since in this case $C^{\alpha,\beta}$ is precisely the Hölder space of order β with respect to the parabolic distance: $d((t,x),(s,y)) = \max\{|t-s|^{1/2},|x-y|\}$.

Note that $C^{\beta/2,\beta+1}([0,T]\times\overline{D}) = C^{(\beta+1)/2,\beta+1}([0,T]\times\overline{D})$. Indeed both spaces consist of bounded functions such that

$$\|f\|_{C^{\beta/2,\beta}} + \sum_{|\gamma|=1}\|\partial^\gamma f\|_{C^{\beta/2,\beta}}$$

is finite.

Given a domain D with a $C^{\beta+k}$ boundary, using local coordinates, we define classically the Hölder spaces on the boundary $C^{\alpha,\beta}([0,T]\times\partial D)$ (see [19], section 3, chapter II. In particular (3.19) and the paragraph above).

Clearly if v is a function in $C^{\alpha+l/2,\beta+k}([0,T]\times\overline{D})$, its restriction to $[0,T]\times\partial D$ is in $C^{\alpha+l/2,\beta+k}([0,T]\times\partial D)$. But it can be much smoother, for instance if v is constant on the boundary.

Note that it holds $C^\alpha([0,T];C^\beta(\overline{D}))\subsetneq C^{\alpha,\beta}([0,T]\times\overline{D})$ and therefore we have to distinguish these two spaces. An example is given in [21]: $\overline{D}=[0,T]$ and $u(t,x)=(x+t)^\alpha$, $\alpha\in(0,1)$.

For $k\in\mathbb{N}_0$ we denote by $C_b^k = C_b^k(\mathbb{R})$ the space of continuous functions that have continuous bounded derivatives up to order k. Note that for $k=0$ it is the space of continuous functions, not necessarily bounded.

In the whole article, C, C_i, K, K_i, κ, ... denote constants. When they depend on some parameters of the problem, this is explicitly stated.

2.2. **Hypotheses.** Let us now introduce the precise setting of (1.1). We work on a finite-time interval $[0,T]$, $T>0$, and on a bounded domain D in $\mathbb{R}^N$ with smooth boundary. We denote by D_T the cylinder $(0,T)\times D$

and by S_T the lateral surface of D_T, that is $S_T = (0,T] \times \partial D$. Concerning the coefficients A, B, F, H, we only state here the basic assumptions that guarantee the existence of a weak solution and are valid throughout the paper. Further regularity hypotheses are necessary in order to obtain better regularity of the weak solution and will be specified later. We assume that the flux function

$$B = (B_1, \dots, B_N) : \mathbb{R} \longrightarrow \mathbb{R}^N$$

is continuous with linear growth. The diffusion matrix

$$A = (A_{ij})_{i,j=1}^N : \mathbb{R} \longrightarrow \mathbb{R}^{N\times N}$$

is supposed to be continuous, symmetric, positive definite and bounded. In particular, there exist constants ν, $\mu > 0$ such that for all $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

$$\nu|\xi|^2 \le A(u)\xi \cdot \xi \le \mu|\xi|^2. \tag{2.4}$$

The drift coefficient $F : \mathbb{R} \to \mathbb{R}$ is continuous with linear growth.

Regarding the stochastic term, let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The driving process W is a cylindrical Wiener process: $W(t) = \sum_{k\ge 1} \beta_k(t) e_k$ with $(\beta_k)_{k\ge 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)_{t\ge 0}$ and $(e_k)_{k\ge 1}$ a complete orthonormal system in a separable Hilbert space K. For each $u \in L^2(D)$ we consider a mapping $H(u) : K \to L^2(D)$ defined by $H(u)\, e_k = H_k(\cdot, u(\cdot))$. In particular, we suppose that $H_k \in C(D \times \mathbb{R})$ and the following linear growth condition holds true

$$\sum_{k\ge 1} |H_k(x,\xi)|^2 \le C\big(1 + |\xi|^2\big), \qquad \forall x \in D,\, \xi \in \mathbb{R}. \tag{2.5}$$

This assumption implies in particular that H maps $L^2(D)$ to $L_2(K; L^2(D))$ where $L_2(K; L^2(D))$ denotes the collection of Hilbert-Schmidt operators from K to $L^2(D)$. Thus, given a predictable process u that belongs to $L^2(\Omega; L^2(0,T; L^2(D)))$, the stochastic integral $t \mapsto \int_0^t H(u)\mathrm{d}W$ is a well defined process taking values in $L^2(D)$ (see [10, Chapter 4] for a thorough exposition).

Later on we are going to estimate the weak solution of (1.1) in certain Bessel potential spaces $H_0^{a,r}$ with $a \ge 0$ and $r \in [2,\infty)$ and therefore we need to ensure the existence of the stochastic integral in (1.1) as an $H_0^{a,r}$-valued process. We recall that the Bessel potential spaces $H_0^{a,r}$ with $a \ge 0$ and $r \in [2,\infty)$ belong to the class of 2-smooth Banach spaces since they are isomorphic to $L^r(0,1)$ according to [23, Theorem 4.9.3] and hence they are well suited for the stochastic Itô integration (see [7], [8] for the precise construction of the stochastic integral). So, let us denote by $\gamma(K,X)$ the space of the γ-radonifying operators from K to a 2-smooth Banach space X. We recall that $\Psi \in \gamma(K,X)$ if the series

$$\sum_{k\ge 0} \gamma_k \Psi(e_k)$$

converges in $L^2(\widetilde{\Omega}, X)$, for any sequence $(\gamma_k)_{k\geq 0}$ of independent Gaussian real-valued random variables on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and any orthonormal basis $(e_k)_{k\geq 0}$ of K. Then, the space $\gamma(K, X)$ is endowed with the norm

$$\|\Psi\|_{\gamma(K,X)} := \left(\widetilde{\mathbb{E}} \left| \sum_{k\geq 0} \gamma_k \Psi(e_k) \right|_X^2 \right)^{1/2}$$

(which does not depend on $(\gamma_k)_{k\geq 0}$, nor on $(e_k)_{k\geq 0}$) and is a Banach space. Now, if $a \geq 0$ and $r \in [2, \infty)$ we denote by $(\mathrm{H}_{a,r})$ the following hypothesis

$$(\mathrm{H}_{a,r}) \qquad \|H(u)\|_{\gamma(K, H_0^{a,r})} \leq \begin{cases} C\big(1 + \|u\|_{H_0^{a,r}}\big), & a \in [0,1], \\ C\big(1 + \|u\|_{H_0^{a,r}} + \|u\|^a_{H_0^{1,ar}}\big), & a > 1, \end{cases}$$

i.e. H maps $H_0^{a,r}$ to $\gamma(K, H_0^{a,r})$ provided $a \in [0,1]$ and it maps $H_0^{a,r} \cap H_0^{1,ar}$ to $\gamma(K, H_0^{a,r})$ provided $a > 1$. The precise values of parameters a and r will be given later in each of our regularity results.

Remark 2.2. We point out that, thanks to the linear growth hypothesis (2.5) on the functions $(H_k)_{k\geq 1}$, one can easily verify that, for all $r \in [2, \infty)$, the bound $(\mathrm{H}_{0,r})$ holds true.

In order to clarify the assumption $(\mathrm{H}_{a,r})$, let us present the main examples we have in mind.

Example 2.3. Let W be a d-dimensional $(\mathscr{F}_t)$-Wiener process, that is $W(t) = \sum_{k=1}^d W_k(t)\, e_k$, where W_k, $k = 1, \dots, d$, are independent standard $(\mathscr{F}_t)$-Wiener processes and $(e_k)_{k=1}^d$ is an orthonormal basis of $K = \mathbb{R}^d$. Then hypothesis $(\mathrm{H}_{a,r})$ is satisfied for $a \geq 0$, $r \in [2, \infty)$ provided the functions $H_1, \dots, H_d$ are sufficiently smooth and respect the boundary conditions in the following sense: if $a > \frac{1}{r}$, then

$$\nabla_x^l H_k(x, 0) = 0, \qquad x \in \partial D, \quad \forall k = 1, \dots, d, \quad \forall l \in \mathbb{N}_0,\, l < a - \frac{1}{r},$$

(for more details we refer the reader to [22]). Note that in this example it is necessary to restrict ourselves to the subspace $H_0^{a,r} \cap H_0^{1,ar}$ of $H_0^{a,r}$ so that the corresponding Nemytskij operators $u \mapsto H_k(\cdot, u(\cdot))$ take values in $H_0^{a,r}$. In fact, if $1 + 1/r \leq a \leq N/r$, $r \in (1, \infty)$, then only linear operators map $H_0^{a,r}$ to itself (see [22]).

Example 2.4. In the case of linear operator H we are able to deal with an infinite dimensional noise. Namely, let W be a $(\mathscr{F}_t)$-cylindrical Wiener process on $K = L^2(D)$, that is $W(t) = \sum_{k\geq 1} W_k(t)\, e_k$, where W_k, $k \geq 1$, are independent standard $(\mathscr{F}_t)$-Wiener processes and $(e_k)_{k\geq 1}$ an orthonormal basis of K. We assume that H is linear of the form $H(u)e_k := u\, Q e_k$, $k \geq 1$, where Q denotes a linear operator from K to K. Then, one can verify that the hypothesis $(\mathrm{H}_{a,r})$ is satisfied for $a \geq 0$, $r \in [2, \infty)$ provided we assume the following regularity property: $\sum_{k\geq 1} \|Q e_k\|^2_{a,\infty} < \infty$. We point out that, in this example, H maps $H_0^{a,r}$ to $\gamma(K, H_0^{a,r})$ for any $a \geq 0$ and $r \in [2, \infty)$.

As we are interested in proving the regularity up to the boundary for weak solutions of (1.1), it is necessary to impose certain compatibility conditions upon the initial data and the null Dirichlet boundary condition. To be more precise, since u_0 can be random in general, let us assume that $u_0 : \Omega \to C(\overline{D})$ is measurable and $u_0 = 0$ on ∂D. Further integrability and regularity assumptions on u_0 will be specified later.

Note that other boundary conditions could be studied with similar arguments, see Remark 2.8 below.

2.3. **Existence of weak solutions.** Let us only give a short comment here as existence is not our main concern and we will only make use of *a priori* estimates for parabolic equations of the form (1.1). In the recent work [11], the authors gave a well-posedness result for degenerate parabolic SPDEs (with periodic boundary conditions) of the form

$$\begin{cases} \mathrm{d}u = \operatorname{div}(B(u))\,\mathrm{d}t + \operatorname{div}(A(u)\nabla u)\,\mathrm{d}t + H(u)\,\mathrm{d}W, \\ u(0) = u_0, \end{cases}$$

where the diffusion matrix was supposed to be positive semidefinite. One can easily verify that the Dirichlet boundary conditions and the drift term $F(u)$ in (1.1) do not cause any additional difficulties in the existence part of the proofs and therefore the corresponding results in [11], namely Section 4 (with the exception of Subsection 4.3) and Proposition 5.1, are still valid in the case of (1.1). In particular, we have the following.

Theorem 2.5. *There exists* $\big((\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \tilde{W}, \tilde{u}\big)$ *which is a weak martingale solution to* (1.1) *and, for all* $p \in [2, \infty)$ *and* $u_0 \in L^p(\tilde{\Omega}; L^p)$

$$\tilde{u} \in L^p(\tilde{\Omega}; C([0,T]; L^2)) \cap L^p(\tilde{\Omega}; L^\infty(0,T; L^p)) \cap L^p(\tilde{\Omega}; L^2(0,T; W^{1,2})).$$

By a weak martingale solution we mean that it is weak in the PDE sense, that is, the equation (1.1) is satisfied in $\mathcal{D}'([0,T) \times D)$. It is also weak in probabilistic sense, that is, the probability space and the driving Wiener process are considered as unknown (see [10, Chapter 8] for precise definition).

In the sequel, we assume the existence of a weak solution (in the PDE sense) on the original probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and show that it possesses regularity that depends on the regularity of coefficients and initial data. We point out that this assumption is taken without loss of generality since pathwise uniqueness can be proved once we have sufficient regularity in hand and hence existence of a pathwise solution can be then obtained by the method of Gyöngy and Krylov [17] (see also [11, Subsection 4.3]).

2.4. **The main result.** To conclude this section let us state our main results to be proved precisely.

Theorem 2.6. *Let* u *be a weak solution to* (1.1) *such that, for all* $p \in [2, \infty)$,

$$u \in L^2(\Omega; C([0,T]; L^2)) \cap L^p(\Omega; L^\infty(0,T; L^p)) \cap L^2(\Omega; L^2(0,T; W_0^{1,2})).$$

Assume that

(i) $u_0 \in L^m(\Omega; C^\iota(\overline{D}))$ *for some* $\iota > 0$ *and all* $m \in [2, \infty)$, *and* $u_0 = 0$ *on* ∂D *a.s.*
(ii) ($\mathrm{H}_{1,2}$) *is fulfilled.*

Then there exists $\eta > 0$ *such that, for all* $m \in [2, \infty)$, *the weak solution* u *belongs to* $L^m(\Omega; C^{\eta/2,\eta}(\overline{D_T}))$.

Next result gives higher regularity. As in the deterministic case, some compatibility conditions are required on the boundary $\{0\} \times \partial D$. To state these, we introduce the notation: $u_0^{(1)} = \operatorname{div}(A(u_0)\nabla u_0) + \operatorname{div}(B(u_0)) + F(u_0)$ and $\mathcal{A}_0 = \sum_{ij} A_{ij}(0)\partial_{ij}$.

Theorem 2.7. *Let* $k = 1, 2, 3$ *or* 4. *Let* u *be a weak solution to* (1.1) *such that, for all* $p \in [2, \infty)$,

$$u \in L^2(\Omega; C([0,T]; L^2)) \cap L^p(\Omega; L^\infty(0,T; L^p)) \cap L^2(\Omega; L^2(0,T; W_0^{1,2})).$$

Assume that

(i) $u_0 \in L^m(\Omega; C^{k+\iota}(\overline{D}))$ *for some* $\iota > 0$ *and all* $m \in [2, \infty)$,
(ii) $u_0 = 0$ *on* ∂D *a.s., and* $u_0^{(1)} = 0$ *on* ∂D *a.s. if* $k = 2, 3, 4$, *and* $2(A'(0)\nabla u_0) \cdot \nabla u_0^{(1)} + B'(0) \cdot \nabla u_0^{(1)} + \mathcal{A}_0 u_0^{(1)} = 0$ *on* ∂D *if* $k = 4$.
(iii) A, $B \in C_b^k$ *and* $F \in C_b^{k-1}$,
(iv) ($\mathrm{H}_{a,r}$) *is fulfilled for all* $a < k + 1$ *and* $r \in [2, \infty)$.

Then for all $\lambda \in (0, 1/2)$ *and all* $m \in [2, \infty)$, *the weak solution* u *belongs to* $L^m(\Omega; C^{\lambda,k+\iota}(\overline{D_T}))$.

Remark 2.8. We could investigate higher regularity. Indeed, the tools required to prove Theorem 2.7 extend without difficulty. In fact, the proofs require different arguments for $k = 1, 2, 3$ and 4. But for $k \geq 5$, the proof is exactly the same as for $k = 4$. This requires just a generalization of Theorem 3.5 below. Note that, as in the deterministic case, this requires stronger assumptions and compatibility conditions. The case of Neumann boundary conditions can be treated similarly.

Our result clearly extends to the case of coefficients A, B, F which may also depend on x under suitable regularity assumptions.

The case of periodic boundary conditions is much easier. Indeed, no compatibility conditions are required. Moreover, the proof of regularity higher than 3 is straightforward by differentiation of the equation.

Remark 2.9. Note, that the condition $m \in [2, \infty)$ in the assumption (i) of Theorem 2.6 and Theorem 2.7 can be weakened to $m \in [2, m_0]$ for some sufficiently large m_0. This would weaken the results accordingly. However, in the proof, we use big vallues of m and the precise value of m_0 is not of great interest. Therefore, we prefered to state our result in the form above.

3. Preliminaries

3.1. **Regularity of the stochastic convolution.** Our proof of Theorem 2.6 and Theorem 2.7 is based on a regularity result that concerns mild solutions to linear SPDEs of the form

$$\begin{cases} \mathrm{d}Z = \Delta_D Z \,\mathrm{d}t + \Psi(t)\,\mathrm{d}W_t, \\ Z(0) = 0, \end{cases} \tag{3.1}$$

where Δ_D is the Laplacian on D with null Dirichlet boundary conditions acting on various Bessel potential spaces.

The solution to (3.1) is given by the stochastic convolution, that is

$$Z(t) = \int_0^t S(t-s)\Psi(s)\,\mathrm{d}W_s, \quad t \in [0,T].$$

In order to describe the connection between its regularity and the regularity of Ψ, we recall the following proposition.

Proposition 3.1. *Let $a \geq 0$ and $r \in [2,\infty)$ and let Ψ be a progressively measurable process in $L^p(\Omega; L^p(0,T;\gamma(K,H_0^{a,r})))$.*

(i) *Let $p \in (2,\infty)$ and $\delta \in [0, 1-2/p)$. Then, for any $\lambda \in [0, 1/2 - 1/p - \delta/2)$, $Z \in L^p(\Omega; C^\lambda(0,T;H_0^{a+\delta,r}))$ and*

$$\mathbb{E}\|Z\|^p_{C^\lambda(0,T;H_0^{a+\delta,r})} \leq C\,\mathbb{E}\|\Psi\|^p_{L^p(0,T;\gamma(K,H_0^{a,r}))}.$$

(ii) *Let $p \in [2,\infty)$ and $\delta \in (0,1)$. Then $Z \in L^p(\Omega; L^p(0,T;H_0^{a+\delta,r}))$ and*

$$\mathbb{E}\|Z\|^p_{L^p(0,T;H_0^{a+\delta,r})} \leq C\,\mathbb{E}\|\Psi\|^p_{L^p(0,T;\gamma(K,H_0^{a,r}))}.$$

Proof. Having established the behavior of the Dirichlet Laplacian and the corresponding semigroup along the fractional power scale (2.2), the proof of (i) is an application of the factorization method and can be found in [7, Corollary 3.5] whereas the point (ii) follows from the Burkholder-Davis-Gundy inequality and regularization properties (2.3) of the semigroup. □

3.2. **Regularity results for deterministic parabolic PDE's.** We now state classical regularity results from [19] and [20]. Since the notations are different and difficult to find in the books, we restate the results with ours and in the situation needed in the sequel. Moreover, the dependence on the coefficients and initial data is not always explicit in these books. We thus precise the bounds, this requires new proofs. Also, we give new results where the regularity is not measured in the classical parabolic scaling.

We first consider a linear parabolic PDE of the form

$$\begin{cases} \partial_t v = \operatorname{div}(a(t,x)\nabla v) + \operatorname{div} g(t,x) + f(t,x), & \text{in } D_T, \\ v = 0, & \text{on } S_T, \\ v(0) = v_0, & \text{in } D, \end{cases} \tag{3.2}$$

and assume that there exist $\nu, \mu > 0$ such that $\nu|\xi|^2 \leq a(t,x)\xi \cdot \xi \leq \mu|\xi|^2$, for all $(t,x) \in D_T$, $\xi \in \mathbb{R}^N$.

This equation is precisely (1.9) with $\mathscr{L}$ defined in (1.1), from [19, Chapter III] with $f_i = g_i$, $i = 1, \dots, N$. Note that a is matrix valued. We have switched the notation for the unknown from u to v and for the initial data from ψ_0 to v_0. Condition (1.2) in [19, Chapter III] is precisely the condition on a just above.

Theorem 3.2. *Assume that $v_0 \in C^\beta(\overline{D})$ for some $\beta > 0$ and $v_0 = 0$ on ∂D. There exists r_0 depending on N; $\alpha \in (0, \beta]$ depending on N, ν, μ, r_0, D and a constant K_1 depending on $N, \nu, \mu, r_0, D, \alpha$ such that if*

(i) *v is a weak solution to (3.2) which is continuous in time with values in L^2 and belongs to $L^2(0,T; W_0^{1,2})$,*
(ii) *$f \in L^{r_0}(D_T)$ and $g \in L^{2r_0}(D_T)$,*

then $v \in C^{\alpha/2,\alpha}(\overline{D_T})$ and the following estimate holds

$$\|v\|_{C^{\alpha/2,\alpha}} \leq K_1\big(\|v_0\|_{C^\alpha(\overline{D})} + \|g\|_{L^{2r_0}(D_T)} + \|f\|_{L^{r_0}(D_T)}\big). \tag{3.3}$$

Proof. This result follows from [19, Theorem 10.1, Chapter III]. Note the compatibility condition $v_0 = 0$ on ∂D implies that the data on the parabolic boundary $\Gamma_T = \{0\} \times D \cup S_T$ is in $C^{\alpha/2,\alpha}(\Gamma_T)$, which is denoted by $H^{\alpha,\alpha/2}(\Gamma_T)$ in [19].

Numbers of equations and theorems below refer to [19, Chapter III].

We first have to check that v is in $L^\infty([0,T] \times D)$. This follows from Theorem 7.1..

We first note that it is always possible to find a number r_0 depending on N such that the couple $(r,q) = (r_0, r_0)$ satisfies (7.2) with some $\kappa_1 \in (0,1)$ if $N \geq 2$ or $\kappa_1 \in (0, \frac{1}{2})$ if $N = 1$. Hypothesis (7.1) with μ_1 depending on r_0 and N then follows. The crucial point is the estimate (7.14) then Theorem 6.1 from Chapter II applies and yields the maximum principle. Moreover, the dependence of the L^∞-bound on the coefficients and the initial data can be seen from (6.2). We obtain

$$\|v\|_{L^\infty(D_T)} \leq C\big(1 + \|v_0\|_{L^\infty}\big)\Big(1 + \|g\|_{L^{2r_0}(D_T)}^{2(1+\frac{N}{2\kappa_1})} + \|f\|_{L^{r_0}(D_T)}^{1+\frac{N}{2\kappa_1}}\Big)$$

where the constant C does not depend on v_0, g, f and depends on a only through its ellipticity constant ν. In other words we have proved that the linear mapping

$$\begin{aligned} L^\infty(D) \times L^{r_0}(D_T) \times L^{2r_0}(D_T) &\to L^\infty(D_T) \\ (v_0, f, g) &\mapsto v \end{aligned} \tag{3.4}$$

is bounded hence continuous and it holds

$$\|v\|_{L^\infty(D_T)} \leq C\big(\|v_0\|_{L^\infty} + \|g\|_{L^{2r_0}(D_T)} + \|f\|_{L^{r_0}(D_T)}\big). \tag{3.5}$$

Since we assume zero Dirichlet boundary conditions we may now apply the second part of Theorem 10.1 and deduce that v is in $C^{\alpha/2,\alpha}(\overline{D_T})$. The

proof is based on estimate (10.6) applied to cylinders $Q(\rho,\tau)$ intersecting the parabolic boundary $\Gamma_T = \{0\} \times D \cup S_T$ and to levels $k \geq \max_{Q(\rho,\tau)\cap\Gamma_T} \pm u$ and follows from Theorem 8.1 in Chapter II. Note that due to Remark 7.1 in Chapter II, the Hölder exponent α depends neither on the L^∞-bound of v nor on the L^{r_0}- and L^{2r_0}-norm of f and g, respectively, but may depend on the C^β-norm of v_0 as can be seen from the proof of Theorem 8.1, Chapter II, namely from (8.6). Nevertheless, one can first take $\|v_0\|_{C^\beta} \leq 1$ and then argue by linearity of the solution map (3.4) to obtain the final estimate (3.3). □

As the next step, we recall the Schauder estimate for equations in divergence form [20, Theorem 6.48].

Theorem 3.3. *Let $\alpha \in (0,1)$, assume that D has a $C^{\alpha+1}$ boundary and*

(i) *$a, g \in C^{\alpha/2,\alpha}(\overline{D_T})$, $f \in L^p(D_T)$ for some $p \geq \frac{N+2}{1-\alpha}$,*
(ii) *$v_0 \in C^{1+\alpha}(\overline{D})$, $v_0 = 0$ on ∂D.*

Then there exists a unique weak solution to (3.2). Moreover, there exists a constant K_2 depending on $N, \nu, \mu, p, D, \alpha$ such that

(3.6)
$$\|v\|_{C^{(1+\alpha)/2,1+\alpha}} \leq K_2 P_1(a_{\alpha/2,\alpha})\big(\|v_0\|_{C^{1+\alpha}(\overline{D})} + \|g\|_{C^{\alpha/2,\alpha}} + \|f\|_{L^p(D_T)}\big),$$

where P_1 is a polynomial and $a_{\alpha/2,\alpha} = \|a\|_{C^{\alpha/2,\alpha}}$.

Proof. According to [20, Theorem 6.48] one obtains the required regularity of v. Note that our compatibility condition $v_0 = 0$ on ∂D implies that the boundary data are in $C^{(1+\alpha)/2,1+\alpha}$ on the parabolic boundary S_T. This parabolic boundary is denoted by $\mathcal{P}\Omega$ in [20] and this Hölder space by $H_{1+\alpha}$. Thus the compatibility condition of [20, Theorem 6.48] is satisfied.

But the dependence on $a_{\alpha/2,\alpha}$ is not obvious. In the following we adapt the technique from [9, Theorem 3.1] and show that the dependence is indeed polynomial.

Take $r \in (0,1]$ to be fixed later. Let $(B_i(r/4))_{i=1,\dots,n}$ be a covering of $\overline{D_T}$ by parabolic cylinders of radius r, i.e. balls of radius r with respect to the parabolic distance

$$d((t,x),(s,y)) = \max\{|t-s|^{1/2}, |x-y|\}.$$

We denote their centers by (t_i, x_i). Let

$$\varphi_i \in C_c^\infty(B_i(r)) \text{ such that } 0 \leq \varphi_i \leq 1 \text{ and } \varphi_i \equiv 1 \text{ on } B_i(r/2).$$

It is possible to choose these functions such that

$$\|\nabla\varphi_i\|_{C^0(\overline{D_T})} \leq Cr^{-1}, \quad \|\partial_t\varphi_i\|_{C^0(\overline{D_T})} \leq Cr^{-2}, \quad \|\nabla\varphi_i\|_{C^{\alpha/2,\alpha}} \leq Cr^{-1-\alpha}.$$

Set $a_i = a(t_i, x_i)$, $v_i = \varphi_i v$, $v_{0,i} = \varphi_i(0)v_0$. Then v_i solves the following parabolic equation

(3.7)
$$\begin{cases} \partial_t v_i - \operatorname{div}(a_i \nabla v_i) = \operatorname{div}[(a-a_i)\nabla v_i] + \varphi_i f + \operatorname{div}(\varphi_i g) - g\nabla\varphi_i \\ \qquad\qquad - a\nabla v \nabla\varphi_i - \operatorname{div}[av\nabla\varphi_i] + v\partial_t\varphi_i, & \text{in } D_T, \\ v_i = 0, & \text{on } S_T, \\ v_i(0) = v_{0,i}, & \text{in } D. \end{cases}$$

This is an equation of the form

$$\begin{cases} \partial_t w - \operatorname{div}(b\nabla w) = h_0 + \operatorname{div} h, & \text{in } D_T, \\ w = 0, & \text{on } S_T, \\ w(0) = w_0, & \text{in } D, \end{cases}$$

where b is a constant matrix. According to [20, Theorem 6.48], it holds true

$$\|w\|_{C^{(1+\alpha)/2,1+\alpha}} \le C\big(\|w_0\|_{C^{1+\alpha}} + \|h\|_{C^{\alpha/2,\alpha}} + \|h_0\|_{M^{1,N+1+\alpha}(D_T)}\big), \tag{3.8}$$

where the constant depends only on N, ν, μ, D, α and $M^{1,N+1+\alpha}(D_T)$ denotes the Morrey space (corresponding to the parabolic distance). It is easy to see that the compatibility conditions are satisfied by (3.7).

By Hölder inequality it follows from the definition of the norm in $M^{1,N+1+\alpha}(D_T)$ that if $p \ge \frac{N+2}{1-\alpha}$ then $L^p(D_T) \hookrightarrow M^{1,N+1+\alpha}(D_T)$. Therefore

$$\|w\|_{C^{(1+\alpha)/2,1+\alpha}} \le C_0\big(\|w_0\|_{C^{1+\alpha}} + \|h\|_{C^{\alpha/2,\alpha}} + \|h_0\|_{L^p(D_T)}\big) \tag{3.9}$$

with a constant depending only on N, ν, μ, D, α.

We apply this bound to (3.7) and obtain

$$\begin{aligned} \|v_i\|_{C^{(1+\alpha)/2,1+\alpha}} \le C_0 \Big[& \|\varphi_i(0)\|_{C^{1+\alpha}} \|v_0\|_{C^{1+\alpha}} + \|(a-a_i)\nabla v_i\|_{C^{\alpha/2,\alpha}} \\ & + \|f\|_{L^p(D_T)} + \|\varphi_i\|_{C^{\alpha/2,\alpha}} \|g\|_{C^{\alpha/2,\alpha}} \\ & + \|\nabla\varphi_i\|_{C^0(\overline{D_T})} \|g\|_{L^p(D_T)} \\ & + \|a\|_{C^0(\overline{D_T})} \|\nabla\varphi_i\|_{C^0(\overline{D_T})} \|\nabla v\|_{L^p(D_T)} \\ & + \|a\|_{C^{\alpha/2,\alpha}} \|v\|_{C^{\alpha/2,\alpha}} \|\nabla\varphi_i\|_{C^{\alpha/2,\alpha}} \\ & + \|v\|_{C^0(\overline{D_T})} \|\partial_t\varphi_i\|_{C^0(\overline{D_T})} \Big]. \end{aligned} \tag{3.10}$$

In the following, $|\cdot|_{\alpha/2,\alpha}$ denotes the α-Hölder seminorm with respect to the parabolic distance, i.e.

$$|w|_{\alpha/2,\alpha} = \sup_{\substack{(t,x),(s,x)\in D_T \\ (t,x)\neq(s,x)}} \frac{|w(t,x) - w(s,x)|}{d((t,x),(s,y))^\alpha}.$$

By interpolation, there exists $C > 0$ such that we have for every $\varepsilon > 0$ (see e.g. [19, Lemma 3.2, Chapter II]):

$$\|v_i\|_{C^{1/2,1}} \le \varepsilon \|v_i\|_{C^{(1+\alpha)/2,1+\alpha}} + \frac{C}{\varepsilon^{1/\alpha}} \|v_i\|_{C^0(\overline{D_T})}.$$

Besides,

$$\|\nabla v_i\|_{C^0(\overline{D_T})} \le \|v_i\|_{C^{1/2,1}}$$
$$\|\nabla v_i\|_{C^0(\overline{D_T})} + |\nabla v_i|_{\alpha/2,\alpha} \le \|v\|_{C^{(1+\alpha)/2,1+\alpha}}$$

hence choosing $\varepsilon = r^\alpha$ we obtain

$$\begin{aligned}\|(a-a_i)\nabla v_i\|_{C^{\alpha/2,\alpha}} &\le \|a-a_i\|_{C^0(\overline{B_i(r)})}\|\nabla v_i\|_{C^0(\overline{D_T})}\\ &\quad + \|a-a_i\|_{C^0(\overline{B_i(r)})}|\nabla v_i|_{\alpha/2,\alpha} + a_{\alpha/2,\alpha}\|\nabla v_i\|_{C^0(\overline{D_T})}\\ &\le a_{\alpha/2,\alpha}r^\alpha\|\nabla v_i\|_{C^0(\overline{D_T})} + a_{\alpha/2,\alpha}r^\alpha|\nabla v_i|_{\alpha/2,\alpha}\\ &\quad + a_{\alpha/2,\alpha}\|\nabla v_i\|_{C^0(\overline{D_T})}\\ &\le a_{\alpha/2,\alpha}\big(2r^\alpha\|v_i\|_{C^{(1+\alpha)/2,1+\alpha}} + Cr^{-1}\|v_i\|_{C^0(\overline{D_T})}\big).\end{aligned}$$

We set $r = \big(4C_0 a_{\alpha/2,\alpha}\big)^{-1/\alpha} \wedge 1$, where C_0 is the constant defined in (3.9), and deduce from (3.10)

$$\begin{aligned}\|v_i\|_{C^{(1+\alpha)/2,1+\alpha}} \le C\Big[&r^{-1-\alpha}\|v_0\|_{C^{1+\alpha}} + a_{\alpha/2,\alpha}r^{-1}\|v_i\|_{C^0(\overline{D_T})} + \|f\|_{L^p(D_T)}\\ &+ r^{-1}\|g\|_{C^{\alpha/2,\alpha}} + a_{\alpha/2,\alpha}r^{-1}\|\nabla v\|_{L^p(D_T)}\\ &+ a_{\alpha/2,\alpha}r^{-1-\alpha}\|v\|_{C^{\alpha/2,\alpha}} + r^{-2}\|v\|_{C^0(\overline{D_T})}\Big].\end{aligned}$$

Next, we estimate the second, the sixth and the seventh term on the right hand side by (3.3) and (3.5) and obtain

$$\begin{aligned}\|v_i\|_{C^{(1+\alpha)/2,1+\alpha}} &\le Cr^{-1-\alpha}\|\nabla v\|_{L^p(D_T)}\\ &\quad + C\big(r^{-1-2\alpha} + r^{-2}\big)\big(\|v_0\|_{C^{1+\alpha}} + \|f\|_{L^p(D_T)} + \|g\|_{C^{\alpha/2,\alpha}}\big).\end{aligned} \tag{3.11}$$

Since $p \ge 2$ we also have the elementary interpolation inequality

$$\|\nabla v\|^p_{L^p(D_T)} \le \|\nabla v\|^{p-2}_{C^0(\overline{D_T})}\|\nabla v\|^2_{L^2(D_T)}$$

as well as the basic energy estimate

$$\|v\|_{L^\infty(0,T;L^2)} + \|\nabla v\|_{L^2(D_T)} \le C\big(\|v_0\|_{L^2} + \|g\|_{L^2(D_T)} + \|f\|_{L^2(D_T)}\big) \tag{3.12}$$

which holds true with a constant that depends on ν but otherwise is independent of the coefficients of (3.2). We deduce

$$\|\nabla v\|^p_{L^p(\overline{D_T})} \le C\|\nabla v\|^{p-2}_{C^0(\overline{D_T})}\big(\|v_0\|_{C^{1+\alpha}} + \|f\|_{L^p(D_T)} + \|g\|_{C^{\alpha/2,\alpha}}\big)^2$$

Therefore, Young's inequality yields

$$\begin{aligned}&Cr^{-1-\alpha}\|\nabla v\|_{L^p(D_T)}\\ &\quad\le Cr^{-1-\alpha}\|\nabla v\|^{\frac{p-2}{p}}_{C^0(\overline{D_T})}\big(\|v_0\|_{C^{1+\alpha}} + \|f\|_{L^p(D_T)} + \|g\|_{C^{\alpha/2,\alpha}}\big)^{\frac{2}{p}}\\ &\quad\le Cr^{-\frac{p}{2}(1+\alpha)}\big(\|v_0\|_{C^{1+\alpha}} + \|f\|_{L^p(D_T)} + \|g\|_{C^{\alpha/2,\alpha}}\big) + \frac{1}{2}\|\nabla v\|_{C^0(\overline{D_T})}.\end{aligned}$$

Hence it follows from (3.11)

(3.13)
$$\|v_i\|_{C^{(1+\alpha)/2,1+\alpha}} \leq \frac{1}{2}\|\nabla v\|_{C^0(\overline{D_T})} + C\big(r^{-1-2\alpha} + r^{-2} + r^{-\frac{p}{2}(1+\alpha)}\big)\big(\|v_0\|_{C^{1+\alpha}} + \|f\|_{L^p(D_T)} + \|g\|_{C^{\alpha/2,\alpha}}\big).$$

Let $(t_0, x_0) \in \overline{D_T}$ such that $|\nabla v(t_0, x_0)| = \|\nabla v\|_{C^0(\overline{D_T})}$ and take i_0 such that $(t_0, x_0) \in B_{i_0}(r/4)$. Then $\nabla v(t_0, x_0) = \nabla v_{i_0}(t_0, x_0)$. Therefore

$$\|\nabla v\|_{C^0(\overline{D_T})} \leq \|\nabla v_{i_0}\|_{C^0(\overline{D_T})}$$

and consequently

(3.14)
$$\|\nabla v\|_{C^0(\overline{D_T})} \leq C\big(r^{-1-2\alpha} + r^{-2} + r^{-\frac{p}{2}(1+\alpha)}\big)\big(\|v_0\|_{C^{1+\alpha}} + \|f\|_{L^p(D_T)} + \|g\|_{C^{\alpha/2,\alpha}}\big).$$

Plugging this back in (3.13) we obtain

$$\|v_i\|_{C^{(1+\alpha)/2,1+\alpha}} \leq C\big(r^{-1-2\alpha} + r^{-2} + r^{-\frac{p}{2}(1+\alpha)}\big)\big(\|v_0\|_{C^{1+\alpha}} + \|f\|_{L^p(D_T)} + \|g\|_{C^{\alpha/2,\alpha}}\big).$$

In order to estimate

$$\|v\|_{(1+\alpha)/2,1+\alpha} = \|v\|_{C^0(\overline{D_T})} + \|\nabla v\|_{C^0(\overline{D_T})} + |\nabla v|_{\alpha/2,\alpha},$$

we apply (3.5), (3.14) and the inequality

$$|\nabla v|_{\alpha/2,\alpha} \leq \max\big\{\max_i |\nabla v_i|_{\alpha/2,\alpha}, 2^{1+2\alpha} r^{-\alpha}\|\nabla v\|_{C^0(\overline{D_T})}\big\}$$

which is easily obtained by considering the points where the maximum in the definition of $|\nabla v|_{\alpha/2,\alpha}$ is reached. Indeed, if (t, x) and (s, y) are such points, then $|\nabla v|_{\alpha/2,\alpha}$ is bounded by

$$2^{1+2\alpha} r^{-\alpha}\|\nabla v\|_{C^0(\overline{D_T})} \quad \text{provided} \quad d((t,x),(s,x)) \geq \frac{r}{4}$$

and by $|\nabla v_i|_{\alpha/2,\alpha}$ otherwise, where i is such that $(t, x), (s, y) \in B_i(r/2)$. We deduce

$$\|v\|_{(1+\alpha)/2,1+\alpha} \leq C(r^{-1-3\alpha} + r^{-2-\alpha} + r^{-\frac{p+\alpha p+2\alpha}{2}})\big(\|v_0\|_{C^{1+\alpha}} + \|f\|_{L^p(D_T)} + \|g\|_{C^{\alpha/2,\alpha}}\big)$$

which implies (3.6). □

We now give higher regularity results for equation in non divergence form. We consider a linear parabolic PDE of the form

$$\begin{cases} \partial_t v = \displaystyle\sum_{i,j=1}^N a_{ij}(t,x)\partial_{i,j} v + f(t,x), & \text{in } D_T, \\ v = 0, & \text{on } S_T, \\ v(0) = v_0, & \text{in } D, \end{cases} \tag{3.15}$$

and assume that the matrix $a(t,x)$ is symmetric and that there exist $\nu, \mu > 0$ such that $\nu|\xi|^2 \leq a(t,x)\xi \cdot \xi \leq \mu|\xi|^2$, for all $(t,x) \in D_T$, $\xi \in \mathbb{R}^N$.

We use a similar approach to obtain regularity as for the proof of Theorem 3.3. In particular, this enables us to quantify the constant with respect to the parameters. We define $\mathcal{A}$ as the second order differential operator: $\mathcal{A}w = \sum_{i,j=1}^N a_{i,j}\partial_{i,j}w$.

We first recall low regularity in the parabolic scaling.

Theorem 3.4. *Let $\alpha \in (0,1)$, $k = 0,1$ and assume that D has a $C^{\alpha+k+2}$ boundary. Consider the equation:*

$$\begin{cases} \partial_t v - \mathcal{A}v = f, & \text{in } D_T, \\ v = \phi, & \text{on } S_T, \\ v(0) = v_0, & \text{in } D. \end{cases} \tag{3.16}$$

Assume that

(i) $f \in C^{(\alpha+k)/2,\alpha+k}(\overline{D_T})$, $a \in C^{(\alpha+k)/2,\alpha+k}(\overline{D_T})$.
(ii) $v_0 \in C^{\alpha+k+2}(\overline{D})$,
(iii) $\phi \in C^{(\alpha+k)/2+1,\alpha+k+2}(\overline{S_T})$
(iv) $v_0 = \phi$, $\partial_t\phi - \mathcal{A}v_0 = f$ on $\{0\} \times \partial D$.

Then there exists a unique solution $v \in C^{(\alpha+k)/2+1,\alpha+k+2}(\overline{D_T})$ and

$$\begin{aligned} \|v\|_{C^{(\alpha+k)/2+1,\alpha+k+2}} \leq K_3 P_2(A_{\alpha+k})\big(&\|f\|_{C^{(\alpha+k)/2,\alpha+k}} + \|v_0\|_{C^{\alpha+k+2}(\overline{D})} \\ &+ \|\phi\|_{C^{(\alpha+k)/2+1,\alpha+k+2}(\overline{S_T})}\big), \end{aligned} \tag{3.17}$$

where K_3 depends only on μ, ν, D, α, P_2 is a polynomial and $A_{\alpha+k} = \|a\|_{C^{(\alpha+k)/2,\alpha+k}}$.

Proof. According to Theorem 5.2, chapter IV in [19], one obtains the required regularity of v. Indeed, $H^{l,l/2}$ in [19] corresponds to $C^{l/2,l}$ in our notations and (iv) is the compatibility condition of order one. But as in Theorem 3.3, the dependence on $A_{\alpha+k}$ is not obvious, we proceed similarly.

Take $r \in (0,1]$ to be fixed. Let $(B_i(r/4))_{i=1,\dots,n}$ be a covering of $\overline{D_T}$ by parabolic cylinders of radius r, i.e. balls of radius r with respect to the parabolic distance, and center (t_i, x_i). Let

$$\varphi_i \in C_c^\infty(B_i(r)) \text{ such that } 0 \leq \varphi_i \leq 1 \text{ and } \varphi_i \equiv 1 \text{ on } B_i(r/2),$$

such that

$$\|\nabla\varphi_i\|_{C^0(\overline{D_T})} \leq Cr^{-1}, \quad \|\partial_t\varphi_i\|_{C^0(\overline{D_T})} \leq Cr^{-2},$$

$$\|\nabla\varphi_i\|_{C^{\alpha/2,\alpha}} \leq Cr^{-1-\alpha}, \quad \|\partial^\gamma\varphi_i\|_{C^{\alpha/2,\alpha}} \leq Cr^{-2-\alpha},$$

for any multi-index γ of length $|\gamma| = 2$.

Set $\mathcal{A}_i = \mathcal{A}(t_i, x_i)$, $v_i = \varphi_i v$, $v_{0,i} = \varphi_i(0)v_0$, $\phi_i = \varphi_i \phi$. Then v_i solves the following parabolic equation

$$\begin{cases} \partial_t v_i - \mathcal{A}_i v_i = v\partial_t \varphi_i + (\mathcal{A} - \mathcal{A}_i)v_i - v\mathcal{A}\varphi_i \\ \qquad\qquad - 2\sum_{l,m=1}^{N} a_{lm}\partial_l v \partial_m \varphi_i + f\varphi_i, & \text{in } D_T, \\ v_i = \phi_i, & \text{on } S_T, \\ v_i(0) = v_{0,i}, & \text{in } D. \end{cases} \tag{3.18}$$

This equation is similar to (3.16) but the operator has constant coefficients. It is easy to see that it satisfies the compatibility condition of order one. Thus, according to Theorem 5.2, chapter IV in [19], we have

$$\begin{aligned} \|v_i\|_{C^{(\alpha+k)/2+1,\alpha+k+2}} \leq C\big(&\|v_{0,i}\|_{C^{\alpha+k+2}(\overline{D})} + \|\phi_i\|_{C^{(\alpha+k)/2+1,\alpha+k+2}(\overline{S_T})} \\ &+ \|v\partial_t\varphi_i + (\mathcal{A} - \mathcal{A}_i)v_i - v\mathcal{A}\varphi_i \\ &- 2\sum_{l,m=1}^{N} a_{lm}\partial_l v\partial_m\varphi_i + f\varphi_i\|_{C^{(\alpha+k)/2,\alpha+k}}\big), \end{aligned}$$

where the constant depends only on N, ν, μ, D, α. To see this, one follows the proof of Theorem 5.2, Chapter IV in [19] for the case of constant coefficients. First, the equation is transformed by a local change of coordinates into a heat equation on the half space. In the case of zero initial data (see the problem (5.3)' in Chapter IV in [19]) one obtains the above estimate in Theorem 6.1, Chapter IV in [19]. Reduction of the problem to the problem with zero initial data is based on Theorem 4.3, Chapter IV in [19] and the method is explained in Section 8, Chapter IV in [19].

The end of the proof is exactly as above for Theorem 3.3 and uses the maximum principle for parabolic equations in non divergence form, see e.g. [6, Theorem 8, Chapter 2]. □

We now study higher regularity. Since the application we have in mind is SPDEs, we do not have more than 1/2 regularity in time. Hence, to increase the spatial regularity, we cannot use the parabolic scaling anymore. Unfortunately, all the results in the literature are in the parabolic scaling. To avoid lengthy technical proofs, we only investigate the first step: spatial regularity of order $k + \alpha$, $k = 4$, $\alpha < 1$. Similar arguments can be used to get higher regularity.

Theorem 3.5. *Let $\alpha \in (0, 1)$, and assume that D has a $C^{\alpha+4}$-boundary. Let v be the solution of*

$$\begin{cases} \partial_t v - \mathcal{A}v = f, & \text{in } D_T, \\ v = 0, & \text{on } S_T, \\ v(0) = v_0, & \text{in } D. \end{cases}$$

Assume that

(i) $f \in C^{\alpha/2,\alpha+2}(\overline{D_T})$, $f|_{\overline{S_T}} \in C^{\alpha/2+1,\alpha+2}(\overline{S_T})$, $a \in C^{\alpha/2,\alpha+2}(\overline{D_T})$, *and* $a|_{\overline{S_T}} \in C^{\alpha/2+1,\alpha+2}(\overline{S_T})$.

(ii) $v_0 \in C^{\alpha+4}(\overline{D})$,

(iii) $v_0 = 0$, $\mathcal{A}v_0 + f = 0$, *and* $\partial_t f + \mathcal{A}^2 v_0 + \mathcal{A}f + (\partial_t \mathcal{A})v_0 = 0$, *on* $\{0\} \times \partial D$.

then $v \in C^{\alpha/2+1,\alpha+4}(\overline{D_T})$ *and*

$$\begin{aligned} \|v\|_{C^{\alpha/2+1,\alpha+4}} \leq K_4 P_3(A_{\alpha+2})\big(&\|f\|_{C^{\alpha/2,\alpha+2}} \\ &+ \|f|_{S_T}\|_{C^{\alpha/2+1,\alpha+2}(\overline{S_T})} + \|v_0\|_{C^{\alpha+4}(\overline{D})}\big), \end{aligned} \tag{3.19}$$

where P_3 *is a polynomial,* $A_{\alpha+2} = \|a\|_{C^{\alpha/2,\alpha+2}} + \|a|_{\overline{S_T}}\|_{C^{\alpha/2+1,\alpha+2}(\overline{S_T})}$ *and* K_4 *depends only on* α, ν, μ, D.

Remark 3.6. It might seem strange that we need to assume more time regularity for f on the boundary S_T. However, it is immediate that a solution in $C^{\alpha/2+1,\alpha+4}(\overline{D}_T)$ satisfies, $-\mathcal{A}v = f$ on S_T. Thus, $f|_{S_T} \in C^{\alpha/2+1,\alpha+2}(\overline{S_T})$. Concerning the compatibility conditions (iii), the first two are already in Theorem 3.4. Since the equation holds up to the boundary, we have $-\mathcal{A}v = f$ on $\overline{S_T}$. Also, the solution is sufficiently regular to differentiate it and get $\partial_t(\mathcal{A}v) - (\partial_t\mathcal{A})v - \mathcal{A}^2 v = \mathcal{A}f$ so that on $\overline{S_T}$: $-\partial_t f - (\partial_t\mathcal{A})v - \mathcal{A}^2 v = \mathcal{A}f$ and for $t = 0$ we see that the third condition is also necessary.

Note that this third compatibility condition is exactly the classical second order condition required to get smoothness in the parabolic scaling (see [19], section 5, chapter IV. In particular (5.6) and below).

We set in the following result: $\mathbb{R}^{N,+} = \{x \in \mathbb{R}^N : x_N \geq 0\}$, $\mathbb{R}^{N,++} = \{x \in \mathbb{R}^N : x_N > 0\}$. Note that its boundary is $\mathbb{R}^{N-1}$. As usual, for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we set $x' = (x_1, \ldots, x_{N-1})$. Also, given a function g defined on $\mathbb{R}^{N,+}$, we denote by $g|_{x_N=0}$ its restriction to $\{x_N = 0\}$.

The proof of Theorem 3.5 follows the lines of the proof of Theorem 5.2, chapter IV in [19]. It is more complicated than the proof of Theorem 3.4 since we cannot base our argument on known results. We start with the case of the Laplace operator on the half space. The following Lemma provides the missing ingredient to reproduce the argument of [19] which is based on inequalities (2.1), (2.2), (2.3) from Chapter IV. Inequality (3.20) is the generalization of these to indices that are not in the parabolic scaling.

Lemma 3.7. *Let* $\alpha \in (0,1)$ *and* v *be the solution of*

$$\begin{cases} \partial_t v - \Delta v = f, & \text{in } \mathbb{R}^{N,++} \times (0,T], \\ v = 0, & \text{on } \mathbb{R}^{N-1} \times (0,T], \\ v(0) = v_0, & \text{in } \mathbb{R}^{N,++}. \end{cases}$$

Assume that

(i) $f \in C^{\alpha/2,\alpha+2}(\mathbb{R}^{N,+} \times [0,T])$ *and* $f|_{x_N=0} \in C^{\alpha/2+1,\alpha+2}(\mathbb{R}^{N-1} \times [0,T])$,

(ii) $v_0 \in C^{\alpha+4}(\mathbb{R}^{N,+})$,

(iii) $v_0 = 0$, $-\Delta v_0 = f$ *and* $\partial_t f + \Delta^2 v_0 + \Delta f = 0$ *for* $x_N = 0$, $t = 0$.

then $v \in C^{\alpha/2+1,\alpha+4}(\mathbb{R}^{N,+} \times [0,T])$ *and*

$$\begin{aligned} \|v\|_{C^{\alpha/2+1,\alpha+4}(\mathbb{R}^{N,+}\times[0,T])} &\leq K_5\big(\|f\|_{C^{\alpha/2,\alpha+2}(\mathbb{R}^{N,+}\times[0,T])} \\ &\quad + \|f\|_{C^{\alpha/2+1,\alpha+2}(\mathbb{R}^{N-1}\times[0,T])} \\ &\quad + \|v_0\|_{C^{\alpha+4}(\mathbb{R}^{N,+})}\big), \end{aligned} \tag{3.20}$$

where K_5 *depends only on* α *and* N.

Proof. We already know from Theorem 3.4 that $v \in C^{\alpha/2+1,\alpha+2}(\mathbb{R}^{N,+} \times [0,T])$.

Let $w = \partial_{ij} v$, with $i \neq N$, $j \neq N$. It satisfies

$$\begin{cases} \partial_t w - \Delta w = \partial_{ij} f, & \text{in } \mathbb{R}^{N,++} \times (0,T], \\ w = 0, & \text{on } \mathbb{R}^{N-1} \times (0,T], \\ w(0) = \partial_{ij} v_0, & \text{in } \mathbb{R}^{N,++}. \end{cases}$$

Clearly, $\partial_{ij} f$ and $\partial_{ij} v_0$ satisfy the assumption of Theorem 3.4. We deduce that $\partial_{ij} v \in C^{\alpha/2+1,\alpha+2}(\mathbb{R}^{N,+} \times [0,T])$.

It remains to prove that $\partial_n^2 v \in C^{1+\alpha/2,\alpha+2}(\mathbb{R}^{N,+} \times [0,T])$. We write $w = \partial_n^2 v$. Note that:

$$w = \partial_t v - \Delta_{x'} v - f.$$

Hence taking the restriction at $x_N = 0$:

$$w|_{x_N=0} = -f.$$

We deduce that w satisties:

$$\begin{cases} \partial_t w - \Delta w = \partial_n^2 f, & \text{in } \mathbb{R}^{N,++} \times (0,T], \\ w = -f, & \text{on } \mathbb{R}^{N-1} \times (0,T], \\ w(0) = \partial_n^2 v_0, & \text{in } \mathbb{R}^{N,++}. \end{cases}$$

Our assumptions imply that $\partial_n^2 f$, $\partial_n^2 v_0$ and $f|_{S_T}$ satisfy the assumptions of Theorem 3.4. We deduce that $w \in C^{\alpha/2+1,\alpha+2}(\mathbb{R}^{N,+} \times [0,T])$ and this finishes the proof. Note that (3.20) follows from the estimate given on $\partial_{ij} v$ and $\partial_n^2 v$ by Theorem 3.4. □

Following the argument in section 6, chapter IV in [19], we generalize Lemma 3.7 to the case of a general elliptic operator with constant coefficient.

Lemma 3.8. *Let* $A = \sum_{ij} \bar{a}_{ij} \partial_{ij}$ *is a second order partial differential operator with constant coefficients such that* $\nu|\xi|^2 \leq \bar{a}\xi \cdot \xi \leq \mu|\xi|^2$, *for all* $\xi \in \mathbb{R}^N$. *Let* $\alpha \in (0,1)$ *and* v *be the solution of*

$$\begin{cases} \partial_t v - Av = f, & \textit{in } \mathbb{R}^{N,++} \times (0,T], \\ v = 0, & \textit{on } \mathbb{R}^{N-1} \times (0,T], \\ v(0) = v_0, & \textit{in } \mathbb{R}^{N,++}. \end{cases}$$

Assume that

(i) $f \in C^{\alpha/2,\alpha+2}(\mathbb{R}^{N,+}\times[0,T])$ *and* $f|_{x_N=0} \in C^{\alpha/2+1,\alpha+2}(\mathbb{R}^{N-1}\times[0,T])$
(ii) $v_0 \in C^{\alpha+4}(\mathbb{R}^{N,+})$, $-Av_0 = f$ *and* $\partial_t f + A^2 v_0 + Af = 0$ *for* $x_N = 0,\ t = 0$.

then $v \in C^{\alpha/2+1,\alpha+4}(D_T)$ *and*

$$\|v\|_{C^{\alpha/2+1,\alpha+4}(\mathbb{R}^{N,+}\times[0,T])} \leq K_6\big(\|f\|_{C^{\alpha/2,\alpha+2}(\mathbb{R}^{N,+}\times[0,T])} + \|f|_{x_N=0}\|_{C^{\alpha/2+1,\alpha+2}(\mathbb{R}^{N-1}\times[0,T])} + \|v_0\|_{C^{\alpha+4}(\mathbb{R}^{N,+})}\big),$$

where K_6 *depends on* α, N, μ *and* ν.

The proof of Theorem 3.5. Follows now exactly the proof given in section 7, chapter IV in [19]. It is quite long and we do not reproduce it here.

Note contrary to Theorem 3.4, we do not know that v has the required regularity. This has to be proved. As above, the proof consists in using a suitable covering of the domain D_T into small sub-domains. On each of these domains intersecting the boundary, the equation is transformed by a local change of coordinate into an equation on the half space. This latter equation is approximated by the same equation with frozen coefficients, whose solution can be estimated thanks to Lemma 3.8. If the sub-domain does not intersect the boundary, an estimate is trivially obtained by differentiation of the equation. Putting together all these local inverses, we show that we obtain an approximate inverse. Than an elementary argument allows to conclude.

Note that contrary to [19], we work with non zero initial data (see the discussion after Theorem 5.3 for the definition of this notion). The price to pay is that we need to pay particular attention to the compatibility conditions but this does not cause any problem.

The polynomial dependence on $A_{\alpha+2}$ comes from similar computations as in the proof of Theorem 3.4. In the notation of section 7, chapter IV in [19], this comes from the choice of τ. It has to be chosen small enough so that $\|T\| < 1/2$ and the proof clearly shows that this condition can be written in terms of $A_{\alpha+2}$ and the local coordinate systems on ∂D.

4. First step in the regularity problem: proof of Theorem 2.6

In this section, we show the first step towards regularity of the weak solution u to (1.1). We consider the auxiliary problem (3.1) with $\Psi = H(u)$, whose solution is given by the stochastic convolution

$$z(t) = \int_0^t S(t-s)H(u_s)\,\mathrm{d}W_s, \quad t \in [0,T]. \tag{4.1}$$

Next, we define the process $y := u - z$. It follows immediately that y solves the following linear parabolic PDE with random coefficients

(4.2)
$$\begin{cases} \partial_t y = \operatorname{div}(A(u)\nabla y) + \operatorname{div}(B(u)) + F(u) + \operatorname{div}((A(u) - \mathrm{I})\nabla z), & \text{in } D_T, \\ y = 0, & \text{on } S_T, \\ y(0) = u_0, & \text{in } D. \end{cases}$$

This way, we have split u into two parts, i.e. y and z, that are much more convenient in order to study regularity.

Proof of Theorem 2.6. Step 1: Regularity of z. According to the hypothesis, the weak solution u to (1.1) belongs to $L^2(\Omega; L^2(0,T; H_0^{1,2}))$ so that, thanks to the hypothesis ($\mathrm{H}_{1,2}$), we have that $H(u)$ belongs to $L^2(\Omega; L^2(0,T; \gamma(K, H_0^{1,2})))$. As a result, with Proposition 3.1 - (ii) and the bound ($\mathrm{H}_{1,2}$) we have that for any $a \in (0,2)$, $z \in L^2(\Omega; L^2(0,T; H_0^{a,2}))$ with

$$\mathbb{E}\|z\|^2_{L^2(0,T;H_0^{a,2})} \leq C\Big(1 + \mathbb{E}\|u\|^2_{L^2(0,T;H_0^{1,2})}\Big) \tag{4.3}$$

and by Proposition 3.1 - (i), for $p > 2$,

$$\mathbb{E}\|z\|^p_{C([0,T];L^2)} \leq C\Big(1 + \mathbb{E}\|u\|^p_{L^p(0,T;L^2)}\Big). \tag{4.4}$$

Besides, since for all $p \in [2,\infty)$, the weak solution u to (1.1) belongs to $L^p(\Omega; L^p(0,T; L^p))$, we obtain, with the hypothesis ($\mathrm{H}_{0,p}$) (see Remark 2.2), that $H(u)$ belongs to $L^p(\Omega; L^p(0,T; \gamma(K, L^p)))$. As a consequence, with Proposition 3.1 - (ii) and the bound ($\mathrm{H}_{0,p}$) we have that for any $b \in (0,1)$, $z \in L^p(\Omega; L^p(0,T; H_0^{b,p}))$ with

$$\mathbb{E}\|z\|^p_{L^p(0,T;H_0^{b,p})} \leq C\Big(1 + \mathbb{E}\|u\|^p_{L^p(0,T;L^p)}\Big).$$

We have proved that for any $a \in (0,2)$ and $b \in (0,1)$, we have $z \in L^2(\Omega; L^2(0,T; H_0^{a,2}))$ and $z \in L^p(\Omega; L^p(0,T; H_0^{b,p}))$. We can now interpolate to obtain that (see [1])

$$z \in L^r(\Omega; L^r(0,T; H_0^{c,r})),$$

where, for $\theta \in (0,1)$,

$$\begin{cases} \dfrac{1}{r} = \dfrac{\theta}{2} + \dfrac{1-\theta}{p}, \\ c = a\theta + b(1-\theta), \end{cases}$$

with the bound

(4.5)
$$\mathbb{E}\|z\|^r_{L^r(0,T;H_0^{c,r})} \leq \Big(\mathbb{E}\|z\|^2_{L^2(0,T;H_0^{a,2})}\Big)^{r\theta/2} \Big(\mathbb{E}\|z\|^p_{L^p(0,T;H_0^{b,p})}\Big)^{r(1-\theta)/p} < \infty.$$

Note that by choosing $\theta \in (0,1)$ and $p \in [2,\infty)$ appropriately, r can be arbitrary in $[2,\infty)$. Furthermore, when $\theta \in (0,1)$ is fixed, it is always

possible to take $(a,b) \in (0,2) \times (0,1)$ such that $c > 1$. As a result, for all $r \in [2,\infty)$, there exists $c_r > 1$ such that

$$z \in L^r(\Omega; L^r(0,T; H_0^{c_r,r})).$$

This gives, for all $r \in [2,\infty)$,

$$\nabla z \in L^r(\Omega; L^r(0,T; L^r)),$$

and, due to the boundedness of the mapping A,

$$(A(u) - \mathrm{I})\nabla z \in L^r(\Omega; L^r(0,T; L^r)),$$

with, thanks to (4.5),

$$\mathbb{E}\|(A(u) - \mathrm{I})\nabla z\|^r_{L^r(0,T;L^r)} \leq C\mathbb{E}\|z\|^r_{L^r(0,T;H_0^{c,r})} < \infty, \tag{4.6}$$

where $C > 0$ depends on μ from (2.4). Note that, thanks to the linear growth property of the coefficients B and F, we obviously have, for all $r \in [2,\infty)$,

$$\mathbb{E}\|B(u)\|^r_{L^r(0,T;L^r)} + \mathbb{E}\|F(u)\|^r_{L^r(0,T;L^r)} \leq C(1 + \mathbb{E}\|u\|^r_{L^r(0,T;L^r)}) < \infty. \tag{4.7}$$

Step 2: Regularity of y. We apply Theorem 3.2 with

$$a = A(u), \quad g = B(u) + ((A(u) - \mathrm{I})\nabla z), \quad f = F(u).$$

The assumptions are satisfied. Indeed (2.4) gives the required uniform ellipticity and boundedness of a. Then sublinear growth of B, F, *Step 1* and (4.3) imply that assumption (ii) is satisfied. By the basic energy estimate for parabolic equations (3.12) we obtain that $y \in L^\infty(0,T;L^2) \cap L^2(0,T;W_0^{1,2})$ a.s. Besides, since $u \in C([0,T];L^2)$ a.s. due to our assumptions and the same is valid for z due to (4.4), assumption (i) holds true as well. We may conclude

$$\begin{aligned} \|y\|_{C^{\alpha/2,\alpha}} \leq K_1\big(&\|u_0\|_{C^\alpha(\overline{D})} \\ &+ \|B(u) + (A(u) - \mathrm{I})\nabla z\|_{L^{2r_0}(D_T)} + \|F(u)\|_{L^{r_0}(D_T)}\big) \end{aligned} \tag{4.8}$$

for $\alpha \leq \iota$ where α is given by Theorem 3.2. Therefore

$$\begin{aligned} \mathbb{E}\|y\|^m_{C^{\alpha/2,\alpha}} \leq K_1\big(&\mathbb{E}\|u_0\|^m_{C^\alpha(\overline{D})} \\ &+ \mathbb{E}\|B(u) + (A(u) - \mathrm{I})\nabla z\|^m_{L^{2r_0}(D_T)} + \mathbb{E}\|F(u)\|^m_{L^{r_0}(D_T)}\big). \end{aligned} \tag{4.9}$$

We now use (4.6)−(4.7) to deduce that the above right hand side is finite thanks to our assumptions on u.

Step 3: Hölder regularity of z. In order to complete the proof it is necessary to improve the regularity of z. Recall that for all $m \in [2,\infty)$, the solution u to (1.1) belongs to $L^m(\Omega; L^m(0,T;L^m))$ and that $H(u)$ belongs to $L^m(\Omega; L^m(0,T;\gamma(K,L^m)))$. We now apply Proposition 3.1 - (i) and $(\mathrm{H}_{0,m})$ to obtain, since $H_0^{a,r} \subset H^{a,r}$, that for $m \in (2,\infty)$, $\delta \in (0, 1 - 2/m)$ and $\gamma \in [0, 1/2 - 1/m - \delta/2)$, $z \in L^m(\Omega; C^\gamma([0,T]; H^{\delta,m}))$ with

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];H^{\delta,m})} \leq C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;L^m)}\right).$$

Note that we can choose δ and γ to be independent of m. For instance, let us suppose in the sequel that $m \geq 3$; then $\delta = 1/6$ and $\gamma = 1/12$ satisfies

the conditions above for any $m \geq 3$. Furthermore, from now on, we also suppose that $m \geq 7N := m_0$. This implies that $m \geq 3$ and $\delta m > N$, so that the following Sobolev embedding holds true

$$H^{\delta,m} \hookrightarrow C^{\lambda}, \quad \lambda := \delta - N/m_0.$$

We conclude that, for all $m \geq m_0$,

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\lambda)} \leq C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;L^m)}\right) < \infty. \tag{4.10}$$

Note that for $m \in [2, m_0)$, we can write with the Hölder inequality

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\lambda)} \leq \left(\mathbb{E}\|z\|^{m_0}_{C^\gamma([0,T];C^\lambda)}\right)^{m/m_0} < \infty. \tag{4.11}$$

Step 4: Conclusion. Finally, we set $\eta := \min(\alpha, 2\gamma, \lambda) > 0$ and we recall that $u = y + z$ so that the conclusion follows from (4.9), (4.10), (4.11) due to the fact that $C^{\eta/2}([0,T]; C^\eta(\overline{D})) \subset C^{\eta/2,\eta}([0,T] \times \overline{D})$. □

5. Increasing the regularity: proof of Theorem 2.7

In this final section, we complete the proof of Theorem 2.7. Having Theorem 2.6 in hand, it is now possible to significantly increase the regularity of u using the results given in Subsection 3.2.

We treat differently the cases $k = 1$, $k = 2$ and $k = 3, 4$.

5.1. The case $k = 1$.

The proof is divided in two parts: we first increase the regularity in space and then in time.

Step 1: Regularity of z. First, we improve the regularity of z that was defined in (4.1). According to Theorem 2.6, there exists $\eta > 0$ such that for all $m \in [2, \infty)$, $u \in L^m(\Omega; C^{\eta/2,\eta}(\overline{D_T}))$. In particular, since u satisfies Dirichlet boundary conditions, this implies that $u \in L^m(\Omega; L^m(0,T; H_0^{\kappa,m}))$ provided $\kappa < \eta$. With ($\mathrm{H}_{\kappa,m}$), we deduce that $H(u) \in L^m(\Omega; L^m(0,T; \gamma(K, H_0^{\kappa,m})))$. An application of Proposition 3.1 yields that $z \in L^m(\Omega; C^\gamma([0,T]; H_0^{\kappa+\delta,m}))$ for every $m \in (2, \infty)$ with

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];H_0^{\kappa+\delta,m})} \leq C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H_0^{\kappa,m})}\right),$$

where $\delta \in (0, 1 - 2/m)$ and $\gamma \in [0, 1/2 - 1/m - \delta/2)$. In the sequel, we assume that $m \geq (N+4)/\kappa := m_0$. Then $\delta = 1 - 3/m_0$ and $\gamma = 1/(4m_0)$ satisfies the conditions above uniformly in $m \geq m_0$. Furthermore, observe that $(\kappa + \delta)m > \kappa m \geq \kappa m_0 \geq N$ so that the following Sobolev embedding holds true

$$H^{\kappa+\delta,m} \hookrightarrow C^\sigma, \quad \sigma = \kappa + \delta - N/m_0.$$

Besides, by definition of m_0, $\sigma = \kappa + 1 - (N+3)/m_0 > 1$.

We deduce that there exists $\gamma > 0$, $\sigma > 1$ such that for all $m \geq m_0$, $z \in L^m(\Omega; C^\gamma([0,T]; C^\sigma(\overline{D})))$ with

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\sigma(\overline{D}))} \leq C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H_0^{\kappa,m})}\right). \tag{5.1}$$

Step 2: Regularity of y. Next, we improve the regularity of y that is given by (4.2).

As a consequence of Theorem 2.6, (4.7) and (5.1), we obtain due to the assumptions upon A, B and F that, for all $m \in [2,\infty)$

$$\begin{aligned} A(u) &\in L^m(\Omega; C^{\alpha/2,\alpha}(\overline{D_T}),\\ B(u) + (A(u) - \mathrm{I})\nabla z &\in L^m(\Omega; C^{\alpha/2,\alpha}(\overline{D_T}),\\ F(u) &\in L^m(\Omega; L^m(0,T; L^m)),\\ u_0 &\in L^m(\Omega; C^{1+\alpha}(\overline{D})), \end{aligned}$$

where $\alpha := \min(\iota, \eta, \sigma - 1, 2\gamma) > 0$. Thus the hypotheses of Theorem 3.3 are fulfilled and we obtain the following (pathwise) estimate

$$\begin{aligned} \|y\|_{C^{(1+\alpha)/2,1+\alpha}} \le K_2\, P_1(a_{\alpha/2,\alpha})\Big(&\|u_0\|_{C^{1+\alpha}(\overline{D})} + \|B(u) + (A(u) - \mathrm{I})\nabla z\|_{C^{\alpha/2,\alpha}}\\ &+ \|F(u)\|_{L^r(0,T;L^r)}\Big), \end{aligned}$$

where $r \in [2,\infty)$ is large enough. We conclude that, for all $m \in [2,\infty)$,

$$y \in L^m(\Omega; C^{(1+\alpha)/2,1+\alpha}(\overline{D_T})) \tag{5.2}$$

which together with (5.1) yields $u \in L^m(\Omega; C^{\gamma,1+\alpha}(\overline{D_T}))$.

Step 3: Time regularity. Having in hand the improved regularity of u, we consider again the stochastic convolution z, repeat the approach from the first step of this proof and obtain due to Theorem 2.6 (with $\delta = 0$) and ($\mathrm{H}_{1+\kappa,m}$)

$$\begin{aligned} &\mathbb{E}\|z\|^m_{C^\lambda([0,T];H_0^{1+\kappa,m})}\\ &\quad \le C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H_0^{1+\kappa,m})} + \mathbb{E}\|u\|^{(1+\kappa)m}_{L^{(1+\kappa)m}(0,T;H_0^{1,(1+\kappa)m})}\right) < \infty, \end{aligned} \tag{5.3}$$

where $\kappa < \alpha$ and $\lambda \in (0, 1/2 - 1/m)$. Therefore for any $\lambda \in (0,1/2)$ there exists m_0 large enough so that (5.3) holds true for any $m \ge m_0$ and the Sobolev embedding then implies that $z \in L^m(\Omega; C^\lambda([0,T]; C^{1+\beta}(\overline{D})))$ for $\beta < \kappa$. Since we already have (5.2) we deduce that $u \in L^m(\Omega; C^{\lambda,1+\alpha}(\overline{D_T}))$ for any $\lambda \in (0,1/2)$ and $m \in \mathbb{N}$.

Step 4: Conclusion. It is now possible to reproduce the 3 steps above. In step 1, we can now take $\kappa < 1 + \alpha$. Then

$$\mathbb{E}\|z\|^m_{C^\gamma([0,T];C^\sigma)} \le C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H_0^{\kappa,m})}\right).$$

with $\sigma = \kappa + \delta - N/m$, $\gamma < 1/2 - 1/m - \delta/2$, $\delta < 1 - 2/m$.

Let $\varepsilon < (1-\iota)/2$, $m \ge \max(4, 2N)1/\varepsilon$ and $\delta = (1-\alpha)/2$, then we can take $\sigma - 1 = 2\gamma = (1-\alpha)/2 - \varepsilon$. Thus in step 2, we can reproduce the argument with α replaced by $\alpha_1 = \min(\iota, (1-\alpha)/2 - \varepsilon)$ and conclude $u \in L^m(\Omega; C^{\lambda,1+\alpha_1}(\overline{D_T}))$ for any $\lambda \in (0,1/2)$ and $m \in \mathbb{N}$. If $\alpha_1 < \iota$, we reproduce this argument and define recursively $\alpha_{n+1} = \min(\iota, (1-\alpha_n)/2 - \varepsilon)$. In a finite number of step, we have $\alpha_n = \iota$.

The proof is complete for $k = 1$.

Remark 5.1. Note that reproducing step 1, we can finally prove that $z \in L^m(\Omega; C^\gamma([0,T]; H_0^{\kappa+\delta,m}))$ for $\kappa < 1+\iota$, $\gamma < 1/2 - 1/m - \delta/2$, $\delta < 1 - 2/m$ and any $m \in \mathbb{N}$. In particular, it is possible to take $\kappa + \delta - 1/m > 2$ and we deduce that z and its first and second derivatives vanish on ∂D.

5.2. **The case $k = 2$.** *Step 1: Regularity of z.* Again we first increase the regularity of z. We know that for any $\lambda \in (0, 1/2)$, there exists $\beta > 0$ such that for all $m \in [2, \infty)$, $u \in L^m(\Omega; C^{\lambda,1+\beta}(\overline{D_T}))$. We deduce

$$\begin{aligned}&\mathbb{E}\|z\|^m_{C^\gamma([0,T];H^{1+\kappa+\delta,m})}\\ &\qquad \le C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H^{1+\kappa,m})} + \mathbb{E}\|u\|^{(1+\kappa)m}_{L^{(1+\kappa)m}(0,T;H^{1,(1+\kappa)m})}\right) < \infty,\end{aligned}$$

where $\kappa < \beta$, $\delta \in (0, 1 - 2/m)$ and $\gamma \in [0, 1/2 - 1/m - \delta/2)$. By a similar reasoning as above we obtain due to the Sobolev embedding that there exist $\gamma > 0$ and $\sigma > 2$ such that $z \in L^m(\Omega; C^\gamma([0,T]; C^\sigma(\overline{D})))$ for $m \in [2, \infty)$.

Step 2: Regularity of y. In order to improve the space regularity of y we use Theorem 3.4. In particular, we set

$$a_{ij} = A_{ij}(u), , \quad f = \sum_{ij} A'_{ij}(u)\partial_i u \partial_j y + \operatorname{div}\big(B(u) + (A(u) - \mathrm{I})\nabla z\big) + F(u).$$

According to the results above, we have

$$\begin{aligned} a_{ij},\ f &\in L^m(\Omega; C^{\alpha/2,\alpha}(\overline{D_T})),\\ u_0 &\in L^m(\Omega; C^{2+\alpha}(\overline{D})),\end{aligned} \tag{5.4}$$

for some $\alpha \in (0, \sigma - 2]$ and all $m \in [2, \infty)$ provided $A, B \in C^2_b$, $F \in C^1_b$. The compatibility conditions (iv) are: $u_0 = 0$ and $-\sum_{ij} A_{ij}(u)\partial_{ij}u = \sum_{ij} A'_{ij}(u)\partial_i u\partial_j y + \operatorname{div}\big(B(u) + (A(u) - \mathrm{I})\nabla z\big) + F(u) = 0$, on $\{0\} \times \partial D$. The first one is clearly satisfied. For the second one, we use Remark 5.1 to rewrite it as: $-\sum_{ij} A_{ij}(0)\partial_{ij}u_0 = \sum_{ij} A'_{ij}(0)\partial_i u_0 \partial_j u_0 + \operatorname{div}\big(B(u_0)\big) + F(u_0) = 0$, rearranging the terms this is exactly $u_0^{(1)}$. Thus, Theorem 3.4 applies and we deduce

$$y \in L^m(\Omega; C^{\alpha/2+1,\alpha+2}(\overline{D_T})),$$

hence

$$u \in L^m(\Omega; C^{\gamma,\alpha+2}(\overline{D_T})).$$

Step 3: Time regularity. Finally, we improve the time regularity of u by considering the stochastic convolution again as in Subsection 5.1. We obtain that for any $\lambda \in (0, 1/2)$ there exists m_0 large enough so that

$$\begin{aligned}&\mathbb{E}\|z\|^m_{C^\lambda([0,T];H^{2+\kappa,m})}\\ &\qquad \le C\left(1 + \mathbb{E}\|u\|^m_{L^m(0,T;H_0^{2+\kappa,m})} + \mathbb{E}\|u\|^{(2+\kappa)m}_{L^{(2+\kappa)m}(0,T;H_0^{1,(2+\kappa)m})}\right),\end{aligned}$$

holds true for any $m \ge m_0$ and the Sobolev embedding then implies that $z \in L^m(\Omega; C^\lambda([0,T]; C^{2+\beta}(\overline{D})))$ for $\beta < \kappa$.

Finally, we iterate the argument as in step 4 of the case $k = 1$ and this completes the proof.

5.3. **The case** $k = 3, 4$**.** The case $k = 3$ is treated exactly as above using Theorem 3.4 except that (5.4) is replaced by

$$\begin{aligned} a_{ij},\ f &\in L^m(\Omega; C^{\alpha/2,\alpha+1}(\overline{D_T})) = L^m(\Omega; C^{(\alpha+1)/2,\alpha+1}(\overline{D_T})), \\ u_0 &\in L^m(\Omega; C^{\alpha+3}(\overline{D})), \end{aligned} \tag{5.5}$$

for some $\alpha \in (0, \sigma - 3]$ where $\sigma > 3$.

For $k = 4$, we argue similarly but apply Theorem 3.5. The only thing we need to check is that f is smooth in time on the boundary and the validity of the third compatibility condition in (iii). Recall that $f = \sum_{ij} A'_{ij}(u)\partial_i u \partial_j y + \operatorname{div}\big(B(u) + (A(u) - \mathrm{I})\nabla z\big) + F(u)$. Since z and its derivatives up to order 4 vanish on ∂D, we have

$$f|_{S_T} = \sum_{ij} A'_{ij}(y)\partial_i y \partial_j y + \operatorname{div}\big(B(y)\big) + F(y).$$

In the proof for $k = 3$, we get in Step 2 that $y \in L^m(\Omega; C^{1+\iota/2,3+\iota}(\overline{D_T}))$, this shows that $f|_{S_T} \in L^m(\Omega; C^{1+\iota/2,2+\iota}(\overline{S_T}))$. Concerning the compatibility condition, we take $\mathcal{A} = \sum_{ij} A_{ij}(u)\partial_{ij}$. Thus, on the boundary $\mathcal{A} = \sum_{ij} A_{ij}(0)\partial_{ij}$. In particular, it is constant in time and the last term of the compatibility condition vanishes. Moreover on $\{0\} \times \partial D$, $\partial_t u = u_0^{(1)} = 0$, we deduce:

$$\partial_t f = 2\sum_{ij} A'_{ij}(0)\partial_i u_0^{(1)} \partial_j u_0 + B'(0) \cdot \nabla u_0^{(1)}.$$

Also, still on $\{0\} \times \partial D$,

$$\mathcal{A} f = \sum_{lk} A_{lk}(0)\partial_{lk} \left(\sum_{ij} A'_{ij}(u_0)\partial_i u_0 \partial_j u_0 + \operatorname{div}\big(B(u_0)\big) + F(u_0) \right)$$

and, since $\mathcal{A} = \mathcal{A}_0$,

$$\mathcal{A}^2 u_0 + \mathcal{A} f = \mathcal{A}_0 \left(\sum_{ij} A_{ij}(u_0)\partial_{ij} u_0 + A'_{ij}(u_0)\partial_i u_0 \partial_j u_0 + \operatorname{div}\big(B(u_0)\big) + F(u_0) \right).$$

Regrouping terms, we obtain:

$$\mathcal{A}^2 u_0 + \mathcal{A} f = \mathcal{A}_0 u_0^{(1)}.$$

Therefore our assumption in Theorem 2.7 implies the compatibility condition.

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