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1 Abstract

We study the asymptotic behaviour of a sequence of Piecewise Constant Markov Processes (in short PDMP) in which three different scales are at work: a rapid, a medium and a slow one. At the limit the rapid scale gives rise to a diffusion part (this is a CLT type regime), the medium scale produces a drift part (this is the law of large numbers type regime) and the slaw rate gives a finite variation jump process. So at the limit we obtain a stochastic differential equation which is similar to the $PDMP$ evolution but now, in-between two jumps the equation evolutes as a general diffusion process including a Brownian part and moreover, an infinity of jumps occur in each finite time interval. This type of equations seems to be new in the literature and our first goal is to prove existence and uniqueness of the solution for them. Afterwords we study the regularity of the semigroup and we use it in order to prove the convergence result mentioned in the beginning.

2 Introduction

In this paper we introduce the following class of jump type stochastic equations:

$$\begin{aligned} X_t &= x+\sum_{l=1}^{m}\int_0^t \sigma_l(X_s)dW_s^l+\int_0^t b(X_s)ds \\ &\quad +\int_0^t\int_E\int_{(0,\infty)} c(z,X_{s-})1_{\{u\leq\gamma(z,X_{s-})\}}N_\mu(ds,dz,du). \end{aligned} \tag{1}$$

Here E is a measurable space, $N_\mu(ds,dz,du)$ is a homogeneous Poisson point measure on $E\times(0,\infty)$ with intensity measure $\mu(dz)\times 1_{(0,\infty)}(u)du$ and the coefficients are $\sigma_l,b:R^d\to R^d$ and $c:R^d\times E\to R^d,\gamma:R^d\times E\to[0,\infty)$. Suppose for a moment that μ is a finite measure and $\sigma_l=0,l=1,...,m$. Then the solution of the above equation is a Piecewise Deterministic Markov Process ($PDMP$ in short) and existence and uniqueness of the solution are well known. But, if μ is an infinite measure and we have a non null diffusion component, this type of equations have not been considered in the literature. So our first aim is to prove that under reasonable hypothesis equation (1) has a unique solution - this is done Theorem 3. The proof is based on some non trivial L^1 estimates (we thank to Nicolas Fournier who gave us an important hint in this direction).

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The equation (1) naturally appears as the limit of sequences of $PDMP's$ with three different regimes, that we describe now. We consider a sequence of processes $X_t^n, n \in N$, which solve

$$X_t^n = x + \sum_{i=1}^{3} \int_0^t \int_E \int_{(0,\infty)} c_{(i),n}(z, X_{s-}^n) 1_{\{u \leq \gamma_{(i),n}(z,X_{s-})\}} N_{\mu_{(i),n}}(ds, dz, du) \tag{2}$$

where $N_{\mu_{(i),n}}(ds, dz, du), i = 1, 2, 3$ are three independent Poisson point measures of intensities

$$\mu_{(i),n}(dz) \times 1_{(0,\infty)}(u)du.$$

Each of them represents a different regime. For $i = 1$ we consider a CLT type regime, described by the following hypothesis:

$$\lim_n \int_E c_{(1),n} c_{(1),n}^*(z, x) \times \gamma_{(1),n}(z, x) d\mu_{(1),n}(z) = \sigma\sigma^*(x) \tag{3}$$

where σ is the matrix with columns $\sigma_l, l = 1, ..., m$ and σ^* designs the transposed matrix. Moreover we assume that

$$\lim_n \int_E \left|c_{(1),n}(z, x)\right|^3 \times \gamma_{(1),n}(z, x) d\mu_{(1),n}(z) = 0. \tag{4}$$

For $i = 2$ we have a Law of Large Numbers type regime: we assume that

$$\lim_n \int_E c_{(2),n}(z, x) \times \gamma_{(2),n}(z, x) d\mu_{(2),n}(z) = b(x) \tag{5}$$

and

$$\lim_n \int_E \left|c_{(2),n}(z, x)\right|^2 \times \gamma_{(2),n}(z, x) d\mu_{(2),n}(z) = 0. \tag{6}$$

Finally, for $i = 3$ we have a "finite variation" type regime: we assume that $\mu_{(3),n}(dz) = 1_{E_n}(z)d\mu(z)$ where μ is the intensity measure which appears in the limit equation (1) and $E_n \uparrow E$ is a sequence of measurable sets such that $\mu(E_n) < \infty$ and

$$\lim_n \int_{E \setminus E_n} |c(z, x)|\gamma(z, x) d\mu(z) = 0. \tag{7}$$

And we assume that

$$\lim_n \int_E c_{(3),n}(z, x)\gamma_{(3),n}(z, x) d\mu_{(3),n}(z) = \int_E c(z, x)\gamma(z, x) d\mu(z) \tag{8}$$

We stress that the convergence in (3),(5) and (8) has to be given in a more precise and quantitative way (see (69)) - here we just give the general direction. Some other technical hypothesis are in force. Then we are able to prove that, for $f \in C_b^3(R^d)$, one has

$$\lim_n E(f(X_t^n)) = E(f(X_t))$$

and to control the speed of convergence.

The weak convergence of Markov chains to diffusion processes has been widely discussed in the literature (see *e.g.* [Kur71], [Kur78], [Kus84], [JS03])) but in our framework we have the following specific difficulty. In the case of standard jump type equations (that is: $1_{\{u \leq \gamma(z,X_{s-})\}}$ does not appear in the equation (1)) the flow $x \to X_t(x)$ is differentiable and consequently, if $f \in C_b^3(R^d)$, then $x \to E(f(X_t(x)))$ is three times differentiable as well. Using this, one proves in a straightforward way the convergence of the semigroups and moreover, obtains an estimate of the error. But, because of the indicator function, this is not true here - and so a key point in our approach is to study the regularity of $x \to E(f(X_t(x)))$. This is done in Theorem 7 and 12.

We also mention that $PDMP's$ with several regimes have recently been considered in the literature for modelling and numerically solving problems in gene networks (see [CDMR12] and [ACT$^+$04], [ACT+05]) and chemical networks (see [BKPR06]). However we do not enter in our paper in the specific framework on such physical phenomenons.

The paper is organized as follows: In Section 3 we prove existence and uniqueness for the solution of equation (1) and in Section 4 we prove the regularity of $x \to E(f(X_t(x)))$. In Section 5 we prove the convergence result and in the Appendix we give some moment inequalities used in the paper.

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3 Existence and uniqueness

3.1 Notation and main result

We consider a measurable space $(E, \mathcal{E})$ and, for a σ finite measure μ on E we denote by N_μ the Poisson point measure on $E \times [0,\infty)$ of compensator $\widehat{N}_\mu(dt,dz,du) = dt \times \mu(dz) \times du$ (we refer to Ikeda Watanabe [IW89] for definitions and notation concerning Poisson point measures). Moreover we consider a m dimensional Brownian motion $W = (W^1, ..., W^m)$ which is independent of the Poisson measure N_μ and we look to the d dimensional stochastic equation

$$\begin{aligned} X_t &= x + \sum_{l=1}^m \int_0^t \sigma_l(X_s) dW_s^l + \int_0^t b(X_s) ds \qquad (9) \\ &\quad + \int_0^{t+} \int_{E\times[0,\infty)} c(z, X_{s-}) 1_{\{u \le \gamma(z, X_{s-})\}} N_\mu(ds, dz, du). \end{aligned}$$

with $\sigma_l, b : R^d \to R^d$ and $c : E \times R^d \to R^d, \gamma : E \times R^d \to [0,\infty)$.

Definition 1 *A process $(X_t)_{t\ge 0}$ is called a L^1 solution of the equation (9) if it is adapted, càdlàg and, for every $T > 0$*

$$\sup_{t\le T} E(|X_t|) < \infty. \qquad (10)$$

Remark 2 *We precise that $X_t, t \ge 0$ is a càdlàg process if it is right continuous and has finite left hand limits almost surely. In particular X_t may not blow up in finite time: if $\tau_R = \inf\{t : |X_t| \ge R\}$ then $\sup_R \tau_R = \infty$ (indeed, if $\sup_R \tau_R = \tau_\infty \le T$, then $X_{T\wedge\tau_\infty -} = \infty$).*

We give now the hypothesis which are needed in order to obtain existence and uniqueness for a L^1 solution of the above equation. We assume that there exist a constant $L \in R_+$ and some functions $l_c, l_\gamma : E \to R_+$ such that, for every $x, y \in R^d$

$$|b(x) - b(y)| + \sum_{l=1}^m |\sigma_l(x) - \sigma_l(y)| \le L\,|x - y| \qquad (11)$$

and for every $x, y \in R^d$ and $z \in E$

$$|c(z,x) - c(z,y)| \le l_c(z)\,|x-y|\,, \quad |\gamma(z,x) - \gamma(z,y)| \le l_\gamma(z)\,|x-y|\,. \qquad (12)$$

Moreover we assume that

$$C_\mu(\gamma,c) := \sup_{x\in R^d}\int_E (l_\gamma(z)\left|c(z,x)\right| + l_c(z)\gamma(z,x))d\mu(z) < \infty. \tag{13}$$

For a measurable set $G\subset E$ and $\Gamma\geq 1$ we denote

$$\lambda(G) = \sup_{x\in R^d}\int_G \left|c(z,x)\right|\gamma(z,x)d\mu(z), \tag{14}$$

$$\beta(\Gamma) = \sup_{x\in R^d}\int_E \left|c(z,x)\right|\gamma(z,x)1_{\{\Gamma\leq\gamma(z,x)\}}d\mu(z) \tag{15}$$

and we assume that

$$\lambda(E) < \infty \quad \textit{and} \quad \lim_{\Gamma\to\infty}\beta(\Gamma) = 0. \tag{16}$$

Our main result is the following:

Theorem 3 *Suppose that (11),(12), (13) and (16) hold. Then the equation (9) has a unique* L^1 *solution.*

3.2 The basic estimate

In this section we give the main estimate which allows to prove Theorem 3. We will work with some truncated versions of the equation (9) that we construct now. We consider a family of smooth functions $\psi_\Gamma : R_+ \to [0,\Gamma]$ such that

$$\begin{aligned}\psi_\Gamma(x) &= x \quad \text{if} \quad x\leq\Gamma-1, \\ &= \Gamma \quad \text{if} \quad x\geq\Gamma\end{aligned} \tag{17}$$

and such that the derivatives of any order of ψ_Γ are bounded, unifromely with respect to Γ. Then we construct

$$\gamma_\Gamma(z,x) = \psi_\Gamma(\gamma(z,x)). \tag{18}$$

This is a smooth version of $\Gamma\wedge\gamma(z,x)$.

For a measurable set $G\subset E$ and a constant $\Gamma>1$ we denote by $X^{G,\Gamma}$ the L^1 solution (if such a solution exists) of the equation

$$\begin{aligned}X_t^{G,\Gamma} &= x + \sum_{l=1}^m\int_0^t \sigma_l(X_s^{G,\Gamma})dW_s^l + \int_0^t b(X_s^{G,\Gamma})ds \\ &\quad + \int_0^{t+}\int_{E\times[0,\infty)} 1_G(z)c(z,X_{s-}^{G,\Gamma})1_{\{u\leq\gamma_\Gamma(z,X_{s-}^{G,\Gamma})\}}N_\mu(ds,dz,du).\end{aligned} \tag{19}$$

Remark 4 *Notice that we accept the case* $G=E$ *and* $\Gamma=\infty$ *and then* $X_t^{G,\Gamma}=X_t$ *the solution of the equation (9).*

Remark 5 *If* $\mu(G)<\infty$ *and* $\Gamma<\infty$ *then it is easy to prove that the equation (19) has a unique* L^1 *solution: indeed if* $T_k, k\in N$ *are the jump times of the Poisson process* $t\to N_\mu(t,G)$ *then, for* $t\in[T_{k-1},T_k)$ *one solves the standard diffusion equation* $dX_s^{G,\Gamma}=\sum_{l=1}^m\sigma_l(X_s^{G,\Gamma})dW_s^l+b(X_s^{G,\Gamma})ds$ *and then defines* $X_{T_k}^{G,\Gamma}=X_{T_k-}^G+c(Z_k,X_{s-}^{G,\Gamma})1_{\{U_k\leq\gamma_\Gamma(z,X_{s-}^G)\}}$ *where* $Z_k\sim\frac{1}{\mu(G)}1_G(z)\mu(dz)$ *and* $U_k\sim\frac{1}{\Gamma}1_{[0,\Gamma]}(u)du$.

Lemma 6 *Suppose that (11),(12),(13) and (16) hold. Let $G_1 \subset G_2 \subset E$ be two measurable sets and $1 < \Gamma_1 \leq \Gamma_2$ (the case $G_1 = G_2 = E$ and $\Gamma_1 = \Gamma_2 = \infty$ is included) and let $X_t^1 := X_t^{G_1,\Gamma_1}$ and $X_t^2 := X_t^{G_2,\Gamma_2}$ be two L^1 solutions of the equation (19) (corresponding to G_1, Γ_1 respectively to G_2, Γ_2). There exists an universal constant C such that for every $T \geq 0$ one has*

$$\sup_{t\leq T} E(\left|X_t^1 - X_t^2\right|) \leq CT\exp(CT(L + C_\mu(\gamma, c))) \times (\beta(\Gamma_1) + \lambda(G_2 \backslash G_1)) \tag{20}$$

with L, $C_\mu(\gamma, c), \beta(\Gamma_1)$ and $\lambda(G_2 \backslash G_1)$ defined in (11), (13),(14) and (15). Moreover for every $\rho > 0$

$$P(\sup_{t\leq T} \left|X_t^1 - X_t^2\right| \geq \rho) \leq \frac{CT}{\rho}\exp(CT(L + C_\mu(\gamma, c))) \times (\beta(\Gamma_1) + \lambda(G_2 \backslash G_1)). \tag{21}$$

Proof. Step 1. We will use a cut-off procedure inspired from [BF11]. Let us introduce some notation. Let $\varphi(x) = \alpha 1_{(-1,1)}(x)\exp(-\frac{1}{1-x^2})$ with α such that $\int \varphi(x)dx = 1$, and, for $\varepsilon > 0$ let $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$. We also denote $h_\varepsilon(x) = 2\varepsilon \vee |x|$ and we define

$$\phi_\varepsilon(x) = h_\varepsilon * \varphi_\varepsilon(x), \quad \textit{and} \quad f_\varepsilon(z) = \phi_\varepsilon(|z|).$$

The basic property of f_ε is the following: there exists an universal constant C such that for every $\varepsilon > 0$

$$\left|\frac{\partial f_\varepsilon}{\partial z_i}(z)\right| \leq C \quad \textit{and} \tag{22}$$

$$\left|\frac{\partial^2 f_\varepsilon}{\partial z_i \partial z_j}(z)\right| \leq \frac{C}{|z|}. \tag{23}$$

Proof. We have

$$\begin{aligned} \frac{\partial f_\varepsilon}{\partial z_i}(z) &= \phi_\varepsilon'(|z|)\frac{z_i}{|z|} \quad \textit{and} \\ \frac{\partial^2 f_\varepsilon}{\partial z_i \partial z_j}(z) &= \left(\phi_\varepsilon''(|z|) - \frac{\phi_\varepsilon'(|z|)}{|z|}\right)\frac{z_i z_j}{|z|^2} + \delta_{i,j}\frac{\phi_\varepsilon'(|z|)}{|z|}. \end{aligned}$$

Since ϕ_ε' is bounded, (22) follows. Let us check (23). If $|z| \leq \varepsilon$ then $\phi_\varepsilon'(|z|) = \phi_\varepsilon''(|z|) = 0$ so $f_\varepsilon(z) = 0$. If $\varepsilon \leq |z| \leq 3\varepsilon$ then $\phi_\varepsilon'(|z|) \leq C$ and $\phi_\varepsilon''(|z|) \leq \frac{1}{\varepsilon}$ so that

$$\left|\frac{\partial^2 f_\varepsilon}{\partial z_i \partial z_j}(z)\right| \leq C(\frac{1}{\varepsilon} + \frac{1}{|z|}) \leq \frac{C}{|z|}.$$

Finally, if $x \geq 3\varepsilon$ then $\phi_\varepsilon'(x) = 1$ and $\phi_\varepsilon''(x) = 0$ so we obtain (23) for $|z| \geq 3\varepsilon$ as well .

Step 2. We denote

$$\begin{aligned} \Delta_j\sigma_t &= \sigma_j(X_t^1) - \sigma_j(X_t^2), \quad \Delta b_t = b(X_t^1) - b(X_t^2) \quad \textit{and} \\ H_t^{\Gamma_1,\Gamma_2}(z,u) &= 1_{G_1}(z)c(z, X_{t-}^1)1_{\{u \leq \gamma_{\Gamma_1}(z, X_{t-}^1)\}} - 1_{G_2}(z)c(z, X_{t-}^2)1_{\{u \leq \Gamma\gamma_{\Gamma_2}(z, X_{t-}^2)\}}. \end{aligned} \tag{24}$$

Then $Z_t := X_t^1 - X_t^2$ verifies the equation

$$Z_t = \sum_{l=1}^m \int_0^t \Delta_l \sigma_s dW_s^l + \int_0^t \Delta b_s ds + \int_0^{t+} \int_{E\times[0,1]} H_{s-}^{\Gamma_1,\Gamma_2}(z,u) N_\mu(ds, dz, du).$$

For $R > 0$ we define the stopping time

$$\tau_R = \inf\{t : \left|X_t^1\right| \vee \left|X_t^2\right| > R\}$$

and we notice that $\lim_{R\to\infty} \tau_R = \infty$ (see Remark 2). We denote

$$Z_t^R = Z_{t\wedge\tau_R}.$$

Using Itô's formula we write

$$f_\varepsilon(Z_t^R) - f_\varepsilon(Z_0^R) = M_\varepsilon(t \wedge \tau_R) + \sum_{i=1}^{3} I_\varepsilon^i(t \wedge \tau_R) \tag{25}$$

with

$$M_\varepsilon(t) = \sum_{i=1}^{d}\sum_{l=1}^{m}\int_0^t \frac{\partial f_\varepsilon}{\partial z_i}(Z_s^R)\Delta_l^i\sigma_s dW_s^l$$

and

$$\begin{aligned}
I_\varepsilon^1(t) &= \frac{1}{2}\sum_{i,j=1}^{d}\sum_{l=1}^{m}\int_0^t \frac{\partial^2 f_\varepsilon}{\partial z_i \partial z_j}(Z_s^R)\Delta_l^i\sigma_s\Delta_l^j\sigma_s ds,\\
I_\varepsilon^2(t) &= \sum_{i=1}^{d}\int_0^t \frac{\partial f_\varepsilon}{\partial z_i}(Z_s^R)\Delta^i b_s ds,\\
I_\varepsilon^3(t) &= \int_0^{t+}\int_{E\times(0,1)}(f_\varepsilon(Z_{s-} + H_{s-}^{\Gamma_1,\Gamma_2}(z,u)) - f_\varepsilon(Z_{s-}))dN(s,z,u).
\end{aligned}$$

Since $\left|\Delta_l^i\sigma_s\right| \le L\left|Z_{s\wedge\tau_R}\right| \le 2LR$ and $\partial_i f_\varepsilon$ is bounded, the process M_ε is a martingale and this gives $E(M_\varepsilon(t\wedge\tau_R)) = 0$. Then (we have $f_\varepsilon(Z_0^R) = f_\varepsilon(0) = \varepsilon$)

$$\left|E(f_\varepsilon(Z_t^R))\right| \le \varepsilon + \sum_{i=1}^{3}\left|E(I_\varepsilon^i(t\wedge\tau_R))\right|.$$

We will prove that

$$\sum_{i=1}^{3} E(\left|I_\varepsilon^i(t\wedge\tau_R)\right|) \le t(\beta(\Gamma_1) + \lambda(G_2\backslash G_1)) + C(L + C_\mu(\gamma,c))\int_0^t E(\left|Z_s^R\right|)ds. \tag{26}$$

We estimate the terms in the *RHS* of the above inequality. Since σ_l is Lipschitz continuous, for $s \le t\wedge\tau_R$ we have $\left|\Delta_l^i\sigma_s\right| \le L\left|Z_s^R\right|$. Then, using (23) we obtain

$$\left|I_\varepsilon^1(t\wedge\tau_R)\right| \le CL\int_0^t \left|Z_s^R\right| ds$$

and using (22) we get a similar upper bound for $\left|I_\varepsilon^2(t\wedge\tau_R)\right|$. So (26) is verified for $i = 1,2$.

We estimate now I_ε^3. Since $N(ds,dz,du)$ is a positive measure and f_ε is Lipschitz continuous

$$\left|I_\varepsilon^3(t\wedge\tau_R)\right| \le C\int_0^{(t\wedge\tau_R)+}\int_{E\times(0,\infty)}\left|H_s^{\Gamma_1,\Gamma_2}(z,u)\right| dN(s,z,u)$$

and then, using the isometry property

$$E(\left|I_\varepsilon^3(t\wedge\tau_R)\right|) \le E(\int_0^{t\wedge\tau_R}\int_{E\times(0,\infty)}\left|H_s^{\Gamma_1,\Gamma_2}(z,u)\right| d\mu(z)duds) \le J_1 + J_2$$

with

$$\begin{aligned}
J_1 &= E(\int_0^{t\wedge\tau_R}\int_{E\times(0,1)}\left|H_s^{\Gamma_1,\Gamma_2}(z,u) - H_s^{\infty,\infty}(z,u)\right| d\mu(z)duds)\\
J_2 &= E(\int_0^{t\wedge\tau_R}\int_{E\times(0,1)}\left|H_s^{\infty,\infty}(z,u)\right| d\mu(z)duds).
\end{aligned}$$

Then

$$\begin{aligned} J_2 &\leq E(\int_0^{t\wedge\tau_R}\int_{G_2\backslash G_1}\left|c(z,X_s^2)\right|\gamma(z,X_s^2)d\mu(z)ds) \\ &\quad +E(\int_0^{t\wedge\tau_R}\int_{G_1}(l_\gamma(z)\left|c(z,X_s^1)\right|+l_c(z)\gamma(z,X_s^2))\left|X_s^2-X_s^1\right|d\mu(z)ds) \\ &\leq t\lambda(G_2\backslash G_1)+C_\mu(\gamma,c)E(\int_0^{t\wedge\tau_R}\left|Z_s^R\right|ds). \end{aligned}$$

And

$$J_1\leq K_1+K_2$$

with

$$\begin{aligned} K_i &\leq E(\int_0^{t\wedge\tau_R}\int_{G_i}\left|c(z,X_s^i)\right|(\gamma(z,X_s^i)-\gamma_{\Gamma_i}(z,X_s^i))d\mu(z)ds) \\ &\leq E(\int_0^{t\wedge\tau_R}\int_{G_i}\left|c(z,X_s^i)\right|\gamma(z,X_s^i)1_{\{\Gamma_i\leq\gamma(z,X_s^i)\}}d\mu(z)ds)\leq\beta(\Gamma_i)t. \end{aligned}$$

So (26) is proved. In particular we obtain

$$\left|E(f_\varepsilon(Z_t^R))\right|\leq\varepsilon+t(\beta(\Gamma_1)+\lambda(G_2\backslash G_1))+C(L+C_\mu(\gamma,c))\int_0^t E(\left|Z_s^R\right|)ds.$$

We have $\lim_{\varepsilon\to0}f_\varepsilon(z)=|z|$, so, using Fatou's lemma

$$E(\left|Z_t^R\right|)\leq t(\beta(\Gamma_1)+\lambda(G_2\backslash G_1))+C(L+C_\mu(\gamma,c))\int_0^t E(\left|Z_s^R\right|)ds.$$

Then, by Gronwall's lemma

$$E(\left|Z_t^R\right|)\leq Ct(\beta(\Gamma_1)+\lambda(G_2\backslash G_1))\exp(Ct(L+C_\mu(\gamma,c))). \tag{27}$$

We recall that $\lim_{R\to\infty}\tau_R=\infty$ so, using again Fatou's lemma, we pass to the limit with $R\to\infty$ and we obtain

$$E(|Z_t|)=E(\underline{\lim}_{R\to\infty}\left|Z_t^R\right|)\leq Ct(\beta(\Gamma_1)+\lambda(G_2\backslash G_1))\exp(Ct(L+C_\mu(\gamma,c)))$$

so (20) is proved.

Step 3. Let us prove (21). Using (25) and the fact that f_ε is Lipschitz continuous, we obtain

$$\begin{aligned} E(|M_\varepsilon(t\wedge\tau_R)|) &\leq E(\left|f_\varepsilon(Z_t^R)-f_\varepsilon(Z_0^R)\right|)+\sum_{i=1}^3 E(\left|I_\varepsilon^i(t\wedge\tau_R)\right|) \\ &\leq CE(\left|Z_t^R\right|)+\sum_{i=1}^3 E(\left|I_\varepsilon^i(t\wedge\tau_R)\right|) \\ &\leq Ct(\beta(\Gamma_1)+\lambda(G_2\backslash G_1))\exp(Ct(L+C_\mu(\gamma,c))) \end{aligned}$$

the last inequality being a consequence of (26) and (27).

We take now $\rho>0$ and we use Doob's inequality and Chebyshev's inequality in order to get

$$\begin{aligned} P(\sup_{t\leq T}\left|f_\varepsilon(Z_t^R)\right| &\geq \rho)\leq P(\sup_{t\leq T}|M_\varepsilon(t\wedge\tau_R)|\geq\frac{\rho}{4})+\sum_{i=1}^3 P(\sup_{t\leq T}\left|I_\varepsilon^i(t\wedge\tau_R)\right|\geq\frac{\rho}{4}) \\ &\leq \frac{C}{\rho}(E(|M_\varepsilon(t\wedge\tau_R)|)+\sum_{i=1}^3 E(\sup_{t\leq T}\left|I_\varepsilon^i(t\wedge\tau_R)\right|) \\ &\leq \frac{Ct}{\rho}(\beta(\Gamma_1)+\lambda(G_2\backslash G_1))\exp(Ct(L+C_\mu(\gamma,c)))). \end{aligned}$$

Using Fatou's lemma we pass to the limit with $\varepsilon\to0$ and with $R\to\infty$ and we obtain(21). $\square$

3.3 Proof of Theorem 3

Uniqueness of the solution immediately follows from (20) with $G_1 = G_2 = E$ and $\Gamma_1 = \Gamma_2 = \infty$. Let us prove existence. We take a sequence of subsets $E_n \uparrow E$ such that $\mu(E_n) < \infty$ and $\Gamma_n = n$. By (16)

$$\lim_{n,m\to\infty} (\beta(\Gamma_n) + \lambda(E_n \setminus E_m)) = 0.$$

Since $\mu(E_n) < \infty, \Gamma_n < \infty$ we may construct a solution $X_t^n := X_t^{E_n,\Gamma_n}$ and then, by (21),

$$\lim_{n,m\to\infty} \sup_{t\leq T} |X_t^n - X_t^m| = 0$$

in probability. Passing to a subsequence, the above convergence holds almost surely, so we may construct a process X_t such that

$$\lim_{n\to\infty} \sup_{t\leq T} |X_t - X_t^n| = 0 \text{ almost surely.}$$

Since X_t^n are adapted and càdlàg processes, so is X_t.

Using (20) we conclude that for every n

$$\sup_{t\leq T} E(|X_t^n|) \leq C\lambda(E_n) \tag{28}$$

and using (20) again we get

$$\sup_{t\leq T} E(|X_t|) \leq C\lambda(E). \tag{29}$$

It remains to check that X_t verifies the equation (9) which reads

$$X_t = x + M(t) + I^1(t) + I^2(t) \tag{30}$$

where

$$\begin{aligned} M(t) &= \sum_{l=1}^m \int_0^t \sigma_l(X_s) dW_s^l, \quad I^1(t) = \int_0^t b(X_s) ds, \\ I^2(t) &= \int_0^{t+} \int_{E\times[0,\infty)} c(z, X_{s-}) 1_{\{u\leq\gamma(z,X_{s-})\}} N_\mu(ds, dz, du). \end{aligned}$$

In a similar way we write

$$X_t^n = x + M_n(t) + I_n^1(t) + I_n^2(t) \tag{31}$$

where

$$\begin{aligned} M_n(t) &= \sum_{l=1}^m \int_0^t \sigma_l(X_s^n) dW_s^l, \quad I_n^1(t) = \int_0^t b(X_s^n) ds, \\ I_n^2(t) &= \int_0^{t+} \int_{E_n\times[0,\infty)} c(z, X_{s-}^n) 1_{\{u\leq\gamma_{\Gamma_n}(z,X_{s-}^n)\}} N_\mu(ds, dz, du). \end{aligned}$$

Since (31) holds true and $X_t^n \to X_t$ almost surely, it remains to prove that the terms in the right side of (31) converge in probability also. We have

$$E(|I^2(t) - I_n^2(t)|) \leq E(\int_0^t \int_{E\times(0,\infty)} |H_s^{\infty,\Gamma_n}(z,u)|\, d\mu(z) du ds)$$

with $H_s^{\infty,\Gamma_n}(z,u)$ defined as in (24) with $G_1 = E, \Gamma_1 = \infty$ and $G_2 = E_n, \Gamma_n = n$. Using (26)

$$E(|I^2(t) - I_n^2(t)|) \leq Ct(\beta(\Gamma_n) + \lambda(E_n^c) + \int_0^t E(|X_s - X_s^n|) ds).$$

since $\lim_n E(|X_s - X_s^n|) = 0$ for every s, we use Lebesgue's theorem (recall (28) and (29)) and we conclude that $I_n^2(t) \to I^2(t)$ in L^1. The same is true for $I_n^1(t)$.

Let us now treat $M_n(t)$. We denote $A_n = \{\sup_{s\leq t} |X_s - X_s^n| \leq 1\}$ and, for $\rho > 0$, we write

$$P(|M_n(t) - M(t)| \geq \rho) \leq P(A_n^c) + P(A_n \cap \{|M_n(t) - M(t)| \geq \rho\}).$$

On A_n we have

$$M(t) - M_n(t) = \sum_{l=1}^{m} \int_0^t (\sigma_l(X_s) - \sigma_l(X_s^n))1_{\{|X_s - X_s^n| \leq 1\}} dW_s^l$$

so that

$$\begin{aligned} P(A_n \cap \{|M_n(t) - M(t)| \quad &\geq \quad \rho\}) \leq \frac{1}{\rho} \sum_{l=1}^{m} E\left(\left|\int_0^t (\sigma_l(X_s) - \sigma_l(X_s^n))1_{\{|X_s - X_s^{E_n}| \leq 1\}} dW_s^l\right|\right) \\ &\leq \frac{C}{\rho}(E(\int_0^t |X_s - X_s^n|^2 \, 1_{\{|X_s - X_s^n| \leq 1\}} ds))^{1/2} \\ &\leq \frac{C}{\rho}(E(\int_0^t |X_s - X_s^n| \, ds))^{1/2} \to 0. \end{aligned}$$

We also have $\lim_{n\to\infty} P(A_n^c) = 0$ so that $\lim_{n\to\infty} P(|M_n(t) - M(t)| \geq \rho) = 0$. □

4 Regularity of the semigroup

Our aim is to study the regularity of the semigroup $P_t f(x) := E(f(X_t(x)))$ where $X_t(x)$ is the solution of the equation (9) which starts from $X_0 = x$. We have to introduce some more notation. For a function $f : R^d \to R$ which is k time differentiable we denote

$$\|f\|_{k,\infty} = \sup_{x\in R^d} \sum_{|\alpha|\leq k} |\partial_x^\alpha f(x)| \,. \tag{32}$$

For a function $f : E \times R^d \to R$,which is q times differentiable with respect to a, for a set $G \subset E$ and for $p \geq 1$ we denote

$$\overline{f}_{q,p}(G) = \sup_{x\in R^d} \sum_{1\leq|\alpha|\leq q} \int_G |\partial_x^\alpha f(z,x)|^p \, \gamma(z,x) d\mu(z) \tag{33}$$

Moreover, for $q \in N$ and $p > 1$, we define (with σ, b and c the coefficients in the equation (9))

$$\theta_{q,p}(G) = 1 + \|\sigma\|_{q,\infty}^{2p} + \|b\|_{q,\infty}^{2p} + \overline{c}_{q,1}(G) + \overline{c}_{q,2p}(G). \tag{34}$$

We also denote

$$\underline{\gamma}(G) = \inf\{\gamma(z,x) : x \in R^d, z \in G\}, \tag{35}$$

$$\overline{\gamma}(G) = \sup_{x\in R^d} \int_G \gamma(z,x) d\mu(x), \tag{36}$$

$$\widetilde{\gamma}_q(G) = \sum_{1\leq|\alpha|\leq q} \frac{1}{\mu(G)} \sup_{x\in R^d} \int_G |\partial_x^\alpha \gamma(z,x)| \, d\mu(x) \tag{37}$$

and

$$\alpha_{q,p}(t,G) = C(t \vee 1)(\frac{\overline{\gamma}_{q,2q}(G)}{\underline{\gamma}^q(G)} + \theta_{q,p}(G)e^{tC\theta_{q,p}(G)}(1 + \frac{\overline{\gamma}(G)}{\underline{\gamma}^q(G)} + \sum_{j=1}^{q} \widetilde{\gamma}_j^{q-j+1}(G))) \tag{38}$$

Here C is an universal constant.

In the following we will repeatedly use the following inequalities:

$$\sup_{x\in R^d}\sum_{1\leq|\alpha|\leq q}\int_G |\partial_x^\alpha f(z,x)|^p\,\gamma_\Gamma(z,x)d\mu(z)\leq \overline{f}_{q,p}(G) \tag{39}$$

and

$$\sum_{1\leq|\alpha|\leq q}\frac{1}{\mu(G)}\sup_{x\in R^d}\int_G |\partial_x^\alpha\gamma_\Gamma(z,x)|\,d\mu(x)\leq C\widetilde{\gamma}_q(G). \tag{40}$$

The first one is a consequence of $\gamma_\Gamma\leq\gamma$ and the second one follows from the definition $\gamma_\Gamma=\psi_\Gamma(\gamma)$ and the fact that ψ_Γ has derivatives which are bounded uniformly with respect to Γ.

Theorem 7 *Let $q\in N$ and $G\subset E$ with $\mu(G)<\infty$. We assume that (11), (12), (13) and (16) hold, that $\sigma\in C_b^{q+1}(R^d), b\in C_b^q(R^d), c(\circ,z)\in C_b^q(R^d),\gamma(\circ,z)\in C_b^q(R^d)$ and $\underline{\gamma}(G)>0$. Let $\Gamma>1$ be such that*

$$m_G(\Gamma):=\frac{1}{\mu(G)}\int_G 1\wedge\frac{\gamma(z,x)}{\Gamma}d\mu(z)<1 \tag{41}$$

and let $P_t^{G,\Gamma}f(x)=E(f(X_t^{G,\Gamma}(x))$ where $X_t^{G,\Gamma}(x)$ is the solution of equation (19) which starts from x. There exits some constants C_q and l_q, depending on q only, such that for every $f\in C_b^q(R^d)$ one has

$$\left\|P_t^{G,\Gamma}f\right\|_{q,\infty}\leq\frac{C_q}{1-m_G(\Gamma)}\alpha_{q,ql_q}(t,G)\left\|f\right\|_{q,\infty} \tag{42}$$

For $q=1,2,3$ we have $l_q=q$.

Remark 8 *Notice that $\alpha_{q,ql_q}(t,G)$ appears as the constant which controls the regularity of $x\to P_t^{G,\Gamma}f(x)$. Roughly speaking we expect that $\alpha_{q,ql_q}(t,G)<\infty$ if $\mu(G)<\infty$. But in the following we will consider a sequence of sets $E_n\uparrow E$ such that $\mu(E_n)<\infty$ and $\alpha_{q,ql_q}(t,E_n)<\infty$ but $\alpha_{q,ql_q}(t,E_n)\uparrow\infty$. So we have regularity for the semigroup of the truncated equations but we loose control when passing with $n\to\infty$. This is the delicate point in our approach. The rate of the blow up $\alpha_{q,ql_q}(t,E_n)\uparrow\infty$ becomes critical ; see also Remark 13.*

In order to prove Theorem 7 we need some preparation. Since $\mu(G)<\infty$ we have an alternative representation of the solution $X_t^{G,\Gamma}$ of the equation (19) by means of a compound Poisson process: we consider a Poisson process J_t with parameter $\mu(G)\Gamma<\infty$ and we denote by $T_k, k\in N$ the jump times of J_t (since G and Γ are fixed, we do not mention them in the notation). Moreover we take a sequence $Z_k,U_k,k\in N$ of independent random variables (which are independent of W and of J) with laws

$$P(Z_k\in dz)=\frac{1}{\mu(G)}1_G(z)\mu(dz),\quad P(U_k\in du)=\frac{1}{\Gamma}1_{(0,\Gamma)}(u)du.$$

Then the equation (19) may be represented as

$$\begin{aligned}X_t^{G,\Gamma}(x) &= x+\sum_{l=1}^m\int_0^t\sigma_l(X_s^{G,\Gamma}(x))dW_s^l+\int_0^t b(X_s^{G,\Gamma}(x))ds \\ &\quad+\sum_{k\leq J_t}c(Z_k,X_{T_k-}^{G,\Gamma}(x))1_{\{U_k\leq\gamma_\Gamma(Z_k,X_{T_k-}^{G,\Gamma}(x))\}}.\end{aligned} \tag{43}$$

We give now a second representation which does no more contain the indicator function $1_{\{U_k\leq\gamma_\Gamma(Z_k,X_{T_k-}^{G,\Gamma}(x))\}}$ and so it is suitable when discussing the regularity with respect to x. We denote by $\Phi_{t,s}(x), 0\leq t\leq s$ the solution of the standard diffusion equation

$$\Phi_{t,s}(x)=x+\sum_{l=1}^m\int_t^s\sigma_l(\Phi_{t,r}(x))dW_r^l+\int_t^s b(\Phi_{t,r}(x))dr. \tag{44}$$

Notice that, since $\sigma \in C_b^{q+1}(R^d)$ and $b \in C_b^q(R^d)$, we may choose a version of Φ which is q times differentiable with respect to x (see [IW89]). Moreover we consider a sequence $(z) := (z_k)_{k\in N}$ with $z_k \in E$, we denote

$$z^k = (z_1, ..., z_k)$$

and we construct a process $x_t(x,(z))$ in the following way: we put $x_0(x) = x$ and, if $x_{T_k-}(x, z^{k-1})$ is given, we define

$$\begin{aligned} x_{T_k}(x, z^k) &= x_{T_k-}(x, z^{k-1}) + c(x_{T_k-}(x, z^{k-1}), z_k)1_G(z_k) \\ x_t(x, z^k) &= \Phi_{T_k,t}(x_{T_k}(x, z^k)) \qquad T_k \le t < T_{k+1}. \end{aligned} \tag{45}$$

Since Φ and $c(\circ, z)$ are q times differentiable with respect to x, so is $x \to x_t(x, z^k)$.

We take now function $\psi : E \to R_+$ such that $\psi(z) = 0$ for $z \in G$ and $\int \psi d\mu = 1$ and we construct the probability density

$$\begin{aligned} q_{G,\Gamma}(z,x) &= \theta_G(x)\psi(z) + \frac{1}{\mu(G)\Gamma}1_G(z)\gamma_\Gamma(z,x) \qquad \textit{with} \\ \theta_{G,\Gamma}(x) &= 1 - \frac{1}{\mu(G)\Gamma}\int_G \gamma_\Gamma(z,x))\mu(dz). \end{aligned} \tag{46}$$

By the very definition of $\theta_{G,\Gamma}(x)$ we have $\int_E q_{G,\Gamma}(z,x)d\mu(z) = 1$. And since $m_G(\Gamma) < 1$ we have $\theta_{G,\Gamma}(x) \ge 1 - m_G(\Gamma) > 0$.

We construct a sequence of random variables $\overline{Z}_k$ in the following way. $\overline{Z}_1$ has conditional law

$$P(\overline{Z}_1 \in dz \mid x_{T_1-}(x) = y) = q_{G,\Gamma}(z,y)\mu(dz).$$

Then, if $\overline{Z}_i, i \le k-1$ are given, we construct $\overline{Z}_k$ to be a random variable with conditional law

$$P(\overline{Z}_k \in dz \mid x_{T_k-}(x, \overline{Z}^{k-1}) = y) = q_{G,\Gamma}(z,y)\mu(dz) \tag{47}$$

where $\overline{Z}^k = (\overline{Z}_1, ..., \overline{Z}_k)$. Notice that the density of the law of $\overline{Z}^n$ with respect to $\mu(dz_1)...\mu(dz_n)$ is given by

$$p_n(x, z_1, ..., z_n) = \prod_{k=1}^n q_G(x_{T_k-}(x, z_1, ..., z_{k-1}), z_k). \tag{48}$$

Finally we define

$$\overline{X}_t^{G,\Gamma}(x) = x_t(x, \overline{Z}^{k-1}), \quad T_{k-1} \le t < T_k \tag{49}$$

and we notice that, according to (45)

$$\begin{aligned} \overline{X}_{T_k}^{G,\Gamma}(x) &= \overline{X}_{T_k-}^{G,\Gamma}(x) + c(\overline{Z}_k, \overline{X}_{T_k-}^{G,\Gamma}(x))1_G(\overline{Z}_k) \\ \overline{X}_t^{G,\Gamma}(x, z^k) &= \Phi_{T_k,t}(\overline{X}_{T_k}^{G,\Gamma}(x)) \qquad T_k \le t < T_{k+1}. \end{aligned} \tag{50}$$

Remark 9 *In mathematical physics the above equation are known as "transport equations" and the equation (43) is called the "fictive chock" representation and the recurrence relation (50) is the "real chock" representation: see [LPS98] pg 49. The above book gives a complete view of the numerical methods used in the Monte Carlo approach to such equations as well as several possible applications.*

Lemma 10 *The law of $X_t^{G,\Gamma}(x)$ coincides with the law of $\overline{X}_t^{G,\Gamma}(x)$. Moreover, for any non negative and measurable function Ψ the law of $S_t = \sum_{k=1}^{J_t} \Psi(Z_k)1_{\{U_k \le \gamma_\Gamma(Z_k, X_k^{G,\Gamma})\}}$ coincides with the law of $\overline{S}_t = \sum_{k=1}^{J_t} \Psi(\overline{Z}_k)$.*

Proof. We have

$$\begin{aligned} E(f(X_{T_j}^{G,\Gamma}) \quad &| \quad X_{T_j-}^{G,\Gamma} = x) \\ &= \quad E(f(x + c(Z_j, x)1_G(Z_j))1_{\{U_j \le \gamma_\Gamma(Z_j,x)\}}) + E(f(x)1_{\{U_j > \gamma_\Gamma(Z_j,x)\}}) \\ &= \quad : I + J. \end{aligned}$$

A simple computation shows that $P(U_j > \gamma(Z_j,x)) = \theta_{G,\Gamma}(x)$ and moreover

$$\begin{aligned} I &= \frac{1}{\Gamma}\int_E\int_0^\Gamma f(x+c(z,x)1_G(z))1_{\{u\leq \gamma_\Gamma(z,x)\}}\frac{1}{\mu(G)}du\mu(dz) \\ &= \int_E f(x+c(z,x)1_G(z))\gamma_\Gamma(z,x)\frac{1}{\Gamma\mu(G)}\mu(dz) \end{aligned}$$

so that

$$\begin{aligned} E(f(X^{G,\Gamma}_{T_j}) &\mid X^{G,\Gamma}_{T_j-}=x) = \int_E f(x+c(z,x)1_G(z))\gamma_\Gamma(z,x)\frac{1}{\Gamma\mu(G)}\mu(dz) \\ +\theta_{G,\Gamma}(x)f(x) &= \int_E f(x+c(z,x)1_G(z))q_{G,\Gamma}(z,x)\mu(dz) = E(f(\overline{X}^{G,\Gamma}_{T_j}) \mid \overline{X}^{G,\Gamma}_{T_j-}=x). \end{aligned}$$

We conclude that the laws of $X^{G,\Gamma}_t$ coincides with the law of $\overline{X}^{G,\Gamma}_t$. In order to check that the law of S_t and of $\overline{S}_t$ are the same, we just use the previous result for the couple $(X^{G,\Gamma}_t, S_t)$ and $(\overline{X}^{G,\Gamma}_t, \overline{S}_t)$. □

The process $\overline{X}^{G,\Gamma}_t(x)$ satisfy the equation:

$$\begin{aligned} \overline{X}^{G,\Gamma}_t(x) &= x+\sum_{l=1}^m\int_0^t \sigma_l(\overline{X}^{G,\Gamma}_s(x))dW^l_s + \int_0^t b(\overline{X}^{G,\Gamma}_s(x))ds \\ &\quad +\sum_{k=1}^{J_t} c(\overline{Z}_k, \overline{X}^{G,\Gamma}_{T_k-}(x))1_G(\overline{Z}_k). \end{aligned} \tag{51}$$

Since $x\to x_t(x,z^k)$is differentiable, so is $x\to \overline{X}^{G,\Gamma}_t(x)$. Our first aim is to estimate the derivatives of this process.

Proposition 11 ***A***. *For every $q,p\in N$ there exists some constants C (depending on q and p), and l_q (depending on q) such that, for every multi-index α with $|\alpha|=q$*

$$E(\left|\partial^\alpha_x \overline{X}^{G,\Gamma}_t(x)\right|^p) \leq C\theta_{q,pl_q}(G)e^{tC\theta_{q,pl_q}(G)} \tag{52}$$

with $\theta_{q,p}(G)$ defined in (34). One has $l_q\leq 2^q$ and, for $q=1,2,3,$ one has $l_q=q$.

B. *Moreover, with $\overline{\gamma}(G)$ defined in (36),*

$$E(\sum_{k=1}^{J_t} 1_G(\overline{Z}_k)\left|\partial^\alpha_x \overline{X}^{G,\Gamma}_t(x)\right|^p) \leq Ct\overline{\gamma}(G)\times\theta_{q,pl_q}(G)e^{tC\theta_{q,pl_q}(G)}. \tag{53}$$

Proof. We treat the first derivatives. We have (with $e_i=(0,...,0,1,0,...,0)$ with 1 on the $i'th$ position)

$$\begin{aligned} \partial_{x^i}\overline{X}^{G,\Gamma}_t(x) &= e_i+\sum_{l=1}^m\int_0^t \left\langle \nabla\sigma_l(\overline{X}^{G,\Gamma}_s(x)), \partial_{x^i}\overline{X}^{G,\Gamma}_s(x)\right\rangle dW^l_s \\ &\quad +\int_0^t \left\langle \nabla b(\overline{X}^{G,\Gamma}_s(x)), \partial_{x^i}\overline{X}^{G,\Gamma}_s(x)\right\rangle ds \\ &\quad +\sum_{k=1}^{J_t}\left\langle \nabla_x c(\overline{X}^{G,\Gamma}_{T_k-}(x),\overline{Z}_k), \partial_{x^i}\overline{X}^{G,\Gamma}_{T_k-}(x)\right\rangle 1_G(\overline{Z}_k). \end{aligned} \tag{54}$$

Using the identity of laws given in Lemma 10 for the system $(\overline{X}^{G,\Gamma}_t(x), \nabla_x\overline{X}^{G,\Gamma}_t(x))_{t\geq 0}$ we conclude that the law of this process coincides with the law of the process $(X^{G,\Gamma}_t(x), V_{(1),t}(x))_{t\geq 0}$ where $X^{G,\Gamma}_t(x)$

is the solution of the equation (19) and $V^i_{(1),t}\in R^d, i=1,...,d$ solves the equation

$$V_{(1),t}(x) = e_i+\sum_{l=1}^m\int_0^t \left\langle \nabla\sigma_l(X_s^{G,\Gamma}(x)),V_{(1),s}(x)\right\rangle dW_s^l \tag{55}$$
$$+\int_0^t \left\langle \nabla b(X_s^{G,\Gamma}(x)),V_{(1),t}(x)\right\rangle ds$$
$$+\sum_{k=1}^{J_t}\left\langle \nabla_x c(Z_k,X_{T_k-}^{G,\Gamma}(x)),V_{(1),T_k-}(x)\right\rangle 1_G(Z_k)1_{\{U_k\leq\gamma_\Gamma(Z_k,X_{T_k-}^{G,\Gamma}(x))\}}.$$

We will use Proposition 21 from the Appendix (with E replaced by G) in order to estimate the moments of $V_{(1),t}(x)$. In order to identify notations we mention that the index set is now $\Lambda=\{1,...,d\}$ and $\alpha=i$. Moreover $V^i_{(1),0}=e_i$ and $H^i=h^i=Q^i=0$ so, in particular, $q=0$ and $R^i=0$. Let us now identify $\widehat{c}_{(1)}(p)$ which is defined in (87):

$$\widehat{c}_{(1)}(p)=\sup_{x\in R^d}\int_G |\nabla_x c(z,x)|)(1+|\nabla_x c(z,x)|)^{2p-1}\gamma_\Gamma(z,x)d\mu(z)\leq\theta_{1,p}(G).$$

Here the lower index in $\widehat{c}_{(1)}(p)$ indicates that we are dealing with the solutions of (55) which concern the first order derivatives. And we have used the inequality $|\partial_x^\alpha\gamma_\Gamma(z,x)|\leq C\,|\partial_x^\alpha\gamma(z,x)|$.

Then, using the identity of law and (88) we obtain

$$E(\left|\partial_{x^i}\overline{X}_t^{G,\Gamma}(x)\right|^{2p})=E(\left|V^i_{(1),t}(x)\right|^{2p})\leq\exp(tC_p\theta_{1,p}(G))$$

so (52) is proved. And

$$\begin{aligned}E(\sum_{k=1}^{J_t}1_G(\overline{Z}_k)\left|\partial_{x^i}\overline{X}_{T_k-}^{G,\Gamma}(x)\right|^p) &= E(\sum_{k=1}^{J_t}1_G(Z_k)\left|V^i_{(1),T_k-}(x)\right|^p 1_{\{U_k\leq\gamma_\Gamma(Z_k,X_{T_k-}^{G,\Gamma}(x))\}})\\ &= E(\int_0^t\int_G\left|V^i_{(1),T_k-}(x)\right|^p\gamma_\Gamma(z,X_{T_k-}^{G,\Gamma}(x))\mu(dz)\\ &\leq \sup_{x\in R^d}\int_G\gamma(z,x)d\mu(z)\int_0^t E(\left|V^i_{(1),s}(x)\right|^{2p})ds\end{aligned}$$

so (53) is also proved (with $l_1=1$).

We estimate now the second order derivatives. We take derivatives in (54) and we obtain

$$\partial_{x^j}\partial_{x^i}\overline{X}_t^{G,\Gamma}(x)=\sum_{l=1}^m\int_0^t\overline{H}_l^{i,j}(s)dW_s^l+\int_0^t\overline{h}^{i,j}(s)ds+\sum_{k=1}^{J_t}\overline{Q}^{i,j}(T_k-,\overline{Z}_k)1_G(\overline{Z}_k) \tag{56}$$
$$+\sum_{l=1}^m\int_0^t\left\langle\nabla\sigma_l(\overline{X}_s^{G,\Gamma}(x)),\partial_{x^j}\partial_{x^i}\overline{X}_s^{G,\Gamma}(x)\right\rangle dW_s^l$$
$$+\int_0^t\left\langle\nabla b(\overline{X}_s^{G,\Gamma}(x)),\partial_{x^j}\partial_{x^i}\overline{X}_s^{G,\Gamma}(x)\right\rangle ds$$
$$+\sum_{k=1}^{J_t^G}\left\langle\nabla_x c(\overline{Z}_k,\overline{X}_{T_k-}^{G,\Gamma}(x)),\partial_{x^j}\partial_{x^i}\overline{X}_{T_k-}^{G,\Gamma}(x)\right\rangle 1_G(\overline{Z}_k).$$

with

$$\overline{H}_l^{i,j}(s) = \sum_{r,r'=1}^d\partial_r\partial_{r'}\sigma_l(\overline{X}_s^G(x))\partial_{x^i}\overline{X}_s^{G,r}(x)\partial_{x^j}\overline{X}_s^{G,r'}(x),$$
$$\overline{h}^{i,j}(s) = \sum_{r,r'=1}^d\partial_r\partial_{r'}b(\overline{X}_s^G(x))\partial_{x^i}\overline{X}_s^{G,r}(x)\partial_{x^j}\overline{X}_s^{G,r'}(x),$$

and

$$\overline{Q}^{i,j}(s,\overline{Z}_k)=\sum_{r,r'=1}^{d}\partial_{x^r}\partial_{x^{r'}}c(\overline{Z}_k,\overline{X}_s^{G,\Gamma}(x))\partial_{x^i}\overline{X}_s^{G,\Gamma,r}(x)\partial_{x^j}\overline{X}_s^{G,\Gamma,r'}(x).$$

Using the identity of laws given in Lemma 10 for the system $(\overline{X}_t^{G,\Gamma}(x),\nabla_x\overline{X}_t^{G,\Gamma}(x),\nabla_x^2\overline{X}_t^{G,\Gamma}(x))_{t\geq 0}$ we conclude that the law of this process coincides with the law of the process $(X_t^{G,\Gamma}(x),V_{(1),t}(x),V_{(2),t}(x))_{t\geq 0}$ where $X_t^{G,\Gamma}(x)$ is the solution of the equation (19), $V_{(1),t}^i\in R^d, i=1,...,d$ solves the equation (55) and $V_{(2),t}^{i,j}(x)\in R^d, i,j=1,...,d$ solves the following equation:

$$\begin{aligned}V_{(2),t}^{i,j}(x) &= \sum_{l=1}^{m}\int_0^t H_l^{i,j}(s)dW_s^l+\int_0^t h^{i,j}(s)ds \\ &+\sum_{k=1}^{J_t^G}Q^{i,j}(T_k-,Z_k)1_G(Z_k)1_{\{U_k\leq\gamma_\Gamma(Z_k,X_{T_k-}^G(x),)\}} \\ &+\sum_{l=1}^{m}\int_0^t\left\langle\nabla\sigma_l(X_s^{G,\Gamma}(x)),V_{(2),s}^{i,j}\right\rangle dW_s^l+\int_0^t\left\langle\nabla b(X_s^{G,\Gamma}(x)),V_{(2),s}^{i,j}\right\rangle ds \\ &+\sum_{k=1}^{J_t^G}\left\langle\nabla_x c(Z_k,X_{T_k-}^{G,\Gamma}(x)),V_{(2),T_k-}^{i,j}\right\rangle 1_G(Z_k)1_{\{U_k\leq\gamma_\Gamma(Z_k,X_{T_k-}^{G,\Gamma}(x))\}}\end{aligned} \tag{57}$$

with

$$\begin{aligned}H_l^{i,j}(s) &= \sum_{r,r'=1}^{d}\partial_r\partial_{r'}\sigma_l(X_s^{G,\Gamma}(x))(V_{(1),s}^i)^r(x)(V_{(1),s}^j)^{r'}(x), \\ h^{i,j}(s) &= \sum_{r,r'=1}^{d}\partial_r\partial_{r'}b(X_s^{G,\Gamma}(x))(V_{(1),s}^i)^r(x)(V_{(1),s}^j)^{r'}(x),\end{aligned}$$

and

$$Q^{i,j}(s,a_k)=\sum_{r,r'=1}^{d}\partial_{x^r}\partial_{x^{r'}}c(Z_k,X_s^{G,\Gamma}(x))(V_{(1),s}^i)^r(x)(V_{(1),s}^j)^{r'}(x).$$

We will again use Proposition 21 (with $E=G$) in order to estimate the moments of $V_{(2),t}(x)$. Now the index set is $\Lambda=\{(i,j);i,j=1,...,d\}$ and $\alpha=(i,j)$. Moreover $V_{(2),0}^{i,j}=0$ and $H^{i,j},h^{i,j},Q^{i,j}$ are given above. In particular we have $\left|Q^{i,j}(s,Z_k)\right|\leq q(Z_k,X_s^G)R^{i,j}(s)$ with $q(z,x)=\sum_{|\alpha|=2}|\partial_x^\alpha c(z,x)|$ and $R_s^{i,j}=\left|V_{(1),s}\right|^2$. So

$$\begin{aligned}\widehat{c}_{(2)}(p) &= \sup_{x\in R^d}\int_G(\sum_{1\leq|\alpha|\leq 2}|\partial_x^\alpha c(z,x)|)(1+\sum_{1\leq|\alpha|\leq 2}|\partial_x^\alpha c(z,x)|)^{2p-1}\gamma_\Gamma(z,x)d\mu(z) \\ &\leq \theta_{2,p}(G)\end{aligned}$$

Moreover, using the estimates for $V_{(1),t}$ we obtain

$$\begin{aligned}&\int_0^t(E(\sum_{l=1}^{m}|H_l^\alpha(s)|^{2p}+|h_l^\alpha(s)|^{2p}+\widehat{c}_{(2)}(p)\left|R_{s-}^\alpha\right|^{2p})ds) \\ &\leq \int_0^t C_p(\|\sigma\|_{2,\infty}^{2p}+\|b\|_{2,\infty}^{2p}+\theta_{2,p}(G))E(\left|V_{(1),s}\right|^{4p})ds \\ &\leq tC_p(\|\sigma\|_{2,\infty}^{2p}+\|b\|_{2,\infty}^{2p}+\theta_{2,p}(G))\exp(tC_p\theta_{1,2p}(G)).\end{aligned} \tag{58}$$

Then

$$E(\left|\partial_{x^j}\partial_{x^i}\overline{X}_t^{G,\Gamma}(x)\right|^{2p})=E(\left|V_t^{i,j}\right|^{2p})\leq tC_p\theta_{2,p}(G)\exp(tC_p\theta_{2,2p}(G)).$$

So the proof of (52) is finished and then (53) follows as above. Notice that $l_2 = 2$ here.

For the third order derivatives the proof is similar: now the set of multi-indexes is $\Lambda = \{\alpha = (i,j,k) : 1 \leq i,j,k \leq d\}$ and $H^\alpha, h^\alpha, Q^\alpha$ are defined in a similar way. Moreover one has $|H^\alpha| + |h^\alpha| \leq C(\|\sigma\|_{3,\infty} + \|b\|_{3,\infty})(\left|V_{(1)}\right|^3 + \left|V_{(2)}\right|\left|V_{(1)}\right|)$ and $|Q^\alpha| \leq C\overline{c}_{[3]}(\left|V_{(1)}\right|^3 + \left|V_{(2)}\right|\left|V_{(1)}\right|)$. Using Proposition 21, Hölder's inequality and the recurrence hypothesis one obtains (52) with $l_3 = 3$. For higher order derivatives the proof is the same, but it is more difficult to give a precise expression for l_q - this is why we keep the bound $l_q \leq 2^q$ which is clearly sufficient. □

We are now ready to give:

Proof of Theorem 7. Recall (45) and (48). Recall also the notation $z^k = (z_1, ..., z_k)$, and recall that J_t represents the number of jumps up to t. Then we write

$$E(f(\overline{X}_t^{G,\Gamma}(x))) = E(\int f(x_t(x, z^{J_t}))p_{J_t}(x, z^{J_t})\mu(dz_1), ..., \mu(dz_{J_t}))$$

where

$$p_{J_t}(x, z^{J_t}) = \prod_{k=1}^{J_t} q_{G,\Gamma}(x_{T_k-}(x, z^{k-1}), z_k).$$

It follows that

$$\partial_{x_i} E(f(\overline{X}_t^{G,\Gamma}(x))) = A + B$$

with

$$\begin{aligned} A &= \sum_{l=1}^d E(\int \partial_l f(x_t(x, z^{J_t}))\partial_{x_i} x_t^l(x, z^{J_t})p_{J_t}(x, z^{J_t})\mu(dz_1), ..., \mu(dz_{J_t})) \\ &= \sum_{l=1}^d E(\partial_l f(x_t(x, \overline{Z}^{J_t}))\partial_{x_i} x_t^l(x, \overline{Z}^{J_t})) \end{aligned}$$

and

$$\begin{aligned} B &= E(\int f(x_t(x, z^{J_t}))\partial_{x_i} p_{J_t}(x, z^{J_t})\mu(dz_1), ..., \mu(dz_{J_t})) \\ &= E(\int f(x_t(x, z^{J_t}))\partial_{x_i} \ln p_{J_t}(x, z^{J_t}) \times p_{J_t}(x, z^{J_t})\mu(dz_1), ..., \mu(dz_{J_t})) \\ &= E(f(x_t(x, \overline{Z}^{J_t}))\partial_{x_i} \ln p_{J_t}(x, \overline{Z}^{J_t})). \end{aligned}$$

Let us estimate A. Using (52) (with $q = 1, l_q = 1$ and $p = 1$)

$$\begin{aligned} |A| &\leq \|f\|_{1,\infty} \sum_{l=1}^d E(\left|\partial_{x_i} x_t^l(x, \overline{Z}^{J_t})\right|) = \|f\|_{1,\infty} E(\left|\nabla_x \overline{X}_t^{G,\Gamma}(x)\right|) \\ &\leq C \|f\|_{1,\infty} \theta_{1,1}(G)e^{tC\theta_{1,1}(G)}. \end{aligned}$$

Let us estimate B. We have

$$\begin{aligned} \partial_{x_i} \ln p_{J_t}(x, z^{J_t}) &= \sum_{k=1}^{J_t} \psi(z_k)\partial_{x_i} \ln \theta_{G,\Gamma}(x_{T_k-}(x, z^{k-1})) \\ &\quad + \sum_{k=1}^{J_t} 1_G(z_k)\partial_{x_i} \ln \gamma_\Gamma(z_k, x_{T_k-}(x, z^{k-1}) \\ &= : S_1(x, z^{J_t}) + S_2(x, z^{J_t}). \end{aligned}$$

Then

$$|B| \leq \|f\|_\infty (E(\left|S_1(x, \overline{Z}^{J_t})\right|) + E(\left|S_2(x, \overline{Z}^{J_t})\right|)$$

Recall that $\theta_{G,\Gamma}(x) \geq 1 - m_G(\Gamma) > 0$ and $|\nabla_x \gamma_\Gamma(z,x)| \leq C\,|\nabla_x \gamma(z,x)|$. It follows that

$$|\nabla_x \theta_{G,\Gamma}(x)| \leq \sup_{x\in R^d} \frac{C}{\mu(G)\Gamma} \int_G |\nabla_x \gamma(z,x)|\, d\mu(z) = \frac{C}{\Gamma} \times \widetilde{\gamma}_1(G).$$

Then

$$\left|\partial_{x_i} \ln \theta_{G,\Gamma}(x_{T_k-}(x,z^{k-1}))\right| \leq \frac{C}{1-m_G(\Gamma)} \widetilde{\gamma}_1(G) \times \left|\nabla_x x_{T_k-}(x,z^{k-1})\right|$$

and consequently

$$\left|S_1(x,\overline{Z}^{J_t})\right| \leq \frac{C}{1-m_G(\Gamma)} \widetilde{\gamma}_1(G) \sum_{k=1}^{J_t} \psi(\overline{Z}_k) \left|\nabla_x x_{T_k-}(x,\overline{Z}^{k-1})\right|.$$

Using (53) we get

$$\begin{aligned} E(\left|S_1(x,\overline{Z}^{J_t})\right|) &\leq \frac{C}{1-m_G(\Gamma)} \widetilde{\gamma}_1(G) \left\|\psi\right\|_\infty E(\sum_{k=1}^{J_t} \left|\nabla_x \overline{X}^{G,\Gamma}_{T_k-}(x)\right|) \\ &\leq \frac{C}{1-m_G(\Gamma)} t\widetilde{\gamma}_1(G) \left\|\psi\right\|_\infty \times \theta_{1,1}(G) e^{tC\theta_{1,1}(G)}. \end{aligned}$$

We estimate now $S_2(x,\overline{Z}^{J_t})$. If $z \in G$ then $\gamma(z,x) \geq \underline{\gamma}(G)$ for every x so that

$$\begin{aligned} E(\left|S_2(x,\overline{Z}^{J_t})\right|) &\leq \frac{1}{\underline{\gamma}(G)} E(\sum_{k=1}^{J_t} 1_G(\overline{Z}_k) \left|\nabla_x \gamma_{\Gamma(G)}(\overline{Z}^k, \overline{X}^{G,\Gamma}_{T_k-}(x))\right| \times \left|\nabla_x \overline{X}^{G,\Gamma}_{T_k-}(x))\right|) \\ &\leq \frac{C}{\underline{\gamma}(G)} (E(\sum_{k=1}^{J_t} 1_G(\overline{Z}_k) \left|\nabla_x \gamma(\overline{Z}^k, \overline{X}^{G,\Gamma}_{T_k-}(x))\right|^2) \\ &\quad + E(\sum_{k=1}^{J_t} 1_G(\overline{Z}_k) \left|\nabla_x \overline{X}^{G,\Gamma}_{T_k-}(x))\right|^2)). \end{aligned}$$

Using the identity of laws

$$\begin{aligned} E(\sum_{k=1}^{J_t} 1_G(\overline{Z}_k) \left|\nabla_x \gamma_\Gamma(\overline{Z}_k, \overline{X}^G_{T_k-}(x))\right|^2) &= E(\sum_{k=1}^{J_t} 1_G(Z_k) \left|\nabla_x \gamma_\Gamma(Z_k, \overline{X}^G_{T_k-}(x))\right|^2 1_{\{U_k \leq \gamma_\Gamma(Z_k, \overline{X}^G_{T_k-}(x))\}}) \\ &\leq \sup_{x\in R^d} \int_G \left|\nabla_x \gamma_\Gamma(z,x)\right|^2 \gamma_\Gamma(z,x) d\mu(z) = \overline{\gamma}_{1,2}(G) \end{aligned}$$

with $\overline{\gamma}_{1,2}(G)$ defined in (33). And using (53)

$$E(\sum_{k=1}^{J_t} 1_G(\overline{Z}_k) \left|\nabla_x \overline{X}^{G,\Gamma}_{T_k-}(x))\right|^2) \leq Ct\overline{\gamma}(G) \times \theta_{1,2}(G) e^{(t\vee 1)C\theta_{1,2}(G)}.$$

So

$$E(\left|S_2(x,\overline{Z}^{J_t})\right|) \leq \frac{C}{\underline{\gamma}(G)} (t\overline{\gamma}(G) \times \theta_{1,2}(G) e^{tC\theta_{1,2}(G)} + \overline{\gamma}_{1,2}(G)).$$

Collecting all these we obtain

$$\begin{aligned} \left|\partial_{x_i} P^G_t f(x)\right| &= \left|\partial_{x_i} E(f(\overline{X}^G_t(x)))\right| \\ &\leq \frac{C(t\vee 1)}{1-m_G(\Gamma)} \left\|f\right\|_{1,\infty} (\frac{\overline{\gamma}_{1,2}(G)}{\underline{\gamma}(G)} + \theta_{1,2}(G) e^{tC\theta_{1,2}(G)} (1 + \frac{\overline{\gamma}(G)}{\underline{\gamma}(G)} + \widetilde{\gamma}_1(G))) \end{aligned}$$

so (42) is proved in the case $q = 1$.

The proof for higher order derivatives is similar but cumbersome, so we live it out. We just precise that in order to obtain the specific powers in the definition of $\alpha_{q,p}(t,G)$ we used the following standard estimate: if $F(x)=\ln f(g(x))$ and $f\geq C_*>0$, then

$$\left\Vert F\right\Vert_{q,\infty}\leq\frac{C}{C_*^q}(\sum_{j=1}^q\left\Vert f\right\Vert_{j,\infty}^{q-j+1})(\sum_{j=1}^q\left\Vert g\right\Vert_{j,\infty}^{q-j+1}).$$

□

Our aim now is to give a regularity criterion for $x\rightarrow P_tf(x)$. We denote $B_R=\{x:\left\vert x\right\vert<R\}$ and $W^{q,p}(B_R)$ is the standard Sobolev space on B_R.

Theorem 12 *We assume that (11),(12),(13) and (16) hold. Moreover we assume that there exists $\varepsilon>0$ such that for every measurable set $G\subset E$ with $\mu(G)<\infty$*

$$\overline{\lim_{\Gamma\rightarrow\infty}}m_G(\Gamma)<1-\varepsilon. \tag{59}$$

where $m_G(\Gamma)$ is given in (41). Let $m\in N_*$ and $q\in N$ be fixed. Suppose that there exists a sequence $E_n\uparrow E$ such that $\mu(E_n)<\infty$ and such that, for some $\eta>\frac{q+1}{m}$, one has

$$\sup_n\alpha_{2m+q,(2m+q)l_{2m+q}}^{\eta}(t,E_n)\times\lambda(E_n^c)<\infty. \tag{60}$$

Then, for every $f\in C_b^{2m+q}(R^d)$ one has $P_tf\in W^{q,p}(B_R)$ for every $p\geq1$ and $R>0$.

Remark 13 *In [Rab15], Rabiet proved that under an uniform ellipticity condition (given in terms of $\gamma\nabla_zc$) one has $P_tf(x)=\int p_t(x,y)f(y)dy$ with $(x,y)\rightarrow p_t(x,y)$ differentiable. A similar result has been obtained before by Bally and Caramellino [BC14] in the particular case $\sigma=0$ (so there is no Brownian part). In contrast, here we assume no ellipticity condition and we study the propagation of regularity only. Notice that there is a significant loss of regularity between the initial condition f and P_tf. This seems rather unusual because, at list under some non-degeneracy conditions, the semigroup has a regularization effect. But here there is no such non-degeneracy condition and this is the only thing that we can prove in this framework (we do not pretend that our result is optimal). We recall that $\alpha_{2m+q,l_{2m+q}}(E_n)$ controls the regularity of $P_t^{E_n}f$ but may blow up as $n\rightarrow\infty$.*

Proof. We will use Theorem 2.3 from [BC14] that we recall here. For a function $\phi:R^d\rightarrow R$ we denote

$$\left\Vert\phi\right\Vert_{2m+q,2m,p}=\sum_{0\leq\left\vert\alpha\right\vert\leq2m+q}(\int_{R^d}(1+\left\vert x\right\vert)^{2m}\left\vert\partial_\alpha f(x)\right\vert^p dx)^{1/p}.$$

Moreover, for $\phi,\psi:R^d\rightarrow R_+$ we consider the Forté Mourier distance

$$d_1(\phi,\psi)=\sup\{\left\vert\int_{R^d}f(x)\phi(x)dx-\int_{R^d}f(x)\psi(x)dx\right\vert:\left\Vert f\right\Vert_\infty+\left\Vert\nabla f\right\Vert_\infty\leq1\}.$$

Then Theorem 2.3 in [BC14] asserts the following: let $q\in N,m\in N_*$ and $p>1$ be given. Suppose that one may find a sequence of functions $\phi_n:R^d\rightarrow R$ such that (with p_* the conjugate of p)

$$\sup_n\left\Vert\phi_n\right\Vert_{2m+q,2m,p}^{\eta}d_1(\phi_n,\phi)<\infty,\quad with\quad\eta>\frac{q+1+d/p_*}{m} \tag{61}$$

Then $\phi\in W^{q,p}(R^d)$.

Now, for each fixed n we choose Γ_n such that $\beta(\Gamma_n)\leq\lambda(E_n^c)$ and $m_{E_n}(\Gamma_n)<1-\varepsilon$ (this is possible by (59)). We will use Theorem 2.3 in [BC14] with

$$\phi(x)=1_{B_R}(x)P_tf(x),\quad and\quad\phi_n(x)=1_{B_R}(x)P_t^{E_n,\Gamma_n}f(x).$$

By (20)

$$d_1(\phi,\phi_n)\leq C(\beta(\Gamma_n)+\lambda(E_n^c))\leq C\lambda(E_n^c)$$

with C a constant which depends on R and t but not on n. Moreover, by (42)

$$\left\| P_t^{E_n,\Gamma_n} f \right\|_{q+2m,\infty} \leq C_q \varepsilon^{-1} \alpha_{q+2m,(q+2m)l_{q+2m}}(t, E_n) t \left\| f \right\|_{q+2m,\infty}$$

and consequently, for every $p \geq 1$

$$\left\| \phi_n \right\|_{q+2m,2m,p} \leq C_{q,p} \varepsilon^{-1} \alpha_{q+2m,(q+2m)l_{q+2m}}(t, E_n) t \left\| f \right\|_{q+2m,\infty}.$$

Now (60) guarantees that (61) is verified and so the conclusion follows. □

5 PDMP's with three regimes: the convergence result

5.1 Main result

In this section we construct a sequence of $PDMP's$ which converge weakly to the solution of our equation (9) which we recall here:

$$\begin{aligned} X_t &= x + \sum_{l=1}^{m} \int_0^t \sigma_l(X_s) dW_s^l + \int_0^t b(X_s) + g(X_s) ds \\ &\quad + \int_0^{t+} \int_{E \times [0,1]} c(z, X_{s-}) 1_{\{u \leq \gamma(z, X_{s-})\}} N_\mu(ds, dz, du). \end{aligned} \tag{62}$$

Notice that instead of the drift coefficient b in (9), here we have $b + g$. This is because b and g appear as a limit of different components.

In order to obtain this convergence result we need an hypothesis on the coefficients which is stronger then the one in Section 3 : we assume

$$C_* := 1 + \sum_{l=1}^{m} \left\| \sigma_l \right\|_{1,\infty} + \left\| b \right\|_{1,\infty} + \left\| g \right\|_{1,\infty} + \sup_{x \in R^d} \int_E (|c| \left| \nabla_x \gamma \right| + \left| \nabla_x c \right| \gamma)(z, x) d\mu < \infty. \tag{63}$$

We construct now the approximation $PDMP's$. We consider two sequences of non negative and finite measures $\nu_n, \eta_n, n \in N$ on E, and a sequence of sets $E_n \uparrow E$ and we denote $\mu_n(dz) = 1_{E_n}(z) d\mu(z)$ where μ is the one which appears in the equation (62). Moreover we consider a sequence of coefficients $b_n : R^d \to R^d, c_n, d_n, e_n : E \times R^d \to R^d$ and $\gamma_n, \xi_n, \beta_n : E \times R^d \to [0, \infty)$ and we denote

$$\begin{aligned} I_{c,\gamma}^n &= 1_{E_n}(\left| \nabla_x \gamma_n \right| |c_n| + (\left| \nabla_x c_n \right| + |c_n| + |c_n|^2) \gamma_n), \\ J_{d,\xi}^n &= \left| \nabla_x \xi_n \right| |d_n|^2 + (\left| \nabla_x d_n \right|^2 + |d_n|^2) \xi_n, \\ K_{e,\beta}^n &= \left| \nabla_x \beta_n \right| |e_n| + (\left| \nabla_x e_n \right| + |e_n| + |e_n|^2) \beta_n. \end{aligned} \tag{64}$$

Then we assume

$$C_n := \left\| b_n \right\|_{1,\infty} + \sup_{x \in R^d} \int_E I_{c,\gamma}^n(z, x) d\mu(z) + \sup_{x \in R^d} \int_E J_{d,\xi}^n(z, x) d\nu_n(z) + \sup_{x \in R^d} \int_E K_{e,\beta}^n(z, x) d\eta_n(z) < \infty. \tag{65}$$

And we associate the equations

$$\begin{aligned} X_t^n &= x + \int_0^t b_n(X_s^n) ds \\ &\quad + \int_0^t \int_{E \times [0,\infty)} d_n(z, X_{s-}^n) 1_{\{u \leq \xi_n(z, X_{s-}^n)\}} \widetilde{N}_{\nu_n}(ds, dz, du) \\ &\quad + \int_0^t \int_{E \times [0,\infty)]} e_n(z, X_{s-}^n,) 1_{\{u \leq \beta_n(z, X_{s-}^n)\}} N_{\eta_n}(ds, dz, du) \\ &\quad + \int_0^t \int_{E \times [0,\infty)} c_n(z, X_{s-}^n,) 1_{\{u \leq \gamma_n(z, X_{s-}^n)\}} N_{\mu_n}(ds, dz, du). \end{aligned} \tag{66}$$

We recall the notation: N_μ is a Poisson point measure on $E\times[0,\infty)$ with compensator $\widehat{N}_\mu(dt,dz,du) = dt\times\mu(dz)\times du$ and $\widetilde{N}_\mu = N_\mu - \widehat{N}_\mu$. We assume that the random measures N_{ν_n}, N_{η_n} and N_{μ_n} are independent. We also assume that

$$\sup_{x\in R^d}\int_E (|\nabla_x\xi_n| + |\xi_n|)(|d_n| + |\nabla_x d_n|)(z,x)d\nu_n(z) < \infty. \tag{67}$$

In particular this means that the integral the with respect to $\widetilde{N}_{\nu_n} = N_{\nu_n} - \widehat{N}_{\nu_n}$ may be splitted. This, together with the assumption $C_n < \infty$ guarantees that the hypothesis (11),(12),(13) and (16) are verified so, for each fixed n, the equation (66) has a unique solution.

We give now the hypothesis which guarantees the weak convergence of X_t^n to X_t. We denote

$$\begin{aligned} a^{i,j}(x) &= \sum_{l=1}^m \sigma_l^i\sigma_l^j(x) \quad and \\ a_n^{i,j}(x) &= \int_E d_n^i(z,x)d_n^j(z,x)\xi_n(z,x)d\nu_n(z), \\ g_n^i(x) &= \int_E e_n^i(z,x)\beta_n(z,x)d\eta_n(z) \end{aligned} \tag{68}$$

and we define

$$\begin{aligned} \varepsilon_0(n) &= \sup_{x\in R^d}\int_E |d_n(z,x)|^3\,\xi_n(z,x)d\nu_n(z) + \sup_{x\in R^d}\int_E |e_n(z,x)|^2\,\beta_n(z,x)d\eta_n(z), \\ \varepsilon_1(n) &= \|a-a_n\|_\infty + \|b-b_n\|_\infty + \|g-g_n\|_\infty \\ \varepsilon_2(n) &= \sup_{x\in R^d}\int_{E_n} (|c|\,(|\gamma-\gamma_n| + \gamma(1+|c|)\,|c-c_n|)(z,x)d\mu(z). \end{aligned} \tag{69}$$

We recall that in (38) we associated to the coefficients σ, b, c, g, γ of the equation (62) the quantity $\alpha_{q,p}(t,G)$ (for a set $G\subset E$ with $\mu(G)<\infty$). We also recall the notation (see (14))

$$\lambda(G) = \sup_{x\in R^d}\int_G |c(z,x)|\,\gamma(z,x)d\mu(z)$$

Then, for every fixed $n\in N$ we construct

$$\varepsilon_*(n) = \inf_{E_n\subset G\subset E}(\lambda(G^c) + \alpha_{3,9}(G)(C_*^2 + C_n^2)(\lambda(G\backslash E_n) + \sum_{i=0}^2\varepsilon_i(n)) \tag{70}$$

with the infimum taken on the sets G with $\mu(G)<\infty$.

Finally we will assume that

$$\sigma\in C_b^4(R^d),\quad b\in C_b^3(R^d),\quad c(\circ,z)\in C_b^3(R^d),\quad \gamma(\circ,z)\in C_b^3(R^d) \tag{71}$$

and

$$\begin{aligned} c_n(\circ,z) &\in C_b^1(R^d),\quad d_n(\circ,z)\in C_b^1(R^d),\quad e_n(\circ,z)\in C_b^1(R^d),\quad \gamma_n(\circ,z)\in C_b^1(R^d) \\ \xi_n(\circ,z) &\in C_b^1(R^d),\quad \beta_n(\circ,z)\in C_b^1(R^d) \end{aligned} \tag{72}$$

Remark 14 *In the case $\alpha_{3,9}(E)<\infty$ one takes $G=E$ and obtains*

$$\varepsilon_*(n) \le \alpha_{3,9}(E)(C_*^2 + C_n^2)\sum_{i=0}^2\varepsilon_i(n).$$

But in the case when $\alpha_{3,9}(E)=\infty$ (and this is the interesting situation) we have to find an equilibrium between $\lambda(G^c)$ (which is small) and $\alpha_{3,9}(G)$ (which is large). This is the idea behind the construction of $\varepsilon_*(n)$. See Example 1.

We are now able to give our main result:

Theorem 15 *We assume that (11), (12), (13),(16),(71),(72) and (67) hold. We also assume that, for every measurable set $G \subset E$ with $\mu(G) < \infty$,*

$$\lim_{\Gamma \to \infty} m_G(\Gamma) = \lim_{\Gamma \to \infty} \frac{1}{\mu(G)} \sup_{x \in R^d} \int_G 1 \wedge \frac{\gamma(z,x)}{\Gamma} d\mu(z) < 1.$$

***A**. There exists an universal constant C such that for every $n \in N$ and every $f \in C_b^3(R^d)$*

$$\|P_t f - P_t^n f\|_\infty \leq Ct \|f\|_{3,\infty} \varepsilon_*(n) \tag{73}$$

where $P_t f(x) = E(f(X_t(x)))$ and $P_t^n f(x) = E(f(X_t^n(x)))$.

***B**. Moreover, if $\lim_{n\to\infty} \varepsilon_*(n) = 0$, then, for every $x \in R^d$ and every $t > 0$, $X_t^n(x)$ converges in law to $X_t(x)$.*

Remark 16 *Notice that if $\alpha_{3,9}(t,G) = \infty$ for every $E_n \subset G \subset E$ then $\varepsilon_*(n) = \infty$ so (73) says nothing.*

Remark 17 *Notice that the estimate (73) is not asymptotic. This in contrast with the assertion **B.** In **B** $P_t^n f$ appears as an approximation of $P_t f$. But in **A** we may think in the converse way: we consider X_t as an approximation of X_t^n obtained by replacing "small jumps" (the one corresponding to ν_n) by the Brownian motion W. This is the point of view in numeric applications (see [AR01])*

5.2 Proof

Before giving the proof of the above theorem we need some preliminary lemmas. We denote

$$\begin{aligned} L_n f(x) &= \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x) a_n^{i,j}(x) + \sum_{i=1}^d \partial_i f(x)(b_n^i(x) + g_n^i(x)) \\ &\quad + \int_{E_n} (f(x + c_n(z,x)) - f(x))\gamma_n(z,x) d\mu(z) \end{aligned} \tag{74}$$

Lemma 18 *There exists an universal constant C such that for every $t > 0$ and every $f \in C_b^3(R^d)$*

$$\|P_t^n f(x) - f(x) - tL_n f\|_\infty \leq CC_n^2 \|f\|_{3,\infty} (t^{1/2} + \varepsilon_0(n)) \times t \tag{75}$$

The proof is rather long and technical so we live it for the appendix.

We fix $\Gamma > 1$ and $G \subset E$ with $\mu(G) < \infty$ and recall that $P_t^{G,\Gamma} f(x) = E(f(X_t^{G,\Gamma}(x))$ where $X_t^{G,\Gamma}(x)$ is the solution of the truncated equation (19). We define

$$\begin{aligned} L^{G,\Gamma} f(x) &= \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x) a^{i,j}(x) + \sum_{i=1}^d \partial_i f(x)(b^i(x) + g^i(x)) \\ &\quad + \int_G (f(x + c(z,x)) - f(x))\gamma_\Gamma(z,x) d\mu(z). \end{aligned}$$

Lemma 19 ***A**. For every $f \in C_b^3(R^d)$*

$$\left\| P_t^{G,\Gamma} f - f - tL^{G,\Gamma} f \right\|_\infty \leq CC_*^2 t^{3/2} \|f\|_{3,\infty} \tag{76}$$

***B**. We also have*

$$\left\| P_t f - P_t^{G,\Gamma} f \right\|_\infty \leq Ct(\beta(\Gamma) + \lambda(G^c)) \|f\|_{1,\infty}. \tag{77}$$

Proof. The proof of **A** is analogues to the proof of (73) so we skip it. And (77) is an immediate consequence of (20). $\square$

Proof of Theorem 15:

Step 1. We fix $n \in N$, a set G with $\mu(G) < \infty$ such that $E_n \subset G$, and $\Gamma > 1$ such that $m_G(\Gamma) < 1$. It is easy to check that

$$\left\| L^{G,\Gamma} f - L_n f \right\|_\infty \leq C \left\| f \right\|_{2,\infty} (\beta(\Gamma) + \lambda(G \backslash E_n) + \sum_{i=0}^{2} \varepsilon_i(n)).$$

This, together with the previous two lemmas give (for every every $\delta > 0$)

$$\left| P_\delta^{G,\Gamma} f(x) - P_\delta^n f(x) \right| \leq C(C_*^2 + C_n^2)\delta(\delta^{1/2} + \beta(\Gamma) + \lambda(G \backslash E_n) + \sum_{i=0}^{2} \varepsilon_i(n)) \left\| f \right\|_{3,\infty} \tag{78}$$

Step 2. Using (77)

$$\left\| P_t f - P_t^n f \right\|_\infty \leq \left\| P_t^{G,\Gamma} f - P_t^n f \right\|_\infty + Ct \left\| f \right\|_{1,\infty} (\beta(\Gamma) + \lambda(G^c).$$

Step 3. Let $\delta > 0, t_k = k\delta$ and $\Delta_\delta f(x) = P_\delta^n f(x) - P_\delta^G f(x)$. We write

$$\left\| P_t^n f(x) - P_t^G f \right\|_\infty \leq \sum_{k \leq t/\delta} \left\| P_{t-t_{k+1}}^n \Delta_\delta P_{t_k}^G f \right\|_\infty \leq \sum_{k \leq t/\delta} \left\| \Delta_\delta P_{t_k}^G f \right\|_\infty .$$

By (78) first and by (42) then

$$\begin{aligned} \left\| \Delta_\delta P_{t_k}^G f \right\|_\infty &\leq C \left\| P_{t_k}^G f \right\|_{3,\infty} (C_*^2 + C_n^2)(\delta^{1/2} + \beta(\Gamma) + \lambda(G \backslash E_n) + \sum_{i=0}^{2} \varepsilon_i(n))\delta \\ &\leq C\alpha_{3,9}(G) \left\| f \right\|_{3,\infty} (C_*^2 + C_n^2)(\delta^{1/2} + \beta(\Gamma) + \lambda(G \backslash E_n) + \sum_{i=0}^{2} \varepsilon_i(n))\delta. \end{aligned}$$

Summing over $k = 1, ..., t/\delta$ we obtain

$$\begin{aligned} \left\| P_t^G f - P_t^n f \right\|_\infty &\leq C\alpha_{3,9}(G) \left\| f \right\|_{3,\infty} (C_*^2 + C_n^2)(\delta^{1/2} + \beta(\Gamma) + \lambda(G \backslash E_n) + \sum_{i=0}^{2} \varepsilon_i(n)) \\ &= C\alpha_{3,9}(G) \left\| f \right\|_{3,\infty} (C_*^2 + C_n^2)(\beta(\Gamma) + \lambda(G \backslash E_n) + \sum_{i=0}^{2} \varepsilon_i(n)) \end{aligned}$$

the last inequality being obtained by taking $\delta^{1/2} = \beta(\Gamma) + \lambda(G \backslash E_n) + \sum_{i=0}^{2} \varepsilon_i(n)$. We conclude that

$$\begin{aligned} \left\| P_t f - P_t^n f \right\|_\infty &\leq C(\lambda(G^c) + \beta(\Gamma)) \\ &\quad + C\alpha_{3,9}(G) \left\| f \right\|_{3,\infty} (C_*^2 + C_n^2)(\lambda(G \backslash E_n) + \beta(\Gamma) + \sum_{i=0}^{2} \varepsilon_i(n)). \end{aligned} \tag{79}$$

This estimate holds for every $E_n \subset G \subset E$, with $\Gamma > 1$ chosen such that $m_G(\Gamma) < 1$ (so Γ depends on G).

Suppose now that $\varepsilon_*(n) < \varepsilon$. Then we may choose a set G_ε such that

$$\lambda(G_\varepsilon^c) + \alpha_{3,9}(G_\varepsilon)(C_*^2 + C_n^2)(\lambda(G_\varepsilon \backslash E_n) + \sum_{i=0}^{2} \varepsilon_i(n)) \leq \varepsilon.$$

Since $\lim_{\Gamma \to \infty} m_{G_\varepsilon}(\Gamma) < 1$ we may chose Γ_ε such that $m_{G_\varepsilon}(\Gamma) < 1$ for $\Gamma \geq \Gamma_\varepsilon$. And, since

$$\lim_{\Gamma \to \infty} \beta(\Gamma) = 0,$$

we pass to the limit with $\Gamma \to \infty$ in (79) (with $G = G_\varepsilon$) and we obtain

$$\begin{aligned}\|P_t f - P_t^n f\|_\infty &\leq C(\lambda(G_\varepsilon^c) + C\alpha_{3,9}(G_\varepsilon)\|f\|_{3,\infty}(C_*^2 + C_n^2)(\lambda(G_\varepsilon \backslash E_n) + \sum_{i=0}^{2}\varepsilon_i(n)) \\ &\leq C\varepsilon.\end{aligned}$$

Then we pass to the limit with $\varepsilon \downarrow \varepsilon_*(n)$ and we conclude.

It is easy to check that the sequence $X_t^n(x), n \in N$ is tight and so, if $\lim_n \varepsilon_*(n) = 0$ the convergence in law follows. □

5.3 Example

We give here the simplest possible example which illustrates the convergence result from the previous section. We consider some $C_b^3(R)$ functions $f, e, c : R \to R$ and $\xi, \beta, \gamma : R \to [0, \frac{1}{2}]$ and we denote

$$Q = \|f\|_{3,\infty} + \|e\|_{3,\infty} + \|c\|_{3,\infty} + \|\xi\|_{3,\infty} + \|\beta\|_{3,\infty} + \|\gamma\|_{3,\infty} < \infty. \tag{80}$$

We also assume that

$$\gamma(x) \geq \underline{\gamma} > 0. \tag{81}$$

We define

$$\begin{aligned}h_n(z,x) &= z^{1/2} f(x) 1_{[\frac{1}{n},\frac{2}{n})}(z) + z e(x) 1_{[\frac{2}{n},\frac{3}{n})}(z) + z^{3/2} c(x) 1_{[\frac{3}{n},1)}(z), \\ \gamma_n(z,x) &= \xi(x) 1_{[\frac{1}{n},\frac{2}{n})}(z) + \beta(x) 1_{[\frac{2}{n},\frac{3}{n})}(z) + \gamma(x) 1_{[\frac{3}{n},1)}(z).\end{aligned}$$

We also take the measure $\mu(dz) = \frac{1}{z^2} 1_{(0,1)}(z)dz$ and we associate the equations

$$\begin{aligned}X_t^n &= x + \int_0^{t+}\int_0^1\int_0^1 h_n(z, X_{s-}^n) 1_{\{u \leq \gamma_n(Z, X_{s-}^n)\}} N_\mu(ds, dz, du) \\ &\quad - \sqrt{2n}(\sqrt{2} - 1)\int_0^t (f \times \xi)(X_s^n) ds\end{aligned} \tag{82}$$

and

$$\begin{aligned}X_t &= x + \ln 2 \int_0^t (f \times \sqrt{\xi})(X_s) dW_s + \ln(\frac{3}{2}) \int_0^t (e \times \beta)(X_s) ds \\ &\quad + \int_0^{t+}\int_0^\infty\int_0^1 z^{3/2} c(X_{s-}) 1_{\{u \leq \gamma(X_{s-})\}} N_\mu(ds, dz, du)\end{aligned} \tag{83}$$

Proposition 20 *Suppose that (80) and (81) hold. There exists an universal constant C such that for every $f \in C_b^3(R)$ and $t > 0$*

$$|E(f(X_t)) - E(f(X_t^n))| \leq \frac{C}{\underline{\gamma}^3}(t \vee 1) Q^{10} e^{CtQ^3} \times \|f\|_{3,\infty} \times \frac{1}{n^{1/14}} \tag{84}$$

Proof. The equation (82) is the equation (66) with

$$\begin{aligned}d_n(z,x) &= z^{1/2} f(x) 1_{[\frac{1}{n},\frac{2}{n})}(z) \quad e_n(z,x) = z e(x) 1_{[\frac{2}{n},\frac{3}{n})}(z) \quad c_n(z,x) = z^{3/2} c(x) 1_{[\frac{3}{n},1)}(z), \\ \xi_n(z,x) &= \xi(x) 1_{[\frac{1}{n},\frac{2}{n})}(z) \quad \beta_n(z,x) = \beta(x) 1_{[\frac{2}{n},\frac{3}{n})}(z) \quad \gamma_n(z,x) = \gamma(x) 1_{[\frac{3}{n},\infty)}(z)\end{aligned}$$

and with measures $\nu_n(dz) = 1_{[\frac{1}{n},\frac{2}{n})}(z)\mu(dz), \eta_n(dz) = 1_{[\frac{2}{n},\frac{3}{n})}(z)\mu(dz)$ and $E_n = [\frac{3}{n}, 1)$. And, with the notation from (68) we have $a(x) = a_n(x) = \ln 2 \times f^2(x)\xi(x), b(x) = b_n(x) = 0, c(z,x) = c_n(z,x) = z^{3/2}c(x)$ and $g(x) = g_n(x) = \ln(\frac{3}{2}) \times e(x)\beta(x)$. Moreover, $\gamma_n(z,x) = \gamma(z,x) = \gamma(x)$. So, with the notation from (69), we have $\varepsilon_1(n) = \varepsilon_2(n) = 0$ and $\varepsilon_0(n) \leq CQ^4 n^{-1/2}$. We also have (see (63) and (65)) $C_* \leq CQ^2$ and $C_n \leq CQ^3$.

Finally, for $\varepsilon < \frac{3}{n}$ we take $G_\varepsilon = (\varepsilon, 1]$ and we estimate $\alpha_{3,3}(t, G_\varepsilon)$ defined in (38) with respect to the coefficients of the equation (83). So we will use the notation from (33), (34), (35), (36) and (37). Notice first that for every $p \geq 1$ and $q \leq 3$ one has $\overline{c}_{q,p}(G_\varepsilon) \leq CQ^3$ so that $\theta_{3,3}(G_\varepsilon) \leq CQ^3$. We also have

$$\widetilde{\gamma}_j(G_\varepsilon) \leq CQ^j \times \int_\varepsilon^1 \frac{dz}{z^2} \leq \frac{CQ^j}{\varepsilon}$$
$$\overline{\gamma}(G_\varepsilon) \leq CQ \times \int_\varepsilon^1 \frac{dz}{z^2} \leq \frac{CQ}{\varepsilon}$$

and in the same way $\overline{\gamma}(G_\varepsilon) \leq CQ\varepsilon^{-1}$ and $\overline{\gamma}_{3,6}(G_\varepsilon) \leq CQ\varepsilon^{-1}$. We conclude (see (38)) that

$$\alpha_{3,3}(t, G_\varepsilon) \leq \frac{C}{\underline{\gamma}^3}(t \vee 1) Q^6 e^{CtQ^3} \times \varepsilon^{-3}.$$

We also have (see (14))

$$\begin{aligned}
\lambda_{\max}(G_\varepsilon^c) &\leq Q \int_0^\varepsilon z^{3/2} \times \frac{dz}{z^2} = CQ\varepsilon^{1/2}, \\
\lambda_{\max}(G_\varepsilon \backslash E_n) &\leq Q \int_\varepsilon^{3/n} z^{3/2} \times \frac{dz}{z^2} = CQn^{-1/2}
\end{aligned}$$

So, with the notation from (70),

$$\begin{aligned}
\varepsilon_*(n) &= \inf_{E_n \subset G_\varepsilon \subset E} (\lambda_{\max}(G^c) + \alpha_{3,3}(G)(C_*^2 + C_n^2)(\lambda_{\max}(G \backslash E_n) + \sum_{i=0}^2 \varepsilon_i(n)) \\
&\leq \frac{C}{\underline{\gamma}^3}(t \vee 1) Q^{10} e^{CtQ^3} \times \inf_{0 < \varepsilon < 3/n} (\varepsilon^{1/2} + \frac{1}{\varepsilon^3} \times n^{1/2}).
\end{aligned}$$

Then we take $\varepsilon = n^{-1/7}$ and we obtain

$$\varepsilon_*(n) \leq \frac{C}{\underline{\gamma}^3}(t \vee 1) Q^{10} e^{CtQ^3} \times \frac{1}{n^{1/14}}.$$

Now (73) yields (84). $\square$

6 Appendix: Moments estimates

We assume in this section that $\mu(E) < \infty$. This is just to simplify notation - in concrete applications we will replace μ by $1_G \mu$ with $\mu(G) < \infty$. Then we consider an indexes set Λ and we denote by α the elements of Λ. Moreover we consider a family of processes $V_t^\alpha \in R^d, \alpha \in \Lambda$ such that

$$\sup_{t \leq T} E(|V_t^\alpha|^{2p}) < \infty \quad \forall p \in N, \forall T > 0$$

and which verify the following equation

$$\begin{aligned}
V_t^\alpha &= V_0^\alpha + \sum_{l=1}^m \int_0^t (H_l^\alpha(s) + \langle \nabla \sigma_l(X_s), V_s^\alpha \rangle) dW_s^l \qquad (85) \\
&+ \int_0^t (h_l^\alpha(s) + \langle \nabla b(X_s), V_s^\alpha \rangle) ds \\
&+ \int_0^t \int_{E \times (0,\Gamma)} (Q^\alpha(s-, z) + \left\langle \nabla_x c(z, X_{s-}), V_{s-}^\alpha \right\rangle) 1_{\{u \leq \gamma(a, X_{s-})\}} dN_\mu(s, u, z).
\end{aligned}$$

Here X_y is the solution of the equation (9) and $H_l^\alpha, h_l^\alpha$and Q^α are previsible processes which verify

$$E(\int_0^T (|H_l^\alpha(s)|^2 + |h_l^\alpha(s)| + \sup_{x \in R^d} \int_E |Q^\alpha(s, z)| \, \gamma(z, x) d\mu(z)) ds) < \infty.$$

So the corresponding stochastic integrals in (85) make sense.

Proposition 21 *We suppose that*

$$|Q^\alpha(s,z)| \leq q(z,X_s)\,|R^\alpha_s| \tag{86}$$

for some previsible processes R^α and some measurable function $q : E\times R^d \to R_+$ and we denote

$$\widehat{c}(p) = \sup_{x\in R^d}\int_E (q(z,x)+|\nabla_x c(z,x)|)(1+q(z,x)+|\nabla_x c(z,x)|)^{2p-1}\gamma(z,x)d\mu(z). \tag{87}$$

For every $p\in N$ and $0\leq t\leq T$ there exists an universal constant C_p such that

$$\begin{aligned} E(|V^\alpha_t|^{2p}) &\leq \exp(C_p t(1+\|\nabla\sigma\|^{2p}_\infty+\|\nabla b\|^{2p}_\infty+\widehat{c}(p))) \\ &\quad \times(|V^\alpha_0|^{2p}+C_p\int_0^t(E(\sum_{l=1}^m|H^\alpha_l(s)|^{2p}+|h^\alpha(s)|^{2p}+\widehat{c}(p)\left|R^\alpha_{s-}\right|^{2p})ds). \end{aligned} \tag{88}$$

Proof. Using Itô's formula for $f(x)=x^{2p}$ we obtain

$$|V^\alpha_t|^{2p} = |V^\alpha_0|^{2p}+M^\alpha_t+I^\alpha_t+J^\alpha_t$$

with

$$M^\alpha_t = \sum_{l=1}^m\int_0^t 2p(V^\alpha_s)^{2p-1}(H^\alpha_l(s)+\langle\nabla\sigma_l(X_s),V^\alpha_s\rangle)dW^l_s,$$

$$\begin{aligned} I^\alpha_t &= \sum_{l=1}^m\int_0^t p(2p-1)(V^\alpha_s)^{2p-2}(\sum_{l=1}^m H^\alpha_l(s)+\langle\nabla\sigma_l(X_s),V^\alpha_s\rangle)^2ds \\ &\quad +2p\int_0^t(V^\alpha_s)^{2p-1}(h^\alpha(s)+\langle\nabla b(X_s),V^\alpha_s\rangle)ds \end{aligned}$$

and

$$J^\alpha_t = \int_0^t\int_{E\times(0,\Gamma)}(\left|V^\alpha_{s-}+Q^\alpha(s-,z)+\left\langle\nabla_x c(z,X_{s-}),V^\alpha_{s-}\right\rangle\right|^{2p}-\left|V^\alpha_{s-}\right|^{2p})1_{\{u\leq\gamma(z,X_{s-})\}}dN_\mu(s,u,z).$$

Using the trivial inequality $a^ub^v\leq a^{u+v}+b^{u+v}$ we obtain

$$\begin{aligned} E(|I^\alpha_t|) &\leq C_p\int_0^t E(\sum_{l=1}^m|H^\alpha_l(s)|^{2p}+|h^\alpha(s)|^{2p})ds \\ &\quad +C_p(1+\|\nabla\sigma\|^{2p}_\infty+\|\nabla b\|^{2p}_\infty)\int_0^t E(|V^\alpha_l(s)|^{2p})ds. \end{aligned}$$

We estimate now J^α_t. We will use the elementary inequality

$$(a+b)^{2p}-a^{2p}\leq C_p\,|b|\,(|a|^{2p-1}+|b|^{2p-1})$$

with

$$a=V^\alpha_{s-},\quad b=Q^\alpha(s-,a)+\left\langle\nabla_x c(a,X_{s-}),V^\alpha_{s-}\right\rangle.$$

Since $|Q^\alpha(s-,z)|\leq q(z,X_{s-})\left|R^\alpha_{s-}\right|$ we have

$$|b|\leq(q(z,X_{s-})+|\nabla_x c(z,X_{s-})|)(\left|R^\alpha_{s-}\right|+\left|V^\alpha_{s-}\right|)$$

so we obtain

$$\begin{aligned} &\left|V^\alpha_{s-}+Q^\alpha(s-,z)+\left\langle\nabla_x c(z,X_{s-}),V^\alpha_{s-}\right\rangle\right|^{2p}-\left|V^\alpha_{s-}\right|^{2p} \\ \leq\ & C_p(q(a,X_{s-})+|\nabla_x c(z,X_{s-})|)(1+q(z,X_{s-})+|\nabla_x c(z,X_{s-})|)^{2p-1} \\ &\times(\left|R^\alpha_{s-}\right|^{2p}+\left|V^\alpha_{s-}\right|^{2p}). \end{aligned}$$

Then

$$\begin{aligned}
E(|J_t^\alpha|) &\leq C_p E(\int_0^t \int_{E\times(0,1)} (q(z,X_{s-}) + |\nabla_x c(z,X_{s-})|)(1+q(z,X_{s-}) + |\nabla_x c(z,X_{s-})|)^{2p-1} \\
&\quad \times(\left|R_{s-}^\alpha\right|^{2p} + \left|V_{s-}^\alpha\right|^{2p}) 1_{\{u\leq\gamma(a,X_s)\}} du d\mu(z) ds \\
&= C_p E(\int_0^t \int_E (q(z,X_{s-}) + |\nabla_x c(z,X_{s-})|)(1+q(z,X_{s-}) + |\nabla_x c(z,X_{s-})|)^{2p-1} \gamma(z,X_s) \\
&\quad \times(\left|R_{s-}^\alpha\right|^{2p} + \left|V_{s-}^\alpha\right|^{2p}) d\mu(z) ds \\
&\leq C_p \widehat{c}(p) \int_0^t E(\left|R_{s-}^\alpha\right|^{2p} + \left|V_{s-}^\alpha\right|^{2p}) ds.
\end{aligned}$$

Since M^α is a martingale we obtain

$$\begin{aligned}
E(|V_t^\alpha|^{2p}) &= |V_0^\alpha|^{2p} + E(I_t^a) + E(J_t^a) \\
&\leq |V_0^\alpha|^{2p} + C_p \int_0^t (E(\sum_{l=1}^m |H_l^\alpha(s)|^{2p} + |h_l^\alpha(s)|^{2p} + \widehat{c}(p) \left|R_{s-}^\alpha\right|^{2p}) ds \\
&\quad + C_p(1 + \|\nabla\sigma\|_\infty^{2p} + \|\nabla b\|_\infty^{2p} + \widehat{c}(p)) \int_0^t E(|V_l{}^\alpha(s)|^{2p} ds
\end{aligned}$$

and Gronwall's lemma gives (88). □

7 Appendix: Proof of Proposition 18

We first notice that the isometry property yields

$$\begin{aligned}
E(|X_t^n - x|^2) &\leq Ct \times (\|b\|_\infty + \sup_{x\in R^d} \int_{E\times[0,1]} (|d_n(z,x)|^2 \xi_n(z,x) d\nu_n(z) \\
&\quad + \sup_{x\in R^d} \int_{E\times[0,1]} |e_n(z,x)| (1 + |e_n(z,x)|) \beta_n(z,x) d\eta_n(z) \\
&\quad + \sup_{x\in R^d} \int_{E\times[0,1]} |c_n(z,x)| (1 + |c_n(z,x)|) \gamma_n(z,x)) d\mu_n(z)) \\
&\leq Ct \times C_n
\end{aligned} \tag{89}$$

We denote

$$\begin{aligned}
k_n(x,z,u) &= d_n(z,x) 1_{\{u\leq\xi_n(z,x)\}}, \quad q_n(x,z,u) = e_n(z,x) 1_{\{u\leq\beta_n(z,x)\}} \\
h_n(x,z,u) &= c_n(z,x) 1_{\{u\leq\gamma_n(z,x)\}}
\end{aligned}$$

Using Itô's formula, for a function $f \in C^2(R)$, we obtain

$$f(X_t^n) = f(x) + M_t^n(f) + I_t^n(f) + J_t^n(f) + H_t^n(f) + D_t^n(f)$$

with

$$\begin{aligned}
M_t^n(f) &= \int_0^t \int_{E\times(0,1)} \left\langle \nabla f(X_{s-}^n), k_n(X_{s-}^n,z,u) \right\rangle \widetilde{N}_{\nu_n}(ds,dz,du), \\
I_t^n(f) &= \int_0^t \int_{E\times(0,1)} f(X_{s-}^n + k_n(X_{s-}^n,z,u)) - f(X_{s-}^n) - \left\langle \nabla f(X_{s-}^n), k_n(X_{s-}^n,z,u) \right\rangle d\nu_n(z) du ds \\
J_t^n(f) &= \int_0^t \int_{E_n\times(0,1)} f(X_{s-}^n + q_n(X_{s-}^n,z,u)) - f(X_{s-}^n) N_{\eta_n}(ds,dz,du) \\
H_t^n(f) &= \int_0^t \int_{E_n\times(0,1)} f(X_{s-}^n + h_n(X_{s-}^n,z,u)) - f(X_{s-}^n) N_{\mu_n}(ds,dz,du) \\
D_t^n(f) &= \int_0^t \left\langle \nabla f(X_s^n), b_n(X_s^n) \right\rangle ds.
\end{aligned}$$

Since $M_t^n(f)$ is a martingale we obtain

$$P_t^n f(x) - f(x) = E(I_t^n(f)) + E(J_t^n(f)) + E(H_t^n(f)) + E(D_t^n(f)).$$

We compute each of these terms. Let us estimate $E(I_t^n(f))$. We denote

$$\begin{aligned} l(x,z,u) &= \frac{1}{2}\sum_{i,j=1}^{d} \partial_i\partial_j f(x) k_n^i(x,z,u) k_n^j(x,z,u) \\ \phi(x,z,u) &= f(x+k_n(x,z,u)) - f(x) - \langle \nabla f(x), k_n(x,z,u)\rangle - l(x,z,u). \end{aligned}$$

Notice that, by the very definition of a_n,

$$\int_{E\times(0,1)} l(x,z,u)dud\nu_n(dz) = \frac{1}{2}\sum_{i,j=1}^{d} \partial_i\partial_j f(x) a_n^{i,j}(x)$$

so that

$$E(I_t^n(f)) = \frac{t}{2}\sum_{i,j=1}^{d} \partial_i\partial_j f(x) a_n^{i,j}(x) + r_1(t,x) + r_2(t,x)$$

with

$$\begin{aligned} r_1(t,x) &= \int_0^t \int_{E\times(0,1)} E(\phi(X_{s-}^n, z, u)))dud\nu_n(dz)ds, \\ r_2(t,x) &= \frac{1}{2}\int_0^t \int_{E\times(0,1)} E(l(X_{s-}^n, z, u) - l(x,z,u))dud\nu_n(z)ds \\ &= \frac{t}{2}\sum_{i,j=1}^{d} \int_0^t E(\partial_i\partial_j f(X_{s-}^n) a_n^{i,j}(X_{s-}^n))ds. \end{aligned}$$

We have

$$|\phi(x,z,u)| \leq C\|f\|_{3,\infty} |k_n(x,z,u)|^3 = C\|f\|_{3,\infty} |d_n(z,x)|^3 1_{\{u\leq \xi_n(z,x)\}}$$

so that

$$\begin{aligned} |r_1(t,x)| &\leq C\|f\|_{3,\infty} \int_0^t \int_E \int_0^1 E(\left|d_n(z,X_{s-}^n)\right|^3 1_{\{u\leq \xi_n(z,X_{s-}^n)\}})dud\nu_n(z)ds \\ &= C\|f\|_{3,\infty} \int_0^t \int_E E(\left|d_n(z,X_{s-}^n)\right|^3 \xi_n(z,X_{s-}^n))d\nu_n(z)ds \\ &\leq C\|f\|_{3,\infty}\, t \sup_{x\in R^d} \int_E |d_n(z,x)|^3 \xi_n(z,x)d\nu_n(z) \leq C\|f\|_{3,\infty}\, t\varepsilon_0(n). \end{aligned}$$

We estimate now r_2. We notice that

$$\begin{aligned} \|a_n\|_{1,\infty} &\leq C\sup_{x\in R^d}\int_E (|\nabla_x d_n(z,x)|^2 + |d_n(z,x)|^2))\xi_n(z,x) + |d_n(z,x)|^2 |\nabla_x \xi_n(z,x)|)d\nu_n(z) \\ &\leq C\times C_n \end{aligned}$$

so, using (89), we obtain

$$\begin{aligned} |r_2(t,x)| &\leq C\|f\|_{3,\infty} C_n \times \int_0^t E(\left|X_{s-}^n(x) - x\right|)ds \\ &\leq C\|f\|_{3,\infty} C_n^2 \times t^{3/2} \end{aligned}$$

We conclude that

$$\left| E(I_t^n(f)) - \frac{t}{2}\sum_{i,j=1}^{d} \partial_i\partial_j f(x) a_n^{i,j}(x)\right| \leq CC_n^2 t\|f\|_{3,\infty}(t^{1/2} + \varepsilon_0(n)).$$

Let us estimate $E(J_t^n(f))$. The strategy is the same as for $I_t^n(f)$. We denote

$$\begin{aligned} l(x,z,u) &= \sum_{i=1}^d \partial_i f(x) q_n^i(x,z,u) \\ \phi(x,z,u) &= f(x+q_n(x,z,u)) - f(x) - l(x,z,u). \end{aligned}$$

By the very definition of g_n,

$$\int_{E\times(0,1)} l(x,z,u)dud\eta_n(dz) = \sum_{i=1}^d \partial_i f(x) g_n^i(x)$$

so that

$$E(J_t^n(f)) = t\sum_{i=1}^d \partial_i f(x) g_n^i(x) + r_1(t,x) + r_2(t,x)$$

with

$$\begin{aligned} r_1(t,x) &= \int_0^t \int_{E\times(0,1)} E(\phi(X_{s-}^n, z, u)))dud\eta_n(dz)ds, \\ r_2(t,x) &= \int_0^t \int_{E\times(0,1)} E(l(X_{s-}^n, z, u) - l(x,z,u))dud\eta_n(z)ds \\ &= \sum_{i=1}^d \int_0^t E(\partial_i f(X_{s-}^n) g_n^i(X_{s-}^n) - \partial_i f(x) g_n^i(x))ds \end{aligned}$$

We have

$$|\phi(x,z,u)| \leq C\left\|f\right\|_{2,\infty} |q_n(x,z,u)|^2 = C\left\|f\right\|_{2,\infty} |e_n(z,x)|^2 1_{\{u\leq \beta_n(z,x)\}}$$

so that

$$\begin{aligned} |r_1(t,x)| &\leq C\left\|f\right\|_{2,\infty} \int_0^t \int_E \int_0^1 E(\left|e_n(z,X_{s-}^n)\right|^2 1_{\{u\leq \beta_n(z,X_{s-}^n)\}})dud\eta_n(z)ds \\ &= C\left\|f\right\|_{2,\infty} \int_0^t \int_E E(\left|e_n(z,X_{s-}^n)\right|^2 \beta_n(z,X_{s-}^n))d\eta_n(z)ds \\ &\leq C\left\|f\right\|_{2,\infty} t \sup_{x\in R^d} \int_E |e_n(z,x)|^2 \beta_n(z,x))d\eta_n(z) \leq C\left\|f\right\|_{2,\infty} t\varepsilon_0(n). \end{aligned}$$

We estimate now r_2. We have

$$\begin{aligned} \left\|g_n\right\|_{1,\infty} &\leq C\sup_{x\in R^d} \int_E (|e_n(z,x)| + |\nabla_x e_n(z,x)|)\beta_n(z,x) + |e_n(z,x)|\left|\nabla_x \beta_n(z,x)\right| d\eta_n(z) \\ &\leq C\times C_n \end{aligned}$$

so, using (89)

$$\begin{aligned} |r_2(t,x)| &\leq Ct\left\|f\right\|_{2,\infty} C_n \times \int_0^t E(\left|X_{s-}^n(x) - x\right|)ds \\ &\leq C\left\|f\right\|_{2,\infty} C_n^2 \times t^{3/2}. \end{aligned}$$

We conclude that

$$\left|E(J_t^n(f)) - t\sum_{i=1}^d \partial_i f(x) g_n^i(x)\right| \leq C\left\|f\right\|_{2,\infty} C_n^2 (t^{1/2} + \varepsilon_0(n)) \times t.$$

We estimate now $H_t^n(f)$. We denote

$$\theta_n(x) = \int_{E_n} (f(x + c_n(z,x)) - f(x))\gamma_n(z,x)d\mu(z)$$

so that

$$E(H_t^n(f)) = \int_0^t \int_{E_n} E(\theta_n(X_{s-}^n))ds.$$

Notice that

$$\theta_n(x) = \sum_{i=1}^d \int_{E_n} d\mu(z)\gamma_n(z,x)c_n^i(z,x) \int_0^1 \partial_i f(x + \lambda c_n(z,x))d\lambda.$$

Then it is easy to check that

$$\begin{aligned}
|\nabla\theta_n(x)| &\leq C \|f\|_{2,\infty} \times \int_{E_n} d\mu(z)(|\nabla_x\gamma_n(z,x)| |c_n(z,x)| + |\nabla_x c_n(z,x)| \gamma_n(z,x) + |c_n(z,x)|^2 \gamma_n(z,x)) \\
&\leq C \|f\|_{2,\infty} \times C_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
|E(H_t^n(f)) - t\theta_n(x)| &\leq \int_0^t \left|E(\theta_n(X_{s-}^n)) - \theta_n(x)\right| ds \\
&\leq C \|f\|_{2,\infty} C_n \times \int_0^t E(\left|X_{s-}^n(x)) - x\right|)ds \\
&\leq Ct^{3/2} \|f\|_{2,\infty} C_n^2.
\end{aligned}$$

Finally

$$\left|E(D_t^n(f)) - \int_0^t \langle\nabla f(x), b_n(x)\rangle\, ds\right| \leq \|b\|_{1,\infty} \|f\|_{2,\infty} \int_0^t E(\left|X_{s-}^n(x)) - x\right|)ds \leq CC_n^2 t^{3/2}.$$

□

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