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Abstract. This paper studies the nonlinear one-dimensional stochastic heat equation driven by a Gaussian noise which is white in time and which has the covariance of a fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$ in the space variable. The existence and uniqueness of the solution u are proved assuming the nonlinear coefficient $\sigma(u)$ is differentiable with a Lipschitz derivative and $\sigma(0) = 0$. In the case of a multiplicative noise, that is, $\sigma(u) = u$, we derive the Wiener chaos expansion of the solution and a Feynman-Kac formula for the moments of the solution. These results allow us to establish sharp lower and upper asymptotic bounds for $\mathbf{E}[u^n(t,x)]$.

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1. Introduction

In this paper we are interested in the one-dimensional stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{W}\,, \quad t \geq 0, \quad x \in \mathbb{R}\,, \tag{1.1}$$

where W is a centered Gaussian process with covariance given by

$$\mathbf{E}\left[W(s,x)W(t,y)\right] = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right)(s\wedge t), \tag{1.2}$$

with $\frac{1}{4} < H < \frac{1}{2}$. That is, W is a standard Brownian motion in time and a fractional Brownian motion with Hurst parameter H in the space variable and $\dot{W} = \frac{\partial^2 W}{\partial t \partial x}$. For this stochastic heat equation with a rough noise in space, understood in the Itô sense, our aim is twofold: on one hand, for a differentiable coefficient σ with a Lipschitz derivative and satisfying $\sigma(0) = 0$, we will obtain the existence and uniqueness of the solution. On the other hand, we shall further investigate the special relevant case $\sigma(u) = u$. We now detail those two main points.

(1) Since the pioneering work by Peszat-Zabczyk [24] and Dalang (see [9]), there has been a lot of interest in stochastic partial differential equations driven by a Brownian motion in time with spatial homogeneous covariance. After more than a decade of investigations, the standard assumptions on W under which existence and uniqueness hold take the following form

(i) $\mathbf{E}[\dot{W}(s,x)\dot{W}(t,y)] = \Lambda(x-y)\,\delta_0(s-t)$, where Λ is a positive distribution of positive type.

(ii) The Fourier transform of the spatial covariance Λ is a tempered measure μ that satisfies the integrability condition $\int_{\mathbb{R}} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$.

In case of the covariance (1.2) under consideration, one can easily compute the measure μ, whose explicit expression is $\mu(d\xi) = c_{1,H}|\xi|^{1-2H}d\xi$, where $c_{1,H}$ is a constant depending on H (see expression (2.2) below). In addition, it is readily checked that μ fulfills the condition $\int_{\mathbb{R}} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$ for all $H \in (0,1)$. However, the corresponding covariance Λ is a distribution which fails to be positive when $H < \frac{1}{2}$, and the covariance of two stochastic integrals with respect to $\dot{W}$ is expressed in terms of fractional derivatives. For this reason, the standard methodology used in the classical references [9, 11, 24] to handle homogeneous spatial covariances does not apply to our case of interest.

In a recent paper, Balan, Jolis and Quer-Sardanyons [3] proved the existence of a unique mild solution for equation (1.1) in the case $\sigma(u) = au+b$, using techniques of Fourier analysis. The method used in [3] cannot be extended to general nonlinear coefficients. Indeed, the isometry property of stochastic integrals with respect to W involves the semi norm

$$\mathcal{N}_{\frac{1}{2}-H,2}u(t,x) = \left(\int_{\mathbb{R}} \mathbf{E}|u(t,x+h)-u(t,x)|^2|h|^{2H-2}dh\right)^{\frac{1}{2}},$$

where $\mathcal{N}_{\beta,p}$ is defined in (3.2). Then, if u and v are two solutions, $\mathcal{N}_{\frac{1}{2}-H,2}(\sigma(u)-\sigma(v))$ cannot be bounded in terms of $\mathcal{N}_{\frac{1}{2}-H,2}(u-v)$, due to the presence of a double increment of the form $\sigma(u(s,z+h)) - \sigma(v(s,z+h)) - \sigma(u(s,z)) + \sigma(v(s,z))$. To overcome this difficulty we shall use a truncation argument to show the uniqueness of mild solutions, inspired by the work of Gyöngy and Nualart in [15] on the stochastic Burgers equation on the whole real line driven by a space-time white noise. The main ingredient is a uniform estimate of the $L^p(\Omega)$-norm of a stochastic convolution (see Lemma 4.2). Due to this argument, the uniqueness is obtained in the space $\mathcal{Z}_T^p$ (see (4.1) for the definition of the norm in $\mathcal{Z}_T^p$), which requires an integrability condition in the space variable.

The existence of a solution is much more involved. The methodology, inspired by the work of Gyöngy in [13] on semi-linear stochastic partial differential equations, consists in taking approximations obtained by regularizing the noise and using a compactness argument on a suitable space of trajectories, together with the strong uniqueness result.

Once existence and uniqueness are obtained, we establish the Hölder continuity of the solution u in both space and time variables. We also derive upper bounds for the moments of the solution using a sharp Burkholder's inequality, as well as the matching lower bounds for the second moment by means of a Sobolev embedding argument. Summarizing, we get a complete basic picture of the solution to equation (1.1) in the case $\frac{1}{4} < H < \frac{1}{2}$. The critical parameter $H = \frac{1}{4}$ is worthwhile noting, since it is also the threshold under which rough differential equations driven by a fractional Brownian motion are ill-defined.

(2) The particular case $\sigma(u) = u$ in equation (1.1) deserves a special attention. Indeed, this linear equation turns out to be a continuous version of the parabolic Anderson model, and is related to challenging systems in random environment like KPZ equation [16, 5] or polymers [1, 6]. The localization and intermittency properties of the linear version of (1.1) have thus been thoroughly studied for equations driven by a space-time white noise (see [21] for a nice survey), while a recent trend consists in extending this kind of result to equations driven by very general Gaussian noises [8, 18, 19, 20].

Nevertheless, the rough noise W with covariance (1.2) presented here is not covered by the aforementioned references, and we wish to fill this gap. We will thus particularize our setting to $\sigma(u) = u$, and first go back to the existence and uniqueness problem. Indeed, in this linear case, one can implement a rather simple procedure involving Fourier transform, as well as a chaos expansion technique, in order to achieve existence and uniqueness of the solution to (1.1). Since this point of view is interesting in its own right and short enough, we develop it at Section 5.1. Moreover in this case we can consider more general initial conditions.

We then move to a Feynman-Kac type representation for the solution: following the approach introduced in [19, 18], we obtain an explicit formula for the kernels of the Wiener chaos expansion and we show its convergence. In fact, we cannot expect a Feynman-Kac formula for the solution, because the covariance is rougher than the space-time white noise case, and this type of formula requires smoother covariance structures (see, for instance, [20]). However, by means of Fourier analysis techniques as in [19, 18], we are able to obtain a Feynman-Kac formula for the moments that involves a fractional derivative of the Brownian local time.

Finally, the previous considerations allow us to handle, in the last section of the paper, the intermittency properties of the solution. More precisely, we show sharp lower bounds for the moments of the solution of the form $\mathbf{E}[u(t,x)^n] \geq \exp(Cn^{1+\frac{1}{H}}t)$, for all $t \geq 0$, $x \in \mathbb{R}$ and $n \geq 2$. These bounds entail the intermittency phenomenon and match the corresponding estimates for the case $H > \frac{1}{2}$ obtained in [18].

The paper is organized as follows. Section 2 contains some preliminaries on stochastic integration with respect to the noise W and elements of Malliavin calculus. Section 3 deals with basic moment estimates and Hölder continuity properties of stochastic convolutions. We establish the uniqueness of the solution in Section 4. To do this, first we derive moment estimates for the supremum norm in space and time for the stochastic convolution. In order to show the existence, we need to introduce several spaces of functions in Subsection 4.2 and derive compactness criteria. Section 5.1 deals with the parabolic Anderson model, that is, the case $\sigma(u) = u$. In Section 5.2, we derive Feynman-Kac type formulas for the moments of the solution which allow us to derive sharp lower and upper moment estimates and intermittency properties.

2. Preliminaries

In this section we introduce some of the functional spaces we will deal with in the remainder of the paper, as well as some general Malliavin calculus tools.

2.1. **Noise structure and stochastic integration.** Our noise W can be seen as a Brownian motion with values in an infinite dimensional Hilbert space. One might thus think that the stochastic integration theory with respect to W can be handled by classical theories (see e.g [7, 9, 12]). However, the spatial covariance function of W, which is formally equal to $H(2H-1)|x-y|^{2H-2}$, is not locally integrable when $H < 1/2$ (in other words, the Fourier transform of $|\xi|^{1-2H}$ is not a function), and W thus lies outside the scope of application of these classical references. Due to this fact, we provide some details about the construction of a stochastic integral with respect to our noise.

Let us start by introducing our basic notation on Fourier transforms of functions. The space of Schwartz functions is denoted by $\mathcal{S}$. Its dual, the space of tempered distributions, is $\mathcal{S}'$. The Fourier transform of a function $u \in \mathcal{S}$ is defined with the normalization

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} e^{-i\xi x} u(x) dx,$$

so that the inverse Fourier transform is given by $\mathcal{F}^{-1}u(\xi) = (2\pi)^{-1}\mathcal{F}u(-\xi)$.

Let $\mathcal{D}((0,\infty)\times\mathbb{R})$ denote the space of real-valued infinitely differentiable functions with compact support on $(0,\infty)\times\mathbb{R}$. Taking into account the spectral representation of the covariance function of the fractional Brownian motion in the case $H < \frac{1}{2}$ proved in [25, Theorem 3.1], we represent our noise W by a zero-mean Gaussian family $\{W(\varphi),\, \varphi \in \mathcal{D}((0,\infty)\times\mathbb{R})\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, whose covariance structure is given by

$$\mathbf{E}\left[W(\varphi)\, W(\psi)\right] = c_{1,H} \int_{\mathbb{R}_+\times\mathbb{R}} \mathcal{F}\varphi(s,\xi)\, \overline{\mathcal{F}\psi(s,\xi)}\, |\xi|^{1-2H}\, ds d\xi, \tag{2.1}$$

where the Fourier transforms $\mathcal{F}\varphi, \mathcal{F}\psi$ are understood as Fourier transforms in space only and

$$c_{1,H} = \frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H)\,. \tag{2.2}$$

The inner product appearing in (2.1) can be expressed in terms of fractional derivatives. Let β be in $(0,1)$. The Marchaud fractional derivative D_-^β of order β with respect to the space variable is defined, for a function $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, as follows

$$D_-^\beta \varphi(s,x) = \lim_{\varepsilon \to 0} D_{-,\varepsilon}^\beta \varphi(s,x)\,, \tag{2.3}$$

where

$$D_{-,\varepsilon}^\beta \varphi(s,x) = \frac{\beta}{\Gamma(1-\beta)} \int_\varepsilon^\infty \frac{\varphi(s,x)-\varphi(s,x+y)}{y^{1+\beta}}\, dy\,.$$

We also define the fractional integral of order β of a function $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by

$$I_-^\beta \psi(s,x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \psi(s,u)(x-u)^{\beta-1} du\,.$$

Note again that here the fractional differentiation and integration are only with respect to space variables. Observe that if $\varphi = I_-^\beta \psi$ for some $\psi \in L^2(\mathbb{R}_+ \times \mathbb{R})$, then by Theorem 6.1 in [26] we have

$$D_-^\beta \varphi = D_-^\beta (I_-^\beta \psi) = \psi$$

and, hence,

$$\int_{\mathbb{R}_+\times\mathbb{R}} \left[D_-^\beta \varphi(s,x) \right]^2 dsdx = \int_{\mathbb{R}_+\times\mathbb{R}} \psi^2(s,x) dsdx < \infty.$$

The previous notions can be related to our noise in the following way: it is known (cf. [25] for further details) that

$$\mathbf{E}\left[W(\varphi)\, W(\psi) \right] = c_{2,H} \int_{\mathbb{R}_+\times\mathbb{R}} D_-^{\frac12 - H} \varphi(s,x) D_-^{\frac12 - H} \psi(s,x) dsdx, \tag{2.4}$$

where

$$c_{2,H} = \left[\Gamma\left(H + \frac12 \right) \right]^2 \left(\int_0^\infty \left((1+s)^{H-\frac12} - s^{H-\frac12} \right)^2 ds + \frac{1}{2H} \right)^{-1}. \tag{2.5}$$

for any $\varphi, \psi \in \mathcal{D}((0,\infty)\times\mathbb{R})$.

Based on the previous observation and relation (2.4), we introduce a new set of function spaces. Indeed, let $\mathfrak{H}$ be the class of functions $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that there exists $\psi \in L^2(\mathbb{R}_+ \times \mathbb{R})$ satisfying $\varphi(s,x) = I_-^{\frac12 - H} \psi(s,x)$. The relation between $\mathfrak{H}$ and our noise W is given in the following proposition.

Proposition 2.1. *The class of functions $\mathfrak{H}$ is a Hilbert space equipped with the inner product*

$$\langle \varphi, \psi \rangle_{\mathfrak{H}} := \ c_{2,H} \int_{\mathbb{R}_+\times\mathbb{R}} D_-^{\frac12 - H} \varphi(s,x) D_-^{\frac12 - H} \psi(s,x) dsdx\,, \tag{2.6}$$

and $\mathcal{D}((0,\infty)\times\mathbb{R})$ is dense in $\mathfrak{H}$. Moreover if $\mathfrak{H}_0$ denotes the class of functions $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R})$ such that $\int_{\mathbb{R}_+\times\mathbb{R}} |\mathcal{F}\varphi(s,\xi)|^2 |\xi|^{1-2H} d\xi ds < \infty$, then $\mathfrak{H}_0$ is not complete and the inclusion $\mathfrak{H}_0 \subset \mathfrak{H}$ is strict. Also for any $\varphi, \psi \in \mathfrak{H}_0$,

$$\langle \varphi, \psi \rangle_{\mathfrak{H}} = c_{1,H} \int_{\mathbb{R}_+\times\mathbb{R}} \mathcal{F}\varphi(s,\xi) \overline{\mathcal{F}\psi(s,\xi)} |\xi|^{1-2H} d\xi ds\,. \tag{2.7}$$

We refer to [25] for the proof of this proposition. Note that in [25], the functions considered there are from $\mathbb{R}$ to $\mathbb{R}$, but by scrutinizing the proofs we see that the results of this paper can be easily extended to our case, i.e. for functions from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$. We omit the details.

Let us now identify our space $\mathfrak{H}$ with another classical space in harmonic analysis. Indeed, according to Proposition 1.37 in [2], for any $\beta \in (0,1)$ the homogeneous Sobolev space $\dot{H}^\beta$ can be defined as the completion of the space of infinitely differentiable functions with compact support with respect to the norm

$$\|f\|_{\dot{H}^\beta}^2 = \int_{\mathbb{R}} |D_-^\beta f(x)|^2 dx = c_{3,\beta}^2 \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x+y) - f(x)|^2 |y|^{-1-2\beta} dx dy \,, \tag{2.8}$$

where $c_{3,\beta}^2 = (1/2-\beta)\beta\, c_{2,\frac{1}{2}-\beta}^{-1}$ and $c_{2,\frac{1}{2}-\beta}$ is defined by (2.5). As a consequence, our Hilbert space $\mathfrak{H}$ can be identified with the homogenous Sobolev space of order $\beta = \frac{1}{2} - H$ of functions with values in $L^2(\mathbb{R}_+)$. Namely $\mathfrak{H} = \dot{H}^{\frac{1}{2}-H}(L^2(\mathbb{R}_+))$, and for any $f \in \mathfrak{H}$ the quantity $\|f\|_{\mathfrak{H}}$ can be represented as

$$\|f\|_{\mathfrak{H}}^2 = c_{3,\frac{1}{2}-H}^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(s,x+y) - f(s,x)|^2 |y|^{2H-2} dx dy ds.$$

From Proposition 2.1, we see that the Gaussian family W can be extended as an isonormal Gaussian process $W = \{W(\phi), \phi \in \mathfrak{H}\}$ indexed by the Hilbert space $\mathfrak{H}$.

Let us now turn to the stochastic integration with respect to W. Since we are handling a Brownian motion in time, one can start by integrating elementary processes.

Definition 2.2. *For any $t \geq 0$, let $\mathcal{F}_t$ be the σ-algebra generated by W up to time t. An elementary process g is a process given by*

$$g(s,x) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j}\, \mathbf{1}_{(a_i,b_i]}(s)\, \mathbf{1}_{(h_j,l_j]}(x),$$

where n and m are finite positive integers, $-\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables for $i = 1, \ldots, n$. The integral of such a process with respect to W is defined as

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x)\, W(ds,dx) &= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j}\, W\left(\mathbf{1}_{(a_i,b_i]} \otimes \mathbf{1}_{(h_j,l_j]}\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \left[W(b_i,l_j) - W(a_i,l_j) - W(b_i,h_j) + W(a_i,h_j)\right]. \end{aligned} \tag{2.9}$$

We can now extend the notion of integral with respect to W to a broad class of adapted processes.

Proposition 2.3. *Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathfrak{H}$ and $\mathbf{E}[\|g\|_{\mathfrak{H}}^2] < \infty$. Then, we have:*

(i) *The space of elementary processes defined in Definition 2.2 is dense in Λ_H.*

(ii) *For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x)\, W(ds,dx)$ is defined as the $L^2(\Omega)$-limit of Riemann sums along elementary processes approximating g, and we have*

$$\mathbf{E}\left[\left(\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x)\, W(ds,dx)\right)^2\right] = \mathbf{E}\left[\|g\|_{\mathfrak{H}}^2\right]. \tag{2.10}$$

Proof. Let us prove item (i). To this aim, consider $g \in \Lambda_H$ and set $\varphi(t,x) = D_-^{\frac{1}{2}-H} g(t,x)$. According to the definition of Λ_H, we have $\mathbf{E}[\int_{\mathbb{R}_+}\int_{\mathbb{R}} |\varphi(s,x)|^2 dxds] < \infty$. Then we will show that $g(t,x)$ can be approximated by elementary processes in $L^2(\Omega;\mathfrak{H})$ in three steps.

Step 1. Recall that we have set $\dot{H}^{\frac{1}{2}-H}$ for the class of functions f, such that there exists $h \in L^2(\mathbb{R})$ satisfying $f = I_-^{1/2-H} h$. We show that the process g can be approximated in $L^2(\Omega;\mathfrak{H})$ by functions of the form

$$\psi_m(s,x;\omega) = \sum_{i=1}^{m} \mathbf{1}_{(a_i,b_i]}(s)\phi_i(x;\omega)\,, \tag{2.11}$$

where for each i, $\phi_i(x;\omega)$ is an $\mathcal{F}_{a_i}$-measurable $L^2(\Omega;\dot{H}^{\frac{1}{2}-H})$-valued random field. To see this, we just set

$$\psi_m(s,x;\omega) = \sum_{k=1}^{m2^m} \mathbf{1}_{((k-1)2^{-m},k2^{-m}]}(s) 2^m \int_{(k-1)2^{-m}}^{k2^{-m}} g(r,x;\omega)dr\,,$$

and we easily get that $D_-^{\frac{1}{2}-H}\psi_m(s,x;\omega) \to D_-^{\frac{1}{2}-H} g(s,x;\omega)$ in $L^2(\Omega\times\mathbb{R}_+\times\mathbb{R})$ as m tends to infinity. In this way we get the desired approximation.

Step 2. We show that each $\psi_m(s,x;\omega)$ of the form (2.11) can be approximated in $L^2(\Omega;\mathfrak{H})$ by a linear combination of elements of the form $X\mathbf{1}_{(a,b]}(s)h(x)$. Indeed, for each $\phi_i(x;\omega)$, we notice that since

$$\mathbf{E}\int_{\mathbb{R}} |D_-^{\frac{1}{2}-H}\phi_i(x;\omega)|^2 dx < \infty\,,$$

the random function $D_-^{\frac{1}{2}-H}\phi_i(x;\omega)$ can be approximated in $L^2(\Omega;L^2(\mathbb{R}))$ by functions of the form $\sum_{j=1}^N X_j h_j(x)$, where each X_j is an $\mathcal{F}_{a_i}$-measurable random variable and each h_j is an element in $L^2(\mathbb{R})$. Thus, it is easily seen that $\phi_i(x;\omega)$ can be approximated by a sequence of functions of the form

$$\sum_{j=1}^{N} X_j I_-^{\frac{1}{2}-H} h_j(x)\,.$$

So we conclude that $\psi_m(s,x;\omega)$ can be approximated in $L^2(\Omega;\mathfrak{H})$ by

$$\sum_{i=1}^{m} \mathbf{1}_{(a_i,b_i]}(s) \sum_{j=1}^{N} X_{i,j} I_-^{\frac{1}{2}-H} h_{i,j}(x)\,,$$

where for each (i,j), $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables and $h_{i,j} \in L^2(\mathbb{R})$.

Step 3. Owing to Theorem 3.3 in [25] we know that

$$\text{Span}\left\{ D_-^{\frac{1}{2}-H}\mathbf{1}_{(h,l]},\; h<l \right\}$$

is dense in $\Lambda_0 := \{D_-^{\frac{1}{2}-H} f : f \in \dot{H}^\beta\}$, in $L^2(\mathbb{R})$ norm. This observation and the results in Step 2 immediately show that $\psi_m(s,x;\omega)$ can be approximated by elementary processes in $L^2(\Omega;\mathfrak{H})$. This completes the proof. □

With this stochastic integral defined, we are ready to state the definition of the solution to equation (1.1).

Definition 2.4. *Let $u = \{u(t,x), 0 \le t \le T, x \in \mathbb{R}\}$ be a real-valued predictable stochastic process such that for all $t \in [0,T]$ and $x \in \mathbb{R}$ the process $\{p_{t-s}(x-y)\sigma(u(s,y))\mathbf{1}_{[0,t]}(s), 0 \le s \le t, y \in \mathbb{R}\}$ is an element of Λ_H, where $p_t(x)$ is the heat kernel on the real line related to $\frac{\kappa}{2}\Delta$. We say that u is a mild solution of (1.1) if for all $t \in [0,T]$ and $x \in \mathbb{R}$ we have*

$$u(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy) \quad a.s., \tag{2.12}$$

where the stochastic integral is understood in the sense of Proposition 2.3.

2.2. **Elements of Malliavin calculus.** We recall that the Gaussian family W can be extended to $\mathfrak{H}$ and this produces an isonormal Gaussian process, where $\mathfrak{H}$ is the Hilbert space introduced in Proposition 2.1. We refer to [23] for a detailed account of the Malliavin calculus with respect to a Gaussian process. On our Gaussian space, the smooth and cylindrical random variables F are of the form

$$F = f(W(\phi_1), \dots, W(\phi_n)),$$

with $\phi_i \in \mathfrak{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (namely f and all its partial derivatives have polynomial growth). For this kind of random variable, the derivative operator D in the sense of Malliavin calculus is the $\mathfrak{H}$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j .$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathfrak{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{\mathbf{E}[F^2] + \mathbf{E}[\|DF\|_{\mathfrak{H}}^2]} .$$

We denote by δ the adjoint of the derivative operator (or divergence) given by the duality formula

$$\mathbf{E}\left[\delta(u)F\right] = \mathbf{E}\left[\langle DF, u\rangle_{\mathfrak{H}}\right], \tag{2.13}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathfrak{H})$ in the domain of δ.

For any integer $n \ge 0$ we denote by $\mathbf{H}_n$ the nth Wiener chaos of W. We recall that $\mathbf{H}_0$ is simply $\mathbb{R}$ and for $n \ge 1$, $\mathbf{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(W(\phi)), \phi \in \mathfrak{H}, \|\phi\|_{\mathfrak{H}} = 1\}$, where H_n is the nth Hermite polynomial. For any $n \ge 1$, we denote by $\mathfrak{H}^{\otimes n}$ (resp. $\mathfrak{H}^{\odot n}$) the nth tensor product (resp. the nth symmetric tensor product) of $\mathfrak{H}$. Then, the mapping $I_n(\phi^{\otimes n}) = H_n(W(\phi))$ can be extended to a linear isometry between $\mathfrak{H}^{\odot n}$ (equipped with the modified norm $\sqrt{n!}\|\cdot\|_{\mathfrak{H}^{\otimes n}}$) and $\mathbf{H}_n$.

Consider now a random variable $F \in L^2(\Omega)$ which is measurable with respect to the σ-field $\mathcal{F}$ generated by W. This random variable can be expressed as

$$F = \mathbf{E}\left[F\right] + \sum_{n=1}^\infty I_n(f_n), \tag{2.14}$$

where the series converges in $L^2(\Omega)$, and the elements $f_n \in \mathfrak{H}^{\odot n}$, $n \ge 1$, are determined by F. This identity is called the Wiener-chaos expansion of F.

The Skorohod integral (or divergence) of a random field u can be computed by using the Wiener chaos expansion. More precisely, suppose that $u = \{u(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ is a random field such that for each (t,x), $u(t,x)$ is an $\mathcal{F}_t$-measurable and square integrable random variable. Then, for each (t,x) we have a Wiener chaos expansion of the form

$$u(t,x) = \mathbf{E}\left[u(t,x)\right] + \sum_{n=1}^{\infty} I_n(f_n(\cdot,t,x)). \tag{2.15}$$

Suppose that $\mathbf{E}[\|u\|_{\mathfrak{H}}^2]$ is finite. Then, we can interpret u as a square integrable random function with values in $\mathfrak{H}$ and the kernels f_n in the expansion (2.15) are functions in $\mathfrak{H}^{\otimes(n+1)}$ which are symmetric in the first n variables. In this situation, u belongs to the domain of the divergence operator (that is, u is Skorohod integrable with respect to W) if and only if the following series converges in $L^2(\Omega)$

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x)\, \delta W(t,x) = W(\mathbf{E}[u]) + \sum_{n=1}^{\infty} I_{n+1}(\widetilde{f}_n(\cdot,t,x)), \tag{2.16}$$

where $\widetilde{f}_n$ denotes the symmetrization of f_n in all its $n+1$ variables. We note that whenever $u \in \Lambda_H$ the integral $\delta(u)$ coincides with the Itô integral.

Along the paper we denote by C a generic constant that may vary from line to line.

3. Moment estimates and Hölder continuity of stochastic convolutions

This section is devoted to a thorough study of the stochastic convolution related to our noise $\dot{W}$, including moment bounds and Hölder continuity estimates.

3.1. **Moment bound of the solution.**

First we introduce some notation, which makes some of our formulae easier to read, and which will prevail until the end of the article. Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$, and let $\beta \in (0,1)$ be a fixed number. For every function $f: \mathbb{R} \to B$, we introduce the function $\mathcal{N}_\beta^B f: \mathbb{R} \to [0,\infty]$ defined by

$$\mathcal{N}_\beta^B f(x) = \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}. \tag{3.1}$$

When $B = \mathbb{R}$, we abbreviate the notation $\mathcal{N}_\beta^{\mathbb{R}} f$ into $\mathcal{N}_\beta f$. With this notation, the norm of the homogeneous Sobolev space $\dot{H}^\beta$ can be written as $c_{3,\beta}\|\mathcal{N}_\beta f\|_{L^2(\mathbb{R})}$. The following technical lemma will be used along the paper.

Lemma 3.1. *For any $\beta \in (0,1)$,*

$$\int_{\mathbb{R}} [\mathcal{N}_\beta p_s(x)]^2 dx \le C_\beta (\kappa s)^{-\frac{1}{2}-\beta}.$$

Proof. Recalling that $\mathcal{F}p_s(\xi) = e^{-\frac{\kappa}{2}s\xi^2}$ and invoking Plancherel's identity we can write

$$\begin{aligned}
\int_{\mathbb{R}} [\mathcal{N}_\beta p_s(x)]^2 dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} |p_s(x+h) - p_s(x)|^2 |h|^{-1-2\beta} dh dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa s|\xi|^2} |e^{-i\xi z} - 1|^2 |z|^{-1-2\beta} dz d\xi \\
&= C_{1,\beta} \int_{\mathbb{R}} e^{-\kappa s \xi^2} |\xi|^{2\beta} d\xi,
\end{aligned}$$

where the second relation is obtained by a scaling $v \equiv \xi z$ in the integral in z and $C_{1,\beta} = \frac{1}{2\pi}\int_{\mathbb{R}} |e^{iv}-1|^2 |v|^{-1-2\beta} dv$, which is easily seen to be a convergent integral. Setting now $\eta = (\kappa s)^{1/2}\xi$ in the integral in ξ, we get

$$\int_{\mathbb{R}} [\mathcal{N}_\beta p_s(x)]^2 dx \le C_\beta (\kappa s)^{-\frac{1}{2}-\beta},$$

where $C_\beta = C_{1,\beta}\int_{\mathbb{R}} e^{-\eta^2}|\eta|^{2\beta} d\eta$. □

The transformation $\mathcal{N}^B_\beta$ can also be defined for functions f defined on $\mathbb{R}_+ \times \mathbb{R}$ acting on the spacial variable, and in this case, $\mathcal{N}^B_\beta f : \mathbb{R}_+ \times \mathbb{R} \to [0,\infty]$. Now fix $p \ge 2$, and suppose that $f = \{f(t,x), t \ge 0, x \in \mathbb{R}\}$ is a random field such that $\mathbf{E}|f(t,x)|^p < \infty$ for all (t,x). Then we can consider f as an $L^p(\Omega)$-valued function and we will denote by $\mathcal{N}_{\beta,p} f$ the transformation introduced in (3.1) for $B = L^p(\Omega)$, that is,

$$\mathcal{N}_{\beta,p} f(t,x) = \left(\int_{\mathbb{R}} \|f(t,x+h) - f(t,x)\|^2_{L^p(\Omega)} |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}. \tag{3.2}$$

With the above notation in mind, the following proposition is essential in our approach.

Proposition 3.2. *Let W be the Gaussian noise defined by the covariance (2.1), and consider a predictable random field $f \in \Lambda_H$. Then, for any $p \ge 2$ we have*

$$\left\| \int_0^t \int_{\mathbb{R}} f(s,y) W(ds,dy) \right\|_{L^p(\Omega)} \le \sqrt{4p}\, c_{3,\frac{1}{2}-H} \left(\int_0^t \int_{\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H,p} f(s,y)]^2 dy ds \right)^{\frac{1}{2}}, \tag{3.3}$$

where $c_{3,\beta}$ is defined by relation (2.8).

Proof. Applying Burkholder's inequality, we have

$$\left\| \int_0^t \int_{\mathbb{R}} f(s,y) W(ds,dy) \right\|_{L^p(\Omega)} \le \sqrt{4p} \left\| \int_0^t \|f(s,\cdot)\|^2_{\dot{H}^{\frac{1}{2}-H}} ds \right\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)}. \tag{3.4}$$

Moreover, using (2.8) we can write

$$\|f(s,\cdot)\|^2_{\dot{H}^{\frac{1}{2}-H}} = c^2_{3,\frac{1}{2}-H} \int_{\mathbb{R}^2} |f(s,y+h) - f(s,y)|^2 |h|^{2H-2} dh dy. \tag{3.5}$$

We now invoke Minkowski's inequality, under the form

$$\left\| \int_S U(\xi) \mu(d\xi) \right\|_{L^q(\Omega)} \le \int_S \|U(\xi)\|_{L^q(\Omega)} \mu(d\xi),$$

for a measure μ on the state space S. Together with (3.5), this yields

$$\begin{aligned} \left\| \int_0^t \|f(s,\cdot)\|^2_{\dot{H}^{\frac{1}{2}-H}} ds \right\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)} &\le c_{3,\frac{1}{2}-H} \int_{\mathbb{R}^2} \left\| (f(s,y+h) - f(s,y))^2 \right\|_{L^{\frac{p}{2}}(\Omega)} |h|^{2H-2} dh dy \\ &= c_{3,\frac{1}{2}-H} \int_{\mathbb{R}^2} \|f(s,y+h) - f(s,y)\|^2_{L^p(\Omega)} |h|^{2H-2} dh dy, \end{aligned}$$

from which identity (3.3) is easily deduced. □

From now on, we fix a finite time horizon T. We introduce the following functions space which plays an important role through out the paper.

Definition 3.3. *Let $\mathfrak{X}_T^\beta(B)$ be the space of all continuous functions $f:[0,T]\times\mathbb{R}\to B$ such that*

$$\|f\|_{\mathfrak{X}_T^\beta(B)} := \sup_{t\in[0,T],x\in\mathbb{R}} \|f(t,x)\| + \sup_{t\in[0,T],x\in\mathbb{R}} \mathcal{N}_\beta^B f(t,x) < \infty,$$

where we recall that $\mathcal{N}_\beta^B$ is defined by (3.1).

We equip $\mathfrak{X}_T^\beta(B)$ with the norm $\|\cdot\|_{\mathfrak{X}_T^\beta(B)}$ defined above. Then $\mathfrak{X}_T^\beta(B)$ is a normed vector space. In fact, the following proposition states that $\mathfrak{X}_T^\beta(B)$ is complete.

Proposition 3.4. *$\mathfrak{X}_T^\beta(B)$ is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{X}_T^\beta(B)$. Since the space $C_b([0,T]\times\mathbb{R};B)$ of bounded continuous functions from $[0,T]\times\mathbb{R}$ to B is complete, there exists a bounded continuous function $f:[0,T]\times\mathbb{R}\to B$ such that

$$\lim_{n\to\infty}\sup_{t\in[0,T],\,x\in\mathbb{R}} \|f_n(t,x)-f(t,x)\| = 0\,.$$

For any $\varepsilon>0$ there exists $n_0>0$ such that

$$\sup_{x\in\mathbb{R}} \mathcal{N}_\beta^B(f_n-f_m)(t,x) < \varepsilon$$

for all $m,n\ge n_0$. It follows from Fatou's lemma that

$$\mathcal{N}_\beta^B(f_n-f)(t,x) \le \liminf_{m\to\infty} \mathcal{N}_\beta^B(f_n-f_m)(t,x) \le \varepsilon$$

for every $t\in[0,T]$, $x\in\mathbb{R}$ and $n\ge n_0$. This implies that $\lim_{n\to\infty}\sup_{t\le T,x\in\mathbb{R}}\mathcal{N}_\beta^B(f_n-f)(t,x)=0$ which means f_n converges to f in $\mathfrak{X}_T^\beta(B)$. □

When $B=L^p(\Omega)$ with $p\in[1,\infty)$, we use the notation $\mathfrak{X}_T^{\beta,p}=\mathfrak{X}_T^\beta(L^p(\Omega))$. A function f in $\mathfrak{X}_T^{\beta,p}$ can be considered as a stochastic process indexed by (t,x) in $[0,T]\times\mathbb{R}$ such that

$$\sup_{t\in[0,T],\,x\in\mathbb{R}} \|f(t,x)\|_{L^p(\Omega)} + \sup_{t\in[0,T],x\in\mathbb{R}} \left(\int_{\mathbb{R}} \|f(t,x+y)-f(t,x)\|_{L^p(\Omega)}^2 |y|^{-2\beta-1} dy\right)^{\frac12} < \infty\,.$$

Next, for $\theta>0$, $\varepsilon>0$ and $\beta\in(0,1)$, we consider the following norm on $\mathfrak{X}_T^{\beta,p}$

$$\|u\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\beta,p}} := \sup_{t\in[0,T],x\in\mathbb{R}} e^{-\theta t}\|u(t,x)\|_{L^p(\Omega)} + \varepsilon \sup_{t\in[0,T],x\in\mathbb{R}} e^{-\theta t}\mathcal{N}_{\beta,p}u(t,x)\,, \tag{3.6}$$

where we recall that $\mathcal{N}_{\beta,p}$ is defined by (3.2).

Remark 3.5. (i) In the case $\varepsilon=1$, we simply write $\|\cdot\|_{\mathfrak{X}_{T,\theta}^{\beta,p}}$.

(ii) The second term in the norm in (3.6) is not invariant by scaling while the first term is. Indeed, denote $f_\lambda(t,x)=f(t,\lambda x)$, then

$$\sup_{x\in\mathbb{R}} \left(\int_{\mathbb{R}} \|f_\lambda(t,x+h)-f_\lambda(t,x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh\right)^{\frac12}$$

$$= \lambda^\beta \sup_{x\in\mathbb{R}} \left(\int_{\mathbb{R}} \|f(t,x+h) - f(t,x)\|^2_{L^p(\Omega)} |h|^{-1-2\beta} dh \right)^{\frac{1}{2}} .$$

This is the very reason why various orders of $(t-s)$ appear in the proof of Proposition 3.6 below. We bypass this technical difficulty by the introduction of an additional scaling factor ε in (3.6).

(iii) Another way to see the role of ε is via dimensional analysis. Suppose that the amplitude of f has unit L, the spatial variable x has unit S, while the randomness ω is dimensionless. Then the first term in (3.6) has unit L while the second term has unit L/S^β. Hence, in order for the two terms to have the same dimension, we multiply the second term with a constant ε having unit of S^β.

(iv) Because T is finite, the norm $\|\cdot\|_{\mathfrak{X}^{\beta,p}_{T,\theta,\varepsilon}}$ defined as above is equivalent to the norm $\|\cdot\|_{\mathfrak{X}^{\beta,p}_T}$.

The next proposition gives a convenient bound on the stochastic convolution in term of the spaces $\mathfrak{X}^{\beta,p}_T$.

Proposition 3.6. *Consider a predictable random field $f \in \Lambda_H$ and define process $\{\Phi(t,x),\ t \geq 0, x \in \mathbb{R}\}$ by*

$$\Phi(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f(s,y) W(ds,dy) \,. \tag{3.7}$$

Then, for any $\beta < H$ and $p \geq 2$, the following inequality holds:

$$\begin{aligned} \|\Phi\|_{\mathfrak{X}^{\beta,p}_{T,\theta,\varepsilon}} &\leq C_0 \sqrt{p} \|f\|_{\mathfrak{X}^{\frac{1}{2}-H,p}_{T,\theta,\varepsilon}} \\ &\quad \times \left(\kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}} + \kappa^{-\frac{1}{4}-\frac{\beta}{2}} \theta^{\frac{\beta}{2}-\frac{1}{4}} + \varepsilon^{-1} \kappa^{-\frac{1}{4}} \theta^{-\frac{1}{4}} + \varepsilon \kappa^{\frac{H}{2}-\frac{\beta}{2}-\frac{1}{2}} \theta^{\frac{\beta}{2}-\frac{H}{2}} \right) , \end{aligned} \tag{3.8}$$

where C_0 is a constant depending only on H and β.

Remark 3.7. According to relation (3.8), the stochastic convolution induces some stability properties in the spaces $\mathfrak{X}^{\beta,p}_{\theta,\varepsilon}$ whenever $\frac{1}{2} - H < \beta < H$. This imposes the restriction $H > \frac{1}{4}$ already at this stage.

Proof of Proposition 3.6. According to our definition (3.6), we have $\|\Phi\|_{\mathfrak{X}^{\beta,p}_{T,\theta,\varepsilon}} = \mathcal{A}_1 + \varepsilon \mathcal{A}_2$, with

$$\mathcal{A}_1 = \sup_{t\in[0,T],x\in\mathbb{R}} e^{-\theta t} \|\Phi(t,x)\|_{L^p(\Omega)}, \quad \text{and} \quad \mathcal{A}_2 = \sup_{t\in[0,T],x\in\mathbb{R}} e^{-\theta t} \mathcal{N}_{\beta,p} \Phi(t,x).$$

We now estimate those terms separately. Along the proof C will denote a generic constant depending only on H and β.

Step 1: Upper bound for $\mathcal{A}_1$. The term $\Phi(t,x)$ is of the form

$$\int_0^t \int_{\mathbb{R}} g_{t,x}(s,y) W(ds,dy), \quad \text{with} \quad g_{t,x}(s,y) = p_{t-s}(x-y) f(s,y).$$

Applying inequality (3.3), we thus have

$$\|\Phi(t,x)\|_{L^p(\Omega)} \leq C\sqrt{p} \left(\int_0^t \int_{\mathbb{R}^2} \|g_{t,x}(s,y+h) - g_{t,x}(s,y)\|^2_{L^p(\Omega)} \, |h|^{2H-2} \, dh \, dy \, ds \right)^{\frac{1}{2}} .$$

A simple decomposition of the increment $g_{t,x}(s,y+h) - g_{t,x}(s,y)$ then yields

$$\|\Phi(t,x)\|_{L^p(\Omega)} \leq C\sqrt{p}\left[\left(\int_0^t J_1(s)ds\right)^{\frac{1}{2}} + \left(\int_0^t J_2(s)\,ds\right)^{\frac{1}{2}}\right],$$

where

$$J_1(s) = \int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y-z) - p_{t-s}(x-y)|^2 \|f(s,y+z)\|_{L^p(\Omega)}^2 |z|^{2H-2} dydz$$

and

$$J_2(s) = \int_{\mathbb{R}}\int_{\mathbb{R}} p_{t-s}^2(x-y)\|f(s,y+z) - f(s,y)\|_{L^p(\Omega)}^2 |z|^{2H-2} dydz\,.$$

To estimate $J_1(s)$, we write

$$J_1(s) \leq \sup_{x\in\mathbb{R}} \|f(s,x)\|_{L^p(\Omega)}^2 \int_{\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(y)]^2 dy.$$

Applying Lemma 3.1 with $\beta = \frac{1}{2} - H$, we obtain

$$J_1(s) \leq C \sup_{x\in\mathbb{R}} \|f(s,x)\|_{L^p(\Omega)}^2 [\kappa(t-s)]^{H-1}\,.$$

Let us now turn to estimate $J_2(s)$. Recalling our notation (3.2), we have

$$\begin{aligned} J_2(s) &= \int_{\mathbb{R}} p_{t-s}^2(x-y)[\mathcal{N}_{\frac{1}{2}-H,p} f(s,y)]^2\,dy \leq \sup_{x\in\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H,p} f(s,x)]^2 \int_{\mathbb{R}} p_{t-s}^2(x-y)\,dy \\ &\leq [2\pi\kappa(t-s)]^{-\frac{1}{2}} \sup_{x\in\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H,p} f(s,x)]^2. \end{aligned} \tag{3.9}$$

Hence, putting together our bounds on J_1 and J_2, we get

$$\begin{aligned} e^{-\theta t} \sup_{x\in\mathbb{R}} \|\Phi(t,x)\|_{L^p(\Omega)} &\leq C\sqrt{p} \sup_{\substack{0\leq s\leq T\\ x\in\mathbb{R}}} e^{-\theta s} \|f(s,x)\|_{L^p(\Omega)} \left(\int_0^t e^{-2\theta(t-s)} [\kappa(t-s)]^{H-1} ds\right)^{\frac{1}{2}} \\ &\quad + C\sqrt{p}\,\varepsilon \sup_{\substack{0\leq s\leq T\\ x\in\mathbb{R}}} e^{-\theta s} \sup_{x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H,p} f(s,x)\, \frac{\left(\int_0^t e^{-2\theta(t-s)} [\kappa(t-s)]^{-\frac{1}{2}} ds\right)^{\frac{1}{2}}}{\varepsilon}, \end{aligned}$$

and some elementary computations for the integrals above yield

$$\mathcal{A}_1 = \sup_{t\in[0,T], x\in\mathbb{R}} e^{-\theta t}\ \|\Phi(t,x)\|_{L^p(\Omega)} \leq C\sqrt{p} \|f\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\frac{1}{2}-H,p}} (\kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}} + \varepsilon^{-1} \kappa^{-\frac{1}{4}} \theta^{-\frac{1}{4}})\,.$$

Step 2: Upper bound for $\mathcal{A}_2$. According to the definition of $\mathcal{A}_2$, we have to bound $\mathcal{N}_{\beta,p}\Phi(t,x)$, where we recall that

$$\mathcal{N}_{\beta,p}\Phi(t,x) = \left(\int_{\mathbb{R}} \|\Phi(t,x+h) - \Phi(t,x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh\right)^{\frac{1}{2}}. \tag{3.10}$$

Furthermore, arguing as in Step 1 above, it is easily seen that

$$\|\Phi(t,x+h) - \Phi(t,x)\|_{L^p(\Omega)} \leq C\sqrt{p}\left[\left(\int_0^t J_1'(s,h)\,ds\right)^{1/2} + \left(\int_0^t (J_2'(s,h))\,ds\right)^{1/2}\right], \tag{3.11}$$

where

$$J_1'(s,h)=\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{t-s}(x+h-y-z)-p_{t-s}(x-y-z)-p_{t-s}(x+h-y)+p_{t-s}(x-y)|^2 \times\|f(s,y+z)\|_{L^p(\Omega)}^2|z|^{2H-2}dydz\,,$$

and

$$J_2'(s,h)=\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{t-s}(x+h-y)-p_{t-s}(x-y)|^2\|f(s,y+z)-f(s,y)\|_{L^p(\Omega)}^2|z|^{2H-2}dydz\,.$$

Plugging (3.11) into (3.10), we end up with

$$\mathcal{N}_{\beta,p}\Phi(t,x)\le C\sqrt{p}\left[\int_0^t\int_{\mathbb{R}}J_1'(s,h)|h|^{-1-2\beta}\,dh\,ds+\int_0^t\int_{\mathbb{R}}J_2'(s,h)|h|^{-1-2\beta}\,dh\,ds\right].$$

In addition, arguing again as in the proof of Lemma 3.1, we can show that

$$\int_{\mathbb{R}}J_1'(s,h)|h|^{-1-2\beta}dh\le C[\kappa(t-s)]^{H-\beta-1}\sup_{x\in\mathbb{R}}\|f(s,x)\|_{L^p(\Omega)}^2.$$

On the other hand, applying Lemma 3.1 leads to

$$\int_{\mathbb{R}}J_2'(s,h)|h|^{-1-2\beta}dh\le C[\kappa(t-s)]^{-\frac12-\beta}\sup_{x\in\mathbb{R}}[\mathcal{N}_{\frac12-H,p}f(s,x)]^2\,.$$

Combining these estimates for J_1', J_2' and resorting to (3.11), similarly as the estimate for $e^{-\theta t}\|\Phi(t,x)\|_{L^p(\Omega)}$, we obtain

$$\mathcal{A}_2\le C\sqrt{p}\left(\|f\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\frac12-H,p}}\kappa^{\frac{H}{2}-\frac{\beta}{2}-\frac12}\theta^{\frac{\beta}{2}-\frac{H}{2}}+\varepsilon^{-1}\|f\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\frac12-H,p}}\kappa^{-\frac14-\frac{\beta}{2}}\theta^{\frac{\beta}{2}-\frac14}\right).$$

Putting together Step 1 and Step 2, our claim (3.8) is now easily checked. □

We conclude this section by a simple remark which is labeled for further use. In the particular case $\beta=\frac12-H$, and using the simplified notation $\|\cdot\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\frac12-H,p}}=\|\cdot\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{p}}$, the estimate (3.8) can be written as

$$\|\Phi\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{p}}\le C_0\sqrt{p}\|f\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{p}}\left(\kappa^{\frac{H}{2}-\frac12}\theta^{-\frac{H}{2}}+\varepsilon^{-1}\kappa^{-\frac14}\theta^{-\frac14}+\varepsilon\kappa^{H-\frac34}\theta^{\frac14-H}\right).\tag{3.12}$$

3.2. **Hölder continuity estimates.** A natural question arising from the definition (3.7) of the process Φ is the derivation of Hölder type exponents in both time and space. Some estimates in this direction are provided in the next proposition. We set $\mathfrak{X}_T^p=\mathfrak{X}_T^{\frac12-H,p}$, and the norm $\|\cdot\|_{\mathfrak{X}_{T,\theta_0}^p}$ is given by (3.6) with $\varepsilon=1$ and $\beta=\frac12-H$.

Proposition 3.8. *Recall that the noise W is given by the covariance (2.1). Consider $p\ge2$ and a predictable random field $f\in\mathfrak{X}_T^p$, where T is a fixed finite time horizon. Let θ_0 be any positive number. We define the random field Φ as in (3.7). Then for every $x,h\in\mathbb{R}$, $t_1,t_2\in[0,T]$ and every $\gamma\in[0,H]$ we have*

$$\|\Phi([t_1,t_2],x+h)-\Phi([t_1,t_2],x)\|_{L^p(\Omega)}\le C\sqrt{p}e^{\theta_0T}\|f\|_{\mathfrak{X}_{T,\theta_0}^p}|t_2-t_1|^{\frac{H-\gamma}{2}}|h|^{\gamma}\,.\tag{3.13}$$

In the above, the constant C depends on T and does not depend on p, and we are using the notation

$$\Phi([t_1,t_2],x)=\Phi(t_2,x)-\Phi(t_1,x)\,.$$

In particular, if we let $t_1=0$, we get the Hölder estimate of the space variable. For the Hölder estimate of the time variable, we have

$$\|\Phi(t_2,x)-\Phi(t_1,x)\|_{L^p(\Omega)}\le C\sqrt{p}e^{\theta_0 T}\|f\|_{\mathfrak{X}^p_{T,\theta_0}}|t_2-t_1|^{\frac{H}{2}}\,. \tag{3.14}$$

Proof. First we prove (3.13). Without loss of generality, we assume $t_1<t_2$ and denote $\Delta t=t_2-t_1$. We also set:

$$V_1(f)=\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\|f(t,x)\|_{L^p(\Omega)}\,,\qquad V_2(f)=\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\mathcal{N}_{H,p}f(t,x), \tag{3.15}$$

and $V(f)=V_1(f)+V_2(f)$. Observe that according to (3.6), we have $V(f)\le\exp(\theta_0T)\|f\|_{\mathfrak{X}^p_{T,\theta_0}}$.

As in the proof of Proposition 3.6, we first write $\Phi([t_1,t_2],x+h)-\Phi([t_1,t_2],x)=\mathcal{A}_1+\mathcal{A}_2$, where

$$\mathcal{A}_1=\int_0^{t_1}\int_{\mathbb{R}}[p_{[t_1-s,t_2-s]}(x+h-y)-p_{[t_1-s,t_2-s]}(x-y)]f(s,y)W(ds,dy)\,,$$

and

$$\mathcal{A}_2=\int_{t_1}^{t_2}\int_{\mathbb{R}}[p_{t_2-s}(x+h-y)-p_{t_2-s}(x-y)]f(s,y)W(ds,dy)\,.$$

We now treat those two terms separately. To alleviate notation we will include the $\sqrt{p}$ into the constant C below.

Step 1: Upper bound for $\mathcal{A}_1$. The computations are carried out analogously to the proof of Proposition 3.6, and we have

$$\|\mathcal{A}_1\|^2_{L^p(\Omega)}\le C\int_0^{t_1}(A_{11}(s)+A_{12}(s))ds,$$

where A_{11} and A_{12} are analogous to J_1, J_2 in the proof of Proposition 3.6, and are respectively defined by

$$A_{11}(s)=\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{[t_1-s,t_2-s]}(x+h-y-z)-p_{[t_1-s,t_2-s]}(x-y-z)\\ -p_{[t_1-s,t_2-s]}(x+h-y)+p_{[t_1-s,t_2-s]}(x-y)|^2\|f(s,y+z)\|^2_{L^p(\Omega)}|z|^{2H-2}dydz\,,$$

and

$$A_{12}(s)=\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{[t_1-s,t_2-s]}(x+h-y)-p_{[t_1-s,t_2-s]}(x-y)|^2\\ \times\|f(s,y+z)-f(s,y)\|^2_{L^p(\Omega)}|z|^{2H-2}dydz\,.$$

Let us now bound A_{11}. Invoking Plancherel's identity with respect to y and the explicit formula for $\mathcal{F}p_t$, we have

$$A_{11}(s)\le CV_1^2(f)\quad\int_{\mathbb{R}}\int_{\mathbb{R}}\Big|p_{[t_1-s,t_2-s]}(h+y-z)-p_{[t_1-s,t_2-s]}(y-z)\\ -p_{[t_1-s,t_2-s]}(h+y)+p_{[t_1-s,t_2-s]}(y)\Big|^2|z|^{2H-2}dydz$$

$$\leq CV_1^2(f)\int_{\mathbb{R}}\int_{\mathbb{R}}|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}|^2|e^{-i\xi z}-1|^2|e^{i\xi h}-1|^2|z|^{2H-2}d\xi dz$$
$$\leq CV_1^2(f)\int_{\mathbb{R}}|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}|^2|e^{i\xi h}-1|^2|\xi|^{1-2H}d\xi\,.$$

Moreover, owing to the inequality

$$\int_0^{t_1}|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}|^2ds\leq\frac{|e^{-\frac{\Delta t\kappa}{2}|\xi|^2}-1|^2}{\kappa|\xi|^2}\,,\tag{3.16}$$

we obtain

$$\begin{aligned}\int_0^{t_1}A_{11}(s)ds&\leq C\kappa^{-1}V_1^2(f)\int_{\mathbb{R}}|e^{-\frac{\Delta t\kappa}{2}|\xi|^2}-1|^2|e^{i\xi h}-1|^2|\xi|^{-1-2H}d\xi\\&\leq C\kappa^{-1}V_1^2(f)\,I\,,\end{aligned}\tag{3.17}$$

where

$$I:=\int_{\mathbb{R}}|1-e^{-\frac{\Delta t\kappa}{2}|\xi|^2}|^2\sin^2(\xi h/2)|\xi|^{-1-2H}d\xi.\tag{3.18}$$

Our next step is to bound I in two elementary and different ways.

(i) The change of variable $h\xi:=\xi$ yields

$$I=|h|^{2H}\int_{\mathbb{R}}\left(1-e^{-\frac{\kappa\Delta t}{2|h|^2}|\xi|^2}\right)^2\sin^2(\xi/2)|\xi|^{-1-2H}d\xi\,,$$

and we then bound $1-e^{-\frac{\kappa\Delta t}{2h^2}}$ by 1 to obtain $I\leq C|h|^{2H}$.

(ii) On the other hand, the change of variable $(\kappa\Delta t)^{1/2}\xi:=\xi$ in (3.18) leads to

$$I=(\kappa\Delta t)^H\int_{\mathbb{R}}\left(1-e^{-\xi^2/2}\right)^2\sin^2\left(\frac{h\xi}{2(\kappa\Delta t)^{1/2}}\right)|\xi|^{-1-2H}d\xi\,,$$

and we bound the trigonometric function $\sin^2$ by 1 to obtain $I\leq C(\kappa\Delta t)^H$.

Interpolating the two estimates we have obtained for I, with a coefficient $\delta=\frac{\gamma}{2H}\in[0,1]$, we see that

$$I\leq C|h|^{2H\delta}\,(\kappa\Delta t)^{H(1-\delta)}\leq C(\kappa\Delta t)^{\frac{2H-\gamma}{2}}|h|^{\gamma}\,.\tag{3.19}$$

Plugging this identity back into (3.17), we have shown

$$\int_0^{t_1}A_{11}(s)ds\leq C\kappa^{-1}(\kappa\Delta t)^{\frac{2H-\gamma}{2}}|h|^{\gamma}V_1^2(f)\,,$$

forall $\gamma\in[0,2H]$. Let us now turn to the estimate for A_{12}. Similarly to what has been done for A_{11} we get

$$\begin{aligned}\int_0^{t_1}A_{12}(s)ds&\leq CV_2^2(f)\int_0^{t_1}\int_{\mathbb{R}}|p_{[t_1-s,t_2-s]}(h+y)-p_{[t_1-s,t_2-s]}(y)|^2dyds\\&\leq CV_2^2(f)\int_{\mathbb{R}}\int_0^{t_1}|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}|^2ds|e^{i\xi h}-1|^2d\xi\,.\end{aligned}$$

Thanks to (3.16), we thus end up with

$$\int_0^{t_1}A_{12}(s)ds\leq C\kappa^{-1}V_2^2(f)\int_{\mathbb{R}}|1-e^{-\frac{\Delta t\kappa}{2}|\xi|^2}|^2\sin^2(h\xi/2)|\xi|^{-2}d\xi\,.$$

In addition, the integral on the right hand side can be estimated as I above, and we get

$$\int_0^{t_1} A_{12}(s)ds \le C\, V_2^2(f)(\kappa\Delta t)^{\frac{1-\gamma'}{2}}|h|^{\gamma'}$$

for all $\gamma' \in [0,1]$. Since $1 > 2H$, we may choose $\gamma' = \gamma$ to obtain

$$\int_0^{t_1} A_{12}(s)ds \le C\kappa^{-1}(\kappa\Delta t)^{\frac{2H-\gamma}{2}}|h|^{\gamma}V_2^2(f)\,,$$

for all $\gamma \in [0,2H]$. Hence, the bounds on A_{11} and A_{12} yield

$$\|\mathcal{A}_1\|^2_{L^p(\Omega)} \le CV^2(f)(\Delta t)^{\frac{2H-\gamma}{2}}h^{\gamma}\,,$$

for all $\gamma \in [0,2H]$.

Step 2: Upper bound for $\mathcal{A}_2$. The term $\|\mathcal{A}_2\|^2_{L^p(\Omega)}$ can be estimated analogously to $\mathcal{A}_1$. Indeed, the reader can check that, owing to inequality (3.3) and Plancherel's identity, we have

$$\|\mathcal{A}_2\|^2_{L^p(\Omega)} \le CV_1^2(f)\int_0^{\Delta t}\int_{\mathbb{R}} e^{-s\kappa|\xi|^2}\sin^2(h\xi/2)(|\xi|^{1-2H}+1)d\xi ds\,,$$

where we recall that V_1 is defined by (3.15). Taking integration in ds first, we see that

$$\|\mathcal{A}_2\|^2_{L^p(\Omega)} \le C\kappa^{-1}V_1^2(f)\int_{\mathbb{R}}(1-e^{-\Delta t\kappa|\xi|^2})\sin^2(h\xi/2)(|\xi|^{-1-2H}+|\xi|^{-2})d\xi\,.$$

These two integrals can be estimated as the term I in (3.19), and we get

$$\|\mathcal{A}_2\|^2_{L^p(\Omega)} \le CV_1^2(f)(\Delta t)^{\frac{2H-\gamma}{2}}|h|^{\gamma}\,,$$

for all $\gamma \in [0,2H]$. Let us remark that the constants in all previous estimates depend only on T, p and κ^{-1}. In addition, as functions of (p,κ^{-1}), these constants have at most polynomial growth. Hence, gathering the estimates for $\|\mathcal{A}_1\|^2_{L^p(\Omega)}$ and $\|\mathcal{A}_2\|^2_{L^p(\Omega)}$ the proof of our claim (3.13) is finished.

Step 3: Proof of (3.14). Again, we assume that $t_1 < t_2$, and we proceed as in the previous steps and the proof of Proposition 3.6. Indeed, we begin by writing

$$\|\Phi(t_2,x)-\Phi(t_1,x)\|_{L^p(\Omega)} \le B_1 + B_2,$$

where

$$B_1 = \left\|\int_0^{t_1}\int_{\mathbb{R}} p_{[t_1-s,t_2-s]}(x-y)f(s,y)W(ds,dy)\right\|_{L^p(\Omega)}$$

and

$$B_2 = \left\|\int_{t_1}^{t_2}\int_{\mathbb{R}} p_{t_2-s}(x-y)f(s,y)W(ds,dy)\right\|_{L^p(\Omega)}.$$

Once again we handle those two terms separately.

For the term B_1, we resort to inequality (3.3) in our usual way. We get

$$B_1 \;\le\; C\left(\int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}} p^2_{[t_1-s,t_2-s]}(x-y)\|f(s,y)-f(s,y+z)\|^2_{L^p(\Omega)}|z|^{2H-2}dzdyds\right)^{\frac{1}{2}}$$

$$+C\Big(\int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{[t_1-s,t_2-s]}(x-y)-p_{[t_1-s,t_2-s]}(x-y-z)|^2 \times\|f(s,y+z)\|^2_{L^p(\Omega)}|z|^{2H-2}dzdyds\Big)^{\frac12}.$$

With the definition (3.15) in mind, it is now readily checked that

$$B_1\le C\left(B_{11}V_2(f)+B_{12}V_1(f)\right)\,, \tag{3.20}$$

with

$$B_{11}=\Big(\int_0^{t_1}\int_{\mathbb{R}}|p_{[t_1-s,t_2-s]}(x-y)|^2dyds\Big)^{\frac12}$$

and

$$B_{12}=\Big(\int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{[t_1-s,t_2-s]}(x-y)-p_{[t_1-s,t_2-s]}(x-y-z)|^2|z|^{2H-2}dzdyds\Big)^{\frac12}.$$

We now appeal to Plancherel's identity to get

$$B_{11}=C\left(\int_0^{t_1}\int_{\mathbb{R}}\left|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\right|^2d\xi ds\right)^{\frac12}=C(t_2-t_1)^{\frac14}\,,$$

and

$$\begin{aligned}B_{12}&=C\left(\int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\right|^2|e^{-i\xi z}-1|^2|z|^{2H-2}dzd\xi ds\right)^{\frac12}\\&=C\left(\int_0^{t_1}\int_{\mathbb{R}}\left|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\right|^2|\xi|^{1-2H}d\xi ds\right)^{\frac12}\\&=C(t_2-t_1)^{\frac H2}\,.\end{aligned}$$

Reporting these estimates in (3.20) and observing that $H<\frac12$, we end up with

$$B_1\le C(t_2-t_1)^{\frac H2}\left[V_1(f)+V_2(f)\right]\le C(t_2-t_1)^{\frac H2}\|f\|_{\mathfrak{X}^p_{T,\theta_0}}e^{\theta_0 T}.$$

The patient reader might check that the same kind of upper bound is valid for B_2, and gathering the estimates for B_1 and B_2 yields inequality (3.14). □

4. Existence and uniqueness of the solution

In this section we will establish a result regarding the uniqueness of the solution. Then we will describe the structure of some new spaces of stochastic processes. Those spaces will finally be used to show the existence of the solution.

4.1. **Uniqueness of the solution.**

In this subsection we give some results about the uniqueness of the solution assuming that the solution has enough regularity. To this end, we first introduce a norm $\|\cdot\|_{\mathcal{Z}^p_T}$ for a random field $v(t,x)$ as follows

$$\|v\|_{\mathcal{Z}^p_T}=\sup_{t\in[0,T]}\|v(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}+\sup_{t\in[0,T]}\mathcal{N}^*_{\frac12-H,p}v(t), \tag{4.1}$$

where $p \geq 2$ and

$$\mathcal{N}^*_{\frac{1}{2}-H,p} v(t) = \left(\int_{\mathbb{R}} \|v(t,\cdot) - v(t,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2} dh \right)^{\frac{1}{2}} . \tag{4.2}$$

Then the space $\mathcal{Z}^p_T$ will consist all the random fields such that the above quantity is finite. Observe that according to the definition (3.1), we have $\mathcal{N}^*_{\frac{1}{2}-H,p} v(t) = \mathcal{N}^{L^p(\Omega\times\mathbb{R})}_{\frac{1}{2}-H,p} v(t)$.

Remark 4.1. A similar quantity to $\mathcal{N}^*_{\frac{1}{2}-H}$ defined in (4.2) appears in the characterization of Besov spaces in finite differences, see [2, Theorem 2.36].

The proof of the uniqueness theorem requires a localization argument, based on uniform estimates (in space and time) of stochastic convolutions. This is provided by the following lemma.

Lemma 4.2. *Suppose that $p > \frac{6}{4H-1}$. Let v be a process in the space $\mathcal{Z}^p_T$. As in (3.7), define*

$$\Phi(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) v(s,y) W(ds,dy) . \tag{4.3}$$

Then, there exists a constant C depending on T, p and H, such that

$$\left\| \sup_{t\in[0,T],\, x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \Phi(t,x) \right\|_{L^p(\Omega)} \leq C \|v\|_{\mathcal{Z}^p_T}, \tag{4.4}$$

where recalling our definition (3.1), we have $\mathcal{N}_{1/2-H}\Phi(t,x) = \mathcal{N}^{\mathbb{R}}_{1/2-H}\Phi(t,x)$.

Remark 4.3. Let us stress the following facts:

(i) In relation (4.4), the operator $\mathcal{N}_{\frac{1}{2}-H}$ (defined in (3.2)) acts on the trajectories of the random field $\Phi(t,x)$. As a consequence, $\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$ is a random variable.

(ii) With respect to Proposition 3.6, inequality (4.4) involves a sup in the variable $x \in \mathbb{R}$ before taking $L^p(\Omega)$ norms. We thus get a stronger result with different kind of assumptions (namely $v \in \mathcal{Z}^p_T$ instead of $v \in \mathfrak{X}^{\frac{1}{2}-H,p}_T$).

Proof of Lemma 4.2. We shall apply the factorization method to handle the stochastic convolution (see, for instance, [12]). Namely, an application of a stochastic version of Fubini's theorem enables to write

$$\Phi(t,x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} p_{t-r}(x-z) Y(r,z) dz dr ,$$

with

$$Y(r,z) = \int_0^r \int_{\mathbb{R}} (r-s)^{-\alpha} p_{r-s}(z-y) v(s,y) W(ds,dy) ,$$

and where $\alpha \in (0,1)$ is a parameter whose value will be chosen later. The proof will be done in two steps.

Step 1: Uniform estimate of $\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$. In order to estimate $\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$, we bound the difference $\Phi(t,x) - \Phi(t,x+h)$ as follows

$$|\Phi(t,x) - \Phi(t,x+h)|$$

$$
\begin{aligned}
&= \frac{\sin(\alpha\pi)}{\pi}\left|\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\left(p_{t-r}(x-z)-p_{t-r}(x+h-z)\right)Y(r,z)dzdr\right| \\
&\le \frac{\sin(\alpha\pi)}{\pi}\int_0^t (t-r)^{\alpha-1}\left\|p_{t-r}(\cdot)-p_{t-r}(\cdot+h)\right\|_{L^q(\mathbb{R})}\left\|Y(r,\cdot)\right\|_{L^p(\mathbb{R})}dr\,,
\end{aligned}
$$

where q satisfies $p^{-1}+q^{-1}=1$. So using Minkowski's integral inequality, we get

$$
\begin{aligned}
&\int_{\mathbb{R}}|\Phi(t,x)-\Phi(t,x+h)|^2|h|^{2H-2}dh \\
&\le C\int_{\mathbb{R}}\left(\int_0^t (t-r)^{\alpha-1}\left\|p_{t-r}(x-\cdot)-p_{t-r}(x+h-\cdot)\right\|_{L^q(\mathbb{R})}\left\|Y(r,\cdot)\right\|_{L^p(\mathbb{R})}dr\right)^2|h|^{2H-2}dh \\
&\le C\left(\int_0^t (t-r)^{\alpha-1}\left\|Y(r,\cdot)\right\|_{L^p(\mathbb{R})}\left[K_t(r)\right]^{1/2}\,dr\right)^2, \qquad (4.5)
\end{aligned}
$$

where we have set

$$
K_t(r):=\int_{\mathbb{R}}\left\|p_{t-r}(x-z)-p_{t-r}(x+h-z)\right\|^2_{L^q(\mathbb{R},dz)}|h|^{2H-2}dh\,.
$$

Now the kernel K_t can be bounded by elementary methods: with the change of variable $z\to\sqrt{t-r}z$ and $h\to\sqrt{t-r}h$, we obtain

$$
\begin{aligned}
K_t(r)&=\int_{\mathbb{R}}\left\|p_{t-r}(x-z)-p_{t-r}(x+h-z)\right\|^2_{L^q(\mathbb{R},dz)}|h|^{2H-2}dh \\
&=C\,(t-r)^{-\frac32+\frac1q+H}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|e^{-\frac{z^2}{2\kappa}}-e^{-\frac{(z+h)^2}{2\kappa}}\right|^q dz\right)^{\frac2q}|h|^{2H-2}dh=C(t-r)^{-\frac12-\frac1p+H}\,,
\end{aligned}
$$

where we have used the fact that $q^{-1}=1-p^{-1}$, and the constant C in the above equation and below in this proof may depend on κ. Going back to (4.5), the following holds true:

$$
\begin{aligned}
\int_{\mathbb{R}}|\Phi(t,x)-\Phi(t,x+h)|^2|h|^{2H-2}dh &\le C\left(\int_0^t (t-r)^{\alpha-1+\frac12(H-\frac1p-\frac12)}\left\|Y(r,\cdot)\right\|_{L^p(\mathbb{R})}dr\right)^2 \\
&\le C\left(\int_0^t (t-r)^{q[\alpha-1+\frac12(H-\frac12-\frac1p)]}dr\right)^{\frac2q}\left(\int_0^t\left\|Y(r,\cdot)\right\|^p_{L^p(\mathbb{R})}dr\right)^{\frac2p}.
\end{aligned}
$$

We can now start to tune our parameters. It is easily checked that the first integral in the right hand side above is finite (uniformly in $0<t\le T$) if and only if

$$
\alpha>\frac{3}{2p}+\frac14-\frac H2\,. \qquad (4.6)
$$

With this choice of α, we get

$$
\int_{\mathbb{R}}|\Phi(t,x)-\Phi(t,x+h)|^2|h|^{2H-2}dh\le C\left(\int_0^t\left\|Y(r,\cdot)\right\|^p_{L^p(\mathbb{R})}dr\right)^{\frac2p},
$$

and since this bound is uniform in x, this yields

$$
\sup_{t\in[0,T],x\in\mathbb{R}}\left[\mathcal{N}_{\frac12-H}\Phi(t,x)\right]^2\le C\left(\int_0^T\left\|Y(r,\cdot)\right\|^p_{L^p(\mathbb{R})}dr\right)^{\frac2p}. \qquad (4.7)
$$

Then, to prove (4.4) it suffices to show that

$$\mathbf{E}\int_{\mathbb{R}}|Y(r,z)|^p\,dz \le C\|v\|^p_{\mathcal{Z}^p_T}. \tag{4.8}$$

Step 2: Proof of (4.8). Set $g_{r,z}(s,y)=(r-s)^{-\alpha}p_{r-s}(z-y)v(s,y)$, so that

$$Y(r,z)=\int_0^r\int_{\mathbb{R}} g_{r,z}(s,y)\,W(ds,dy).$$

Then applying the Burkholder type inequality (3.3), plus an elementary decomposition of the increments of $g_{r,z}$, we obtain

$$\mathbf{E}\int_{\mathbb{R}}|Y(r,z)|^p\,dz \le C\,\left[D_1(r)+D_2(r)\right],$$

where

$$D_1(r)=\int_{\mathbb{R}}\Big(\int_0^r\int_{\mathbb{R}^2}(r-s)^{-2\alpha}\left|p_{r-s}(y)-p_{r-s}(y+h)\right|^2 \times \|v(s,y+z+h)\|^2_{L^p(\Omega)}\,|h|^{2H-2}dhdyds\Big)^{\frac{p}{2}}dz$$

and

$$D_2(r)=\int_{\mathbb{R}}\Big(\int_0^r\int_{\mathbb{R}^2}(r-s)^{-2\alpha}\left|p_{r-s}(y)\right|^2 \times \|v(s,y+z+h)-v(s,y+z)\|^2_{L^p(\Omega)}\,|h|^{2H-2}dhdyds\Big)^{\frac{p}{2}}dz.$$

Let us now bound the term D_1. Invoking Minkowski's integral inequality, it is easily seen that

$$D_1(r)\le\Big(\int_0^r\int_{\mathbb{R}^2}(r-s)^{-2\alpha}\left|p_{r-s}(y)-p_{r-s}(y+h)\right|^2\|v(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}\,|h|^{2H-2}dhdyds\Big)^{\frac{p}{2}}.$$

Integrating this identity in h and y, we end up with

$$D_1(r)\le C\Big(\int_0^r(r-s)^{-2\alpha+H-1}\|v(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}\,ds\Big)^{\frac{p}{2}}.$$

Similarly we get the following estimate for $D_2(r)$

$$\begin{aligned} D_2(r) &\le C\left(\int_0^r\int_{\mathbb{R}}(r-s)^{-2\alpha-\frac12}\,\|v(s,\cdot+h)-v(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}\,|h|^{2H-2}dhds\right)^{\frac{p}{2}} \\ &= C\left(\int_0^r(r-s)^{-2\alpha-\frac12}\left[\mathcal{N}^*_{\frac12-H,p}v(s)\right]^2ds\right)^{\frac{p}{2}}. \end{aligned}$$

Combining the estimates for $D_1(r)$ and $D_2(r)$ we obtain

$$\mathbf{E}\int_{\mathbb{R}}|Y(r,z)|^p\,dz\le C\left(\int_0^r\Big[(r-s)^{-2\alpha+H-1}\|v(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} + (r-s)^{-2\alpha-\frac12}\left[\mathcal{N}^*_{\frac12-H,p}v(s)\right]^2\Big]\,ds\right)^{\frac{p}{2}}. \tag{4.9}$$

Let us go back now to the values of our parameters α, p. One can check that the two singularities in the integrals on the right hand side above are non divergent whenever $\alpha < \frac{H}{2}$. Combining this condition with the restriction (4.6), we end up with the relation

$$\frac{3}{2p} + \frac{1}{4} - \frac{H}{2} < \alpha < \frac{H}{2}. \tag{4.10}$$

Those two conditions can be jointly met if and only if $H > \frac{1}{4}$ and $p > \frac{6}{4H-1}$. This completes the proof of the lemma. □

Remark 4.4. Notice that the previous lemma implies that for any process $v \in \mathcal{Z}_T^p$, the random variable $\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$ is finite almost surely, if Φ is given by (4.3).

We can now turn to the uniqueness result for equation (1.1).

Theorem 4.5. *Assume the following conditions hold true:*

(1) For $p > \frac{6}{4H-1}$, the initial condition u_0 is in $L^p(\mathbb{R})$ and

$$\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|^2_{L^p(\mathbb{R})} |h|^{2H-2} dh < \infty\,. \tag{4.11}$$

(2) σ is differentiable, its derivative is Lipschitz and $\sigma(0) = 0$.

(3) u and v are two solutions of (1.1) and $u, v \in \mathcal{Z}_T^p$.

Then for every $t \in [0,T]$ and $x \in \mathbb{R}$, $u(t,x) = v(t,x), a.s.$

Remark 4.6. This is the first occurrence of the hypothesis $\sigma(0) = 0$, and one might wonder about the necessity of this assumption. To this respect, let us mention that if we define Φ as in (3.7) for $f \equiv \mathbf{1}$, then Φ does not belong to $\mathcal{Z}_T^p$.

Proof. Assume that u solves (1.1) and $u \in \mathcal{Z}_T^p$. From the mild formulation of the solution we have

$$u(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)\,. \tag{4.12}$$

We claim that

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} u(t,x) < \infty, \quad \text{a.s.} \tag{4.13}$$

This follows from the decomposition (4.12). Indeed, on one hand, (4.11) implies that, if $g(t,x) = p_t u_0(x)$, then

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} g(t,x) < \infty.$$

On the other hand, from the properties of σ, it follows that if $u \in \mathcal{Z}_T^p$, then $\sigma(u)$ also belongs to $\mathcal{Z}_T^p$ (notice that to estimate the first term of (4.1) for $\sigma(u)$, we need to assume $\sigma(0) = 0$). Hence, Remark 4.4 entails

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \sigma(u)(t,x) < \infty, \quad \text{a.s.}$$

If v is another solution of equation (1.1) belonging also to $\mathcal{Z}_T^p$, then (4.13) also holds for v. In this way, we can define the stopping times

$$T_k = \inf\left\{0 \le t \le T : \sup_{0\le s\le t, x\in\mathbb{R}} \int_{\mathbb{R}} |u(s,x) - u(s,x+h)|^2 |h|^{2H-2} dh \ge k\right.$$

$$\text{or} \quad \sup_{0\le s\le t, x\in\mathbb{R}} \int_{\mathbb{R}} |v(s,x)-v(s,x+h)|^2|h|^{2H-2}dh \ge k \Big\},$$

and $T_k \uparrow T$, almost surely, as k tends to infinity. Our strategy will be to control the two following quantities

$$I_1(t,x) = \mathbf{E}\left[\mathbf{1}_{\{t<T_k\}}|u(t,x)-v(t,x)|^2\right]$$

and

$$I_2(t,x) = \mathbf{E}\left[\int_{\mathbb{R}} \mathbf{1}_{\{t<T_k\}} \left|u(t,x)-v(t,x)-u(t,x+h)+v(t,x+h)\right|^2 |h|^{2H-2}dh\right].$$

We also set $\mathcal{I}_j(t) = \sup_{x\in\mathbb{R}} I_j(t,x)$ for $j=1,2$.

In order to bound I_1, let us first use elementary properties of Itô's integral, which yield

$$\mathbf{1}_{\{t<T_k\}} \left(u(t,x)-v(t,x)\right) = \mathbf{1}_{\{t<T_k\}} \int_0^{t\wedge T_k} \int_{\mathbb{R}} p_{t-s}(x-y)\left[\sigma(u(s,y))-\sigma(v(s,y))\right] W(ds,dy)$$

$$= \mathbf{1}_{\{t<T_k\}} \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\mathbf{1}_{\{s<T_k\}}\left[\sigma(u(s,y))-\sigma(v(s,y))\right] W(ds,dy).$$

We thus get $I_1(t,x) \le C(I_{11}(t,x)+I_{12}(t,x))$, where

$$I_{11}(t,x) = \mathbf{E}\int_0^t \int_{\mathbb{R}^2} |p_{t-s}(x-y)-p_{t-s}(x-y-h)|^2 \times \mathbf{1}_{\{s<T_k\}} \left|\sigma(u(s,y+h))-\sigma(v(s,y+h))\right|^2 |h|^{2H-2}dhdyds\,,$$

and

$$I_{12}(t,x) = \mathbf{E}\int_0^t \int_{\mathbb{R}^2} p_{t-s}^2(x-y)\mathbf{1}_{\{s<T_k\}}\Big|\sigma(u(s,y))-\sigma(v(s,y)) - \sigma(u(s,y+h))+\sigma(u(s,y+h))\Big|^2|h|^{2H-2}dhdyds\,.$$

Next we bound the term $I_{11}(t,x)$ as follows

$$I_{11}(t,x) \le C\,\mathbf{E}\int_0^t \int_{\mathbb{R}^2} |p_{t-s}(x-y)-p_{t-s}(x-y-h)|^2 \times \mathbf{1}_{\{s<T_k\}}|u(s,y+h)-v(s,y+h)|^2|h|^{2H-2}dhdyds \le C\int_0^t (t-s)^{H-1}\mathcal{I}_1(s)\,ds,$$

where we recall that $\mathcal{I}_1(t) = \sup_{x\in\mathbb{R}} I_1(t,x)$, and the constant C in the above inequality and below in this proof may depend on κ. Let us now invoke the following elementary bound on the rectangular increments of σ, valid whenever σ' is Lipschitz

$$|\sigma(a)-\sigma(b)-\sigma(c)+\sigma(d)| \le C|a-b-c+d| + C|a-b|(|a-c|+|b-d|)\,,$$

With this additional ingredient, and along the same lines as for $I_{11}(t,x)$, we get

$$I_{12}(t,x) \le Ck\int_0^t (t-s)^{-\frac{1}{2}}\left[\mathcal{I}_1(s)+\mathcal{I}_2(s)\right]\,ds\,.$$

Finally, gathering our estimates on I_{11} and I_{12} we end up with

$$\mathcal{I}_1(t) \le Ck\int_0^t (t-s)^{H-1}\left[\mathcal{I}_1(s)+\mathcal{I}_2(s)\right]\,ds\,.$$

The term $I_2(t,x)$ above is dealt with exactly the same way, and we leave to the reader the task of showing that

$$\mathcal{I}_2(t) \le Ck \int_0^t (t-s)^{2H-\frac{3}{2}} \left[\mathcal{I}_1(s) + \mathcal{I}_2(s)\right] ds\,.$$

As a consequence,

$$\mathcal{I}_1(t) + \mathcal{I}_2(t) \le Ck \int_0^t (t-s)^{2H-\frac{3}{2}} \left[\mathcal{I}_1(s) + \mathcal{I}_2(s)\right] ds\,,$$

which implies $\mathcal{I}_1(t) + \mathcal{I}_2(t) = 0$ for all $t \in [0,T]$. In particular,

$$\mathbf{E}\left[\mathbf{1}_{\{t<T_k\}} |u(t,x) - v(t,x)|^2\right] = 0\,,$$

which implies $u(t,x) = v(t,x)$ a.s. on $\{t < T_k\}$ for all $k \ge 1$ and $t \in [0,T]$. Therefore, taking into account that $T_k \uparrow \infty$ a.s. as k tends to infinity, we conclude that $u(t,x) = v(t,x)$ a.s. for all $(t,x) \in [0,T] \times \mathbb{R}$. This proves the uniqueness. □

4.2. **Space-time function spaces.** We introduce here the function spaces which form the underlying spaces of our treatment for the existence of the solution. Since these spaces do not belong to standard classes of function spaces, we describe them in detail.

We denote by $C_{\mathrm{uc}}([0,T]\times\mathbb{R})$ the space of all real-valued continuous functions on $[0,T]\times\mathbb{R}$ equipped with the topology of convergence uniformly over compact sets. Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$. Let $\beta \in (0,1)$ be a fixed number. For every $\delta \in (0,\infty]$ and every function $f : \mathbb{R} \to B$, we introduce the function $\mathcal{N}_\beta^{B,(\delta)} f : \mathbb{R} \to [0,\infty]$ defined by

$$\mathcal{N}_\beta^{B,(\delta)} f(x) = \left(\int_{|h|\le\delta} \|f(x+h) - f(x)\|^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}\,. \tag{4.14}$$

Notice that for $\delta = \infty$, the quantity (4.14) coincides with the function $\mathcal{N}_\beta^{B,(\infty)} f = \mathcal{N}_\beta^B f$ introduced in (3.1). As our usual practice, when $B = \mathbb{R}$ we omit the dependence of $\mathbb{R}$ in $\mathcal{N}_\beta^{\mathbb{R},(\delta)}$ and simply write $\mathcal{N}_\beta^{(\delta)}$.

As we will see later along the development of the paper, $\mathcal{N}_\beta^{B,(\delta)} f$ plays a role analogous to the modulus of continuity of f near x. It follows from the triangular inequality, that $\mathcal{N}$ satisfies

$$|\mathcal{N}_\beta^{B,(\delta)} f(x) - \mathcal{N}_\beta^{B,(\delta)} g(x)| \le \mathcal{N}_\beta^{B,(\delta)} (f-g)(x) \tag{4.15}$$

for all $\delta \in (0,\infty]$, functions f, g and x in $\mathbb{R}$. Thus, $\mathcal{N}$ is a seminorm.

Suppose, for instance, that a function f has modulus of continuity $|h|^\beta \omega(h)$ at x, for any $|h| \le \delta$. Then $[\mathcal{N}_\beta^{B,(\delta)} f(x)]^2$ is majorized by $2\int_0^\delta \omega^2(h) h^{-1} dh$. Thus, for $\mathcal{N}_\beta^{B,(\delta)} f(x)$ to be finite, it is sufficient that $\omega^2(h)h^{-1}$ is integrable near 0. On the other hand, if $\mathcal{N}_\beta^{B,(\delta)} f$ is bounded over a domain, the following proposition asserts that f is necessarily Hölder continuous.

Proposition 4.7. *Let I be a non-empty open interval of $\mathbb{R}$ and $\delta \in (0,\infty]$. Let f be a function on $\mathbb{R}$ such that $\sup_{x\in\bar I}\mathcal{N}^{B,(\delta)}_\beta f(x)$ is finite. Then*

$$\sup_{x\in I, |y|\le\frac{\delta}{3}\wedge \mathrm{dist}(x,\partial I)} \frac{\|f(x+y)-f(x)\|}{|y|^\beta} \le c(\beta)\sup_{x\in\bar I}\mathcal{N}^{B,(\delta)}_\beta f(x) \tag{4.16}$$

for some finite constant $c(\beta)$ which depends only on β.

Proof. For every $x\in I$ and positive R, $R\le\delta$, we denote $f_{x,R}=\frac{1}{2R}\int_{-R}^R f(y+x)dy$. We first estimate $\|f(x)-f_{x,R}\|$ as follows

$$\begin{aligned}\|f(x)-f_{x,R}\| &\le \frac{1}{2R}\int_{-R}^R \|f(x)-f(x+y)\|dy\\ &\le \frac{1}{2R}\left(\int_{-R}^R \|f(x)-f(x+y)\|^2|y|^{-1-2\beta}dy\right)^{\frac12}\left(\int_{-R}^R |y|^{1+2\beta}dy\right)^{\frac12}\\ &\le \frac{R^\beta}{2\sqrt{(1+\beta)}}\sup_{x\in\bar I}\mathcal{N}^{B,(\delta)}_\beta f(x)\,. \end{aligned}\tag{4.17}$$

Let us now fix $x\in I$ and $y\in\mathbb{R}$ such that $|y|\le\delta/3\wedge\mathrm{dist}(x,\partial I)$. We also choose $R=|y|$. It follows from triangle inequality that

$$\|f(x+y)-f(x)\|\le\|f(x+y)-f_{x+y,R}\|+\|f_{x+y,R}-f_{x,R}\|+\|f(x)-f_{x,R}\|\,. \tag{4.18}$$

For the second term, we apply Minkowski's inequality to get

$$\|f_{x+y,R}-f_{x,R}\|\le\frac{1}{4R^2}\int_{-R}^R\int_{-R}^R\|f(x+y+z)-f(x+w)\|\,dzdw,$$

and invoking Cauchy-Schwarz' inequality this yields

$$\begin{aligned}\|f_{x+y,R}-f_{x,R}\|\le\frac{1}{4R^2}\int_{-R}^R&\left(\int_{-R}^R\|f(x+y+z)-f(x+w)\|^2|y+z-w|^{-2\beta-1}dz\right)^{\frac12}\\ &\times\left(\int_{-R}^R|y+z-w|^{2\beta+1}dz\right)^{\frac12}dw\,.\end{aligned}$$

Notice that because of the restrictions on the variables, the domain of integration above satisfies $|y+z-w|\le 3R\le\delta$ and $x+w\in\bar I$. Hence

$$\|f_{x+y,R}-f_{x,R}\|\le C_\beta\sup_{y\in\bar I}\mathcal{N}^{B,(\delta)}_\beta f(y)R^\beta\,.$$

We can now conclude our proof as follows: the first and third terms on the right hand side of (4.18) are estimated in (4.17). Combining these estimates within (4.18) yields (4.16). □

Remark 4.8. In the same way as for the quantities $\mathcal{N}^B_\beta f$, the function $\mathcal{N}^{B,(\delta)}_\beta f$ can be defined for functions defined on $\mathbb{R}_+\times\mathbb{R}$. In this case, we have $\mathcal{N}^{B,(\delta)}_\beta f:\mathbb{R}_+\times\mathbb{R}\to[0,\infty]$.

Whenever σ is an affine function (i.e. $\sigma(u)=au+b$ for some constants a,b), the spaces $\mathfrak{X}^{\beta,p}_T$ are sufficient to show existence and uniqueness for equation (1.1). On the other hand, the case of general Lipschitz function σ leads to the consideration of additional spaces, which we are going to study now.

For every $h \in \mathbb{R}$, let τ_h be the translation map in the spatial variable, that is $\tau_h f(t,x) = f(t, x-h)$.

Definition 4.9. *Let X_T^β be the space of all real-valued continuous functions f on $[0,T]\times\mathbb{R}$ such that*

(i) $(t,x)\mapsto \mathcal{N}_\beta^{(1)} f(t,x)$ *is finite and continuous on* $[0,T]\times\mathbb{R}$.

(ii) $\lim_{h\downarrow 0}\sup_{t\in[0,T],\, x\in[-R,R]} \mathcal{N}_\beta^{(1)}(\tau_h f - f)(t,x) = 0$ *for every positive R.*

We equip X_T^β with the following topology. A sequence $\{f_n\}$ in X_T^β converges to f in X_T^β if for all $R>0$, the sequences $\{f_n\}$ and $\{\mathcal{N}_\beta^{(1)}(f_n - f)\}$ converge uniformly on $[0,T]\times[-R,R]$ to f and 0 respectively. We define a metric on X_T^β as follows

$$d_\beta(f,g) = \sum_{n=1}^\infty 2^{-n} \frac{\|f-g\|_{n,\beta}}{1+\|f-g\|_{n,\beta}}\,, \tag{4.19}$$

where $\|\cdot\|_{n,\beta}$ is the seminorm

$$\|f\|_{n,\beta} := \sup_{t\in[0,T],\, x\in[-n,n]} |f(t,x)| + \sup_{t\in[0,T],\, x\in[-n,n]} \mathcal{N}_\beta^{(1)} f(t,x)\,.$$

Since functions in X_T^β are locally bounded, the topology of X_T^β is not altered if in the previous definition $\mathcal{N}_\beta^{(1)} f$ is replaced by $\mathcal{N}_\beta^{(\delta)} f$ for some *finite* and positive δ. We emphasize that replacing δ by ∞ would create a strictly smaller space.

Remark 4.10. The space which satisfies only condition (i) in Definition 4.9 would be too big and fail to be separable. Analogous situations occur frequently in analysis. In the study of Morrey spaces, this fact was first observed by Zorko in [27]. Continuity spatial translations with respect to a norm is therefore sometimes called Zorko condition.

Proposition 4.11. *X_T^β is a complete metric space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in X_T^β. Since the space $C_{\mathrm{uc}}([0,T]\times\mathbb{R})$ is complete, there exists continuous function $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ such that for all compact intervals I,

$$\lim_{n\to\infty} \sup_{t\in[0,T],\, x\in I} |f_n(t,x) - f(t,x)| = 0\,.$$

Let us fix a compact interval $I=[-N,N]$, and $\varepsilon>0$. There exists $n_0>0$ such that

$$\sup_{t\in[0,T],\, x\in I} \mathcal{N}_\beta^{(1)}(f_n - f_m)(t,x) < \varepsilon$$

for all $m,n\ge n_0$. It follows from Fatou's lemma that

$$\mathcal{N}_\beta^{(1)}(f_n - f)(t,x) \le \liminf_{m\to\infty} \mathcal{N}_\beta^{(1)}(f_n - f_m)(t,x) \le \varepsilon,$$

for every $t\in[0,T]$, $x\in I$ and $n\ge n_0$. This implies that $\mathcal{N}_\beta^{(1)}(f_n - f)$ converges to 0 uniformly on $[0,T]\times I$. In addition, from (4.15), it follows that $\mathcal{N}_\beta^{(1)} f_n$ converges to $\mathcal{N}_\beta^{(1)} f$ uniformly on $[0,T]\times I$, thus the continuity of $\mathcal{N}_\beta^{(1)} f_n$ implies that of $\mathcal{N}_\beta^{(1)} f$.

It remains to check that f satisfies the condition (ii) of Definition 4.9. For every $\varepsilon > 0$ and $|h| \leq 1$, choose n sufficiently large so that $\sup_{t\in[0,T],\, x\in[N-1,N+1]} \mathcal{N}_\beta^{(1)}(f_n - f)(t,x) < \varepsilon$. Applying Minkowski's inequality, for every $(t,x) \in [0,T] \times [-N,N]$, we have

$$\begin{aligned}\mathcal{N}_\beta^{(1)}(\tau_h f - f)(t,x) &\leq \mathcal{N}_\beta^{(1)}(\tau_h f - \tau_h f_n)(t,x) + \mathcal{N}_\beta^{(1)}(\tau_h f_n - f_n)(t,x) + \mathcal{N}_\beta^{(1)}(f_n - f)(t,x)\\ &\leq 2\varepsilon + \mathcal{N}_\beta^{(1)}(\tau_h f_n - f_n)(t,x).\end{aligned}$$

Since f_n belongs to X_T^β, $\lim_{h\to 0}\sup_{t\in[0,T],\, x\in[-N,N]} \mathcal{N}_\beta^{(1)}(\tau_h f_n - f_n)(t,x) = 0$ which implies f belongs to X_T^β. □

The next results give some characterizations of the space X_T^β.

Lemma 4.12. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $t \mapsto \mathcal{N}_\beta^{(1)} f(t,x)$ is continuous for every fixed x. Suppose in addition that for every $R > 0$,*

$$\lim_{\delta\downarrow 0} \sup_{t\in[0,T],\, x\in[-R,R]} \int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy = 0\,.$$

Then $\mathcal{N}_\beta^{(1)} f$ is continuous and f belongs to X_T^β.

Proof. Fix $R, \varepsilon > 0$, and choose δ such that

$$\sup_{t\in[0,T],\, x\in[-R-1,R+1]} \int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy < \varepsilon\,.$$

Then for every $t \in [0,T]$, $x \in [-R,R]$ and $|h| \leq 1$

$$[\mathcal{N}_\beta^{(1)}(\tau_h f - f)(t,x)]^2 \leq 2\varepsilon + \sup_{t\in[0,T],\, x\in[-R-1,R+1]} 2|\tau_h f(t,x) - f(t,x)|^2 \int_{|y|>\delta} |y|^{-2\beta-1} dy\,.$$

Since f is continuous, $\lim_{h\to 0}\sup_{t\in[0,T], x\in[-R-1,R+1]} |\tau_h f(t,x) - f(t,x)| = 0$. Together with the previous estimate, this yields $\lim_{h\to 0}\sup_{t\in[0,T],\, x\in[-R,R]} \mathcal{N}_\beta^{(1)}(\tau_h f - f)(t,x) = 0$ which on one hand, together with (4.15) implies the continuity of $\mathcal{N}_\beta^{(1)} f$. On the other hand, it obviously implies $f \in X_T^\beta$. □

Proposition 4.13. *Let $\phi \in C^\infty(\mathbb{R})$ be supported in $[-1,1]$, such that $\int_{\mathbb{R}} \phi(x)dx = 1$ and $0 \leq \phi \leq 1$. Set $\phi_n(x) = n\phi(nx)$. Then*

(1) If $f \in X_T^\beta$, then $f * \phi_n \to f$ in X_T^β as $n \to \infty$, where $*$ denotes the convolution with respect to the space variable.

(2) $C^{0,1}([0,T] \times \mathbb{R})$ i.e., the space of functions which are continuous in time and continuously differentiable in space, is dense in X_T^β.

(3) Suppose that f is a continuous function on $[0,T] \times \mathbb{R}$ such that $t \mapsto \mathcal{N}_\beta^{(1)} f(t,x)$ is finite and continuous in time for every fixed $x \in \mathbb{R}$. Then f belongs to X_T^β if and only if for every $R > 0$

$$\lim_{\delta\downarrow 0} \sup_{t\in[0,T],\, x\in[-R,R]} \int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy = 0\,. \tag{4.20}$$

Proof. We denote $f_n = f * \phi_n$. To show (1), we observe that

$$f_n(t,x+y) - f_n(t,x) - f(t,x+y) + f(t,x) = \int_{\mathbb{R}} [\tau_h f(t,x+y) - \tau_h f(t,x) - f(t,x+y) + f(t,x)]\phi_n(h)dh$$

and hence, for every $x \in [-R,R]$, applying Jensen's inequality, we get

$$\begin{aligned}\int_{-1}^1 &|f_n(t,x+y) - f_n(t,x) - f(t,x+y) + f(t,x)|^2 |y|^{-2\beta-1} dy \\ &\le \int_{\mathbb{R}} \int_{-1}^1 |\tau_h f(t,x+y) - \tau_h f(t,x) - f(t,x+y) + f(t,x)|^2 |y|^{-2\beta-1} \phi_n(h) dh dy \\ &\le \sup_{r\in[0,T],\, z\in[-R-1,R+1]} \sup_{|h|\le \frac{1}{n}} [\mathcal{N}_\beta^{(1)}(\tau_h f - f)(r,z)]^2 .\end{aligned}$$

By assumption f belongs to X_T^β. Therefore, owing to condition (ii) in Definition 4.9, this integral converges to 0 when $n \to \infty$. This proves item (1).

To show (2), we first prove that X_T^β contains $C^{0,1}([0,T]\times\mathbb{R})$. Indeed, if g is a function in $C^{0,1}([0,T]\times\mathbb{R})$, by dominated convergence theorem, it is easy to show that $\mathcal{N}_\beta^{(1)} g(t,x)$ is finite and continuous in time for every fixed x. Moreover, for every $R>0$, we have

$$\sup_{t\in[0,T],\, x\in[-R,R]} \int_{-\delta}^{\delta} |g(t,x+y) - g(t,x)|^2 |y|^{-2\beta-1} dy \le \sup_{x\in[-R,R]} \|\partial_x g\|_\infty \int_{|y|\le\delta} |y|^{1-2\beta} dy . \quad (4.21)$$

Since $\lim_{\delta\to 0}\int_{|y|\le\delta} |y|^{1-2\beta}dy = 0$, Lemma 4.12 implies that g belongs to X_T^β. We have thus proved that $C^{0,1} \subset X_T^\beta$. Together with item (1), this yields item (2).

The sufficiency of (3) is in fact the content of Lemma 4.12. We focus on the necessity of (4.20). Assume that f belongs to X_T^β. Fix $R>0$, $\varepsilon>0$ and choose g in $C^{0,1}$ so that

$$\sup_{t\in[0,T],\, x\in[-R,R]} \mathcal{N}_\beta^{(1)}(f-g)(t,x) < \varepsilon .$$

Then for every $\delta > 0$ we have

$$\begin{aligned}\sup_{t\in[0,T],\, |x|\le R} &\int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy \\ &\le 2\varepsilon^2 + 2 \sup_{t\in[0,T],\, |x|\le R} \int_{-\delta}^{\delta} |g(t,x+y) - g(t,x)|^2 |y|^{-2\beta-1} dy . \quad (4.22)\end{aligned}$$

Since g is $C^{0,1}$, the last term converges to 0 when $\delta \downarrow 0$ (see relation (4.21)). Due to the fact that ε can be chosen arbitrarily small, this implies that f satisfies the condition (4.20). □

Corollary 4.14. *X_T^β is a Polish (complete and separable) space.*

Proof. Completeness comes from Proposition 4.11. For separability, we invoke Proposition 4.13(2) and the fact that the functions in $C^{0,1}([0,T]\times\mathbb{R})$ can be approximated by polynomials with rational coefficients, using a truncation argument. □

Proposition 4.15. *The inclusion $X_T^\beta \subset X_T^\alpha$ holds continuously for $\beta > \alpha$.*

Proof. Suppose f belongs to X^β_T. Fix $n \geq 1$. By Proposition 4.7, we see that

$$\sup_{t\in[0,T],\,|x|\leq n} |f(t,x+y)-f(t,x)| \leq C \sup_{t\in[0,T],\,|x|\leq n+1} \mathcal{N}^{(3)}_\beta f(t,x)|y|^\beta$$

for every $|y|\leq 1$. Hence for every $t \leq T$, $|x|\leq n$ and $\alpha<\beta$ we have:

$$\int_{|y|\leq 1} |f(t,x+y)-f(t,x)|^2|y|^{-2\alpha-1}dy \leq C \sup_{t\in[0,T,\,|x|\leq n+1} \mathcal{N}^{(3)}_\beta f(t,x)\,,$$

which is a finite quantity. The continuity of $(t,x)\mapsto \int_{|y|\leq 1}|f(t,x+y)-f(t,x)|^2|y|^{-2\alpha-1}dy$ follows at once from dominated convergence theorem. □

We state an analogous result for $\mathfrak{X}^{\beta,p}_T$ without proof.

Proposition 4.16. *The inclusion $\mathfrak{X}^{\beta,p}_T \subset \mathfrak{X}^{\alpha,q}_T$ holds continuously for $\beta>\alpha$ and $p\geq q$.*

Next, we derive a compactness criterion for X^β_T. We first recall some well-known definitions and facts. An ε-*cover* of a metric space is a cover of the space consisting of sets of diameter at most ε. A metric space is called *totally bounded* if it admits a finite ε-cover for every $\varepsilon>0$. It is well known that a metric space is compact if and only if it is complete and totally bounded. The following lemma is the key ingredient for many compactness results.

Lemma 4.17. *Let X be a metric space. Assume that, for every $\varepsilon>0$, there exists a $\delta>0$, a metric space W, and a mapping $\Phi: X\to W$ such that $\Phi(X)$ is totally bounded, and for all $x,y\in X$ with $d(\Phi(x),\Phi(y))<\delta$, we have $d(x,y)<\varepsilon$. Then X is totally bounded.*

The proof of this lemma is elementary, we refer readers to Lemma 1 in [17] for details. The following result provides sufficient conditions for relative compactness in X^β_T.

Proposition 4.18. *A set $\mathfrak{F}$ in X^β_T is relatively compact if*

[A1] $\sup_{f\in\mathfrak{F}}|f(0,0)|$ *is finite.*

[A2] *For every fixed $x\in\mathbb{R}$, $\{f(\cdot,x): f\in\mathfrak{F}\}$ is equicontinuous in time.*

[A3] *For every $R>0$, $\lim_{\delta\downarrow 0}\sup_{f\in\mathfrak{F}}\sup_{t\in[0,T]}\sup_{x\in[-R,R]}\int_{-\delta}^{\delta}\frac{|f(t,x+y)-f(t,x)|^2}{|y|^{1+2\beta}}dy=0$.*

Proof. Suppose that $\mathfrak{F}$ satisfies the three conditions. We first observe that condition [A3] together with (4.16) implies the following equicontinuity property. For every $R>0$ and $\varepsilon>0$, there exists $\eta>0$ such that

$$\sup_{t\in[0,T]}|f(t,x)-f(t,y)|<\varepsilon$$

whenever $f\in\mathfrak{F}$ and $x,y\in[-R,R]$ satisfy $|x-y|<\eta$. Together with [A2], this implies equicontinuity for $\mathfrak{F}$ in $(t,x)\in[0,T]\times[-R,R]$. Indeed, take N to be a sufficiently large integer, and set $x_i=-R+\frac{j}{N}R$, $j=0,1,\dots,2N$. According to [A2], $\{f(\cdot,x_i): f\in\mathfrak{F}\}$ is equicontinuous in time, uniformly for $j=0,1,\dots,2N$. By writing

$$|f(t,x)-f(s,x)|\leq|f(t,x)-f(t,x_i)|+|f(t,x_i)-f(s,x_i)|+|f(s,x_i)-f(s,x)|\,,$$

where x_i is chosen in such a way that $|x-x_i|<\eta$, this shows the uniformity in x.

Fix now $R>0$ and $\varepsilon>0$. From [A3], we can choose a positive number $\delta_1=\delta_1(\varepsilon)$, such that $\delta_1<1$ and

$$2\sup_{f\in\mathfrak{F}}\sup_{t\in[0,T],\,x\in[-R,R]}\int_{-\delta_1}^{\delta_1}\frac{|f(t,x+y)-f(t,x)|^2}{|y|^{1+2\beta}}\,dy<\varepsilon^2\,.$$

We now choose $\delta_2\le\varepsilon$ satisfying

$$2(3\delta_2)^2\int_{|y|>\delta_1}\frac{dy}{|y|^{1+2\beta}}<\varepsilon^2\,.$$

By the equicontinuity, we can also choose a positive number $\eta=\eta(\varepsilon)$, $\eta<1$, such that

$$\|f(t,x)-f(s,y)\|<\delta_2, \tag{4.23}$$

whenever $f\in\mathfrak{F}$ and $(t,x),(s,y)\in[0,T]\times[-R-2,R+2]$ satisfy $|t-s|+|x-y|<\eta$. Since $[0,T]\times[-R-2,R+2]$ is compact, we can find a finite set of points $\{(t_a,x_i):1\le a,i\le n\}$ in $[0,T]\times[-R-2,R+2]$ such that for every $(t,x)\in[0,T]\times[-R-1,R+1]$, there is some (t_a,x_j) so that $|t-t_a|+|x-x_j|<\eta$ and $[x_j-1,x_j+1]\subset[-R-2,R+2]$.

Define $\Phi:\mathfrak{F}\to\mathbb{R}^{n^2}$ by

$$\Phi(f)=(f(t_a,x_i):1\le a,i\le n)\,.$$

Condition [A1] and equicontinuity imply that the image $\Phi(\mathfrak{F})$ is bounded and thus totally bounded in $\mathbb{R}^{n^2}$. Furthermore, consider $f,g\in\mathfrak{F}$ with $\|\Phi(f)-\Phi(g)\|_\infty<\delta_2$. Resorting to the fact that for any $(t,x)\in[0,T]\times[-R-1,R+1]$ there are some a,j so that $|t-t_a|+|x-x_j|<\eta$, we can write

$$|f(t,x)-g(t,x)|\le|f(t,x)-f(t_a,x_j)|+|f(t_a,x_j)-g(t_a,x_j)|+|g(t_a,x_j)-g(t,x)|\le3\delta_2\,,$$

where we bounded the first and third term on the right hand side thanks to (4.23), and the second one according to the fact that $\|\Phi(f)-\Phi(g)\|_\infty<\delta_2$. We end up with

$$\sup_{t\in[0,T],\,x\in[-R-1,R+1]}|f(t,x)-g(t,x)|\le3\delta_2\le3\varepsilon\,.$$

In addition, for every $(t,x)\in[0,T]\times[-R,R]$ we have

$$\begin{aligned}[\mathcal{N}_\beta(f-g)(t,x)]^2 \;&\le\; 2\sup_{h\in\{f,g\}}\int_{|y|\le\delta_1}|h(t,x+y)-h(t,x)|^2\frac{dy}{|y|^{1+2\beta}}\\ &\quad+2\sup_{r\in[0,T],\,z\in[-R-1,R+1]}|f(r,z)-g(r,z)|^2\int_{|y|>\delta_1}\frac{dy}{|y|^{1+2\beta}}\le2\varepsilon^2\,.\end{aligned}$$

Therefore, by the definition of the metric on X_T^β (see (4.19)) and Lemma 4.17, the set $\mathfrak{F}$ is totally bounded in X_T^β. □

A useful consequence of the previous proposition is the following corollary.

Corollary 4.19. *Suppose $\alpha>\beta$. Let $\mathfrak{F}$ be a subset of X_T^α such that $\mathfrak{F}$ is equicontinuous in time for every fixed x, $\sup_{f\in\mathfrak{F}}|f(0,0)|<\infty$ and $\sup_{f\in\mathfrak{F}}\sup_{t\in[0,T],\,|x|\le R}\mathcal{N}_\alpha^{(1)}f(t,x)<\infty$ for every positive R. Then $\mathfrak{F}$ is relatively compact in X_T^β.*

Proof. It suffices to check that $\mathfrak{F}$ satisfies condition [A3]. Applying (4.16), for δ small enough, the assumption on $\mathfrak{F}$ implies

$$\sup_{f\in\mathcal{F}}\sup_{t\in[0,T],\,|x|\le R}|f(t,x+y)-f(t,x)|\le C|y|^{\alpha}\,,$$

for all $|y|\le\delta$. Hence,

$$\sup_{f\in\mathcal{F}}\sup_{t\in[0,T],\,|x|\le R}\int_{|y|\le\delta}|f(t,x+y)-f(t,x)|^2|y|^{-2\beta-1}dy\le C\int_{|y|\le\delta}|y|^{2(\alpha-\beta)-1}dy\,,$$

which clearly implies [A3] since $\alpha>\beta$. □

The following result provides sufficient conditions for relative compactness in $\mathfrak{X}^\beta_T(B)$. Its proof is completely analogous to that of Proposition 4.18 and is omitted for the sake of conciseness.

Proposition 4.20. *Suppose that a set $\mathfrak{F}$ in $\mathfrak{X}^\beta_T(B)$ satisfies the following properties.*

(1) For every $t\in[0,T]$ and $x\in\mathbb{R}$, $\mathfrak{F}(t,x):=\{f(t,x):f\in\mathfrak{F}\}$ is relatively compact in the Banach space B.
(2) For every fixed $x\in\mathbb{R}$, $\{f(\cdot,x):f\in\mathfrak{F}\}$ is equicontinuous in time.
(3) For every $R>0$, we have

$$\lim_{\delta\downarrow0}\sup_{f\in\mathfrak{F}}\sup_{t\in[0,T],\,x\in[-R,R]}\int_{-\delta}^{\delta}\frac{\|f(t,x+y)-f(t,x)\|^2}{|y|^{1+2\beta}}\,dy=0\,.$$

Then $\mathfrak{F}$ is relatively compact in $\mathfrak{X}^\beta_T(B)$.

In order to handle the nonlinearity in equation (1.1), the following composition rule is crucial.

Proposition 4.21 (Left composition)**.** *Let σ be a Lipschitz function on $\mathbb{R}$ and let f be a function in X^β_T. Suppose that for every fixed x, the map $t\mapsto\mathcal{N}^{(1)}_\beta\sigma(f)(t,x)$ is continuous. Then $\sigma(f)$ belongs to X^β_T. Furthermore, if f_n is a sequence converging to f in X^β_T, then for every positive R and for any $\delta>0$, we have*

$$\lim_{n\to\infty}\sup_{t\in[0,T],\,|x|\le R}\mathcal{N}^{(\delta)}_\beta(\sigma(f_n)-\sigma(f))(t,x)=0\,.$$

Proof. We first show that $\sigma(f)$ belongs to X^β_T. For any $\delta>0$ we have

$$\int_{|y|\le\delta}|\sigma(f(t,x+y))-\sigma(f(t,x))|^2|y|^{-2\beta-1}dy\le\|\sigma\|^2_{\mathrm{Lip}}[\mathcal{N}^{(\delta)}_\beta f(t,x)]^2$$

which together with the criterion (3) in Proposition 4.13 implies that $\sigma(f)$ belongs to X^β_T.

For the second assertion, for every positive R and any $\varepsilon>0$, we can choose $\delta_0>0$ and $n_0>0$, so that, for any $n\ge n_0$,

$$\sup_{t\in[0,T],\,|x|\le R}\mathcal{N}^{(\delta_0)}_\beta(\sigma(f_n)-\sigma(f))(t,x)\le\varepsilon\,. \tag{4.24}$$

Indeed, it is easily seen that

$$\mathcal{N}^{(\delta_0)}_\beta(\sigma(f_n)-\sigma(f))(t,x)\le\mathcal{N}^{(\delta_0)}_\beta\sigma(f_n)(t,x)+\mathcal{N}^{(\delta)}_\beta\sigma(f)(t,x)$$

$$\le \|\sigma\|_{\rm Lip}\left(\mathcal{N}_\beta^{(\delta_0)} f_n(t,x) + \mathcal{N}_\beta^{(\delta_0)} f(t,x)\right)$$
$$\le \|\sigma\|_{\rm Lip}\left(\mathcal{N}_\beta^{(\delta_0)} (f_n - f)(t,x) + 2\mathcal{N}_\beta^{(\delta_0)} f(t,x)\right),$$

and the last term is readily bounded by ε if δ_0 is chosen small enough. Now with (4.24) in hand we obtain, for any $\delta > 0$,

$$\sup_{t\in[0,T],\,|x|\le R} \mathcal{N}_\beta^{(\delta)}(\sigma(f_n) - \sigma(f))(t,x)$$
$$\le C\,\varepsilon + C\,\|\sigma\|_{\rm Lip} \sup_{t\in[0,T],\,|x|\le R+1} |f_n(t,x) - f(t,x)| \left(\int_{|y|>\delta_0} |y|^{-2\beta-1} dy\right)^{\frac{1}{2}}.$$

We conclude the proof by taking the limit as n tends to infinity. □

The next lemma gives a criterion for a process in $\mathfrak{X}_T^{\alpha,p}$ to have its paths almost surely lie in the space X_T^β for a certain value of β.

Lemma 4.22. *Let f be a stochastic process in $\mathfrak{X}_T^{\alpha,p}$ with $p\alpha > 1$. Assume that for any $R > 0$,*

$$\sup_{s,t\in[0,T]} \sup_{|x|\le R} \|f(t,x) - f(s,x)\|_{L^p(\Omega)} \le C_R |t-s|^\lambda, \tag{4.25}$$

where $\lambda > p^{-1}$. Then f has a version $\tilde f$ such that with probability one, $\tilde f$ belongs to X_T^β for every $\beta < \alpha - \frac{1}{p}$.

Proof. Since f belongs to $\mathfrak{X}_T^{\alpha,p}$, inequality (4.16) implies

$$\sup_{t\in[0,T]} \sup_{x,y\in\mathbb{R}} \frac{\|f(t,x+y) - f(t,x)\|_{L^p(\Omega)}}{|y|^\alpha} \le C \sup_{t\in[0,T],\,x\in\mathbb{R}} \int_{\mathbb{R}} \|f(t,x+y) - f(t,x)\|^2_{L^p(\Omega)} |y|^{-2\alpha-1} dy\,.$$

Then by Kolmogorov continuity criterion, f has a version $\tilde f$ such that with probability one, $\tilde f$ satisfies

$$\sup_{s,t\in[0,T],\,|x|\le R} |\tilde f(t,x+y) - \tilde f(s,x)| \le C(|y|^{\beta'} + |t-s|^{\lambda'})$$

for every R and $|y| \le 1$, where β' and λ' are fixed and such that $\beta < \beta' < \alpha - 1/p$ and $\lambda < \lambda' < \lambda - 1/p$. This implies that a.s. $\mathcal{N}_\beta^{(1)} f(t,x)$ is finite and $\mathcal{N}_\beta^{(\delta)}$ satisfies condition (4.20). The continuity of $\mathcal{N}_\beta^{(1)} f$ follows from dominated convergence theorem. These facts imply that $\tilde f$ belongs to X_T^β almost surely. □

4.3. **Probability measures on X_T^β.**

To show the existence of solution to equation (1.1) we need some tightness arguments for some probability measures defined on X_T^β. We have the following result towards this aim.

Theorem 4.23. *Let $\{\mathbf{P}_n,\, n \ge 1\}$ be a sequence of probability measures on X_T^β. This sequence is tight if the following three conditions hold:*

(1) For each positive η, there exist a and n_0 such that for all $n \ge n_0$

$$\mathbf{P}_n(f \in X_T^\beta : |f(0,0)| \ge a) \le \eta\,. \tag{4.26}$$

(2) For every $x \in \mathbb{R}$, and every positive ε and η, there exist δ satisfying $0 < \delta < 1$, and n_0 such that for all $n \geq n_0$

$$\mathbf{P}_n\left(f \in X_T^\beta : \sup_{s,t\leq T, |t-s|<\delta} |f(t,x) - f(s,x)| \geq \varepsilon\right) \leq \eta\,. \tag{4.27}$$

(3) For every $R > 0$, for each positive ε and η, there exist $\delta \in (0,1)$ and n_0 such that for all $n \geq n_0$

$$\mathbf{P}_n\left(f \in X_T^\beta : \sup_{t\in[0,T],\,|x|\leq R} \int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy \geq \varepsilon\right) \leq \eta\,. \tag{4.28}$$

Proof. Without loss of generality we assume $n_0 = 1$. For a given $\eta > 0$, we choose a so that $\mathbf{P}_n(B^c) \leq \eta$ for all $n \geq 1$, where

$$B = \left\{f \in X_T^\beta : |f(0,0)| < a\right\}\,.$$

According to condition (3), for any integer k, N, we also choose and fix $\delta_{k,N}$ such that $\mathbf{P}_n(A_{k,N}^c) \leq \eta 2^{-k-N}$ for all $n \geq 1$, where

$$A_{k,N} = \left\{f \in X_T^\beta : \sup_{t\in[0,T],\,|x|\leq N} \int_{-\delta_{k,N}}^{\delta_{k,N}} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy \leq \frac{1}{k^2}\right\}\,.$$

Then for each $\tilde{x} \in [-N,N] \cap \frac{\delta_{k,N}}{3}\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers (note that the number of such $\tilde{x}$ has order $\frac{N}{\delta_{k,N}}$), we choose $\delta'_{k,N}(\tilde{x})$ according to condition (2) such that $\mathbf{P}_n(B_{k,N}^c(\tilde{x})) \leq \delta_{k,N}\eta 2^{-k-N}$, where

$$B_{k,N}(\tilde{x}) = \left\{f \in X_T^\beta : \sup_{t,s,\leq T, |t-s|\leq \delta'_{k,N}(\tilde{x})} |f(t,\tilde{x}) - f(s,\tilde{x})| \leq \frac{1}{k^2}\right\}\,.$$

Consider now $B_{k,N} = \cap_{\tilde{x}\in[-N,N]\cap\frac{\delta_{k,N}}{3}\mathbb{Z}} B_{k,N}(\tilde{x})$. It is easy to see that

$$\mathbf{P}_n(B_{k,N}^c) \leq \sum_{\tilde{x}\in[-N,N]\cap\frac{\delta_{k,N}}{3}\mathbb{Z}} \mathbf{P}_n(B_{k,N}^c(\tilde{x})) \leq C\frac{N}{\delta_{k,N}}\eta\delta_{k,N}2^{-k-N} = C\eta 2^{-k-N}N\,.$$

We thus set $A = \cap_{k,N}(A_{k,N} \cap B_{k,N}) \cap B$. Then according to Proposition 4.18 we see that the closure of A is compact in X_T^β, and $\mathbf{P}_n(A) \geq 1 - C\eta$. This shows the tightness of $\mathbf{P}_n$. ☐

The following proposition states that under some moment conditions, a sequence of processes u_n can be regarded as a tight sequence of probability measures on the space X_T^β.

Proposition 4.24. *Assume that $\alpha, \lambda \in (0,1)$ and $p \geq 1$ satisfy $p\alpha > 1$, $p\lambda > 1$ and $\beta < \alpha - 1/p$. Let $\{u_n,\, n \geq 1\}$ be a sequence of stochastic processes such that*

(1) $\lim_{\delta\to\infty} \limsup_n \mathbf{P}(|u_n(0,0)| > \delta) = 0\,,$

(2) For every $R > 0$, $\sup_n \sup_{s,t\in[0,T],|x|\leq R} \|u_n(t,x) - u_n(s,x)\|_{L^p(\Omega)} \leq C_R|t-s|^\lambda\,,$

(3) $\sup_n \|u_n\|_{\mathfrak{X}_T^{\alpha,p}}$ *is finite.*

From Lemma 4.22, the law of u_n can be considered as a probability measure on X_T^β. In addition, as probability measures on X_T^β, the sequence $\{u_n,\, n\geq 1\}$ is tight.

Proof. This proposition can be easily proved using the same ideas as in the proof of Lemma 4.22 and Theorem 4.23, we omit the details. ☐

4.4. **Existence of the solution.** The main result of this subsection is the existence of a solution for equation (1.1). The methodology, inspired by the work of Gyöngy [13] on semilinear stochastic partial differential equations, consists in proving tightness of a sequence of solutions obtained by regularizing the noise, and then using the uniqueness result. The space $\mathcal{Z}_T^p$, where we proved our uniqueness result, consists of $L^p(\mathbb{R})$-valued processes, and it is not clear how to characterize compactness of probability laws on the space of trajectories of these processes. For this reason, we prove the existence of a solution with paths in the space $X_T^{\frac12-H}$ introduced in Definition 4.9, equipped with the metric (4.19).

Theorem 4.25. *Assume that for equation (1.1) the following conditions hold:*

(1) For some $\beta_0>\frac12-H$ and some $p>\max(\frac{6}{4H-1},\frac{1}{\beta_0+H-1/2})$, the initial condition u_0 is in $L^p(\mathbb{R})\cap L^\infty(\mathbb{R})$ and

$$\sup_{x\in\mathbb{R}}\mathcal{N}_{\beta_0}u_0(x)+\left(\int_{\mathbb{R}}\|u_0(\cdot)-u_0(\cdot+h)\|^2_{L^p(\mathbb{R})}|h|^{2H-2}dh\right)^{\frac12}<\infty\,. \tag{4.29}$$

(2) σ is differentiable and the derivative of σ is Lipschitz and $\sigma(0)=0$.

Then there exists a solution u to (1.1) in $\mathcal{Z}_T^p\cap\mathfrak{X}_T^{\frac12-H,p}$. In addition, the solution has sample paths in the space $X_T^{\frac12-H}$.

Proof. As mentioned above, we follow the methodology developed in [13] and we consider a regularization of the noise in space. Indeed, for $\varepsilon>0$ and $\varphi\in\mathfrak{H}$, we define

$$W_\varepsilon(\varphi)=\int_0^t\int_{\mathbb{R}}[\rho_\varepsilon*\varphi](s,x)W(ds,dy)=\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(s,x)\rho_\varepsilon(x-y)W(ds,dy)dx\,, \tag{4.30}$$

where $\rho_t(x)=(2\pi t)^{-\frac12}e^{-x^2/2t}$. Notice that relation (4.30) can be also read (either in Fourier or direct coordinates) as:

$$\begin{aligned}\mathbf{E}\left[W_\varepsilon(\varphi)W_\varepsilon(\psi)\right] &= c_{1,H}\int_0^t\int_{\mathbb{R}}\mathcal{F}\varphi(s,\xi)\,\overline{\mathcal{F}\psi(s,\xi)}\,e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi ds \\ &= c_{1,H}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(s,x)f_\varepsilon(x-y)\psi(s,y)\,dxdyds,\end{aligned} \tag{4.31}$$

where f_ε is given by $f_\varepsilon(x)=\mathcal{F}^{-1}(e^{-\varepsilon|\xi|^2}|\xi|^{1-2H})$. In other words, our noise is still a white noise in time but its space covariance is now given by f_ε. Note that f_ε is a real positive definite function, but is not necessarily positive. As assessed by (4.31), we however have

$$\mathbf{E}\left[|W_\varepsilon(\varphi)|^2\right]\leq\mathbf{E}\left[|W(\varphi)|^2\right]\,, \tag{4.32}$$

for all φ in $\mathfrak{H}$.

For every fixed $\varepsilon > 0$, the noise W_ε induces an approximation to equation (2.12), namely

$$u_\varepsilon(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_\varepsilon(s,y))\, W_\varepsilon(ds,dy), \tag{4.33}$$

where the integral is understood in the Itô sense. Since $|\xi|^{1-2H}e^{-\varepsilon|\xi|^2}$ is in $L^1(\mathbb{R})$, $|f_\varepsilon|$ is bounded. Thus, using Picard iteration, it is easy to see that (4.33) has a unique random field solution, and by estimating the pth moment of $|u_\varepsilon(t,x) - u_\varepsilon(t,x')|$, we see that each solution $u_\varepsilon(t,x)$ is Hölder continuous in space with order β for all $\beta \in (0,1)$. Therefore we conclude that u_ε is in $\mathfrak{X}_T^{\beta,p}$ for all $\beta \in (0,1)$. We remark that $\|u_\varepsilon\|_{\mathfrak{X}_T^{\beta,p}}$ may not be bounded uniformly in ε as seen from this procedure. However, using (4.32), (3.12) for suitable parameters yielding a contraction, plus the norm equivalence stated in Remark 3.5 item (iv), we obtain the following uniform bound:

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathfrak{X}_T^{\beta,p}} < \infty\,,$$

for all $\beta \le \beta_0$ and $\beta < H$. In particular, because $\frac{1}{2} - H < \beta_0 - \frac{1}{p}$, we can choose β such that $\frac{1}{2} - H < \beta - \frac{1}{p}$. In addition, we can show that u_ε satisfies Condition (2) in Proposition 4.24. With these properties, we can check that the three conditions in Proposition 4.24 are satisfied. Hence the laws of the processes u_ε, considered as probability measures on the space $X_T^{\frac{1}{2}-H}$, are tight and hence weakly relatively compact.

We now base our final considerations on the forthcoming Lemmas 4.26 - 4.29. Fix a sequence ε_n converging to zero and set $u_n = u_{\varepsilon_n}$. We shall hinge on Lemma 4.27 in order to prove that the sequence u_n actually converges in probability. To apply this lemma, we consider now two sequences $u_{m(n)}$ and $u_{l(n)}$, where $\{m(n), n \ge 1\}$ and $\{l(n), n \ge 1\}$ are strictly increasing sequences of positive integers. For each $n \ge 1$, the triplet $(u_{m(n)}, u_{l(n)}, W)$ defines probability measure on the space

$$\mathcal{B} := X_T^{\frac{1}{2}-H} \times X_T^{\frac{1}{2}-H} \times C_{\mathrm{uc}}([0,T]\times\mathbb{R}).$$

Since the family $\{u_\varepsilon, \varepsilon > 0\}$ is weakly relatively compact, there exists a subsequence of the form $\{(u_{m(n_k)}, u_{l(n_k)}, W), k \ge 1\}$ which converges in distribution as k tends to infinity. Thus, by Skorokhod embedding theorem, there is a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$ and a sequence of random elements $z_k = (u'_{m(n_k)}, u'_{l(n_k)}, W')$ with values on $\mathcal{B}$ such that z_k has the same distribution as $(u_{m(n_k)}, u_{l(n_k)}, W)$ and z_k converges almost surely (in the topology of $\mathcal{B}$) to (u', v', W'). By Lemma 4.29 we see that both u' and v' are solutions to equation (2.12), with W replaced by W'. Then by Lemma 4.28 and the uniqueness result Theorem 4.5 we thus get that $u' = v'$ in $X_T^{\frac{1}{2}-H}$. We can now apply Lemma 4.27 in order to assert that u_n converges to some random field u in $X_T^{\frac{1}{2}-H}$, in probability. Moreover, taking a subsequence if necessary, we see that u_n converges to u in $X_T^{\frac{1}{2}-H}$ a.s. Hence, thanks to another application of Lemma 4.29 we see that u satisfies equation (2.12). This proves the existence of the solution. □

We now state the lemmae on which the proof of Theorem 4.25 relies. The first lemma is a version of Gronwall's lemma, borrowed from [9, Lemma 15], and the correction [10] to this paper.

Lemma 4.26. *Let $g \in L^1([0,T];\mathbb{R}_+)$ and consider a sequence of functions $\{f_n;\, n \geq 0\}$ with $f_n : [0,T] \to \mathbb{R}_+$, such that f_0 is bounded and for all $n \geq 1$*

$$f_n(t) \leq c_1 + c_2 \int_0^t g(t-s)\, f_{n-1}(s)\, ds, \tag{4.34}$$

for two positive constants c_1, c_2. Then $\sup_{n\geq 1} f_n$ is bounded. If we assume moreover that $c_1 = 0$ in inequality (4.34), we obtain that $\sum_{n\geq 0} f_n^{1/p}$ converges uniformly in $[0,T]$, for all $1 \leq p < \infty$.

The second lemma is a general result on convergence of random variables borrowed from [14, 13].

Lemma 4.27. *Let $\mathbb{E}$ be a Polish space equipped with the Borel σ-algebra. A sequence of $\mathbb{E}$-valued random elements z_n converges in probability if and only if for every pair of subsequences $z_{l(n)}$, $z_{m(n)}$ there exists a subsequence $w_k := (z_{l(n_k)}, z_{m(n_k)})$ converging weakly to a random element w supported on the diagonal $\{(x,y) \in \mathbb{E}\times\mathbb{E} : x = y\}$.*

The next result asserts that the approximate solution to the stochastic heat equation is uniformly bounded in the space $\mathcal{Z}_T^p$ defined by (4.2).

Lemma 4.28. *The approximate solutions u_ε satisfy the condition*

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathcal{Z}_T^p} < \infty\,. \tag{4.35}$$

Furthermore, if $u_\varepsilon \to u$ in $X_T^{\frac12 - H}$ a.s., as ε tends to zero, then u is also in $\mathcal{Z}_T^p$.

Proof. We will use Picard iteration to show that for each ε, $u_\varepsilon \in \mathcal{Z}_T^p$. Then we will use Gronwall's lemma to show that the processes u_ε are uniformly (in ε) bounded in $\mathcal{Z}_T^p$. To this end, we first define

$$u_\varepsilon^0(t,x) = p_t u_0(x)\,,$$

and recursively

$$u_\varepsilon^{n+1}(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_\varepsilon^n(s,y)) W_\varepsilon(ds,dy)\,.$$

We wish to bound $\|u_\varepsilon^n\|_{\mathcal{Z}_T^p}$ uniformly in n. First recall that

$$\|u_\varepsilon^n\|_{\mathcal{Z}_T^p} = \sup_{t\in[0,T]} \|u_\varepsilon^n(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac12 - H,p} u_\varepsilon^n(t),$$

where $\mathcal{N}^*_{\frac12-H,p}$ is defined in (4.2). Let us now bound the terms $\|u_\varepsilon^n(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ and $\mathcal{N}^*_{\frac12-H,p} u_\varepsilon^n(t)$.

Step 1. We shall bound $\|u_\varepsilon^n(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ uniformly in n by considering the differences of Picard's iterations. Indeed, by Burkholder's inequality we have

$$\begin{aligned}
&\mathbf{E}|u_\varepsilon^{n+1}(t,x) - u_\varepsilon^n(t,x)|^p \\
&= \mathbf{E}\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)[\sigma(u_\varepsilon^n(s,y)) - \sigma(u_\varepsilon^{n-1}(s,y))] W_\varepsilon(ds,dy)\right|^p \\
&\leq C_p\, \mathbf{E}\Big| \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) p_{t-s}(x-z)[\sigma(u_\varepsilon^n(s,y)) - \sigma(u_\varepsilon^{n-1}(s,y))]
\end{aligned}$$

$$\times\, [\sigma(u_\varepsilon^n(s,z)) - \sigma(u_\varepsilon^{n-1}(s,z))] f_\varepsilon(y-z) dydzds \Big|^{\frac{p}{2}}.$$

Thus, since $\|f_\varepsilon\|_\infty \le C_\varepsilon$ and owing to the fact that σ is a Lipschitz function, we have

$$\begin{aligned} &\mathbf{E}|u_\varepsilon^{n+1}(t,x) - u_\varepsilon^n(t,x)|^p \\ \le\ & C_\varepsilon\, \mathbf{E} \left| \int_0^t \left(\int_{\mathbb{R}} p_{t-s}(y) |u_\varepsilon^n(s,x+y) - u_\varepsilon^{n-1}(s,x+y)| dy \right)^2 ds \right|^{\frac{p}{2}}, \end{aligned}$$

where C_ε denotes a generic constant depending on ε and p. We now integrate with respect to the space variable and invoke Minkowski's inequality. In this way we obtain

$$\begin{aligned} &\mathbf{E} \left\| u_\varepsilon^{n+1}(t,\cdot) - u_\varepsilon^n(t,\cdot) \right\|_{L^p(\mathbb{R})}^p \\ \le\ & C_\varepsilon \mathbf{E} \left\| \int_0^t \left(\int_{\mathbb{R}} p_{t-s}(y) |u_\varepsilon^n(s,y+\cdot) - u_\varepsilon^{n-1}(s,y+\cdot)| dy \right)^2 ds \right\|_{L^{\frac{p}{2}}(\mathbb{R})}^{\frac{p}{2}} \\ \le\ & C_\varepsilon \mathbf{E} \left(\int_0^t \left(\int_{\mathbb{R}} p_{t-s}(y) \left\| u_\varepsilon^n(s,\cdot) - u_\varepsilon^{n-1}(s,\cdot) \right\|_{L^p(\mathbb{R})} dy \right)^2 ds \right)^{\frac{p}{2}} \\ \le\ & C_\varepsilon \left(\int_0^t \left\| u_\varepsilon^n(s,\cdot) - u_\varepsilon^{n-1}(s,\cdot) \right\|_{L^p(\Omega\times\mathbb{R})}^2 ds \right)^{\frac{p}{2}}. \end{aligned}$$

This relation easily entails

$$\left\| u_\varepsilon^{n+1}(t,\cdot) - u_\varepsilon^n(t,\cdot) \right\|_{L^p(\Omega\times\mathbb{R})}^2 \le C_\varepsilon \int_0^t \left\| u_\varepsilon^n(s,\cdot) - u_\varepsilon^{n-1}(s,\cdot) \right\|_{L^p(\Omega\times\mathbb{R})}^2 ds,$$

and a direct application of Gronwall's lemma as stated in Lemma 4.26 yields that the quantity $\sup_n \sup_{t\in[0,T]} \|u_\varepsilon^n(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ is finite for each fixed $\varepsilon > 0$. This implies that $\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} < \infty$ for each fixed $\varepsilon > 0$.

Step 2. Next we estimate $\mathcal{N}^*_{\frac12-H,p} u_\varepsilon(t)$, and observe that we are able to handle this term directly (namely without invoking Picard's iterations). We can write

$$\begin{aligned} \int_{\mathbb{R}} \mathbf{E}|u_\varepsilon(t,x) - u_\varepsilon(t,x+h)|^p dx &\le C \int_{\mathbb{R}} |p_t u_0(x) - p_t u_0(x+h)|^p dx \\ &\quad + C_\varepsilon \int_{\mathbb{R}} \mathbf{E} \left| \int_0^t \left(\int_{\mathbb{R}} |p_{t-s}(y) - p_{t-s}(y+h)| |u_\varepsilon(s,y+x)| dy \right)^2 ds \right|^{\frac{p}{2}} dx \\ \le C \int_{\mathbb{R}} |p_t u_0(x) - p_t u_0(x+h)|^p dx & \\ &\quad + C_\varepsilon \left(\int_0^t \left(\int_{\mathbb{R}} |p_{t-s}(y) - p_{t-s}(y+h)| dy \right)^2 \|u_\varepsilon(s,\cdot)\|_{L^p(\Omega\times\mathbb{R})}^2 ds \right)^{\frac{p}{2}}. \end{aligned}$$

We thus end up with

$$\mathcal{N}^*_{\frac12-H,p} u_\varepsilon(t) = \int_{\mathbb{R}} \frac{\|u_\varepsilon(t,\cdot) - u_\varepsilon(t,\cdot+h)\|_{L^p(\Omega\times\mathbb{R})}^2}{|h|^{2-2H}} dh$$

$$\leq \quad C \int_{\mathbb{R}} \frac{\|p_t u_0(\cdot) - p_t u_0(\cdot + h)\|_{L^p(\mathbb{R})}^2}{|h|^{2-2H}} dh + C_\varepsilon \sup_{s\in[0,T]} \|u_\varepsilon(s,\cdot)\|_{L^p(\Omega\times\mathbb{R})}^2 \times \int_0^t \int_{\mathbb{R}} \frac{\left(\int_{\mathbb{R}} |p_{t-s}(y) - p_{t-s}(y+h)|dy\right)^2}{|h|^{2-2H}} dhds,$$

and the right-hand side in the above inequality is easily seen to be finite. Putting together the last two steps, we can conclude that for each fixed ε, $u_\varepsilon \in \mathcal{Z}_T^p$.

Step 3: Uniform bounds in ε. To prove the norms of u_ε in $\mathcal{Z}_T^p$ are uniformly bounded in ε, we note that u_ε satisfies the equation

$$u_\varepsilon(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} [(p_{t-s}(x-\cdot)\sigma(u_\varepsilon(s,\cdot))) * \rho_\varepsilon](y) W(ds,dy)\,.$$

Hence we have

$$\begin{aligned}\mathbf{E}|u_\varepsilon(t,x)|^p &\leq C|p_t u_0(x)|^p + C\mathbf{E}\left(\int_0^t \int_{\mathbb{R}} |\mathcal{F}\left(p_{t-s}(x-\cdot)\sigma(u_\varepsilon(s,\cdot))\right)(\xi)|^2 e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} d\xi ds\right)^{\frac{p}{2}} \\ &\leq C|p_t u_0(x)|^p + C\mathbf{E}\left(\int_0^t \int_{\mathbb{R}} |\mathcal{F}\left(p_{t-s}(x-\cdot)\sigma(u_\varepsilon(s,\cdot))\right)(\xi)|^2 |\xi|^{1-2H} d\xi ds\right)^{\frac{p}{2}}. \qquad (4.36)\end{aligned}$$

Going back from Fourier to direct coordinates, one can check that

$$\mathbf{E}|u_\varepsilon(t,x)|^p \leq C\,|p_t u_0(x)|^p + \mathcal{D}_1(t) + \mathcal{D}_2(t)\,,$$

with

$$\mathcal{D}_1(t) = \Big(\int_0^t \int_{\mathbb{R}^2} |p_{t-s}(y) - p_{t-s}(y+h)|^2 \, \|u_\varepsilon(s, y+x+h)\|_{L^p(\Omega)}^2 \, |h|^{2H-2} dhdyds\Big)^{\frac{p}{2}}$$

and

$$\mathcal{D}_2(t) = \Big(\int_0^t \int_{\mathbb{R}^2} |p_{t-s}(y)|^2 \, \|u_\varepsilon(s, y+x+h) - u_\varepsilon(s, y+x)\|_{L^p(\Omega)}^2 \, |h|^{2H-2} dhdyds\Big)^{\frac{p}{2}}.$$

These terms are treated exactly as the terms D_1, D_2 in the proof of Lemma 4.2, except for the fact that $\alpha = 0$ in the current situation. We obtain

$$\begin{aligned}\|u_\varepsilon(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}^2 &\leq C\,\|u_0\|_{L^p(\mathbb{R})}^2 + C\int_0^t (t-s)^{H-1} \|u_\varepsilon(s,\cdot)\|_{L^p(\Omega\times\mathbb{R})}^2 ds \\ &+ C\int_0^t (t-s)^{-\frac{1}{2}} \int_{\mathbb{R}} \|u_\varepsilon(s,\cdot) - u_\varepsilon(s,\cdot+h)\|_{L^p(\Omega\times\mathbb{R})}^2 |h|^{2H-2} dhds\,. \qquad (4.37)\end{aligned}$$

Similarly we get (see also the bounds for the terms $\mathcal{I}_1, \mathcal{I}_2$ in the proof of Theorem 4.5)

$$\begin{aligned}[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon(t)]^2 &\leq C\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot+h)\|_{L^p(\mathbb{R})}^2 |h|^{2H-2} dh \\ &+ C\int_0^t (t-s)^{2H-\frac{3}{2}} \|u_\varepsilon(s,\cdot)\|_{L^p(\Omega\times\mathbb{R})}^2 ds \\ &+ C\int_0^t \int_{\mathbb{R}} (t-s)^{H-1} \|u_\varepsilon(s,\cdot) - u_\varepsilon(s,\cdot+l)\|_{L^p(\Omega\times\mathbb{R})}^2 |l|^{2H-2} dlds\,. \qquad (4.38)\end{aligned}$$

Set

$$\Psi(t) = \|u_\varepsilon(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} + [\mathcal{N}^*_{\frac{1}{2}-H,p}u_\varepsilon(t)]^2.$$

Thus combining the estimates (4.37) and (4.38) yields

$$\Psi(t) \le C\,\|u_0\|^2_{L^p(\mathbb{R})} + C\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot+h)\|^2_{L^p(\mathbb{R})}|h|^{2H-2}dh + C\int_0^t (t-s)^{2H-\frac{3}{2}}\Psi(s)ds\,.$$

Since we have shown that for each fixed ε, $\|u_\varepsilon\|_{\mathcal{Z}^p_T} < \infty$, we can apply the Gronwall type Lemma 4.26 to the above inequality to show that

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathcal{Z}^p_T} < \infty\,.$$

Step 4: u is an element of $\mathcal{Z}^p_T$. Recall once again that we have decomposed $\|u\|_{\mathcal{Z}^p_T}$ according to relation (4.1). We now bound $\|u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}$ and $\mathcal{N}^*_{\frac{1}{2}-H,p}u(t)$ in this decomposition.

Since u_ε converges to u in $X_T^{\frac{1}{2}-H}$ a.s., we have $u_\varepsilon(t,x) \to u(t,x)$ a.s. for each $(t,x) \in \mathbb{R}_+\times\mathbb{R}$. Thus by Fatou's lemma,

$$\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} = \left(\mathbf{E}\int_{\mathbb{R}} \lim_{\varepsilon\to0}|u_\varepsilon(t,x)|^p dx\right)^{\frac{1}{p}} \le \varliminf_{\varepsilon\to0}\left(\mathbf{E}\int_{\mathbb{R}}|u_\varepsilon(t,x)|^p dx\right)^{\frac{1}{p}} \le C\,.$$

Therefore we conclude that $\sup_{t\in[0,T]}\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ is finite. On the other hand, for each x and h we have $|u_\varepsilon(t,x+h) - u_\varepsilon(t,x)|^2 \to |u(t,x+h) - u(t,x)|^2$ a.s., so by Fatou's lemma again we obtain

$$\begin{aligned}\int_{|h|\le1}\frac{\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh &\le \int_{|h|\le1}\frac{\varliminf_{\varepsilon\to0}\|u_\varepsilon(t,\cdot+h)-u_\varepsilon(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh\\ &\le \varliminf_{\varepsilon\to0}\int_{|h|\le1}\frac{\|u_\varepsilon(t,\cdot+h)-u_\varepsilon(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh\,.\end{aligned}$$

The desired bound on $\mathcal{N}^*_{\frac{1}{2}-H,p}u(t)$ is obtained from the inequality above, by handling the integral on the domains $|h|\le1$ and $|h|>1$. In the latter case, we simply bound $\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}$ by $2\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$. By doing so, we conclude that

$$\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}u(t) = \sup_{t\in[0,T]}\int_{\mathbb{R}}\frac{\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh < \infty\,.$$

Together with the previous estimate on $\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$, we conclude that $u\in\mathcal{Z}^p_T$. □

We now state a convergence result for stochastic integrals, with respect to the approximating noise W_ε.

Lemma 4.29. *Let $u_n(t,x)$ be a solution to the equation*

$$u_n(t,x) = p_t u_0(x) + \int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_n(s,y))W_n(ds,dy)\,,$$

where we have set $W_n = W_{\varepsilon_n}$ (recall that W_ε is defined by (4.30)) for a sequence $\{\varepsilon_n,\, n\ge1\}$ satisfying $\lim_{n\to\infty}\varepsilon_n = 0$. We assume the following conditions:

(i) with probability one, u_n converges to u in $X_T^{\frac{1}{2}-H}$,

(ii) $\sup_n \|u_n\|_{\mathfrak{X}_T^{\beta,p}} < \infty$, *with* $\frac{1}{2} - H < \beta < H$ *and* $p > \frac{2}{H}$.

Then the process u *belongs to* $\mathfrak{X}_T^{\frac{1}{2}-H,2}$. *Furthermore, for any fixed* $t \le T$ *and* $x \in \mathbb{R}$, *the random variable* $\Phi^n(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_n(s,y))W_n(ds,dy)$ *converges a.s. to* $\Phi(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)$, *as* $n \to \infty$.

Proof. We focus on the convergence part and decompose the difference $\Phi(t,x) - \Phi^n(t,x)$ into $(\Phi(t,x) - \Phi^{n,1}(t,x)) + (\Phi^{n,1}(t,x) - \Phi^n(t,x))$, where

$$\Phi^{n,1}(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W^n(ds,dy) \,.$$

Now we note that $\Phi(t,x) - \Phi^{n,1}(t,x)$ can be expressed as

$$\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy) - \int_0^t \int_{\mathbb{R}} \left[(p_{t-s}(x-\cdot)\sigma(u(s,\cdot))) * \rho_{\varepsilon_n}\right](y)W(ds,dy),$$

and thus

$$\begin{aligned}\mathbf{E}\left|\Phi(t,x) - \Phi^{n,1}(t,x)\right|^2 \\ = C\mathbf{E}\int_0^t \int_{\mathbb{R}} \left|e^{-\frac{\varepsilon_n|\xi|^2}{2}} - 1\right|^2 \left|\mathcal{F}\left(p_{t-s}(x-\cdot)\sigma\left(u(s,\cdot)\right)\right)(\xi)\right|^2 |\xi|^{1-2H} d\xi ds\,.\end{aligned}$$

The latter quantity obviously converges to 0 as ε_n goes to 0 because of the finiteness of

$$\mathbf{E}\int_0^t \int_{\mathbb{R}} \left|\mathcal{F}\left(p_{t-s}(x-\cdot)\sigma\left(u(s,\cdot)\right)\right)(\xi)\right|^2 |\xi|^{1-2H} d\xi ds,$$

which can be seen by an application of Fatou's lemma (as in Step 4 of the proof of Lemma 4.28).

It remains to show that $\lim_{n\to\infty} \mathbf{E}|\Phi^{n,1}(t,x) - \Phi^n(t,x)|^2 = 0$. However, similarly to (4.36), we have

$$\mathbf{E}\left[|\Phi^{n,1}(t,x) - \Phi^n(t,x)|^2\right] \le \mathbf{E}\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)f_n(s,y)W(ds,dy)\right|^2,$$

where we have set $f_n = \sigma(u_n) - \sigma(u)$. Furthermore, appealing to Proposition 4.21, we see that f_n converges to 0 in $X_T^{\frac{1}{2}-H}$. We will verify that f_n satisfies the conditions (C1)-(C3) of Lemma 4.30 below. Indeed, (C1) is verified by assumption (i). (C2) is verified by assumption (ii) and the estimate (3.14). (C3) is readily assumption (ii). Then an application of Lemma 4.30 completes the proof. □

Lemma 4.30. *Suppose that* $\{f_n, n \ge 1\}$ *is a sequence of stochastic processes belonging to* $\mathfrak{X}_T^{\beta,p}$ *with* $\frac{1}{2} - H < \beta < H$ *and* $p > \frac{2}{H}$. *Assume that the following conditions hold:*

(C1) With probability one, f_n *converges uniformly to* 0 *over compact sets of* $[0,T] \times \mathbb{R}$.
(C2) For every $R > 0$, $\sup_n \sup_{s,t\in[0,T],|x|\le R} \mathbf{E}|f_n(t,x) - f_n(s,x)|^p \le C|t-s|^{p\frac{H}{2}}$.
(C3) $\sup_n \|f_n\|_{\mathfrak{X}_T^{\beta,p}} \le M$, *where* M *is a finite number.*

Then for every $t \le T$ and $x \in \mathbb{R}$ the random variable $Y_n(t,x)$ defined by:

$$Y_n(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f_n(s,y) W(ds,dy)$$

converges to 0 in $L^2(\Omega)$.

Proof. We first observe that Proposition 4.16 asserts that f_n belongs to $\mathfrak{X}_T^{\frac{1}{2}-H,2}$. Next, we show that $\{f_n,\, n \ge 1\}$ is relatively compact and converges to 0 in $\mathfrak{X}_T^{\frac{1}{2}-H,2}$. For this purpose, we verify the three conditions (1)-(3) of Proposition 4.20. Condition (2) in Proposition 4.20 is evident from (C2). Condition (3) in Proposition 4.20 follows from the following inequality, where $\delta \le 1$

$$\int_{|y|\le\delta} \frac{\|f(t,x+y)-f(t,x)\|^2_{L^2(\Omega)}}{|y|^{2-2H}} dy \le \sup_{|y|\le 1} \frac{\|f(t,x+y)-f(t,x)\|^2_{L^2(\Omega)}}{|y|^{2\beta}} \int_{|y|\le\delta} |y|^{2\beta+2H-2} dy.$$

In fact, the first factor on the right side of the above inequality is uniformly bounded in $(t,x) \in [0,T]\times\mathbb{R}$ because of inequality (4.16) and the fact that f_n is bounded in $\mathfrak{X}_T^{\beta,2}$ by condition (C3). Taking into account that $\beta > 1/2 - H$, the second factor converges to zero as δ tends to zero. To verify condition (1) in Proposition 4.20, we fix t,x and note that (C1) implies $f_n(t,x)$ converges almost surely to 0. On the other hand, $\mathbf{E}|f_n(t,x)|^p$ is uniformly bounded, where $p>2$. These two facts imply $\{f_n(t,x)\}$ converges to 0 in $L^2(\Omega)$, thus condition (1) in Proposition 4.20 is verified. Furthermore, condition (C1) ensures that 0 is the only possible limit point of $\{f_n\}$ in $\mathfrak{X}_T^{1/2-H,2}$. We conclude that f_n converges to 0 in $\mathfrak{X}_T^{1/2-H,2}$.

Let us now prove that $Y_n(t,x)$ converges to 0 in $L^2(\Omega)$. Applying (3.3) we get $\mathbf{E}|Y_n(t,x)|^2 \le C\,(J_1(t)+J_2(t))$ with

$$J_1(t) = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{t-s}(x-y-z) - p_{t-s}(x-y)|^2 \, \mathbf{E} f_n^2(s,y+z) |z|^{2H-2} dydzds$$

and

$$J_2(t) = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{t-s}(x-y)|^2 \, \mathbf{E}|f_n(s,y+z) - f_n(s,y)|^2 |z|^{2H-2} dydzds\,.$$

Now for every fixed $\varepsilon > 0$ we choose $R>0$ sufficiently large such that

$$\int_0^t \int_{|y|>R} [|p_{t-s}(y)|^2 + [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(x-y)]^2] dyds < \varepsilon\,.$$

With this choice of R we choose n so that

$$\sup_{s\in[0,T],\,|y|\le R} \mathbf{E} f_n^2(s,y) + \sup_{s\in[0,T],\,|y|\le R} \int_{\mathbb{R}} \mathbf{E}|f_n(s,y+z)-f_n(s,y)|^2 |y|^{2H-2} dy < \varepsilon\,.$$

By making a shift in y, we end up with

$$\begin{aligned} J_1(t) &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{t-s}(x-y) - p_{t-s}(x-y+z)|^2 \mathbf{E} f_n^2(s,y) |z|^{2H-2} dydzds \\ &\le \int_0^t \sup_{|y|\le R} \mathbf{E} f_n^2(s,y) \int_{|y-x|\le R} [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(x-y)]^2 dyds \end{aligned}$$

$$+ \sup_{r\in[0,T],\, w\in\mathbb{R}} \mathbf{E} f_n^2(r,w) \int_0^t \int_{|y-x|>R} [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(x-y)]^2 dyds$$

$$\leq C\varepsilon + CM \int_0^t \int_{|y|>R} [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(x-y)]^2 dyds\,.$$

Similarly,

$$J_2(t) \leq C\varepsilon + CM \int_0^t \int_{|y|>R} p_{t-s}^2(y) dyds\,.$$

Then $\mathbf{E}|Y_n(t,x)|^2 \leq C\varepsilon$ for n sufficiently large. This implies the result. □

Finally, the techniques we have designed to get existence and uniqueness for equation (1.1) also allow us to obtain the following moment bound for the solution.

Theorem 4.31. *There are some changes in the formulae for this theorem. Assuming the conditions in Theorem 4.25, then a solution of (1.1) satisfies following moment bounds*

$$\sup_{x\in\mathbb{R}} \|u(t,x)\|_{L^p(\Omega)} \leq 2\|u_0\|_{\varepsilon_0} \exp\{\theta_0 p^{\frac{1}{H}} t\}, \tag{4.39}$$

and

$$\sup_{x\in\mathbb{R}} \mathcal{N}_{1/2-H,p} u(t,x) \leq 2\|u_0\|_{\varepsilon_0} \varepsilon_0^{-1} \exp\{\theta_0 p^{\frac{1}{H}} t\}\,,$$

where we recall that $\mathcal{N}_{1/2-H,p}$ is defined by (3.2), and where for any $\varepsilon > 0$ we have:

$$\|u_0\|_\varepsilon := \sup_{x\in\mathbb{R}} |u_0(x)| + \varepsilon \sup_{x\in\mathbb{R}} \left(\int_{\mathbb{R}} |u_0(x+h) - u_0(x)|^2 |h|^{2H-2} dh \right)^{\frac{1}{2}}\,,$$

In the formulae above, C_0 is as defined in (3.12), and we have $\theta_0 = (6C_0)^{\frac{2}{H}} \kappa^{1-\frac{1}{H}} \|\sigma\|_{\mathrm{Lip}}^{\frac{2}{H}}$, and $\varepsilon_0 = (6C_0)^{1-\frac{1}{2H}} \kappa^{\frac{1}{4H}-\frac{1}{2}} p^{\frac{1}{2}-\frac{1}{4H}} \|\sigma\|_{\mathrm{Lip}}^{1-\frac{1}{2H}}$. In addition, from Proposition 4.7, we see that the initial condition u_0 is Hölder continuous with order β_0, then by Proposition 3.8 we have

$$\|u(t,x) - u(s,y)\|_{L^p(\Omega)} \leq C(|t-s|^{\frac{H}{2}\wedge\frac{\beta_0}{2}} + |x-y|^{H\wedge\beta_0}) \tag{4.40}$$

for all $s,t\in[0,T]$ and $x,y\in\mathbb{R}$.

Proof. We will apply Proposition 3.6 by taking f to be the solution u to equation (1.1), and combine it with the mild formulation of the solution. For every fixed $\varepsilon > 0$, by noticing that $\|p_t u_0\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}} \leq \|u_0\|_\varepsilon$, we get the following bound

$$\|u\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}} \leq \|u_0\|_\varepsilon + C_0 \|\sigma\|_{\mathrm{Lip}} \sqrt{p} \|u\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}} \left(\kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}} + \varepsilon^{-1} \kappa^{-\frac{1}{4}} \theta^{-\frac{1}{4}} + \varepsilon \kappa^{H-\frac{3}{4}} \theta^{\frac{1}{4}-H} \right)\,.$$

We optimize the formula above by choosing $\varepsilon = \kappa^{\frac{1}{4}-\frac{H}{2}} \theta^{-\frac{1}{4}+\frac{H}{2}}$, in order to obtain

$$\|u\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}} \leq \|u_0\|_\varepsilon + 3C_0 \|\sigma\|_{\mathrm{Lip}} \sqrt{p} \|u\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}} \kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}}\,,$$

then choose $\theta = \theta_0$ so that $3C_0 \|\sigma\|_{\mathrm{Lip}} \sqrt{p} \kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}} = \frac{1}{2}$, that is

$$\theta_0 = (6C_0)^{\frac{2}{H}} p^{\frac{1}{H}} \kappa^{1-\frac{1}{H}} \|\sigma\|_{\mathrm{Lip}}^{\frac{2}{H}}\,, \quad \text{and take} \quad \varepsilon = \varepsilon_0 := (6C_0)^{1-\frac{1}{2H}} \kappa^{\frac{1}{4H}-\frac{1}{2}} p^{\frac{1}{2}-\frac{1}{4H}} \|\sigma\|_{\mathrm{Lip}}^{1-\frac{1}{2H}}\,.$$

Plugging this choice into the above inequality gives the bound

$$\|u\|_{\mathfrak{X}^p_{T,\theta_0,\varepsilon_0}} \le 2\|u_0\|_{\varepsilon_0}\,.$$

from which our claims are easily deduced by noticing that the constant C_0 does not depend on T. □

We now show the matching lower bound in term of κ and t for the second moment.

Proposition 4.32. *Under the conditions of Theorem 4.25, let u be a solution to the equation*

$$u(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy). \tag{4.41}$$

Suppose that u_0 is a bounded nontrivial function and there is a positive constant σ_* such that $|\sigma(z)| \ge \sigma_*|z|$ for all $z \in \mathbb{R}$. Then there exist some universal constants C and L such that

$$\mathbf{E}|u(t,x)|^2 \ge C\frac{|p_t u_0(x)|^3}{\|u_0\|_{L^\infty}} \exp\{L\sigma_*^{\frac{2}{H}}\kappa^{1-\frac{1}{H}}t\}\,. \tag{4.42}$$

Proof. Applying Itô isometry to equation (2.12), we see that

$$\mathbf{E}|u(t,x)|^2 = |p_t u_0(x)|^2 + c_{1,H}\mathbf{E}\int_0^t \|p_{t-s}(x-y)\sigma(u(s,y))\|^2_{\dot{H}^{\frac{1}{2}-H}} ds\,. \tag{4.43}$$

Let us recall the well-known Sobolev embedding inequality

$$\|g\|_{\dot{H}^{\frac{1}{2}-H}} \ge c\|g\|_{L^{\frac{1}{H}}}\,, \quad \forall g \in \dot{H}^{\frac{1}{2}-H}(\mathbb{R})\,.$$

Hence, together with our assumption on σ, it follows that there exists some positive constant b such that

$$\mathbf{E}|u(t,x)|^2 \ge |p_t u_0(x)|^2 + b\sigma_*^2\mathbf{E}\int_0^t \|p_{t-s}(x-\cdot)u(s,\cdot)\|^2_{L^{\frac{1}{H}}(\mathbb{R})} ds\,.$$

Since $2H < 1$, applying Jensen inequality we see that

$$\begin{aligned}\|p_{t-s}(x-\cdot)u(s,\cdot)\|^2_{L^{\frac{1}{H}}(\mathbb{R})} &= \left(\int_{\mathbb{R}} p_{t-s}^{\frac{1}{H}-1}(x-y)|u(s,y)|^{\frac{1}{H}} p_{t-s}(x-y)dy\right)^{2H} \\ &\ge \int_{\mathbb{R}} p_{t-s}^{3-2H}(x-y)|u(s,y)|^2 dy\,.\end{aligned}$$

It follows that

$$\mathbf{E}|u(t,x)|^2 \ge |p_t u_0(x)|^2 + b\sigma_*^2 \int_0^t \int_{\mathbb{R}} p_{t-s}^{3-2H}(x-y)\mathbf{E}|u(s,y)|^2 dyds\,.$$

Iterating the previous inequality yields

$$\mathbf{E}|u(t,x)|^2 \ge |p_t u_0(x)|^2 + \sum_{n=1}^\infty (b\sigma_*^2)^n I_n(t,x)\,. \tag{4.44}$$

In the above, we have adopted the notation

$$I_n(t,x) = \int_{T_n(t)} \int_{\mathbb{R}^n} p_{t-s_n}^{3-2H}(x-y_n)\cdots p_{s_2-s_1}^{3-2H}(y_2-y_1)|p_{s_1}u_0(y_1)|^2 d\bar{y}d\bar{s}$$

where $T_n(t)=\{(s_1,\dots,s_n)\in[0,t]^n : 0<s_1<\cdots<s_n<t\}$ and $d\bar y=dy_1\cdots dy_n$, $d\bar s=ds_1\cdots ds_n$. Note that for every $x,z\in\mathbb{R}$ and $a,b>0$, the following identity holds

$$\int_{\mathbb{R}} p_a^{3-2H}(x-y)p_b^{3-2H}(y-z)dy=(3-2H)^{-\frac12}\left(\frac{2\pi\kappa ab}{a+b}\right)^{H-1}p_{a+b}^{3-2H}(x-z)\,.$$

We thus can compute $I_n(t,x)$ by integrating y_j's in descending order starting from y_n. This procedure yields

$$I_n(t,x)=(3-2H)^{-\frac{n-1}{2}}\times \int_{T_n(t)}\left(\frac{t-s_n}{t-s_1}\prod_{j=2}^n 2\pi\kappa(s_j-s_{j-1})\right)^{H-1}\int_{\mathbb{R}} p_{t-s_1}^{3-2H}(x-y_1)|p_{s_1}u_0(y_1)|^2dy_1d\bar s\,. \quad (4.45)$$

On the other hand, for every fixed $R>0$, applying Jensen inequality, we see that

$$\begin{aligned}\int_{\mathbb{R}} p_{t-s_1}^{3-2H}(x-y_1)|p_{s_1}u_0(y_1)|^2dy_1 &\ge p_{t-s_1}^{1-2H}(R)\int_{|x-y_1|<R}p_{t-s_1}^2(x-y_1)|p_{s_1}u_0(y_1)|^2dy_1\\ &\ge p_{t-s_1}^{1-2H}(R)R^{-1}\left(\int_{|x-y_1|<R}p_{t-s_1}(x-y_1)p_{s_1}*u_0(y_1)dy_1\right)^2\,. \end{aligned}\quad (4.46)$$

The integral on the right side can be rewritten as

$$p_tu_0(x)-\int_{|x-y_1|\ge R}p_{t-s_1}(x-y_1)p_{s_1}*u_0(y_1)dy_1\,.$$

Since u_0 is bounded, we see that $|p_{s_1}*u_0(y_1)|\le\|u_0\|_{L^\infty}$ and hence

$$\begin{aligned}|\int_{|x-y_1|\ge R}p_{t-s_1}(x-y_1)p_{s_1}*u_0(y_1)dy_1| &\le \|u_0\|_{L^\infty}\int_{|y|>R}p_{t-s_1}(y)dy\\ &=\|u_0\|_{L^\infty}\pi^{-\frac12}\int_{|z|>\frac{R}{\sqrt{2\kappa(t-s_1)}}}e^{-z^2}dz\,.\end{aligned}$$

For every fixed ϵ in $(0,1)$, we now choose $R=M\sqrt{2\kappa(t-s_1)}$ where M is such that $e^{-(1-2H)M^2}M^{-1}=\epsilon$. It follows that

$$p_{t-s_1}^{1-2H}(R)R^{-1}=\pi^{H-\frac12}(2\kappa(t-s_1))^{H-1}e^{-(1-2H)M^2}M^{-1}$$

and

$$|\int_{|x-y_1|<R}p_{t-s_1}(x-y_1)p_{s_1}*u_0(y_1)dy_1|\ge|p_tu_0(x)|-\|u\|_\infty e^{-M^2}M^{-1}\,.$$

Together with (4.46), we see that

$$\int_{\mathbb{R}}p_{t-s_1}^{3-2H}(x-y_1)|p_{s_1}u_0(y_1)|^2dy_1\ge ce^{-M^2}M^{-1}(\kappa(t-s_1))^{H-1}\left(|p_tu_0(x)|-e^{-M^2}M^{-1}\|u_0\|_{L^\infty}\right)^2$$

for some universal constant c. Hence, upon combining the previous estimate and (4.45), we arrive at

$$I_n(t,x)\ge\epsilon c^n\kappa^{(H-1)n}\int_{T_n(t)}\prod_{j=2}^{n+1}(s_j-s_{j-1})^{H-1}d\bar s\left(|p_tu_0(x)|-\epsilon\|u_0\|_{L^\infty}\right)^2$$

where $s_{n+1} = t$ and c is some universal constant. It is elementary (see Lemma 5.3 below) to compute

$$\int_{T_n(t)} \prod_{j=2}^{n+1} (s_j - s_{j-1})^{H-1} d\bar{s} = \frac{\Gamma(H)^n t^{nH}}{\Gamma(nH+1)}.$$

Therefore, together with (4.44), we obtain

$$\mathbb{E}|u(t,x)|^2 \geq \epsilon \left(|p_t u_0(x)| - \epsilon \|u_0\|_{L^\infty}\right)^2 \sum_{n=0}^{\infty} (cb\Gamma(H))^n \frac{(\sigma_*^{\frac{2}{H}} \kappa^{1-\frac{1}{H}} t)^{nH}}{\Gamma(nH+1)}.$$

We now recall the elementary bound $\sum_{n\geq 0} x^n/(n!)^a \leq 2\exp(cx^{1/a})$, which can be found e.g in [4, Lemma A.1]. Together with the previous inequality, this yields:

$$\mathbf{E}|u(t,x)|^2 \geq C\epsilon \left(p_t u_0(x) - \epsilon \|u_0\|_{L^\infty}\right)^2 e^{L\sigma_*^{\frac{2}{H}} \kappa^{1-\frac{1}{H}} t}. \tag{4.47}$$

By choosing $\epsilon = \frac{|p_t u_0(x)|}{3\|u_0\|_{L^\infty}}$, we conclude the proof. □

Remark 4.33. (i) We can add a drift $b(u(t,x))$ in equation (1.1), and if the function b is Lipschitz continuous with $b(0) = 0$, the results we have obtained on the existence and uniqueness of a solution can be extended to equations with drift.

(ii) If we only assume that the initial condition u_0 is bounded and

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |u_0(x) - u_0(x+h)|^2 |h|^{2H-2} dh < \infty, \tag{4.48}$$

and we only assume that σ is Lipschitz. Then from the proof of Theorem 4.25 it is easy to see that we have the weak existence of a solution to equation (1.1). The assumption (1) in Theorem 4.25 and the condition that the derivative of σ is Lipschitz and $\sigma(0) = 0$ are only used to show the uniqueness.

5. The Anderson model

In this section we will study the special case of equation (1.1) when the function σ is the identity function. This is a continuous version of the so-called parabolic Anderson model. In this case equation (1.1) is reduced to

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2} \frac{\partial^2 u}{\partial x^2} + u\,\dot{W} \tag{5.1}$$

with deterministic initial condition $u(0,x) = u_0(x)$. This reduced form allows for some simplified versions of the existence-uniqueness theorems, and also some Feynman-Kac representation which is useful for intermittency estimates.

5.1. **Existence and uniqueness.**

With some restrictions on the initial condition $u_0(x)$, the existence and uniqueness of the solution to this linear equation stems directly from Theorems 4.5 and 4.25. However, we shall prove this result again by means of two different methods: one is via Fourier transform and the other is via chaos expansion. We include these methods here for two reasons: first, they lead to proofs which are shorter and more elegant than in the case of a general coefficient σ; secondly, the assumptions on initial conditions are different.

5.1.1. *Existence and uniqueness via Fourier transform.* In this subsection we discuss the existence and uniqueness of equation (5.1) using techniques of Fourier analysis.

We recall that $\dot H^{\frac12-H}$ is the class of functions $f:\mathbb{R}\to\mathbb{R}$ such that there exists $g\in L^2(\mathbb{R})$ such that $f=I_-^{1/2-H}g$. Let $\dot H_0^{\frac12-H}$ be the set of functions $f\in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}}|\mathcal{F}f(\xi)|^2|\xi|^{1-2H}d\xi<\infty$. These spaces are the time independent analogues to the spaces $\mathfrak{H}$ and $\mathfrak{H}_0$ introduced in Proposition 2.1. We know that the inclusion $\dot H_0^{\frac12-H}\subset \dot H^{\frac12-H}$ is strict and $\dot H_0^{\frac12-H}$ is not complete with the seminorm $\left[\int_{\mathbb{R}}|\mathcal{F}f(\xi)|^2|\xi|^{1-2H}d\xi\right]^{\frac12}$ (see [25]). However, it is not difficult to check that the space $\dot H_0^{\frac12-H}$ is complete for the seminorm

$$\|f\|^2_{\mathcal{V}(H)}:=\int_{\mathbb{R}}|\mathcal{F}f(\xi)|^2(1+|\xi|^{1-2H})d\xi.$$

In the next theorem we show the existence and uniqueness result assuming that the initial condition belongs to $\dot H_0^{\frac12-H}$ and using estimates based on the Fourier transform in the space variable. To this purpose, we introduce the space $\mathcal{V}_T(H)$ as the completion of the set of elementary $\dot H_0^{\frac12-H}$-valued stochastic processes $\{u(t,\cdot),t\in[0,T]\}$ with respect to the seminorm

$$\|u\|^2_{\mathcal{V}_T(H)}:=\sup_{t\in[0,T]}\mathbf{E}\|u(t,\cdot)\|^2_{\mathcal{V}(H)}. \tag{5.2}$$

We now state a convolution lemma.

Proposition 5.1. *Consider a function $u_0\in\dot H_0^{\frac12-H}$ and $\frac14<H<\frac12$. For any $v\in\mathcal{V}_T(H)$ we set $\Gamma(v)=V$ in the following way:*

$$\Gamma(v):=V(t,x)=p_tu_0(x)+\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)v(s,y)W(ds,dy),\quad t\in[0,T],\ x\in\mathbb{R}.$$

Then Γ is well-defined as a map from $\mathcal{V}_T(H)$ to $\mathcal{V}_T(H)$. Furthermore, there exist two positive constants c_1,c_2 such that the following estimate holds true on $[0,T]$:

$$\|V(t,\cdot)\|^2_{\mathcal{V}(H)}\le c_1\|u_0\|^2_{\mathcal{V}(H)}+c_2\int_0^t(t-s)^{2H-3/2}\|v(s,\cdot)\|^2_{\mathcal{V}(H)}\,ds\,. \tag{5.3}$$

Proof. Let v be a process in $\mathcal{V}_T(H)$ and set $V=\Gamma(v)$. We focus on the bound (5.3) for V.

Notice that the Fourier transform of V can be computed easily. Indeed, setting $v_0(t,x)=p_tu_0(x)$ and invoking a stochastic version of Fubini's theorem, we get

$$\mathcal{F}V(t,\xi)=\mathcal{F}v_0(t,\xi)+\int_0^t\int_{\mathbb{R}}\left(\int_{\mathbb{R}}e^{ix\xi}\,p_{t-s}(x-y)\,dx\right)v(s,y)W(ds,dy)\,.$$

According to the expression of $\mathcal{F}p_t$, we obtain

$$\mathcal{F}V(t,\xi)=\mathcal{F}v_0(t,\xi)+\int_0^t\int_{\mathbb{R}}e^{-i\xi y}e^{-\frac{\kappa}{2}(t-s)\xi^2}v(s,y)W(ds,dy)\,.$$

We now evaluate the quantity $\mathbf{E}[\int_{\mathbb{R}}|\mathcal{F}V(t,\xi)|^2|\xi|^{1-2H}d\xi]$ in the definition of $\|u_n\|_{\mathcal{V}_T(H)}$ given by (5.2). We thus write

$$\mathbf{E}\left[\int_{\mathbb{R}} |\mathcal{F}V(t,\xi)|^2|\xi|^{1-2H}d\xi\right] \leq 2\int_{\mathbb{R}} |\mathcal{F}v_0(t,\xi)|^2|\xi|^{1-2H}d\xi$$
$$+ 2\int_{\mathbb{R}} \mathbf{E}\left[\Big|\int_0^t\int_{\mathbb{R}} e^{-i\xi y}e^{-\frac{\kappa}{2}(t-s)\xi^2}v(s,y)W(ds,dy)\Big|^2\right]|\xi|^{1-2H}d\xi := 2\,(I_1+I_2)\;,$$

and we handle the terms I_1 and I_2 separately.

The term I_1 can be easily bounded by using that $u_0 \in \dot{H}_0^{\frac{1}{2}-H}$ and recalling $v_0 = p_t u_0$. That is,

$$I_1 = \int_{\mathbb{R}} |\mathcal{F}u_0(\xi)|^2 e^{-\kappa t|\xi|^2}|\xi|^{1-2H}d\xi \leq C\,\|u_0\|^2_{\mathcal{V}(H)}.$$

We thus focus on the estimation of I_2, and we set $f_\xi(s,\eta) = e^{-i\xi\eta}e^{-\frac{\kappa}{2}(t-s)\xi^2}v(s,\eta)$. Applying the isometry property (2.10) together with the Fourier transform expression for $\|h\|_{\dot{H}^{\frac{1}{2}-H}}$ in (2.7), we have:

$$\mathbf{E}\left[\Big|\int_0^t\int_{\mathbb{R}} e^{-i\xi y}e^{-\frac{\kappa}{2}(t-s)\xi^2}v(s,y)W(ds,dy)\Big|^2\right] = c_{1,H}\int_0^t\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}_\eta f_\xi(s,\eta)|^2\right]|\eta|^{1-2H}\,dsd\eta,$$

where $\mathcal{F}_\eta$ is the Fourier transform with respect to η. It is obvious that the Fourier transform of $e^{-i\xi y}V(y)$ is $\mathcal{F}V(\eta+\xi)$. Thus we have

$$\begin{aligned} I_2 &= C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2}\,\mathbf{E}\left[|\mathcal{F}v(s,\eta+\xi)|^2\right]|\eta|^{1-2H}|\xi|^{1-2H}\,d\eta d\xi ds \\ &= C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2}\,\mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2\right]|\eta-\xi|^{1-2H}|\xi|^{1-2H}\,d\eta d\xi ds\,. \end{aligned}$$

We now bound $|\eta-\xi|^{1-2H}$ by $|\eta|^{1-2H}+|\xi|^{1-2H}$, which yields $I_2 \leq I_{21}+I_{22}$ with:

$$\begin{aligned} I_{21} &= C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2}\,\mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2\right]|\eta|^{1-2H}|\xi|^{1-2H}\,d\eta d\xi ds \\ I_{22} &= C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2}\,\mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2\right]|\xi|^{2-4H}\,d\eta d\xi ds\,. \end{aligned}$$

Performing the change of variable $\xi \to (t-s)^{1/2}\xi$ and then trivially bounding the integrals of the form $\int_{\mathbb{R}}|\xi|^\beta e^{-\kappa\xi^2}d\xi$ by constants, we end up with

$$\begin{aligned} I_{21} &\leq C\int_0^t (t-s)^{H-1}\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2\right]|\eta|^{1-2H}\,d\eta\,ds \\ I_{22} &\leq C\int_0^t (t-s)^{2H-3/2}\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2\right]\,d\eta\,ds. \end{aligned}$$

Observe that for $H \in (\frac{1}{4},\frac{1}{2})$ the term $(t-s)^{2H-3/2}$ is more singular than $(t-s)^{H-1}$, but we still have $2H-\frac{3}{2} > -1$. Summarizing our consideration up to now, we have thus obtained

$$\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}V(t,\xi)|^2\right]|\xi|^{1-2H}d\xi$$
$$\leq C_{1,T}\,\|u_0\|^2_{\mathcal{V}(H)} + C_{2,T}\int_0^t (t-s)^{2H-3/2}\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}v(s,\xi)|^2\right]\,(1+|\xi|^{1-2H})\,d\xi\,ds, \quad (5.4)$$

for two strictly positive constants $C_{1,T}, C_{2,T}$.

The term $\mathbf{E}[\int_{\mathbb{R}} |\mathcal{F}V(t,\xi)|^2 d\xi]$ in the definition of $\|V\|_{\mathcal{V}_T(H)}$ can be bounded with the same computations as above, and we find

$$\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}V(t,\xi)|^2\right] d\xi \\ \leq C_{1,T}\,\|u_0\|^2_{\mathcal{V}(H)} + C_{2,T}\int_0^t (t-s)^{H-1}\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}v(s,\xi)|^2\right]\,(1+|\xi|^{1-2H})\,d\eta\,ds, \quad (5.5)$$

Hence, gathering our estimates (5.4) and (5.5), our bound (5.3) is easily obtained, which finishes the proof. □

As in the general case, Proposition 5.1 is the key to the existence and uniqueness result for equation (5.1).

Theorem 5.2. *Suppose that u_0 is an element of $\dot{H}_0^{\frac{1}{2}-H}$ and $\frac{1}{4} < H < \frac{1}{2}$. Fix $T > 0$. Then there is a unique process u in the space $\mathcal{V}_T(H)$ such that for all $t \in [0,T]$,*

$$u(t,\cdot) = p_t u_0 + \int_0^t \int_{\mathbb{R}} p_{t-s}(\cdot - y)u(s,y)W(ds,dy). \quad (5.6)$$

Proof. The proof follows from the standard Picard iteration scheme, where we just set $u_{n+1} = \Gamma(u_n)$. Details are left to the reader for sake of conciseness. □

5.1.2. *Existence and uniqueness via chaos expansions.* Next, we provide another way to prove the existence and uniqueness of the solution to equation (5.1), by means of chaos expansions. This will enable us to obtain moment estimates. Before stating our main theorem in this direction, let us label an elementary lemma borrowed from [18] for further use.

Lemma 5.3. *For $m \geq 1$ let $\alpha \in (-1+\varepsilon, 1)^m$ with $\varepsilon > 0$ and set $|\alpha| = \sum_{i=1}^m \alpha_i$. For $t \in [0,T]$, the m-th dimensional simplex over $[0,t]$ is denoted by $T_m(t) = \{(r_1, r_2, \ldots, r_m) \in \mathbb{R}^m : 0 < r_1 < \cdots < r_m < t\}$. Then there is a constant $c > 0$ such that*

$$J_m(t,\alpha) := \int_{T_m(t)} \prod_{i=1}^m (r_i - r_{i-1})^{\alpha_i} dr \leq \frac{c^m t^{|\alpha|+m}}{\Gamma(|\alpha|+m+1)},$$

where by convention, $r_0 = 0$.

Let us now state a new existence and uniqueness theorem for our equation of interest.

Theorem 5.4. *Suppose that $\frac{1}{4} < H < \frac{1}{2}$ and that the initial condition u_0 satisfies*

$$\int_{\mathbb{R}} (1+|\xi|^{\frac{1}{2}-H})|\mathcal{F}u_0(\xi)|d\xi < \infty\,. \quad (5.7)$$

Then there exists a unique solution to equation (5.1), that is a process $u \in \Lambda_H$ (remember that Λ_H is defined in Proposition 2.3) such that for any $(t,x) \in [0,T]\times\mathbb{R}$, relation (2.12) holds true.

Remark 5.5. (i) The formulation of Theorem 5.4 yields the definition of our solution u for all $(t,x) \in [0,T]\times\mathbb{R}$. This is in contrast with Theorem 5.2 which gives a solution sitting in $\dot{H}_0^{\frac{1}{2}-H}$ for every value of t, and thus defined a.e. in x only.

(ii) Condition (5.7) is satisfied by constant functions.

Proof of Theorem 5.4. Suppose that $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a solution to equation (2.12) in Λ_H. Then according to (2.14), for any fixed (t,x) the random variable $u(t,x)$ admits the following Wiener chaos expansion

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)), \tag{5.8}$$

where for each (t,x), $f_n(\cdot,t,x)$ is a symmetric element in $\mathfrak{H}^{\otimes n}$. Furthermore, we have seen that Itô and Skorohod's integral coincide for processes in Λ_H. Hence, thanks to (2.16) and using an iteration procedure, one can find an explicit formula for the kernels f_n for $n \geq 1$. Indeed, we have:

$$\begin{aligned} &f_n(s_1,x_1,\dots,s_n,x_n,t,x) \\ &\qquad = \frac{1}{n!} p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)}) p_{s_{\sigma(1)}} u_0(x_{\sigma(1)}), \end{aligned} \tag{5.9}$$

where σ denotes the permutation of $\{1,2,\dots,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$ (see, for instance, formula (4.4) in [19] or formula (3.3) in [18]). Then, to show the existence and uniqueness of the solution it suffices to prove that for all (t,x) we have

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot,t,x)\|^2_{\mathfrak{H}^{\otimes n}} < \infty. \tag{5.10}$$

The remainder of the proof is devoted to prove relation (5.10).

Starting from relation (5.9), some elementary Fourier computations show that

$$\begin{aligned} \mathcal{F} f_n(s_1,\xi_1,\dots,s_n,\xi_n,t,x) &= \frac{c_H^n}{n!} \int_{\mathbb{R}} \prod_{i=1}^n e^{-\frac{\kappa}{2}(s_{\sigma(i+1)}-s_{\sigma(i)})|\xi_{\sigma(i)}+\cdots+\xi_{\sigma(1)}-\zeta|^2} \\ &\quad \times e^{-ix(\xi_{\sigma(n)}+\cdots+\xi_{\sigma(1)}-\zeta)} \mathcal{F}u_0(\zeta) e^{-\frac{\kappa s_{\sigma(1)}|\zeta|^2}{2}} d\zeta, \end{aligned}$$

where we have set $s_{\sigma(n+1)} = t$. Hence, owing to formula (2.7) for the norm in $\mathfrak{H}$ (in its Fourier mode version), we have

$$\begin{aligned} n! \|f_n(\cdot,t,x)\|^2_{\mathfrak{H}^{\otimes n}} = \frac{c_H^{2n}}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^n} \bigg| \int_{\mathbb{R}} \prod_{i=1}^n e^{-\frac{\kappa}{2}(s_{\sigma(i+1)}-s_{\sigma(i)})|\xi_i+\cdots+\xi_1-\zeta|^2} e^{-ix(\xi_{\sigma(n)}+\cdots+\xi_{\sigma(1)}-\zeta)} \\ \mathcal{F}u_0(\zeta) e^{-\frac{\kappa s_{\sigma(1)}|\zeta|^2}{2}} d\zeta \bigg|^2 \times \prod_{i=1}^n |\xi_i|^{1-2H} d\xi ds, \end{aligned} \tag{5.11}$$

where $d\xi$ denotes $d\xi_1 \cdots d\xi_n$ and similarly for ds. Then using the change of variable $\xi_i + \cdots + \xi_1 = \eta_i$, for all $i = 1,2,\dots,n$ and a linearization of the above expression, we obtain

$$\begin{aligned} n! \|f_n(\cdot,t,x)\|^2_{\mathfrak{H}^{\otimes n}} = \frac{c_H^{2n}}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^2} \prod_{i=1}^n e^{-\frac{\kappa}{2}(s_{\sigma(i+1)}-s_{\sigma(i)})(|\eta_i-\zeta|^2+|\eta_i-\zeta'|^2)} \mathcal{F}u_0(\zeta) \overline{\mathcal{F}u_0(\zeta')} \\ \times e^{ix(\zeta-\zeta')} e^{-\frac{\kappa s_{\sigma(1)}(|\zeta|^2+|\zeta'|^2)}{2}} \prod_{i=1}^n |\eta_i - \eta_{i-1}|^{1-2H} d\zeta d\zeta' d\eta ds, \end{aligned}$$

where we have set $\eta_0 = 0$. Then we use Cauchy-Schwarz inequality and bound the term $\exp(-\kappa s_{\sigma(1)}(|\zeta|^2 + |\zeta'|^2)/2)$ by 1 to get

$$n!\|f_n(\cdot,t,x)\|^2_{\mathfrak{H}^{\otimes n}} \le \frac{c_H^{2n}}{n!}\int_{\mathbb{R}^2}\left(\int_{[0,t]^n}\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)}-s_{\sigma(i)})|\eta_i-\zeta|^2}\prod_{i=1}^n|\eta_i-\eta_{i-1}|^{1-2H}d\eta ds\right)^{\frac12}$$
$$\times\left(\int_{[0,t]^n}\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)}-s_{\sigma(i)})|\eta_i-\zeta'|^2}\prod_{i=1}^n|\eta_i-\eta_{i-1}|^{1-2H}d\eta ds\right)^{\frac12}|\mathcal{F}u_0(\zeta)|\,|\mathcal{F}u_0(\zeta')|\,d\zeta d\zeta'.$$

Arranging the integrals again, performing the change of variables $\eta_i := \eta_i - \zeta$ and invoking the trivial bound $|\eta_i - \eta_{i-1}|^{1-2H} \le |\eta_{i-1}|^{1-2H} + |\eta_i|^{1-2H}$, this yields

$$n!\|f_n(\cdot,t,x)\|^2_{\mathfrak{H}^{\otimes n}} \le \frac{c_H^{2n}}{n!}\left(\int_{\mathbb{R}} L_{n,t}^{\frac12}(\zeta)\,|\mathcal{F}u_0(\zeta)|\,d\zeta\right)^2, \tag{5.12}$$

where

$$L_{n,t}(\zeta)$$
$$=\int_{[0,t]^n}\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)}-s_{\sigma(i)})|\eta_i|^2}(|\zeta|^{1-2H}+|\eta_1|^{1-2H})\times\prod_{i=2}^n(|\eta_i|^{1-2H}+|\eta_{i-1}|^{1-2H})d\eta ds.$$

Let us expand the product $\prod_{i=2}^n(|\eta_i|^{1-2H}+|\eta_{i-1}|^{1-2H})$ in the integral defining $L_{n,t}(\zeta)$. We obtain an expression of the form $\sum_{\alpha\in D_n}\prod_{i=1}^n|\eta_i|^{\alpha_i}$, where D_n is a subset of multi-indices of length $n-1$. The complete description of D_n is omitted for sake of conciseness, and we will just use the following facts: $\mathrm{Card}(D_n) = 2^{n-1}$ and for any $\alpha \in D_n$ we have

$$|\alpha| \equiv \sum_{i=1}^n \alpha_i = (n-1)(1-2H), \quad \text{and} \quad \alpha_i \in \{0, 1-2H, 2(1-2H)\}, \quad i = 1,\dots,n.$$

This simple expansion yields the following bound

$$L_{n,t}(\zeta) \le |\zeta|^{1-2H}\sum_{\alpha\in D_n}\int_{[0,t]^n}\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)}-s_{\sigma(i)})|\eta_i|^2}\prod_{i=1}^n|\eta_i|^{\alpha_i}d\eta ds$$
$$+\sum_{\alpha\in D_n}\int_{[0,t]^n}\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)}-s_{\sigma(i)})|\eta_i|^2}|\eta_1|^{1-2H}\prod_{i=1}^n|\eta_i|^{\alpha_i}d\eta ds\,.$$

Perform the change of variable $\xi_i = (\kappa(s_{\sigma(i+1)} - s_{\sigma(i)}))^{1/2}\eta_i$ in the above integral, and notice that $\int_{\mathbb{R}} e^{-\xi^2}|\xi|^{\alpha_i}d\xi$ is bounded by a constant for $\alpha_i > -1$. Changing the integral over $[0,t]^n$ into an integral over the simplex, we get

$$L_{n,t}(\zeta) \;\le\; C|\zeta|^{1-2H}n!c_H^n\sum_{\alpha\in D_n}\int_{T_n(t)}\prod_{i=1}^n(\kappa(s_{i+1}-s_i))^{-\frac12(1+\alpha_i)}ds.$$
$$+Cn!c_H^n\sum_{\alpha\in D_n}\int_{T_n(t)}(\kappa(s_2-s_1))^{-\frac{2-2H+\alpha_1}{2}}\prod_{i=2}^n(\kappa(s_{i+1}-s_i))^{-\frac12(1+\alpha_i)}ds.$$

We observe that whenever $\frac{1}{4} < H < \frac{1}{2}$, we have $\frac{1}{2}(1+\alpha_i) < 1$ for all $i = 2, \dots n$, and it is easy to see that α_1 is at most $1-2H$ so $\frac{1}{2}(2-2H+\alpha_1) < 1$. Thanks to Lemma 5.3 and recalling that $\sum_{i=1}^n \alpha_i = (n-1)(1-2H)$ for all $\alpha \in D_n$, we thus conclude that

$$L_{n,t}(\zeta) \leq \frac{C(t^{H-\frac{1}{2}}\kappa^{H-\frac{1}{2}} + |\zeta|^{1-2H}) n! c_H^n t^{nH} \kappa^{nH-n}}{\Gamma(nH+1)}.$$

Plugging this expression into (5.12), we end up with

$$n! \|f_n(\cdot, t, x)\|^2_{\mathfrak{H}^{\otimes n}} \leq \frac{C c_H^n t^{nH} \kappa^{nH-n}}{\Gamma(nH+1)} \left(\int_{\mathbb{R}} (t^{H-\frac{1}{2}}\kappa^{H-\frac{1}{2}} + |\zeta|^{\frac{1}{2}-H}) \, |\mathcal{F}u_0(\zeta)| \, d\zeta \right)^2. \tag{5.13}$$

The proof of (5.10) is now easily completed thanks to the asymptotic behavior of the Gamma function and our assumption of u_0, and this finishes the existence and uniqueness proof. □

5.2. **Moment bounds.** In this section we derive the upper and lower bounds for the moments of the solution to equation (5.1) which allow us to conclude on the intermittency of the solution. We proceed by first getting an approximation result for u, and then deriving the upper and lower bounds for the approximation.

5.2.1. *Approximation of the solution.* The approximation of the solution we consider is based on an approximation of the noise W, which is defined in (4.30). The noise W_ε induces an approximation to the mild formulation of equation (5.1), namely

$$u_\varepsilon(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) u_\varepsilon(s,y) \, W_\varepsilon(ds, dy), \tag{5.14}$$

where the integral is understood (as in Section 5.1.1) in the Itô sense. We will start by a formula for the moments of u_ε.

Proposition 5.6. *Let W_ε be the noise defined by (4.30), and assume $\frac{1}{4} < H < \frac{1}{2}$. Assume u_0 is such that $\int_{\mathbb{R}} (1+|\xi|^{\frac{1}{2}-H}) |\mathcal{F}u_0(\xi)| d\xi < \infty$. Then*
(i) *Equation (5.14) admits a unique solution.*
(ii) *For any integer $n \geq 2$ and $(t,x) \in [0,T] \times \mathbb{R}$, we have*

$$\mathbf{E}\left[u_\varepsilon^n(t,x)\right] = \mathbf{E}_B \left[\prod_{j=1}^n u_0(x + B^j_{\kappa t}) \exp\left(c_{1,H} \sum_{1 \leq j \neq k \leq n} V^{\varepsilon, j, k}_{t,x} \right) \right], \tag{5.15}$$

with

$$V^{\varepsilon,j,k}_{t,x} = \int_0^t f_\varepsilon(B^j_{\kappa r} - B^k_{\kappa r}) dr = \int_0^t \int_{\mathbb{R}} e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} e^{i\xi(B^j_{\kappa r} - B^k_{\kappa r})} \, d\xi dr. \tag{5.16}$$

In formula (5.16), $\{B^j; j = 1, \dots, n\}$ is a family of n independent standard Brownian motions which are also independent of W and $\mathbf{E}_B$ denotes the expected value with respect to the randomness in B only.
(iii) *The quantity $\mathbf{E}[(u_\varepsilon(t,x))^n]$ is uniformly bounded in ε. More generally, for any $a > 0$ we have*

$$\sup_{\varepsilon > 0} \mathbf{E}_B \left[\exp\left(a \sum_{1 \leq j \neq k \leq n} V^{\varepsilon,j,k}_{t,x} \right) \right]. \equiv c_a < \infty$$

Proof. The proof of item (i) is almost identical to the proof of Theorem 5.4, and is omitted for sake of conciseness. Moreover, in the proof of (ii) and (iii), we may take $u_0(x) \equiv 1$ for simplicity.

In order to check item (ii), set

$$A^\varepsilon_{t,x}(r,y) = \rho_\varepsilon(B^x_{\kappa(t-r)} - y), \quad \text{and} \quad \alpha^\varepsilon_{t,x} = \|A^\varepsilon_{t,x}\|^2_{\mathfrak{H}}. \tag{5.17}$$

Then one can prove, similarly to Proposition 5.2 in [19], that u_ε admits a Feynman-Kac representation of the form

$$u_\varepsilon(t,x) = \mathbf{E}_B\left[\exp\left(W(A^\varepsilon_{t,x}) - \frac{1}{2}\alpha^\varepsilon_{t,x}\right)\right]. \tag{5.18}$$

Now fix an integer $n \geq 2$. According to (5.18) we have

$$\mathbf{E}\left[u^n_\varepsilon(t,x)\right] = \mathbf{E}_W\left[\prod_{j=1}^n \mathbf{E}_B\left[\exp\left(W(A^{\varepsilon,B^j}_{t,x}) - \frac{1}{2}\alpha^{\varepsilon,B^j}_{t,x}\right)\right]\right],$$

where for any $j = 1,\dots,n$, $A^{\varepsilon,B^j}_{t,x}$ and $\alpha^{\varepsilon,B^j}_{t,x}$ are evaluations of (5.17) using the Brownian motion B^j. Therefore, since $W(A^{\varepsilon,B^j}_{t,x})$ is a Gaussian random variable conditionally on B, we obtain

$$\begin{aligned}
\mathbf{E}\left[u^n_\varepsilon(t,x)\right] &= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^n A^{\varepsilon,B^j}_{t,x}\|^2_{\mathfrak{H}} - \frac{1}{2}\sum_{j=1}^n \alpha^{\varepsilon,B^j}_{t,x}\right)\right] \\
&= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^n A^{\varepsilon,B^j}_{t,x}\|^2_{\mathfrak{H}} - \frac{1}{2}\sum_{j=1}^n \|A^{\varepsilon,B^j}_{t,x}\|^2_{\mathfrak{H}}\right)\right] \\
&= \mathbf{E}_B\left[\exp\left(\sum_{1\leq i<j\leq n}\langle A^{\varepsilon,B^i}_{t,x}, A^{\varepsilon,B^j}_{t,x}\rangle_{\mathfrak{H}}\right)\right].
\end{aligned}$$

The evaluation of $\langle A^{\varepsilon,B^i}_{t,x}, A^{\varepsilon,B^j}_{t,x}\rangle_{\mathfrak{H}}$ easily yields our claim (5.15), the last details being left to the patient reader.

Let us now prove item (iii), namely

$$\sup_{\varepsilon>0}\ \sup_{t\in[0,T],x\in\mathbb{R}} \mathbf{E}\left[u^n_\varepsilon(t,x)\right] < \infty. \tag{5.19}$$

To this aim, observe first that we have obtained an expression (5.15) which does not depend on $x \in \mathbb{R}$, so that the $\sup_{t\in[0,T],x\in\mathbb{R}}$ in (5.19) can be reduced to a sup in t only. Next, still resorting to formula (5.15), it is readily seen that it suffices to show that for two independent Brownian motions B and $\tilde{B}$, we have

$$\sup_{\varepsilon>0,t\in[0,T]} \mathbf{E}_B\left[\exp\left(c\,F^\varepsilon_t\right)\right] < \infty, \quad \text{with} \quad F^\varepsilon_t \equiv \int_0^t\int_{\mathbb{R}} e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}e^{i\xi(B_{\kappa r}-\tilde{B}_{\kappa r})}d\xi dr, \tag{5.20}$$

for any positive constant c. In order to prove (5.20), we expand the exponential and write:

$$\mathbf{E}_B\left[\exp(c\,F^\varepsilon_t)\right] = \sum_{l=0}^\infty \frac{\mathbf{E}_B\left[(c\,F^\varepsilon_t)^l\right]}{l!}. \tag{5.21}$$

Next, we have

$$\mathbf{E}_B\left[(F_t^\varepsilon)^l\right] = \mathbf{E}_B\left[\int_{[0,t]^l}\int_{\mathbb{R}^l}\prod_{j=1}^l e^{-i\xi_j(B_{\kappa r_j}-\tilde{B}_{\kappa r_j})-\varepsilon|\xi_j|^2}|\xi_j|^{1-2H}d\xi dr\right]$$
$$\leq \int_{[0,t]^l}\int_{\mathbb{R}^l}\prod_{j=1}^l e^{-\kappa(t-r_{\sigma(l)})|\xi_l+\cdots+\xi_1|^2}\,|\xi_j|^{1-2H}\,d\xi dr\,,$$

where σ is the permutation on $\{1,2,\dots,l\}$ such that $t\geq r_{\sigma(l)}\geq\cdots\geq r_{\sigma(1)}$. We have thus gone back to an expression which is very similar to (5.11). We now proceed as in the proof of Theorem 5.4 to show that (5.19) holds true from equation (5.21). ☐

Starting from Proposition 5.6, let us take limits in order to get the moment formula for the solution u to equation (5.1).

Theorem 5.7. *Assume $\frac{1}{4}<H<\frac{1}{2}$ and consider $n\geq 1$, $j,k\in\{1,\dots,n\}$ with $j\neq k$. For $(t,x)\in[0,T]\times\mathbb{R}$, denote by $V_{t,x}^{j,k}$ the limit in $L^2(\Omega)$ as $\varepsilon\to 0$ of*

$$V_{t,x}^{\varepsilon,j,k}=\int_0^t\int_{\mathbb{R}}e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}e^{i\xi(B_{\kappa r}^j-B_{\kappa r}^k)}d\xi dr.$$

Then $\mathbf{E}\left[u_\varepsilon^n(t,x)\right]$ converges as $\varepsilon\to 0$ to $\mathbf{E}[u^n(t,x)]$, which is given by

$$\mathbf{E}[u^n(t,x)]=\mathbf{E}_B\left[\prod_{j=1}^n u_0(B_{\kappa t}^j+x)\exp\left(c_{1,H}\sum_{1\leq j\neq k\leq n}V_{t,x}^{j,k}\right)\right]. \tag{5.22}$$

Proof. As in Proposition 5.6, we will prove the theorem for $u_0\equiv 1$ for simplicity. For any $p\geq 1$ and $1\leq j<k\leq n$, we can easily prove that $V_{t,x}^{\varepsilon,j,k}$ converges in $L^p(\Omega)$ to $V_{t,x}^{j,k}$ defined by

$$V_{t,x}^{j,k}=\int_0^t\int_{\mathbb{R}}|\xi|^{1-2H}e^{i\xi(B_{\kappa r}^j-B_{\kappa r}^k)}d\xi dr. \tag{5.23}$$

Indeed, this is due to the fact that $e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}e^{i\xi(B_{\kappa r}^j-B_{\kappa r}^k)}$ converges to $|\xi|^{1-2H}e^{i\xi(B_{\kappa r}^j-B_{\kappa r}^k)}$ in the $d\xi\otimes dr\otimes d\mathbf{P}$ sense, plus standard uniform integrability arguments. Now, taking into account relation (5.15), Proposition 5.6 and the fact that $V_{t,x}^{\varepsilon,j,k}$ converges to $V_{t,x}^{j,k}$ in $L^2(\Omega)$ as $\varepsilon\to 0$, we obtain

$$\lim_{\varepsilon\to 0}\mathbf{E}\left[u_\varepsilon^n(t,x)\right] = \lim_{\varepsilon\to 0}\mathbf{E}_B\left[\exp\left(c_{1,H}\sum_{1\leq j\neq k\leq n}V_{t,x}^{\varepsilon,j,k}\right)\right]$$
$$= \mathbf{E}_B\left[\exp\left(c_{1,H}\sum_{1\leq j\neq k\leq n}V_{t,x}^{j,k}\right)\right]. \tag{5.24}$$

To end the proof, let us now identify the right hand side of (5.24) with $\mathbf{E}[u^n(t,x)]$, where u is the solution to equation (5.1). For $\varepsilon,\varepsilon'>0$ we write

$$\mathbf{E}\left[u_\varepsilon(t,x)\,u_{\varepsilon'}(t,x)\right]=\mathbf{E}_B\left[\exp\left(\,\langle A_{t,x}^{\varepsilon,B^1},A_{t,x}^{\varepsilon',B^2}\rangle_{\mathfrak{H}}\right)\right],$$

where we recall that $A^{\varepsilon,B}_{t,x}$ is defined by relation (5.17). As before we can show that this converges as $\varepsilon, \varepsilon'$ tend to zero. So, $u_\varepsilon(t,x)$ converges in L^2 to some limit $v(t,x)$, and the limit is actually in L^p , for all $p \geq 1$. Moreover, $\mathbf{E}[v^k(t,x)]$ is equal to the right hand side of (5.24). Finally, for any smooth random variable F which is a linear combination of $W(\mathbf{1}_{[a,b]}(s)\varphi(x))$, where φ is a C^∞ function with compact support, using the fact that Itô's and Skorohod's integrals coincide on the set Λ_H, plus the duality relation (2.13), we have

$$\mathbf{E}\left[F u_\varepsilon(t,x)\right] = \mathbf{E}\left[F\right] + \mathbf{E}\left[\langle Y^\varepsilon, DF\rangle_{\mathfrak{H}}\right], \tag{5.25}$$

where

$$Y^{t,x}(s,z) = \left(\int_{\mathbb{R}} p_{t-s}(x-y)\, p_\varepsilon(y-z) u_\varepsilon(s,y)\, dy\right) \mathbf{1}_{[0,t]}(s).$$

Letting ε tend to zero in equation (5.25), after some easy calculation we get

$$\mathbf{E}[F v_{t,x}] = \mathbf{E}[F] + \mathbf{E}\left[\langle DF, v p_{t-\cdot}(x-\cdot)\rangle_{\mathfrak{H}}\right] .$$

This equation is valid for any $F \in \mathbb{D}^{1,2}$ by approximation. So the above equation implies that the process v is the solution of equation (5.1), and by the uniqueness of the solution we have $v = u$. □

5.2.2. *Intermittency estimates.* In this section we prove some upper and lower bounds on the moments of the solution which entail the intermittency phenomenon.

Theorem 5.8. *Let $\frac{1}{4} < H < \frac{1}{2}$, and consider the solution u to equation (5.1). For simplicity we assume that the initial condition is $u_0(x) \equiv 1$. Let $n \geq 2$ be an integer, $x \in \mathbb{R}$ and $t \geq 0$. Then there exist some positive constants c_1, c_2, c_3 independent of n, t and κ with $0 < c_1 < c_2$ satisfying*

$$\exp(c_1 n^{1+\frac{1}{H}} \kappa^{1-\frac{1}{H}} t) \leq \mathbf{E}\left[u^n(t,x)\right] \leq c_3 \exp\left(c_2 n^{1+\frac{1}{H}} \kappa^{1-\frac{1}{H}} t\right). \tag{5.26}$$

Remark 5.9. Observe that the upper bound in (5.26) has already been proven for a general coefficient σ (see (4.39)).

Proof of Theorem 5.8. We divide this proof into upper and lower bound estimates.

Step 1: Upper bound. Recall from equation (5.8) that for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $u(t,x)$ can be written as: $u(t,x) = \sum_{m=0}^\infty I_m(f_m(\cdot,t,x))$. Moreover, as a consequence of the hypercontractivity property on a fixed chaos we have (see [23, p. 62])

$$\|I_m(f_m(\cdot,t,x))\|_{L^n(\Omega)} \leq (n-1)^{\frac{m}{2}} \|I_m(f_m(\cdot,t,x))\|_{L^2(\Omega)} ,$$

and substituting the above right hand side by the bound (5.13), we end up with

$$\|I_m(f_m(\cdot,t,x))\|_{L^n(\Omega)} \leq n^{\frac{m}{2}} \|I_m(f_m(\cdot,t,x))\|_{L^2(\Omega)} \leq \frac{c^{\frac{n}{2}} n^{\frac{m}{2}} t^{\frac{mH}{2}} \kappa^{\frac{Hm-m}{2}}}{\left[\Gamma(mH+1)\right]^{\frac{1}{2}}} .$$

Therefore recalling again the elementary bound $\sum_{n\geq 0} x^n/(n!)^a \leq 2\exp(cx^{1/a})$, we get:

$$\|u(t,x)\|_{L^n(\Omega)} \leq \sum_{m=0}^\infty \|J_m(t,x)\|_{L^n(\Omega)} \leq \sum_{m=0}^\infty \frac{c^{\frac{m}{2}} n^{\frac{m}{2}} t^{\frac{mH}{2}} \kappa^{\frac{Hm-m}{2}}}{\left(\Gamma(mH+1)\right)^{\frac{1}{2}}} \leq c_1 \exp\left(c_2 t n^{\frac{1}{H}} \kappa^{\frac{H-1}{H}}\right),$$

from which the upper bound in our theorem is easily deduced.

Step 2: Lower bound for u_ε. For the lower bound, we start from the moment formula (5.15) for the approximate solution, and write

$$\mathbf{E}\left[u_\varepsilon^n(t,x)\right] = \mathbf{E}_B\left[\exp\left(c_{1,H}\left[\int_0^t\int_{\mathbb{R}} e^{-\varepsilon|\xi|^2}\left|\sum_{j=1}^n e^{-iB^j_{\kappa r}\xi}\right|^2 |\xi|^{1-2H}d\xi dr - nt\int_{\mathbb{R}} e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi\right]\right)\right].$$

In order to estimate the expression above, notice first that the obvious change of variable $\lambda=\varepsilon^{1/2}\xi$ yields $\int_{\mathbb{R}} e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi = C\varepsilon^{-(1-H)}$ for some constant C. Now for an additional arbitrary parameter $\eta>0$, consider the set

$$A_\eta=\left\{\omega;\ \sup_{1\le j\le n}\sup_{0\le r\le t}|B^j_{\kappa r}(\omega)|\le\frac{\pi}{3\eta}\right\}.$$

Observe that classical small balls inequalities for a Brownian motion (see (1.3) in [22]) yield $\mathbf{P}(A_\eta)\ge c_1e^{-c_2\eta^2 n\kappa t}$ for a large enough η. In addition, if we assume that A_η is realized and $|\xi|\le\eta$, some elementary trigonometric identities show that the following deterministic bound hold true: $|\sum_{j=1}^n e^{-iB^j_{\kappa r}\xi}|\ge\frac{n}{2}$. Gathering those considerations, we thus get

$$\begin{aligned}\mathbf{E}\left[u_\varepsilon^n(t,x)\right] &\ge \exp\left(c_1n^2\int_0^t\int_0^\eta e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi dr - c_2nt\varepsilon^{H-1}\right)\mathbf{P}\left(A_\eta\right)\\ &\ge C\exp\left(c_1n^2t\varepsilon^{-(1-H)}\int_0^{\varepsilon^{1/2}\eta}e^{-|\xi|^2}|\xi|^{1-2H}d\xi - c_2nt\varepsilon^{-(1-H)} - c_3n\kappa t\eta^2\right).\end{aligned}$$

We now choose the parameter η such that $\kappa\eta^2=\varepsilon^{-(1-H)}$, which means in particular that $\eta\to\infty$ as $\varepsilon\to0$. It is then easily seen that $\int_0^{\varepsilon^{1/2}\eta}e^{-|\xi|^2}|\xi|^{1-2H}d\xi$ is of order $\varepsilon^{H(1-H)}$ in this regime, and some elementary algebraic manipulations entail

$$\mathbf{E}\left[u_\varepsilon^n(t,x)\right]\ge C\exp\left(c_1n^2t\kappa^{H-1}\varepsilon^{-(1-H)^2}-c_2nt\varepsilon^{-(1-H)}\right)\ge C\exp\left(c_3t\kappa^{1-\frac{1}{H}}n^{1+\frac{1}{H}}\right),$$

where the last inequality is obtained by choosing $\varepsilon^{-(1-H)}=c\,\kappa^{\frac{H-1}{H}}n^{\frac{1}{H}}$ in order to optimize the second expression. We have thus reached the desired lower bound in (5.26) for the approximation u^ε in the regime $\varepsilon=c\,\kappa^{\frac{1}{H}}n^{-\frac{1}{H(1-H)}}$.

Step 3: Lower bound for u. To complete the proof, we need to show that for all sufficiently small ε, $\mathbf{E}\left[u_\varepsilon^n(t,x)\right]\le\mathbf{E}[u^n(t,x)]$. We thus start from equation (5.15) and use the series expansion of the exponential function as in (5.21). We get

$$\mathbf{E}\left[u_\varepsilon^n(t,x)\right]=\sum_{m=0}^\infty\frac{c_H^m}{m!}\mathbf{E}_B\left[\left(\sum_{1\le j\ne k\le n}V_{t,x}^{\varepsilon,j,k}\right)^m\right],\tag{5.27}$$

where we recall that $V_{t,x}^{\varepsilon,j,k}$ is defined by (5.16). Furthermore, expanding the mth power above, we have

$$\mathbf{E}_B\left[\left(\sum_{1\le j\ne k\le n}V_{t,x}^{\varepsilon,j,k}\right)^m\right]=\sum_{\alpha\in K_{n,m}}\int_{[0,t]^m}\int_{\mathbb{R}^m}e^{-\varepsilon\sum_{l=1}^m|\xi_l|^2}\mathbf{E}_B\left[e^{iB^\alpha(\xi)}\right]\prod_{l=1}^m|\xi_l|^{1-2H}\,d\xi dr\,,$$

where $K_{n,m}$ is a set of multi-indices defined by

$$K_{n,m} = \left\{\alpha = (j_1, \dots, j_m, k_1, \dots, k_m) \in \{1, \dots, n\}^{2m};\ j_l < k_l \text{ for all } l = 1, \dots, n\right\},$$

and $B^\alpha(\xi)$ is a shorthand for the linear combination $\sum_{l=1}^m \xi_l (B^{j_l}_{\kappa r_l} - B^{k_l}_{\kappa r_l})$. The important point here is that $E_B e^{iB^\alpha(\xi)}$ is positive for any $\alpha \in K_{n,m}$. We thus get the following inequality, valid for all $m \geq 1$

$$\begin{aligned} \mathbf{E}_B\left[\left(\sum_{1\leq j\neq k\leq n} V^{\varepsilon,j,k}_{t,x}\right)^m\right] &\leq \sum_{\alpha\in K_{n,m}} \int_{[0,t]^m}\int_{\mathbb{R}^m} \mathbf{E}_B\left[e^{iB^\alpha(\xi)}\right]\prod_{l=1}^m |\xi_l|^{1-2H}\, d\xi dr \\ &= \mathbf{E}_B\left[\left(\sum_{1\leq j\neq k\leq n} V^{j,k}_{t,x}\right)^m\right], \end{aligned}$$

where $V^{j,k}_{t,x}$ is defined by (5.23). Plugging this inequality back into (5.27) and recalling expression (5.22) for $\mathbf{E}[u^n(t,x)]$, we easily deduce that $\mathbf{E}[(u^n_\varepsilon(t,x)] \leq \mathbf{E}[u^n(t,x)]$, which finishes the proof. □

References

[1] Alberts, T., Khanin, K., Quastel, J. The continuum directed random polymer. *J. Stat. Phys.* **154** (2014), no. 1-2, 305-326.

[2] Bahouri, H., Chemin, J., and Danchin, R. Fourier analysis and nonlinear partial differential equations. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343. *Springer, Heidelberg*, 2011. xvi+523 pp.

[3] Balan, R., Jolis, M. and Quer-Sardanyons, L. SPDEs with fractional noise in space with index $H < 1/2$. *arXiv* preprint.

[4] Balan, R., Conus, D. Intermittency for the wave and heat equations with fractional noise in time. *arXiv* preprint. 2013.

[5] Bertini, L., Cancrini, N. The stochastic heat equation: Feynman- Kac formula and intermittence. *J. Statist. Phys.* **78** (1995), no. 5-6, 1377-1401.

[6] Bezerra, S., Tindel, S., Viens, F. Superdiffusivity for a Brownian polymer in a continuous Gaussian environment. *Ann. Probab.* **36** (2008), no. 5, 1642-1675.

[7] Brzeźniak, Z., Peszat, S. Space-time continuous solutions to SPDE's driven by a homogeneous Wiener process. *Studia Math.* **137** (1999), no. 3, 261-299.

[8] Chen, X. Spatial asymptotics for the parabolic Anderson models with generalized time-space Gaussian noise. preprint.

[9] Dalang, R. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* **4** (1999), no. 6, 29 pp.

[10] Dalang, R. Corrections to: "Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s" [*Electron J. Probab.* **4** (1999), no. 6, 29 pp. *Electron J. Probab.* **6** (2001), no. 6, 5 pp.

[11] Dalang, R., Quer-Sardanyons, L. Stochastic integrals for spde's: a comparison. *Expo. Math.* **29** (2011), no. 1, 67-109.

[12] Da Prato,G., Zabczyk, J. *Stochastic Equations in Infinite Dimensions* (Cambridge University Press, 1992).

[13] Gyöngy, I. Existence and uniqueness results for semilinear stochastic partial differential equations. *Stochastic Process. Appl.* **73** (1998), no. 2, 271-299.

[14] Gyöngy, I., Krylov, N. Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields* **105** (1996), no. 2, 143-158.

[15] Gyöngy, I., Nualart, D. On the stochastic Burgers' equation in the real line. *Ann. Probab.* **27** (1999), no. 2, 782-802.

[16] Hairer, M. Solving the KPZ equation. *Ann. of Math.* (2) **178** (2013), no. 2, 559-664.
[17] Hanche-Olsen, H. Holden, H. The Kolmogorov-Riesz compactness theorem. *Expo. Math.* **28** (2010), no. 4, 385-394.
[18] Hu, Y., Huang, J., Nualart, D., Tindel, S. Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. Accepted by *Electron. J. Probab. arXiv* preprint, 2014.
[19] Hu, Y., Nualart, D. Stochastic heat equation driven by fractional noise and local time. *Probab. Theory Related Fields* **143** (2009), no. 1-2, 285-328.
[20] Hu, Y., Nualart, D., Song, J. Feynman-Kac formula for heat equation driven by fractional white noise. *Ann. Probab.* **30**, 291-326.
[21] Khoshnevisan, D. Analysis of stochastic partial differential equations. *CBMS Regional Conference Series in Mathematics*, 119. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2014. viii+116 pp.
[22] Li, W. V., Shao, Q.-M. Gaussian processes: inequalities, small ball probabilities and applications. *Stochastic processes: theory and methods, Handbook of Statist.* **19**, North-Holland, Amsterdam, 2001.
[23] Nualart, D. The Malliavin calculus and related topics. Second edition. Probability and its Applications (New York). *Springer-Verlag, Berlin*, 2006. xiv+382 pp.
[24] Peszat, S., Zabczyk, J. Stochastic evolution equations with a spatially homogeneous Wiener process. *Stochastic Process. Appl.* **72** (1997), no. 2, 187-204.
[25] Pipiras, V., Taqqu, M. Integration questions related to fractional Brownian motion. *Probab. Theory Related Fields* **118** (2000), no. 2, 251-291.
[26] Samko, S., Kilbas, A., Marichev. O. Fractional integrals and derivatives. Theory and applications. *Gordon and Breach Science Publishers, Yverdon*, 1993. xxxvi+976 pp.
[27] Zorko, C. T. Morrey space. *Proc. Amer. Math. Soc.* **98** (1986), no. 4, 586-592.

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