1998

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Report Number:
98-001

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CSD-TR #98-001
January 1998

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Abstract. Piecewise Hermite cubics have been widely used in a variety of ways for solving partial differential equations. For a number of these techniques, knowledge about the Hermite cubic collocation approximations to the spectrum of the Laplace operator is often very useful, for error analysis and, *a fortiori*, possible iteration parameters. To this end, we give here explicit closed-form expressions for the Hermite cubic approximations to both the eigenvalues and the eigenfunctions of the Laplace operator for both the Dirichlet and the Neumann problems. Moreover, for the Dirichlet case, we show that optimal approximations are obtained using the Gauss points for collocation points. For both cases, we give numerical examples that verify our theoretical results.

Key words. eigenvalues and eigenfunctions, elliptic boundary value problems, Hermite cubic collocation, generalized eigenvalue problem, Laplace operator.

AMS(MOS) subject classifications. 65F15, 65L60, 65M27, 65N22, 65N25, 65N30, 65N35.

1. Introduction. Piecewise Hermite cubics have proven to be very useful for a variety of numerical applications. Dyksen, *et al*, have demonstrated that Hermite cubics are particularly effective for approximating solutions to partial differential equations [11].

As is typical, the discrete problem arising from Hermite cubic collocation results in a large, sparse linear system whose unknowns represent the coefficients of the Hermite cubic basis functions. Dyksen and Rice have shown that, with the proper ordering and proper scaling, the Hermite collocation equations are numerically stable and can be accurately solved using conventional direct methods [9, 10].

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However, even though Hermite cubics produce relatively accurate solutions to partial differential equations, the size of the problems that are solvable using direct methods is rather limited. Starting in 1984, a variety of iterative techniques for the Hermite collocation equations arising from large classes of separable elliptic problems have been introduced by a number of people including those by Dyksen [12, 13, 14], Cooper and Prenter [4], Bialecki, *et al* [2], Sun [16, 17], and Russell and Sun [15].

As one would expect for these methods, knowledge of the spectrum of the Hermite cubic approximation to the spectrum of the Laplace operator is very useful for error analysis and, *a fortiori*, for iteration parameters. In fact, if one knows the complete set of eigenvalues, iteration parameters can often be chosen that make the iterative technique exact in a finite number of iterations; that is, the iterative method becomes a direct method in theory. To that end, we give here explicit closed-form expressions for the Hermite cubic approximations to both the eigenvalues and the eigenfunctions of the Laplace operator for both the Dirichlet and the Neumann problems.

We briefly review Hermite cubic collocation and we introduce our notation in Section 2. In Section 3, we derive the formulas for the eigenvalues and eigenfunctions for the Dirichlet problem. Moreoever, we show that an optimal approximation is obtained using the Gauss points for collocation points. We give three numerical examples that verify our theoretical work. The complete Neumann problem is considered in a similar manner in Section 4. We conclude with Section 5.

2. Hermite Cubic Collocation. For a fixed positive integer N, we divide up the unit interval into N equal subintervals, each of length $h = 1/N$. To each of the $N+1$ grid points $x_k = kh$ there are associated two Hermite cubic polynomials defined by

(2.1a)
$$\Phi_k(x) = \begin{cases} 0 & x \le x_{k-1}, x_{k+1} \le x \\ -2\left[\dfrac{x-x_{k-1}}{x_k-x_{k-1}}\right]^3 + 3\left[\dfrac{x-x_{k-1}}{x_k-x_{k-1}}\right]^2 & x_{k-1} \le x \le x_k \\ -2\left[\dfrac{x_{k+1}-x}{x_{k+1}-x_k}\right]^3 + 3\left[\dfrac{x_{k+1}-x}{x_{k+1}-x_k}\right]^2 & x_k \le x \le x_{k+1} \end{cases}$$

$$
\text{(2.1b)} \qquad \Psi_k(x)=\begin{cases} 0 & x \le x_{k-1}, x_{k+1} \le x \\ \left[\dfrac{x-x_{k-1}}{x_k-x_{k-1}}\right]^2 (x-x_k) & x_{k-1} \le x \le x_k \\ \left[\dfrac{x_{k+1}-x}{x_{k+1}-x_k}\right]^2 (x-x_k) & x_k \le x \le x_{k+1}. \end{cases}
$$

The grid points x_k are often called the "knots" of the piecewise polynomial since they are the points where it is "tied together".

The Hermite cubic basis functions are particularly effective for interpolating Dirichlet or Neumann boundary conditions since they are the dual basis with respect to function and derivative evaluation at the grid points x_k. To see this, note that

$$
\Phi_k(x_{k-1})=\Phi_k(x_{k+1})=0, \quad \Phi_k(x_k)=1, \quad \Phi_k'(x_{k-1})=\Phi_k'(x_k)=\Phi_k'(x_{k+1})=0,
$$

$$
\Psi_k(x_{k-1})=\Psi_k(x_k)=\Psi_k(x_{k+1})=0, \quad \Psi_k'(x_{k-1})=\Psi_k'(x_{k+1})=0, \quad \Psi_k'(x_k)=1.
$$

Hence, an arbitrary cubic polynomial p defined on $[0,1]$ may be written as

$$
p(x)=\sum_{k=0}^{N} p(x_k)\Phi_k(x)+p'(x_k)\Psi_k(x).
$$

Graphs of Φ_1 and Ψ_1 are given below in Figure 2.1 for the case $N=2$. For a complete treatment of Hermite cubics, see [5].

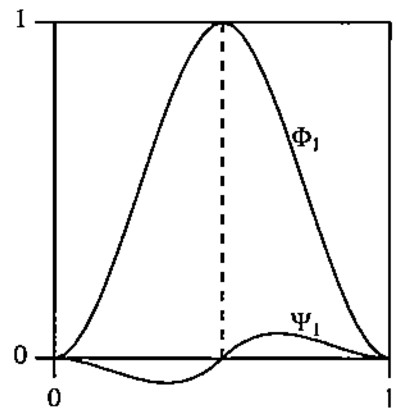

Figure 2.1 The Hermite cubic polynomials $\Phi_1(x)$ and $\Psi_1(x)$ for the case $N=2$.

3. The Dirichlet Problem. Consider the classical Dirichlet eigenvalue problem

$$
\begin{aligned}
&u''(x)=\lambda u(x), \quad x \in (0,1) \\
&u(0)=u(1)=0.
\end{aligned}
\tag{3.1}
$$

We divide the unit interval into N equal subintervals of length $h=1/N$. We approximate an eigenfunction u of (3.1) by

$$U(x)=\sum_{i=1}^{2N} c_i \phi_i(x)$$

for some constants c_i, where the ϕ_i are the $2N$ Hermite cubics

$$\{\phi_i\}_{i=1}^{2N}=\{\Psi_0,\Phi_1,\Psi_1,\ldots,\Phi_{N-1},\Psi_{N-1},\Psi_N\}. \tag{3.2}$$

Note that since we have discarded Φ_0 and Φ_N, it follows that $\phi_i(0)=\phi_i(1)=0$ and hence $U(0)=U(1)=0$. Also, note that the Φ_k and Ψ_k are ordered in a natural way from left to right, corresponding to their support.

Now, in order to determine the $2N$ unknowns c_i, we choose $2N$ distinct points $\{\tau_l\}_{l=1}^{2N}$ in (0,1), and collocate the equations in (3.1) at these points. In particular, for a fixed parameter $0<\theta<1/2$, we place in each subinterval (x_k, x_{k+1}) two collocation points,

$$
\begin{aligned}
\tau_{2k+1} &= 1/2(x_k+x_{k+1})-\theta h, \\
\tau_{2k+2} &= 1/2(x_k+x_{k+1})+\theta h.
\end{aligned}
\tag{3.3}
$$

Substituting U into (3.1) and collocating at the τ_l, we obtain the generalized eigenvalue problem

$$A\,\mathbf{c}=\lambda B\,\mathbf{c}, \tag{3.4}$$

where

$$A_{lj}=\phi_j''(\tau_l), \quad B_{lj}=\phi_j(\tau_l), \quad \begin{matrix} l=1,\ldots,2N \\ j=1,\ldots,2N. \end{matrix}$$

The generalized eigenvalues and eigenvectors of (3.4) give the Hermite cubic collocation approximations to the eigenvalues and eigenvectors of (3.1). Since the support of each Hermite cubic function ϕ_i spans at most two subintervals, it follows that A and B are band matrices with bandwidth two.

Next, we give below in the following theorem explicit closed-form expressions for the generalized eigenvalues of (3.4). We note here that the results of Theorem 3.1 along with a proof were first given in [13]. We give a new proof here for two reasons. First, the original proof in [13] was incomplete and contained some errors. Second, the derivation of the eigenvectors given below in Theorem 3.3 (and not given in [13]) requires in detail both the notation and the machinery developed in the proof of Theorem 3.1.

THEOREM 3.1. *The* $2N$ *generalized eigenvalues of the discrete Dirichlet problem* $A\mathbf{c}=\lambda B\mathbf{c}$ *in* (3.4) *are given by*

$$\lambda_0=\frac{6}{h^2(\theta^2-1/4)} \tag{3.5a}$$

$$\lambda_N=\frac{2}{h^2(\theta^2-1/4)} \tag{3.5b}$$

$$\lambda_l^{\pm}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\,,\quad l=1,\dots,N-1 \tag{3.5c}$$

where

$$a=h^4\left[(16\theta^4-16\theta^2+3)d-8\theta^2+2\right], \tag{3.6a}$$

$$b=h^2\left[(-128\theta^2+48)d+48\right], \tag{3.6b}$$

$$c=192d, \tag{3.6c}$$

and where

$$d=\tan^2\left(\frac{l\pi}{2N}\right) \tag{3.6d}$$

Proof. Let P be the Hermite cubic collocation approximation of the eigenfunction of (3.1) corresponding to the approximate eigenvalue λ. Since $h=1/N$, P consists of N pieces, each of which has support in (x_k, x_{k+1}). For simplicity, we assume that each polynomial piece is centered at the midpoint of its corresponding interval which gives

$$p_k(x)=\alpha_k+\beta_k(x-\bar{x}_k)+\gamma_k\frac{(x-\bar{x}_k)^2}{2}+\delta_k\frac{(x-\bar{x}_k)^3}{6}, \quad x_k \le x \le x_{k+1},\ k=0,\ldots,N-1,$$

where $\bar{x}_k = \frac{1}{2}(x_k+x_{k+1})$. To simplify even further, we write this as

$$p_k(y)=\alpha_k+\beta_k y+\gamma_k\frac{y^2}{2}+\delta_k\frac{y^3}{6},\ -\frac{h}{2}\le y\le\frac{h}{2},\ k=0,\ldots,N-1. \tag{3.7}$$

First, we relate the α_k's to the γ_k's and the β_k's to the δ_k's by using the eigenvalue problem. Since P satisfies $P''=\lambda P$ at the collocation points, we have $p_k''(\pm\theta h)=\lambda p_k(\pm\theta h)$, or equivalently,

$$\gamma_k \pm \delta_k\theta h=\lambda\left[\alpha_k\pm\beta_k\theta h+\gamma_k\frac{\theta^2h^2}{2}\pm\delta_k\frac{\theta^3h^3}{6}\right]. \tag{3.8}$$

Adding and subtracting the equations in (3.8), we obtain, respectively,

$$\begin{aligned}\gamma_k&=\lambda\left[\alpha_k+\gamma_k\frac{\theta^2h^2}{2}\right],\\ \delta_k&=\lambda\left[\beta_k+\delta_k\frac{\theta^2h^2}{6}\right].\end{aligned} \tag{3.9}$$

If $\lambda=0$, then it follows from (3.9) that $\gamma_k=\delta_k=0$, so that (3.7) reduces to $p_k(y)=\alpha_k+\beta_k y$. Now, since each piece p_k is linear and since P is continuous, we must have $P(x)=\alpha+\beta x$. Moreover, since $P(0)=P(1)=0$, it follows that $P\equiv 0$, which is not an eigenfunction of (3.1). Thus, $\lambda=0$ is not an eigenvalue of (3.4).

Now, for the case $\lambda\ne 0$, (3.9) gives

$$\begin{aligned}\alpha_k&=\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{2}\right]\gamma_k,\\ \beta_k&=\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}\right]\delta_k,\end{aligned}$$

so that (3.7) simplifies to

$$p_k(\lambda;y)=\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{2}+\frac{y^2}{2}\right]\gamma_k+y\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{6}\right]\delta_k, \tag{3.10}$$

from which it follows that

(3.11) $$p_k'(\lambda;y)=y\gamma_k+\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{2}\right)\delta_k.$$

Next, we relate the γ_k's to the δ_k's by using the continuity of P and P'. Since P is continuous, we have $p_k(\lambda;+h/2)=p_{k+1}(\lambda;-h/2)$. From (3.10), it follows that

$$r\gamma_k+s\delta_k=r\gamma_{k+1}-s\delta_{k+1},$$

where

$$r=\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{2}+\frac{h^2}{8}\right),$$

$$s=\frac{h}{2}\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{h^2}{24}\right).$$

We obtain

(3.12) $$r(-\gamma_k+\gamma_{k+1})=s(\delta_k+\delta_{k+1}).$$

Furthermore, since P' is continuous, we have $p_k'(\lambda;+h/2)=p_{k+1}'(\lambda;-h/2)$. From (3.11), it follows that

$$\gamma_k+t\delta_k=-\gamma_{k+1}+t\delta_{k+1},$$

where

$$t=\frac{2}{h}\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{h^2}{8}\right).$$

We obtain

(3.13) $$\gamma_k+\gamma_{k+1}=t(-\delta_k+\delta_{k+1}).$$

Now, using (3.12) and (3.13), we show that the γ_k's and δ_k's both satisfy the same difference equation. We consider (3.12) and the equation obtained from it by replacing k by $k-1$. We obtain

(3.14) $$\begin{aligned} r(-\gamma_k+\gamma_{k+1})&=s(\delta_k+\delta_{k+1}) \\ r(-\gamma_{k-1}+\gamma_k)&=s(\delta_{k-1}+\delta_k), \end{aligned}$$

which, if substracted, yield

$$r(\gamma_{k-1}-2\gamma_k+\gamma_{k+1})=s(-\delta_{k-1}+\delta_{k+1}). \tag{3.15}$$

Similarly, from (3.13) we obtain

$$\begin{aligned}\gamma_k+\gamma_{k+1}&=t(-\delta_k+\delta_{k+1})\\ \gamma_{k-1}+\gamma_k&=t(-\delta_{k-1}+\delta_k),\end{aligned} \tag{3.16}$$

which, if added, yield

$$\gamma_{k-1}+2\gamma_k+\gamma_{k+1}=t(-\delta_{k-1}+\delta_{k+1}). \tag{3.17}$$

Substituting (3.17) into (3.15) gives

$$rt(\gamma_{k-1}-2\gamma_k+\gamma_{k+1})=s(\gamma_{k-1}+2\gamma_k+\gamma_{k+1}). \tag{3.18}$$

If we add the equations in (3.14) and subtract the equations in (3.16), we obtain, respectively,

$$\begin{aligned}r(-\gamma_{k-1}+\gamma_{k+1})&=s(\delta_{k-1}+2\delta_k+\delta_{k+1})\\ -\gamma_{k-1}+\gamma_{k+1}&=t(\delta_{k-1}-2\delta_k+\delta_{k+1}),\end{aligned}$$

which gives

$$rt(\delta_{k-1}-2\delta_k+\delta_{k+1})=s(\delta_{k-1}+2\delta_k+\delta_{k+1}). \tag{3.19}$$

Now, since the γ_k's and δ_k's satisfy the difference equations in (3.19) and (3.18), respectively, we may in the usual way set

$$\begin{aligned}\gamma_k&=\hat{A}_\lambda\zeta^k+\hat{C}_\lambda\zeta^{-k}\\ \delta_k&=\hat{B}_\lambda\zeta^k+\hat{D}_\lambda\zeta^{-k}\end{aligned} \tag{3.20}$$

for arbitrary constants $\hat{A}_\lambda$, $\hat{B}_\lambda$, $\hat{C}_\lambda$, and $\hat{D}_\lambda$ that depend on λ.

To find appropriate values for ζ, we again impose the boundary conditions $P(0)=P(1)=0$. Since $P(0)=0$, we may extend P on $0\le x\le 1$ to $-1\le x\le 0$ as an odd function by $P(x)=-P(-x)$ for $-1\le x\le 0$. Since both P and P' are continuous on $0\le x\le 1$, it follows that P and P' are continuous on $-1\le x\le 1$. Similarly, we can extend P to $1\le x\le 2$ by $P(x)=-P(2-x)$ for $1\le x\le 2$. In particular, we now have the pieces

$$p_{-1}(y) = -p_0(-y) \quad \text{and} \quad p_N(y) = -p_{N-1}(-y).$$

Now, since $P(0)=0$, it follows from (3.10) and (3.20) that

$$p_0(\lambda; -h/2) = r(\hat{A}_\lambda + \hat{C}_\lambda) - s(\hat{B}_\lambda + \hat{D}_\lambda) = 0,$$

which, provided $(r\hat{A}_\lambda - s\hat{B}_\lambda) \neq 0$, gives

$$-\frac{(r\hat{C}_\lambda - s\hat{D}_\lambda)}{(r\hat{A}_\lambda - s\hat{B}_\lambda)} = 1. \tag{3.21}$$

Similarly, since $P(1)=0$, we have

$$p_{N-1}(\lambda; +h/2) = -p_N(\lambda; -h/2) = -r\left[\hat{A}_\lambda \zeta^N + \hat{C}_\lambda \zeta^{-N}\right] + s\left[\hat{B}_\lambda \zeta^N + \hat{D}_\lambda \zeta^{-N}\right] = 0. \tag{3.22}$$

Solving for ζ and applying (3.21), we obtain

$$\zeta^{2N} = -\frac{(r\hat{C}_\lambda - s\hat{D}_\lambda)}{(r\hat{A}_\lambda - s\hat{B}_\lambda)} = 1 = e^{2\pi i l}$$

from which it follows that

$$\zeta = e^{\frac{l\pi i}{N}} \tag{3.23}$$

for any integer l; we take $l = 0, \ldots, N$.

To simplify subsequent derivations, we modify the arbitrary contants in (3.20) by taking

$$\begin{aligned}
\gamma_k &= \bar{A}_\lambda e^{\frac{l\pi i}{2N}} \zeta^k + \bar{C}_\lambda e^{-\frac{l\pi i}{2N}} \zeta^{-k} \\
&= \bar{A}_\lambda e^{\frac{(k+\frac{1}{2})l\pi}{N}} + \bar{C}_\lambda e^{-\frac{(k+\frac{1}{2})l\pi}{N}} \\
&= (\bar{A}_\lambda + \bar{C}_\lambda)\cos\left(\frac{(k+\frac{1}{2})l\pi}{N}\right) + i(\bar{A}_\lambda - \bar{C}_\lambda)\sin\left(\frac{(k+\frac{1}{2})l\pi}{N}\right),
\end{aligned}$$

which we write as

$$\gamma_k = A_\lambda \sin\left(\frac{(k+\frac{1}{2})l\pi}{N}\right) + C_\lambda \cos\left(\frac{(k+\frac{1}{2})l\pi}{N}\right). \tag{3.24a}$$

A similar modification with δ_k gives

$$\delta_k = B_\lambda \sin\left[\frac{(k+\frac{1}{2})l\pi}{N}\right] + D_\lambda \cos\left[\frac{(k+\frac{1}{2})l\pi}{N}\right]. \tag{3.24b}$$

Substituting (3.24a) and (3.24b) into (3.18) and (3.19), respectively, and simplifying, we obtain

$$\begin{aligned} rt\left[-4\sin^2\left(\frac{l\pi}{2N}\right)\gamma_k\right] &= s\left[4\cos^2\left(\frac{l\pi}{2N}\right)\gamma_k\right] \\ rt\left[-4\sin^2\left(\frac{l\pi}{2N}\right)\delta_k\right] &= s\left[4\cos^2\left(\frac{l\pi}{2N}\right)\delta_k\right]. \end{aligned} \tag{3.25}$$

Since r, s, and t depend on λ, and since we cannot have $\gamma_k = \delta_k = 0$, it follows from (3.25) that the eigenvalues of (3.4) satisfy

$$rt\sin^2\left(\frac{l\pi}{2N}\right) + s\cos^2\left(\frac{l\pi}{2N}\right) = 0. \tag{3.26}$$

We can now obtain the formulas given in (3.5) by considering (3.26) for various values of l. If $l = 0$, then (3.26) reduces to

$$s = \frac{h}{2}\left(\frac{1}{\lambda} - \frac{\theta^2 h^2}{6} + \frac{h^2}{24}\right) = 0, \tag{3.27}$$

so that

$$\lambda_0 = \frac{6}{h^2(\theta^2 - \frac{1}{4})} \tag{3.28}$$

is a potential eigenvalue of (3.4). Note that (3.28) may also be written as

$$6(\pm\theta h) = \lambda(\pm\theta h)(\theta^2 h^2 - h^2/4),$$

which shows that the approximate eigenfunction associated with $\lambda = \lambda_0$ is, up to a multiplicative constant, given by $p_k(\lambda_0; y) = y(y^2 - h^2/4)$. Since $p_k(\lambda_0; y)$ satisfies the boundary conditions $p_0(\lambda_0; -h/2) = 0$ and $p_{N-1}(\lambda_0; +h/2) = 0$, it follows that λ_0 is indeed an eigenvalue of (3.4), which gives the desired result in (3.5a). Note that for $\lambda = \lambda_0$, P is a piecewise approximation to the eigenfunction $\sin(2N\pi x)$ of (3.1).

If $l=N$, then (3.26) implies $rt=0$ so that either

(3.29)
$$r=\frac{1}{\lambda}-\frac{\theta^2h^2}{2}+\frac{h^2}{8}=0$$

or

(3.30)
$$t=\frac{2}{h}\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{h^2}{8}\right]=0.$$

From (3.29) it follows that

(3.31)
$$\lambda_N=\frac{2}{h^2(\theta^2-{}^1\!/\!_4)}$$

is a potential eigenvalue of (3.4). The approximate eigenfunction corresponding to λ_N is, up to a multiplicative constant, given by $p_k(\lambda_N;y)=(y^2-h^2/4)$. Since $p_k(\lambda_N;y)$ satisfies the boundary conditions $p_0(\lambda_N;-h/2)=0$ and $p_{N-1}(\lambda_N;+h/2)=0$, it follows that λ_N is indeed an eigenvalue of (3.4), which gives the desired result in (3.5b). Note that for $\lambda=\lambda_N$, P is a piecewise approximation to the eigenfunction $\sin(N\pi x)$ of (3.1).

From (3.30) it follows that

(3.32)
$$\tilde{\lambda}_N=\frac{6}{h^2(\theta^2-{}^3\!/\!_4)}$$

is a potential eigenvalue of (3.4) with corresponding approximate eigenfunction $p_k(\tilde{\lambda}_N;y)=y(y^2-3h^2/4)$. However, since $p_k(\tilde{\lambda}_N;\pm h/2)\neq 0$, $\tilde{\lambda}_N$ is **not** an eigenvalue of (3.4).

Finally, for $l=1,\ldots,N-1$, we have from (3.26) that

(3.33)
$$\lambda^2rt\tan^2\left[\frac{l\pi}{2N}\right]+\lambda^2s=0,$$

which is a quadratic equation in λ. If simplified, (3.33) may be written as

(3.34)
$$a\lambda^2+b\lambda+c=0,$$

where a, b, c and d are given in (3.6). Thus, for each of $l=1,\ldots,N-1$, (3.34) represents two eigenvalues of (3.4), which gives the desired results in (3.5c).

Example 3.1. The Generalized Eigenvalues of the Discrete Dirichlet Problem $A\mathbf{c}=\lambda B\mathbf{c}$.

In order to verify the results of Theorem 3.1, we compute the generalized eigenvalues of $A\mathbf{c}=\lambda B\mathbf{c}$ in (3.4) using the LAPACK routine SGEGV [1]. We then compare these computed results with those obtained by using the formulas of Theorem 3.1 given in (3.5).

Recall that in the continuous case, the eigenvalues are of the form $-k^2\pi^2$, $k=1,2,\ldots$. Thus, if divided by $-\pi^2$, we expect the generalized eigenvalues of (3.4) to approximate k^2, $k=1,2,\ldots$. Now, for the case $N=4$ and $\theta=\frac{1}{2\sqrt{3}}$, we obtain the results given below in Table 3.1. We see from Table 3.1 that the formulas of Theorem 3.1 agree up to round-off with the computed results from LAPACK.

Table 3.1

Eigenvalues of the discrete Dirichlet problem $A\mathbf{c}=\lambda B\mathbf{c}$ divided by $-\pi^2$ for the case $N=4$ and $\theta=1/(2\sqrt{3})$.

λ	Theorem 3.1	LAPACK
λ_1^+	1.00017e+00	1.00017e+00
λ_2^+	4.00902e+00	4.00902e+00
λ_3^+	9.06012e+00	9.06012e+00
λ_4	1.94537e+01	1.94537e+01
λ_3^-	2.77562e+01	2.77562e+01
λ_2^-	4.04565e+01	4.04565e+01
λ_1^-	5.28336e+01	5.28337e+01
λ_0	5.83610e+01	5.83610e+01

By varying the free parameter $0<\theta<\frac{1}{2}$ in Theorem 3.1, we can vary the location of the $2N$ collocation points τ_l, thereby affecting the accuracy of the approximations to the eigenvalues of (3.1). As expected, we see in the following corollary that optimal approximations are obtained using the Gauss points [6, 7, 8].

Corollary 3.2. *If $0<\theta<1/2$, then λ_1^+ is at least an $O(h^2)$ approximation to the eigenvalue of smallest magnitude of* (3.1), $-\pi^2$. *If* $\theta=\frac{1}{2\sqrt{3}}$, *then* $\lambda_1^+=-\pi^2+O(h^4)$.

Proof. From Theorem 3.1 we have

$$\lambda_1^+=\frac{-b+\sqrt{b^2-4ac}}{2a}, \tag{3.35}$$

where a, b and c and given in (3.6) and where $d=\tan^2(h\pi/2)$. Expanding the right side of (3.35) in a Taylor series with respect to h using Mathematica [18], we obtain

$$\lambda_1^+=-\pi^2-\frac{1}{24}(12\theta^2-1)\pi^4h^2-\frac{1}{2880}(720\theta^4-200\theta^2+13)\pi^6h^4+O(h^6) \tag{3.36}$$

so that $\lambda_1^+=-\pi^2+O(h^2)$.

Setting $12\theta^2-1=0$, we obtain $\theta=\frac{\pm 1}{2\sqrt{3}}$ which are the Gauss points in (0,1). Substituting $\theta=\frac{1}{2\sqrt{3}}$ into (3.36), we obtain the desired result,

$$\lambda_1^+=-\pi^2-\frac{\pi^6h^4}{2160}+O(h^6),$$

which is approximately

$$\lambda_1^+\cong-\pi^2-0.445h^4+O(h^6), \tag{3.37}$$

so that

$$\lambda_1^+=-\pi^2+O(h^4).$$

Example 3.2. Convergence of λ_1^+ to $-\pi^2$.

To numerically verify the results of Corollary 3.2, we compute λ_1^+ in double precision using (3.35) with $\theta=\frac{1}{2\sqrt{3}}$ for $N=4, 8, 16, \ldots, 128$. Note that $-\pi^2\cong-9.86960440109$. We obtain the following results:

Table 3.2
Convergence of λ_l^+.

$N = 1/h$	λ_1^+	$\lvert\lambda_1^+ + \pi^2\rvert$
4	-9.8712579554	1.6536E-03
8	-9.8697117358	1.0733E-04
16	-9.8696111718	6.7707E-06
32	-9.8696048252	4.2414E-07
64	-9.8696044276	2.6524E-08
128	-9.8696044027	1.6580E-09

A logarithmic least squares fit of these data shows that $\lvert\lambda_1^+ + \pi^2\rvert \cong 0.424h^{3.99}$, which agrees with the theoretical result in (3.37).

As a companion to Theorem 3.1, we give below explicit, closed-form expressions for the generalized eigenvectors of (3.4).

THEOREM 3.3. *The k^{th} piece of the Hermite cubic approximate eigenfunction of the Dirichlet problem* (3.1) *associated with the approximate eigenvalues λ_0, λ_N, $\lambda_l^\pm$ of Theorem 3.1 is given by*

(3.38a) $$p_k(\lambda_0;x) = D_{\lambda_0}(x-\bar{x}_k)\left[(x-\bar{x}_k)^2 - h^2/4\right],$$

(3.38b) $$p_k(\lambda_N;x) = A_{\lambda_N}\left[(x-\bar{x}_k)^2 - h^2/4\right](-1)^k, \text{ and}$$

(3.38c) $$p_k(\lambda_l^\pm;x) = A_{\lambda_l^\pm}\left[\frac{1}{\lambda_l^\pm} - \frac{\theta^2 h^2}{2} + \frac{(x-\bar{x}_k)^2}{2}\right]\sin\left[\frac{(k+\frac{1}{2})l\pi}{N}\right]$$

$$+ D_{\lambda_l^\pm}(x-\bar{x}_k)\left[\frac{1}{\lambda_l^\pm} - \frac{\theta^2 h^2}{6} + \frac{(x-\bar{x}_k)^2}{6}\right]\cos\left[\frac{(k+\frac{1}{2})l\pi}{N}\right],$$

for $l = 1,\ldots,N-1$,

where $\bar{x}_k = \frac{1}{2}(x_k + x_{k+1})$, D_{λ_0}, A_{λ_N}, $D_{\lambda_l^\pm}$ are arbitrary nonzero constants, and where

$$A_{\lambda_l^\pm} = -\frac{2}{h}\left[\frac{1}{\lambda_l^\pm} - \frac{\theta^2 h^2}{6} + \frac{h^2}{8}\right]\tan\left[\frac{l\pi}{2N}\right] D_{\lambda_l^\pm}.$$

Proof.

Recall from (3.10) and (3.24) that we have

$$p_k(\lambda;y)=\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{2}+\frac{y^2}{2}\right]\gamma_k+y\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{6}\right]\delta_k, \tag{3.39}$$

where

$$\begin{aligned}\gamma_k&=A_\lambda\sin\left[\frac{(k+1/2)l\pi}{N}\right]+C_\lambda\cos\left[\frac{(k+1/2)l\pi}{N}\right]\\ \delta_k&=B_\lambda\sin\left[\frac{(k+1/2)l\pi}{N}\right]+D_\lambda\cos\left[\frac{(k+1/2)l\pi}{N}\right].\end{aligned} \tag{3.40}$$

Consider first the special case $l=0$ $(\lambda=\lambda_0)$. From (3.28), we have $\frac{1}{\lambda_0}=\frac{\theta^2h^2}{6}-\frac{h^2}{24}$. If we substitute $l=0$ into (3.40), then (3.39) reduces to

$$p_k(\lambda_0;y)=\left[\frac{y^2}{2}-\frac{h^2}{24}-\frac{\theta^2h^2}{3}\right]C_{\lambda_0}+\frac{1}{6}y\left[y^2-\frac{h^2}{4}\right]D_{\lambda_0}.$$

Enforcing the boundary condition $P(0)=0$, we obtain

$$p_0(\lambda_0;-h/2)=\frac{h^2}{12}(1-4\theta^2)C_{\lambda_0}=0, \tag{3.41}$$

which implies that $C_{\lambda_0}=0$, so that

$$p_k(\lambda_0;y)=\frac{1}{6}y\left[y^2-\frac{h^2}{4}\right]D_{\lambda_0}. \tag{3.42}$$

Note that $p_{N-1}(\lambda_0;+h/2)=0$ so that $P(1)=0$. The desired result in (3.38a) follows from (3.42) by incorporating the factor 1/6 into the arbitrary constant D_{λ_0}, and by replacing y by $(x-\bar{x}_k)$.

Second, consider the special case $l=N$ $(\lambda=\lambda_N)$. From (3.31) we have $\frac{1}{\lambda_N}=\frac{\theta^2h^2}{2}-\frac{h^2}{8}$. Substituting $l=N$ into (3.40), we obtain $\sin((k+1/2)\pi)=(-1)^k$ and $\cos((k+1/2)\pi)=0$, so that (3.39) reduces to

$$p_k(\lambda_N;y)=\frac{1}{2}\left[y^2-\frac{h^2}{4}\right](-1)^kA_{\lambda_N}+y\left[\frac{y^2}{6}-\frac{h^2}{8}+\frac{\theta^2h^2}{3}\right](-1)^kB_{\lambda_N}. \tag{3.43}$$

Enforcing the boundary condition $P(0)=0$ gives

$$p_0(\lambda_N;-h/2)=\frac{h^3}{24}\left\{1-4\theta^2\right\}(-1)^k B_{\lambda_N}=0,$$

which implies $B_{\lambda_N}=0$ so that

$$p_k(\lambda_N;y)=\frac{1}{2}\left\{y^2-\frac{h^2}{4}\right\}(-1)^k A_{\lambda_N}. \tag{3.44}$$

Note that $p_{N-1}(\lambda_N;+h/2)=0$ so that $P(1)=0$. The desired result in (3.38b) follows from (3.44) by incorporating the factor 1/2 into the arbitrary constant A_{λ_N}, and by replacing y by $(x-\bar{x}_k)$.

Finally, we consider the remaining cases $l=1,\ldots,N-1$. First we show that $B_\lambda=C_\lambda=0$ for all corresponding eigenvalues λ.

For $k=0$, it follows from (3.39) and (3.40) that

$$p_0(\lambda;-h/2)=r\left[A_\lambda\sin\left(\frac{l\pi}{2N}\right)+C_\lambda\cos\left(\frac{l\pi}{2N}\right)\right]-s\left[B_\lambda\sin\left(\frac{l\pi}{2N}\right)+D_\lambda\cos\left(\frac{l\pi}{2N}\right)\right] \tag{3.45}$$

where

$$r=\frac{1}{\lambda}-\frac{\theta^2h^2}{2}+\frac{h^2}{8}$$

$$s=\frac{h}{2}\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{h^2}{24}\right).$$

Moreover, substituting $l=N-1$ into (3.40), we obtain

$$\cos\left(\frac{(N-\frac{1}{2})l\pi}{N}\right)=(-1)^l\cos\left(\frac{l\pi}{2N}\right) \quad \text{and} \quad \sin\left(\frac{(N-\frac{1}{2})l\pi}{N}\right)=-(-1)^l\sin\left(\frac{l\pi}{2N}\right),$$

so that

$$p_{N-1}(\lambda;+h/2)=(-1)^l\left\{r\left[-A_\lambda\sin\left(\frac{l\pi}{2N}\right)+C_\lambda\cos\left(\frac{l\pi}{2N}\right)\right]+s\left[-B_\lambda\sin\left(\frac{l\pi}{2N}\right)+D_\lambda\cos\left(\frac{l\pi}{2N}\right)\right]\right\}. \tag{3.46}$$

From the boundary conditions $p_0(\lambda;-h/2)=0$ and $p_{N-1}(\lambda;+h/2)=0$, we have from (3.45) and (3.46), respectively,

$$r\left[A_\lambda \sin\left(\frac{l\pi}{2N}\right)+C_\lambda \cos\left(\frac{l\pi}{2N}\right)\right]-s\left[B_\lambda \sin\left(\frac{l\pi}{2N}\right)+D_\lambda \cos\left(\frac{l\pi}{2N}\right)\right] = 0$$

$$r\left[-A_\lambda \sin\left(\frac{l\pi}{2N}\right)+C_\lambda \cos\left(\frac{l\pi}{2N}\right)\right]+s\left[-B_\lambda \sin\left(\frac{l\pi}{2N}\right)+D_\lambda \cos\left(\frac{l\pi}{2N}\right)\right] = 0.$$

Adding and simplifying, we obtain

(3.47)
$$rC_\lambda = sB_\lambda \tan\left(\frac{l\pi}{2N}\right).$$

Now, from (3.39) it follows that

$$p_k'(\lambda;y)=y\gamma_k+\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{2}\right)\delta_k,$$

from which we obtain

(3.48)
$$p_0'(\lambda;-h/2)=\frac{h}{2}\left\{-\left[A_\lambda \sin\left(\frac{l\pi}{2N}\right)+C_\lambda \cos\left(\frac{l\pi}{2N}\right)\right]+t\left[B_\lambda \sin\left(\frac{l\pi}{2N}\right)+D_\lambda \cos\left(\frac{l\pi}{2N}\right)\right]\right\}$$

and

(3.49)
$$p_{-1}'(\lambda;+h/2)=\frac{h}{2}\left\{\left[-A_\lambda \sin\left(\frac{l\pi}{2N}\right)+C_\lambda \cos\left(\frac{l\pi}{2N}\right)\right]+t\left[-B_\lambda \sin\left(\frac{l\pi}{2N}\right)+D_\lambda \cos\left(\frac{l\pi}{2N}\right)\right]\right\}$$

where

$$t=\frac{2}{h}\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{h^2}{8}\right).$$

Now, since P' is continuous, we must have $p_{-1}'(\lambda;+h/2)=p_0'(\lambda;-h/2)$. Equating (3.48) and (3.49) and simplifying, we obtain

(3.50)
$$-C_\lambda + tB_\lambda \tan\left(\frac{l\pi}{2N}\right)=0.$$

Combining (3.47) and (3.50), we obtain

$$-sB_\lambda \tan\left(\frac{l\pi}{2N}\right)+rtB_\lambda \tan\left(\frac{l\pi}{2N}\right)=0,$$

or, since $\tan\left[\frac{l\pi}{2N}\right] \neq 0$,

$$B_\lambda(rt - s) = 0. \tag{3.51}$$

Now, in order to show that $B_\lambda = 0$, suppose that $B_\lambda \neq 0$. From the above, we have $rt = s$ so that the difference equations in (3.18) and (3.19) reduce, respectively, to

$$\gamma_{k-1} - 2\gamma_k + \gamma_{k+1} = \gamma_{k-1} + 2\gamma_k + \gamma_{k+1},$$

and

$$\delta_{k-1} - 2\delta_k + \delta_{k+1} = \delta_{k-1} + 2\delta_k + \delta_{k+1},$$

However, this yields $\gamma_k = \delta_k = 0$, which results in $p_k \equiv 0$, or $P \equiv 0$. Thus, we must have $B_\lambda = 0$. Furthermore, (3.50) shows that $C_\lambda = 0$ as well. Thus, we must have $B_\lambda = C_\lambda = 0$ for all remaining eigenvalues λ.

Now, with $B_\lambda = C_\lambda = 0$, the expression for p_k in (3.39) reduces to

$$p_k(\lambda; y) = \left[\frac{1}{\lambda} - \frac{\theta^2 h^2}{2} + \frac{y^2}{2}\right] A_\lambda \sin\left[\frac{(k+1/2)l\pi}{N}\right] + y\left[\frac{1}{\lambda} - \frac{\theta^2 h^2}{6} + \frac{y^2}{6}\right] D_\lambda \cos\left[\frac{(k+1/2)l\pi}{N}\right] \tag{3.52}$$

where A_λ and D_λ are arbitrary constants.

We next chose A_λ and B_λ so that P satisfies the boundary conditions $P(0) = P(1) = 0$. To this end, we set

$$p_0(\lambda; -h/2) = rA_\lambda \sin\left[\frac{l\pi}{2N}\right] - sD_\lambda \cos\left[\frac{l\pi}{2N}\right] = 0, \tag{3.53}$$

from which it follows that we must have

$$A_\lambda = \frac{s}{r} cot\left[\frac{l\pi}{2N}\right] D_\lambda. \tag{3.54}$$

From (3.26), we have

$$rt \sin^2\left[\frac{l\pi}{2N}\right] + s\cos^2\left[\frac{l\pi}{2N}\right] = 0,$$

which gives

$$\frac{s}{r}cot\left(\frac{l\pi}{2N}\right)=-t\tan\left(\frac{l\pi}{2N}\right). \tag{3.55}$$

Substituting (3.55) into (3.54), we have

$$A_\lambda=-\frac{2}{h}\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{h^2}{8}\right]\tan\left(\frac{l\pi}{2N}\right). \tag{3.56}$$

Now, since

$$p_{N-1}(\lambda;+h/2)=-(-1)^l\left[rA_\lambda\sin\left(\frac{l\pi}{2N}\right)-sD_\lambda\cos\left(\frac{l\pi}{2N}\right)\right],$$

it follows from (3.56) (equivalently (3.53)) that $p_{N-1}(\lambda;+h/2)=0$ as well. Thus, we have $P(0)=P(1)=0$.

The desired results in (3.38c) follow from (3.52) and (3.56) by replacing λ by $\lambda_l^{\pm}$ and y by $(x-\bar{x}_k)$.

EXAMPLE 3.3. The Hermite Cubic Eigenfunctions $P(\lambda_1^+;x)$ and $P(\lambda_2^+;x)$.

Graphs of the Hermite cubic eigenfunctions $P(\lambda_1^+;x)$ and $P(\lambda_2^+;x)$ for the case $N=4$ are given in Figure 3.1.

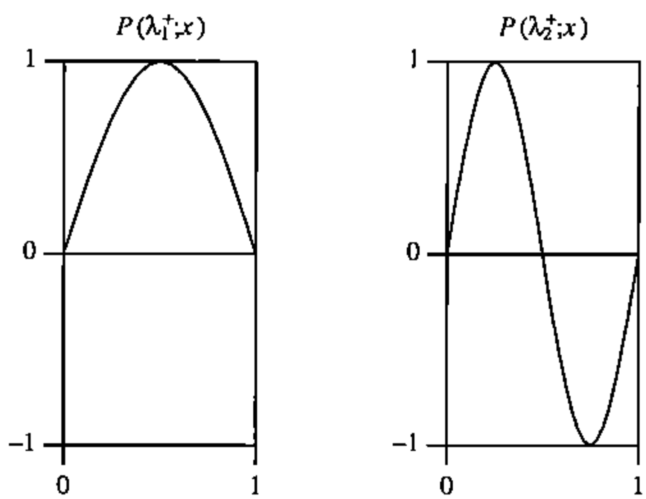

Figure 3.1 The Hermite cubic eigenfunctions $P(\lambda_1^+;x)$ and $P(\lambda_2^+;x)$ approximating $\sin(\pi x)$ and $\sin(2\pi x)$, respectively, for the case $N=4$.

The discrete eigenfunctions $P(\lambda_1^+;x)$ and $P(\lambda_2^+;x)$ are approximations to the continuous eigenfunctions $\sin(\pi x)$ and $\sin(2\pi x)$, respectively. Note that each eigenfunction consists of four pieces, and that the arbitrary constants $D_{\lambda_1^+}$ and $D_{\lambda_2^+}$ are chosen so the functions have maximum absolute value one.

4. The Neumann Problem. Consider the classical Neumann eigenvalue problem

$$
\begin{aligned}
&u''(x)=\lambda u(x), \quad x \in (0,1) \\
&u'(0)=u'(1)=0.
\end{aligned} \tag{4.1}
$$

We again divide the unit interval into N equal subintervals of length $h=1/N$ and approximate an eigenfunction of (4.1) by

$$U(x)=\sum_{i=1}^{2N} c_i \phi_i(x)$$

for some constants c_i. For the Neumann case, we choose the ϕ_i to be the $2N$ Hermite cubics

$$\{\phi_i\}_{i=1}^{2N}=\{\Phi_0,\Phi_1,\Psi_1,\ldots,\Phi_{N-1},\Psi_{N-1},\Phi_N\}. \tag{4.2}$$

Since we have discarded Ψ_0 and Ψ_N, it follows that $\phi_i'(0)=\phi_i'(1)=0$ and hence $U'(0)=U'(1)=0$.

As before, we substitute U into (4.1) and collocate at the points $\{\tau_l\}_{l=1}^{2N}$ in (0,1) defined in (3.3). We obtain the generalized eigenvalue problem

$$A\,\mathbf{c}=\lambda B\,\mathbf{c}, \tag{4.3}$$

where

$$A_{lj}=\phi_j''(\tau_l), \quad B_{lj}=\phi_j(\tau_l), \quad \begin{array}{l} l=1,\ldots,2N \\ j=1,\ldots,2N. \end{array}$$

The generalized eigenvalues and eigenvectors of (4.3) give the Hermite cubic collocation approximations to the eigenvalues and eigenvectors of (4.1).

As in (3.4), the matrices A and B will be banded with bandwidth two. Moreover, since only the differences between the Hermite cubics in (3.2) and (4.2) are the first and last basis functions ϕ_1 and ϕ_{2N}, it follows that the only differences between the matrices in (3.4) and (4.3) are the first and the last columns.

Theorem 4.1. *The* $2N$ *generalized eigenvalues of the discrete Neumann problem* $A\mathbf{c}=\lambda B\mathbf{c}$ *in* (4.3) *are given by*

$$\lambda_0=0 \tag{4.4a}$$

$$\lambda_N=\frac{6}{h^2(\theta^2-3/4)} \tag{4.4b}$$

$$\lambda_l^{\pm}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\ ,\ \ l=1,\ldots,N-1 \tag{4.4c}$$

where

$$a=h^4\left[(16\theta^4-16\theta^2+3)d-8\theta^2+2\right], \tag{4.5a}$$

$$b=h^2\left[(-128\theta^2+48)d+48\right], \tag{4.5b}$$

$$c=192d, \tag{4.5c}$$

and where

$$d=\tan^2\left(\frac{l\pi}{2N}\right). \tag{4.5d}$$

Proof. Proceeding as in the proof of Theorem 3.1, we have from (3.7) and (3.9) that

$$p_k(y)=\alpha_k+\beta_k y+\gamma_k\frac{y^2}{2}+\delta_k\frac{y^3}{6}\ ,\ -\frac{h}{2}\le y\le\frac{h}{2},\ \ k=0,\ldots,N-1, \tag{4.6}$$

and

$$\begin{aligned}\gamma_k&=\lambda\left(\alpha_k+\gamma_k\frac{\theta^2h^2}{2}\right),\\ \delta_k&=\lambda\left(\beta_k+\delta_k\frac{\theta^2h^2}{6}\right).\end{aligned} \tag{4.7}$$

If $\lambda=0$, then it follows from (4.7) that $\gamma_k=\delta_k=0$ so that (4.6) reduces to $p_k(y)=\alpha_k+\beta_k y$. Now, since each piece p_k is linear and since P is continuous, we must have $P(x)=\alpha+\beta x$. Moreover, since $P'(1)=\beta=0$, it follows that

(4.8) $$p_k(\lambda_0;y)=\alpha,$$

where α is an arbitrary constant. Thus, $\lambda_0=0$ is an eigenvalue of (4.3), which gives the desired result in (4.4a). Note that $P\equiv\alpha$ is, up to a multiplicative constant, a piecewise approximation to the eigenfunction $\cos(0\pi x)\equiv 1$ of (4.1).

If $\lambda\neq 0$, then we may continue as in the proof of Theorem 3.1 to show that γ_k and δ_k satisfy

(4.9) $$\gamma_k=\hat{A}_\lambda\zeta^k+\hat{C}_\lambda\zeta^{-k},$$
$$\delta_k=\hat{B}_\lambda\zeta^k+\hat{D}_\lambda\zeta^{-k}.$$

To find appropriate values for ζ, we impose the boundary conditions $P'(0)=P'(1)=0$. We again extend P, only this time as an even function by $P(x)=P(-x)$, $-1\leq x\leq 0$, and $P(x)=P(2-x)$, $1\leq x\leq 2$, to obtain the pieces

(4.10) $$p_{-1}(\lambda;y)=p_0(-y)\quad\text{and}\quad p_N(\lambda;y)=p_{N-1}(-y).$$

Recall from (3.10) that

(4.11) $$p_k(\lambda;y)=\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{2}+\frac{y^2}{2}\right]\gamma_k+y\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{6}\right]\delta_k,$$

so that

(4.12) $$p_k'(\lambda;y)=y\,\gamma_k+\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{2}\right]\delta_k.$$

Enforcing the boundary condition $P'(0)=0$, (4.12) and (4.9) give

(4.13) $$p_0'(\lambda;-h/2)=\frac{h}{2}\left[-(\hat{A}_\lambda-t\hat{B}_\lambda)-(\hat{C}_\lambda-t\hat{D}_\lambda)\right]=0$$

where

$$t=\frac{2}{h}\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{h^2}{8}\right].$$

Now, if $\hat{A}_\lambda-t\hat{B}_\lambda\neq 0$, then (4.13) yields

$$\frac{\hat{C}_\lambda - t\hat{D}_\lambda}{\hat{A}_\lambda - t\hat{B}_\lambda} = -1. \tag{4.14}$$

Similarly, enforcing the boundary condition $P'(1)=0$, it follows from (4.10), (4.12), and (4.9) that

$$p'_{N-1}(\lambda;+h/2) = -p'_N(\lambda;-h/2) = -\frac{h}{2}\left[-(\hat{A}_\lambda - t\hat{B}_\lambda)\zeta^N - (\hat{C}_\lambda - t\hat{D}_\lambda)\zeta^{-N}\right] = 0. \tag{4.15}$$

Solving (4.15) for ζ and using the result in (4.14), we obtain

$$\zeta^{2N} = -\frac{\hat{C}_\lambda - t\hat{D}_\lambda}{\hat{A}_\lambda - t\hat{B}_\lambda} = 1 = e^{2\pi i l},$$

from which it follows that

$$\zeta = e^{\frac{l\pi i}{N}} \tag{4.16}$$

for any integer l; we take $l=0,\ldots,N$.

Now, since (4.16) is the same as (3.23), the subsequent analysis in the proof of Theorem 3.1 may be applied here as well. However, note that the eigenvalues may be different as a result of the different boundary conditions.

From (3.28) we have that

$$\tilde{\lambda}_0 = \frac{6}{h^2(\theta^2 - 1/4)} \tag{4.17}$$

is a potential eigenvalue of (4.3) with corresponding eigenfunction, given up to a multiplicative constant, by $p_k(\tilde{\lambda}_0;y) = y(y^2 - h^2/4)$. However, since $p'_0(\tilde{\lambda}_0;-h/2) = p'_{N-1}(\tilde{\lambda}_0;+h/2) = h^2/2 \neq 0$, it follows that $\tilde{\lambda}_0$ is not an eigenvalue of the discrete Neumann problem (4.3).

Similarly, from (3.31) we have that

$$\tilde{\lambda}_N = \frac{2}{h^2(\theta^2 - 1/4)} \tag{4.18}$$

is a potential eigenvalue with eigenfunction $p_k(\tilde{\lambda}_N;y) = (y^2 - h^2/4)$. Here, $p'_0(\tilde{\lambda}_N;-h/2) = -5h^2/4 \neq 0$ and $p'_{N-1}(\tilde{\lambda}_N;+h/2) = 3h^2/4 \neq 0$ so that (4.18) is also not an eigenvalue of (4.3).

Now, from (3.32) we have that

$$\lambda_N = \frac{6}{h^2(\theta^2 - 3/4)} \tag{4.19}$$

is a potential eigenvalue of (4.3) with $p_k(\lambda_N; y) = y(y^2 - 3h^2/4)$. Since $p_0'(\lambda_N; -h/2) = 0$ and $p_{N-1}'(\lambda_N; +h/2) = 0$, it follows that λ_N is an eigenvalue of (4.3) thereby giving the desired result in (4.4c). Note that $p_k(\lambda_N; y)$ is, up to a multiplicative constant, a piecewise approximation to the eigenfunction $\cos((N+1)\pi x)$ of (4.1).

The remaining analysis in the proof of Theorem 3.1 applies directly here without modification, which gives the remaining desired eigenvalues in (4.4c).

EXAMPLE 4.1. The Generalized Eigenvalues of the Discrete Neumann Problem $A\mathbf{c} = \lambda B\mathbf{c}$.

In order to verify the results of Theorem 4.1, we compute the generalized eigenvalues of $A\mathbf{c} = \lambda B\mathbf{c}$ in (4.3) using the LAPACK routine SGEGV [1]. We then compare these computed results with those obtained by using the formulas of Theorem 4.1 given in (4.4).

Recall that in the continuous case, the eigenvalues are of the form $-k^2\pi^2$, $k = 0, 1, \ldots$. Thus, if divided by $-\pi^2$, we expect the generalized eigenvalues of (4.3) to approximate k^2, $k = 0, 1, \ldots$. Now, for the case $N = 4$ and $\theta = \frac{1}{2\sqrt{3}}$, we obtain the results given below in Table 4.1. We see from Table 4.1 that the formulas of Theorem 4.1 agree up to round-off with the computed results from LAPACK.

Table 4.1

Eigenvalues of the discrete Neumann problem $A\mathbf{c}=\lambda B\mathbf{c}$ divided by $-\pi^2$ for the case $N=4$ and $\theta=1/(2\sqrt{3})$.

λ	Theorem 4.1	LAPACK
λ_0	0.00000E+00	-3.18064E-07
λ_1^+	1.00017E+00	1.00017E+00
λ_2^+	4.00902E+00	4.00901E+00
λ_3^+	9.06012E+00	9.06012E+00
λ_4	1.45902E+01	1.45902E+01
λ_3^-	2.77562E+01	2.77562E+01
λ_2^-	4.04565E+01	4.04565E+01
λ_1^-	5.28336E+01	5.28336E+01

THEOREM 4.2. *The k^{th} piece of the Hermite cubic approximate eigenfunction of the Neumann problem* (4.1) *associated with the approximate eigenvalues λ_0, λ_N, $\lambda_l^\pm$ of Theorem 4.1 is given by*

(4.20a) $$p_k(\lambda_0;x)=C_{\lambda_0},$$

(4.20b) $$p_k(\lambda_N;x)=B_{\lambda_N}(x-\bar{x}_k)\left[(x-\bar{x}_k)^2-3h^2/4\right](-1)^k,\ \textit{and}$$

(4.20c) $$p_k(\lambda_l^\pm;x)=C_{\lambda_l^\pm}\left[\frac{1}{\lambda_l^\pm}-\frac{\theta^2h^2}{2}+\frac{(x-\bar{x}_k)^2}{2}\right]\cos\left[\frac{(k+1/2)l\pi}{N}\right]$$

$$+\,B_{\lambda_l^\pm}(x-\bar{x}_k)\left[\frac{1}{\lambda_l^\pm}-\frac{\theta^2h^2}{6}+\frac{(x-\bar{x}_k)^2}{6}\right]\sin\left[\frac{(k+1/2)l\pi}{N}\right],$$

for $l=1,\ldots,N-1$,

where $\bar{x}_k=1/2(x_k+x_{k+1})$, C_{λ_0}, B_{λ_N}, $B_{\lambda_l^\pm}$ are arbitrary nonzero constants, and where

$$C_{\lambda_l^\pm}=\frac{2}{h}\left[\frac{1}{\lambda_l^\pm}-\frac{\theta^2h^2}{6}+\frac{h^2}{8}\right]\tan\left[\frac{l\pi}{2N}\right]B_{\lambda_l^\pm}.$$

Proof. Consider first the case of the eigenfunction corresponding to $\lambda=\lambda_0=0$. The desired result in (4.20a) follows directly from (4.8) by replacing α by C_{λ_0}.

Now, for the case $\lambda \neq 0$, we again have from (3.10) and (3.24) that

$$p_k(\lambda;y)=\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{2}+\frac{y^2}{2}\right]\gamma_k+y\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{6}\right]\delta_k, \tag{4.21}$$

where

$$\begin{aligned}\gamma_k&=A_\lambda\sin\left[\frac{(k+1/2)l\pi}{N}\right]+C_\lambda\cos\left[\frac{(k+1/2)l\pi}{N}\right]\\ \delta_k&=B_\lambda\sin\left[\frac{(k+1/2)l\pi}{N}\right]+D_\lambda\cos\left[\frac{(k+1/2)l\pi}{N}\right].\end{aligned} \tag{4.22}$$

For the special case $l=N$ $(\lambda=\lambda_N)$, we have from (4.19) that $\frac{1}{\lambda_N}=\frac{\theta^2h^2}{6}+\frac{h^2}{8}$. Substituting $l=N$ into (4.22), we obtain $\sin((k+1/2)\pi)=(-1)^k$ and $\cos((k+1/2)\pi)=0$, so that (4.21) reduces to

$$p_k(\lambda_N;y)=\left[\frac{y^2}{2}-\frac{\theta^2h^2}{3}-\frac{h^2}{8}\right](-1)^kA_{\lambda_N}+\frac{1}{6}y\left[y^2-\frac{3h^2}{4}\right](-1)^kB_{\lambda_N}. \tag{4.23}$$

Note that

$$p_k'(\lambda_N;y)=y(-1)^kA_{\lambda_N}+1/2(y^2-h^2/4)(-1)^kB_{\lambda_N}. \tag{4.24}$$

Enforcing the boundary condition $P'(0)=0$ gives

$$p_0'(\lambda_N;-h/2)=-\frac{h}{2}(-1)^kA_{\lambda_N}=0$$

which implies that $A_{\lambda_N}=0$. Thus, (4.23) reduces to

$$p_k(\lambda_N;y)=\frac{1}{6}y\left[y^2-\frac{3h^2}{4}\right](-1)^kB_{\lambda_N}. \tag{4.25}$$

A simple calculation shows that $p_{N-1}'(\lambda_N;+h/2)=0$ so that $P'(1)=0$ as well. The desired result in (4.20b) follows from (4.25) by incorporating the factor 1/6 into the arbitrary constant B_{λ_N} and by replacing y by $(x-\bar{x}_k)$.

Finally, we consider the remaining cases $l=1,\ldots,N-1$. Analogous to the Dirichlet problem, we first show that $A_\lambda=D_\lambda=0$ for all corresponding eigenvalues λ.

From (4.21) we have

$$p_k'(\lambda;y)=y\gamma_k+\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{2}\right]\delta_k \tag{4.26}$$

For $k=0$, it follows from (4.26) and (4.22) that

$$p_0'(\lambda;-h/2)=\frac{h}{2}\left[-A_\lambda\sin\left(\frac{l\pi}{2N}\right)-C_\lambda\cos\left(\frac{l\pi}{2N}\right)+tB_\lambda\sin\left(\frac{l\pi}{2N}\right)+tD_\lambda\cos\left(\frac{l\pi}{2N}\right)\right] \tag{4.27}$$

where

$$t=\frac{2}{h}\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{h^2}{8}\right).$$

Moreover, substituting $k=N-1$ into (4.22), we obtain

$$\cos\left(\frac{(N-\frac{1}{2})l\pi}{N}\right)=(-1)^l\cos\left(\frac{l\pi}{2N}\right) \quad\text{and}\quad \sin\left(\frac{(N-\frac{1}{2})l\pi}{N}\right)=-(-1)^l\sin\left(\frac{l\pi}{2N}\right),$$

so that

$$p_{N-1}'(\lambda;-h/2)=(-1)^l\frac{h}{2}\left[-A_\lambda\sin\left(\frac{l\pi}{2N}\right)+C_\lambda\cos\left(\frac{l\pi}{2N}\right)-tB_\lambda\sin\left(\frac{l\pi}{2N}\right)+tD_\lambda\cos\left(\frac{l\pi}{2N}\right)\right]. \tag{4.28}$$

Now, from the boundary conditions $P'(0)=P'(1)=0$, we have $p_0'(\lambda;-h/2)=0$ and $p_{N-1}'(\lambda;+h/2)=0$, so that (4.27) and (4.28) give

$$\begin{aligned}
-A_\lambda\sin\left(\frac{l\pi}{2N}\right)-C_\lambda\cos\left(\frac{l\pi}{2N}\right)+tB_\lambda\sin\left(\frac{l\pi}{2N}\right)+tD_\lambda\cos\left(\frac{l\pi}{2N}\right) &= 0\\
-A_\lambda\sin\left(\frac{l\pi}{2N}\right)+C_\lambda\cos\left(\frac{l\pi}{2N}\right)-tB_\lambda\sin\left(\frac{l\pi}{2N}\right)+tD_\lambda\cos\left(\frac{l\pi}{2N}\right) &= 0,
\end{aligned}$$

respectively, which when added yield

$$tD_\lambda-A_\lambda\tan\left(\frac{l\pi}{2N}\right)=0. \tag{4.29}$$

Also, from (4.21) and (4.22), we have

$$(4.30)\qquad p_0(\lambda;-h/2)=rA_\lambda \sin\left(\frac{l\pi}{2N}\right)+rC_\lambda \cos\left(\frac{l\pi}{2N}\right)-sB_\lambda \sin\left(\frac{l\pi}{2N}\right)-sD_\lambda \cos\left(\frac{l\pi}{2N}\right)$$

and

$$(4.31)\qquad p_{-1}(\lambda;+h/2)=-rA_\lambda \sin\left(\frac{l\pi}{2N}\right)+rC_\lambda \cos\left(\frac{l\pi}{2N}\right)-sB_\lambda \sin\left(\frac{l\pi}{2N}\right)+sD_\lambda \cos\left(\frac{l\pi}{2N}\right),$$

respectively, where

$$r = \frac{1}{\lambda}-\frac{\theta^2h^2}{2}+\frac{h^2}{8}$$

$$s = \frac{h}{2}\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{h^2}{24}\right).$$

Now, since $p_{-1}(\lambda;+h/2)=p_0(\lambda;-h/2)$, we equate (4.30) to (4.31) and simplify to obtain

$$(4.32)\qquad rA_\lambda \tan\left(\frac{l\pi}{2N}\right)-sD_\lambda=0.$$

From (4.29), we have $rA_\lambda \tan\left(\frac{l\pi}{2N}\right)=rtD_\lambda$ so that (4.32) reduces to

$$(rt-s)D_\lambda=0.$$

Now, in order to show that $D_\lambda=0$, suppose that $D_\lambda \neq 0$. From the above, we have $rt=s$ so that the difference equations in (3.18) and (3.19) reduce, respectively, to

$$\gamma_{k-1}-2\gamma_k+\gamma_{k+1}=\gamma_{k-1}+2\gamma_k+\gamma_{k+1},$$

and

$$\delta_{k-1}-2\delta_k+\delta_{k+1}=\delta_{k-1}+2\delta_k+\delta_{k+1},$$

However, this yields $\gamma_k=\delta_k=0$, which results in $p_k\equiv 0$, or $P\equiv 0$. Thus, we must have $D_\lambda=0$. Furthermore, since $r\neq 0$ and $\tan\left(\frac{l\pi}{2N}\right)\neq 0$, (4.32) shows that $A_\lambda=0$ as well. Thus, we must have $A_\lambda=D_\lambda=0$ for all remaining eigenvalues λ.

Now, with $A_\lambda=D_\lambda=0$, the expression for p_k in (4.21) reduces to

$$(4.33)\quad p_k(\lambda;y)=\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{2}+\frac{y^2}{2}\right)C_\lambda \cos\left(\frac{(k+1/2)l\pi}{N}\right)+y\left(\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{6}\right)B_\lambda \sin\left(\frac{(k+1/2)l\pi}{N}\right)$$

where C_λ and B_λ are arbitrary constants. Note that

$$p_k'(\lambda;y)=yC_\lambda\cos\left(\frac{(k+1/2)l\pi}{N}\right)+\left[\frac{1}{\lambda}-\frac{\theta^2h^2}{6}+\frac{y^2}{2}\right]B_\lambda\sin\left(\frac{(k+1/2)l\pi}{N}\right).$$

We next choose C_λ and B_λ so that P satisfies the boundary conditions $P'(0)=P'(1)=0$. To this end, we set

$$p_0'(\lambda;-h/2)=\frac{h}{2}\left[tB_\lambda\sin\left(\frac{l\pi}{2N}\right)-C_\lambda\cos\left(\frac{l\pi}{2N}\right)\right]=0,$$

from which it follows that we must have

(4.34) $$C_\lambda=tB_\lambda\tan\left(\frac{l\pi}{2N}\right).$$

Also, since

$$p_{N-1}'(\lambda;+h/2)=(-1)^l\frac{h}{2}\left[C_\lambda\cos\left(\frac{l\pi}{2N}\right)-tB_\lambda\sin\left(\frac{l\pi}{2N}\right)\right]$$

it follows from (4.34) that $p_{N-1}'(\lambda;+h/2)=0$ as well. Thus, $P'(0)=P'(1)=0$.

The desired results in (4.20c) follow from (4.33) and (4.34) by replacing λ by $\lambda_l^\pm$, and by replacing y by $(x-\bar{x}_k)$.

EXAMPLE 4.2. The Hermite Cubic Eigenfunctions $P(\lambda_1^+;x)$ and $P(\lambda_2^+;x)$ for the Discrete Neumann Problem.

Graphs of the Hermite cubic eigenfunctions $P(\lambda_1^+;x)$ and $P(\lambda_2^+;x)$ for the case $N=4$ are given in Figure 4.1.

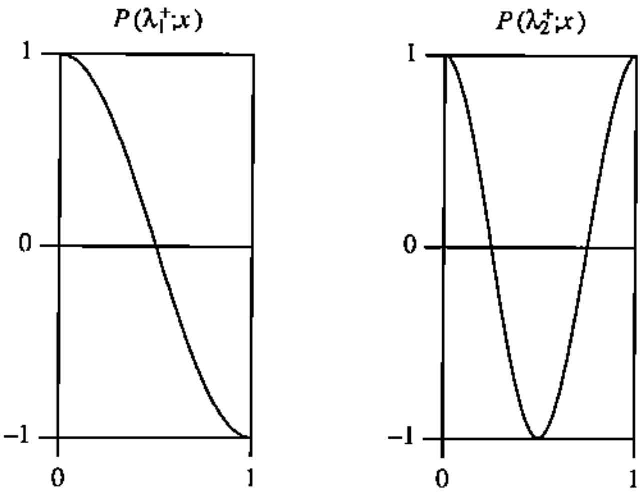

Figure 4.1 The Hermite cubic eigenfunctions $P(\lambda_1^+;x)$ and $P(\lambda_2^+;x)$ approximating $\cos(\pi x)$ and $\cos(2\pi x)$, respectively, for the case $N=4$.

The discrete eigenfunctions $P(\lambda_1^+;x)$ and $P(\lambda_2^+;x)$ are approximations to the continuous eigenfunctions $\cos(\pi x)$ and $\cos(2\pi x)$, respectively. Note that each eigenfunction consists of four pieces, and that the arbitrary constants $B_{\lambda_1^+}$ and $B_{\lambda_2^+}$ are chosen so the functions have maximum absolute value one.

5. Conclusions. We have given explicit closed-form expressions for the Hermite cubic approximations to both the eigenvalues and the eigenfunctions of the Laplace operator for both the Dirichlet and the Neumann problems. Moreover, for the Dirichlet case, we have shown that optimal approximations are obtained using the Gauss points for collocation points. For both cases, we have given numerical examples that verify our theoretical results. Our results apply directly to a number of iterative techniques used to solve the linear system arising from Hermite cubic approximations to large classes of separable, elliptic partial differential equations.

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