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Chandrajit L. Bajaj

Jindon Chen

Robert J. Holt

Arun N. Netravali

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# Energy Formulations of A-Splines*

Chandrajit L. Bajaj Jindon Chen
Department of Computer Sciences
Purdue University
West Lafayette, IN 47907

Robert J. Holt Arun N. Netravali
AT&T Bell Laboratories
Murray Hill, NJ 07974

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**Abstract**

A-splines are implicit real algebraic curves in Bernstein-Bézier (BB) form that are smooth. We develop a spectrum of physical A-spline curve models using various energy formulations based on the theory of elasticity. Our bending and stretching energy formulations yield interactive physically based modeling with A-splines while preserving the advantages of free-form modeling of complex shapes.

## 1 Introduction

An A-spline is a smooth zero contour curve of a bivariate polynomial in Bernstein-Bézier (BB) form defined within a triangle [BX92], where the "A" stands for algebraic. Solutions to the problem of constructing a $C^1$ chain of implicit algebraic splines based on a polygon $\mathcal{P}$ have been given by [BX92] and for dense image or sparse scattered data by [BX94].

In this paper we develop a spectrum of physical A-spline curve models using various energy formulations based on the theory of elasticity[LL59]. Several applications in image processing and computer graphics have been shown to be enhanced by using physically based modeling (see [TPBF87, KWT88, ?, ?, ?, ?] and several others).

The search for minimal energy curves has a history stretching back to Euler (curves he termed as "elastica") and more recently by pure and applied mathematicians [BB65, BG86, GJ82], computer vision experts [Hor83, Mum94] and geometric designers [?]. Dynamic A-splines differ from the above as they are based on a piecewise implicit polynomial curve representation. There are several advantages of this representation:

- A-splines can model arbitrary closed curves with quadratic and cubic polynomial degree segments and various inter-segment continuity

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- The class of objects represented by A-splines contains the class of objects representable by polynomial or rational B-splines

The rest of this paper is as follows. Section 2 gives preliminary information about A-splines as well as the theory of elasticity. Section 3 develops the elastic strain energy model for an A-spline as well an energy simplification for computational efficiency. Section 4 describes algorithms that minimize the different energy formulations of the previous section and also presents example case studies.

## 2 Notation and Preliminary Details

Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3 \in I\!R^2$ be non-collinear. Then the triangle (or two-dimensional simplex) with vertices $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, is $T = [\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$. Let $\mathbf{p} = (x, y)^{\mathsf{T}}$, $\mathbf{p}_i = (x_i, y_i)^{\mathsf{T}}$. Then for any $\mathbf{p} = \sum_{i=1}^{3} \alpha_i \mathbf{p}_i \in I\!R^2$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^{\mathsf{T}}$ is the *barycentric coordinate* of $\mathbf{p}$, where the Cartesian and barycentric coordinates are related by

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} . \tag{2.1}$$

The non-collinearity of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ guarantees that the $3 \times 3$ matrix in (2.1) is non-singular and that the barycentric coordinates are well defined.

Any polynomial $F(\mathbf{p})$ of degree $d$ can be expressed in BB form over $T$ as $F(\mathbf{p}) = \sum_{|\boldsymbol{\lambda}|=d} b_{\boldsymbol{\lambda}}\, B^d_{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$, $\boldsymbol{\lambda} \in \mathcal{Z}^3_+$ where $B^d_{\boldsymbol{\lambda}}(\boldsymbol{\alpha}) = (d!/\lambda_1!\lambda_2!\lambda_3!)\, \alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}$ are the bivariate Bernstein polynomials for $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^{\mathsf{T}}$, and $|\boldsymbol{\lambda}|$ is defined to be $\sum_{i=1}^{3} \lambda_i$. Also, $b_{\boldsymbol{\lambda}} = b_{\lambda_1\lambda_2\lambda_3}$ (as a subscript, we simply write $\boldsymbol{\lambda}$ as $\lambda_1\lambda_2\lambda_3$) are called control points, and $\mathcal{Z}^3_+$ stands for the set of all three-dimensional vectors with nonnegative integer components. Let

$$F(\boldsymbol{\alpha}) = \sum_{|\boldsymbol{\lambda}|=d} b_{\boldsymbol{\lambda}} B^d_{\boldsymbol{\lambda}}(\boldsymbol{\alpha}), \quad |\boldsymbol{\alpha}| = 1, \tag{2.2}$$

be a given polynomial of degree $d$ on the triangle $T = \{(\alpha_1, \alpha_2, \alpha_3)^{\mathsf{T}} \in I\!R^3 : \sum_{i=1}^{3} \alpha_i = 1, \quad \alpha_i \geq 0\}$. The curve segment within the triangle is defined by $S(\alpha_1, \alpha_2, \alpha_3) = 0$.

Collecting the base functions $B^d_{\boldsymbol{\lambda}}$'s into a vector $\mathbf{B}^d$ and the coefficients $b_{\boldsymbol{\lambda}}$'s into a vector $\mathbf{b}$, equation 2.2 is rewritten as

$$F(\boldsymbol{\alpha}) = \mathbf{b}^{\mathsf{T}}\mathbf{B}^d$$

Equation (2.1) may be rewritten without the use of $\alpha_3$ as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \tag{2.3}$$

$\mathbf{J}$ is the Jacobian of $\boldsymbol{\alpha}$ in terms of $x$ and $y$:

$$\mathbf{J} = \frac{\partial(\alpha_1, \alpha_2)}{\partial(x, y)} = \frac{1}{\Delta} \begin{bmatrix} y_2 - y_3 & x_3 - x_2 \\ y_3 - y_1 & x_1 - x_3 \end{bmatrix} , \tag{2.4}$$

where

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \quad . \tag{2.5}$$

When we speak of functions of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, it should be noted that these are really functions of two independent variables since $\alpha_3 = 1 - \alpha_1 - \alpha_2$. The curve segment within the triangle $T$ may thus be expressed as

$$S(\alpha_1, \alpha_2) = F(x(\boldsymbol{\alpha}), y(\boldsymbol{\alpha})) \ .$$

We also have these relationships involving the two sets of coordinates for the gradient

$$\nabla F(x,y) = \mathbf{J}^{\mathsf{T}} \nabla S(\alpha_1, \alpha_2) \ , \tag{2.6}$$

and the Hessian

$$\begin{aligned} \nabla^2 F(x,y) &= \mathbf{H}(\boldsymbol{\alpha}) \\ &= \mathbf{J}^{\mathsf{T}} \nabla^2 S(\alpha_1, \alpha_2) \mathbf{J} \\ &= \begin{bmatrix} \mathbf{J_1}^{\mathsf{T}} \nabla^2 S \mathbf{J}_1 & \mathbf{J_1}^{\mathsf{T}} \nabla^2 S \mathbf{J}_2 \\ \mathbf{J_2}^{\mathsf{T}} \nabla^2 S \mathbf{J}_1 & \mathbf{J_2}^{\mathsf{T}} \nabla^2 S \mathbf{J}_2 \end{bmatrix} \ , \end{aligned} \tag{2.7}$$

where $\mathbf{J}_i$ is the $i^{\text{th}}$ column of $\mathbf{J}$.

## 2.1 Differential Geometry of Implicit Functions

### 2.1.1 Arc Length

Given an implicit function $F(x,y) = 0$, we have $dF = 0$, or

$$F_x dx + F_y dy = 0 \tag{2.8}$$

Assuming that in a small neighborhood of point $(x,y)$, $y$ is a function of the independent variable $x$, from (2.8) we have

$$y_x \ = \ -\frac{F_x}{F_y} \tag{2.9}$$

For a curve $F(x,y) = 0$ in Cartesian space, let $S(\alpha_1, \alpha_2) = 0$ be its BB form in some local coordinates. Then the arc length $s$ of the curve may be expressed as

$$\frac{ds}{dx} = \sqrt{1 + y_x^2} = \frac{(F_x^2 + F_y^2)^{1/2}}{|F_y|} = \frac{(\nabla F^{\mathsf{T}} \nabla F)^{1/2}}{|F_y|} = \frac{(\nabla S^{\mathsf{T}} \mathbf{J} \mathbf{J}^{\mathsf{T}} \nabla S)^{1/2}}{\left|\mathbf{J_2}^{\mathsf{T}} \nabla S\right|} \ . \tag{2.10}$$

### 2.1.2 Curvature

By differentiating (2.9) with respect to $x$, we obtain

$$y_{xx} = -\frac{\partial}{\partial x}\left(\frac{F_x}{F_y}\right) = -\frac{F_{yy}F_x^2 + F_{xx}F_y^2 - 2F_{xy}F_xF_y}{F_y^3} \ .$$

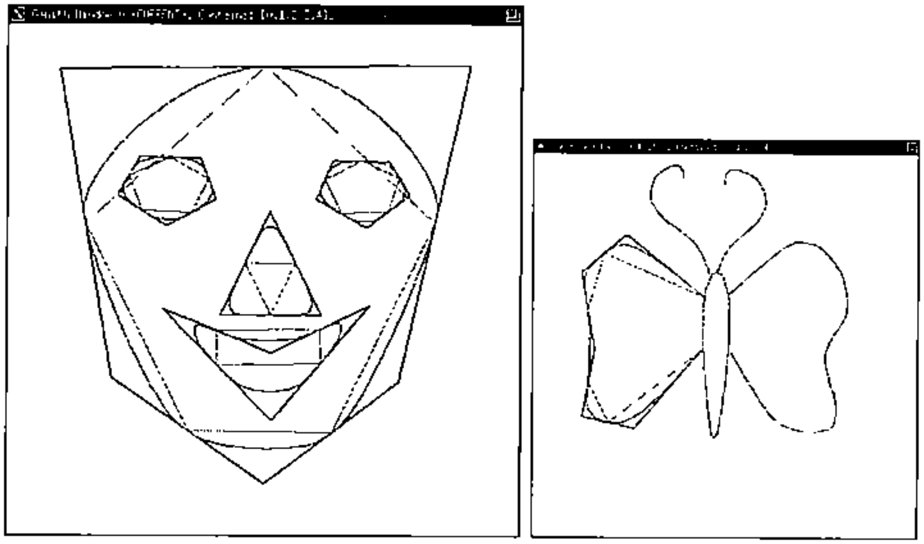


Figure 2.1: Interactively designed contours with cubic A-splines

This allows us to express the curvature of the function $F(x,y)=0$, or $S(\alpha_1,\alpha_2)=0$ in BB form, as

$$\begin{aligned} \kappa &= \frac{|y_{xx}|}{(1+y_x^2)^{3/2}} \\ &= \frac{\left|\nabla F^{\mathsf{T}}\mathbf{P}^{\mathsf{T}}\nabla^2 F\mathbf{P}\nabla F\right|}{(\nabla F^{\mathsf{T}}\nabla F)^{3/2}} \\ &= \frac{\left|\nabla S^{\mathsf{T}}\mathbf{J}\mathbf{P}^{\mathsf{T}}\mathbf{J}^{\mathsf{T}}\nabla^2 S\mathbf{J}\mathbf{P}\mathbf{J}^{\mathsf{T}}\nabla S\right|}{(\nabla S^{\mathsf{T}}\mathbf{J}\mathbf{J}^{\mathsf{T}}\nabla S)^{3/2}} \end{aligned} \tag{2.11}$$

where the permutation matrix $\mathbf{P}$ is defined as

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

## 2.2 A-splines

An A-spline is a smooth zero contour curve of a bivariate polynomial in BB form defined within a triangle [BX92]. Figure 2.1 shows examples of A-splines that are defined in [BX92].

The papers [BX92, BX94] explore the possibilities of building piecewise smooth curves that interpolate or approximate given polygonal data sets.

For further formulation of elastic models we briefly describe the A-spline in the following way. The formulation includes the A-spline [BX92], the 2D counterpart of the A-spline, which treats piecewise implicit curves. A piecewise A-spline curve consists of the zero contour of some piecewise smooth BB polynomials defined over a simplicial hull $\Sigma$, or a triangulation of a connected region

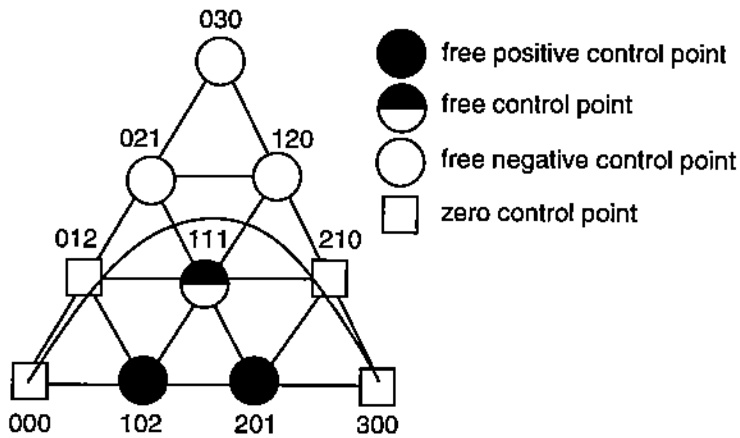



Figure 2.2: Sign inequalities for a $C^1$ cubic A-spline. The symbol labeled by $ijk$ indicates the sign restriction, if any, for the coefficient $b_{ijk}$.

of the space. In particular,

$$F^{(i)}(\alpha) = \sum_{|\lambda|=d} b^{(i)}_{\lambda} B^d_{\lambda}(\alpha) = \mathbf{b}^{(i)^{\mathsf{T}}} \mathbf{B}^d = 0$$

is the zero contour of a BB polynomial defined within the $i^{\text{th}}$ simplex in $\Sigma$. The base functions $B^d_{\lambda}$ are grouped into a vector $\mathbf{B}$ and the coefficients $b_{\lambda}$ into $\mathbf{b}$. Smoothness of certain degrees and local interpolation of certain degrees are enforced by some linear equality constraints

$$\mathbf{b}^{\mathsf{T}}\mathbf{C}(\mathbf{p}) = \mathbf{0} \;,$$

and connectedness of the curve is enforced by additional linear sign inequalities

$$\mathbf{b}^{\mathsf{T}}\mathbf{S} > \mathbf{0} \;.$$

See Figure 2.2 for an example of sign equalities and inequalities of a $C^1$ cubic A-spline segment.

Vector $\mathbf{b}$ is a global collection of the coefficient vector $\mathbf{b}^{(i)}$ of all simplexes in $\Sigma$, and $\mathbf{S}$ and $\mathbf{C}(\mathbf{p})$ are defined explicitly for A-Splines with $C^k$ continuity in [BX92].

The partial derivatives of a BB polynomial $F(\alpha) = \mathbf{b}^{\mathsf{T}}\mathbf{B}$ are given by

$$\begin{aligned} F_{\alpha_j} &= \mathbf{b}^{\mathsf{T}}\mathbf{B}^d_{\alpha_j}(\alpha) &&= \mathbf{b}^{\mathsf{T}}\mathcal{D}^d_j\mathbf{B}^{d-1}(\alpha) \\ F_{\alpha_j\alpha_k} &= \mathbf{b}^{\mathsf{T}}\mathbf{B}_{\alpha_j\alpha_k}(\alpha) &&= \mathbf{b}^{\mathsf{T}}\mathcal{D}^d_k\mathcal{D}^{d-1}_j\mathbf{B}^{d-2}(\alpha) \end{aligned}$$

The $\mathcal{D}^d_j$ are matrices that relate the derivatives of the basis elements of $\mathbf{B}^d$ to the basis elements of $\mathbf{B}^{d-1}$. The $\mathcal{D}^d_j$ are independent of $\mathbf{b}$ and have dimensions $\binom{d+2}{2} \times \binom{d+1}{2}$. For example, with $d = 2$, we have $\mathbf{B}^d = [\alpha_1^2, 2\alpha_1\alpha_2, \alpha_2^2, 2\alpha_1(1-\alpha_1-\alpha_2), 2\alpha_2(1-\alpha_1-\alpha_2), (1-\alpha_1-\alpha_2)^2]^{\mathsf{T}}$, $\mathbf{B}^{d-1} = [\alpha_1, \alpha_2, 1-\alpha_1-\alpha_2]^{\mathsf{T}}$, $\mathbf{B}^d_{\alpha_1} = [2\alpha_1, 2\alpha_2, 0, -4\alpha_1-2\alpha_2+2, -2\alpha_2, 2\alpha_1+2\alpha_2-2]^{\mathsf{T}}$, $\mathbf{B}^d_{\alpha_2} = [0,$

$2\alpha_1, 2\alpha_2, -2\alpha_1, -2\alpha_1 - 4\alpha_2 + 2, 2\alpha_1 + 2\alpha_2\ -2]^{\mathsf{T}}$, and

$$\mathcal{D}_1^2 = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathcal{D}_2^2 = 2\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

# 3 Energy Models for A-splines

## 3.1 Elastic curves

Let $a$ be the intrinsic or material coordinate of a point on a plane curve $C$ with parameterization $\mathbf{w}(a)$. The natural undeformed state is specified by $\mathbf{w}^0(a)$. A reasonable strain energy for elastic curves is given by

$$E_{curve}(\mathbf{w}) = \int_C \left[\beta(s - s^0)^2 + \gamma(\kappa - \kappa^0)^2\right] da \ .$$

This norm measures the amount of deformation away from the natural undeformed state and consists of a stretching component and a bending component. Note $s^0$ and $\kappa^0$ are the arc length and curvature, respectively, of the undeformed curve as defined in Section 2.1.

Most of the elastic curves that are built on parameterizations other than the intrinsic, are referred to as "elastica" [JH90, Mum94].

For a parametric representation $\mathbf{w}(a)$, the elastic potential energy of a deformable curve is as follows:

$$E = \int_C [\beta(a)|\mathbf{w}_a|^2 + \gamma(a)|\mathbf{w}_{aa}|^2]\, da \tag{3.1}$$

where $a$ is the intrinsic or material coordinate of the curve.

The first step of defining the elastic energy of a geometric entity is actually the mapping between the material coordinates and some parameterization of the geometric entity. For a piecewise representation, the joining points of the pieces could be identified as such a parameterization. Namely, each joining point is associated with fixed material coordinates during a deformation process. However, such a parameterization is too coarse. A second level parameterization is needed to describe detail changes, especially when there is substantial freedom within each entity. In the following energy curves, we assume that the material is uniformly distributed along the curve.

## 3.2 Elastic strain energy model of A-splines

Let $S(\alpha_1, \alpha_2) = 0$ be an A-spline defined within a triangle $\Delta ABC$ (see figure 3.1(a)). The curved piece interpolates $B$, $C$ and is tangent to $BA$ and $CA$ at $B$ and $C$ respectively. Let $F(x, y) = 0$ be its representation in the Cartesian coordinates whose $x$-axis coincides with edge $BC$. The Cartesian coordinates of $A$, $B$ and $C$ are $(x_A, y_A)$, $(0, 0)$ and $(x_C, 0)$, respectively, and their local barycentric coordinates are $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 0, 0)$, respectively. Alternatively, we may suppress the third coordinate $\alpha_3 = 1 - \alpha_1 - \alpha_2$ and express the barycentric coordinates of these three points as $(0, 1)$, $(0, 0)$, and $(1, 0)$. With the assumption that the projection of $A$ onto the $x$-axis falls between

Figure 3.1: Relationship of local BB coordinates, Cartesian coordinates and material coordinates. The $X$-axis coincides with the $O\alpha_1$-axis of the BB coordinates. The material coordinate begins with $a_i$ at $B$ and ends with $a_{i+1}$ at $C$.

$B$ and $C$, $y$ is a function of $x$ throughout this piece of curve. Let the material coordinates of $B$ and $C$ be fixed as $a_i$ and $a_{i+1}$, and the piece of the curve between these points be $C_i$.

We assume that no matter how the A-spline curve deforms, the "material" is always "uniformly distributed" over the piece $C_i$, i.e. $|\mathbf{w}_a(a)|$, or $ds/da$, is always constant throughout $C_i$. Therefore, we have

$$|\mathbf{w}_a(a)| = \frac{ds}{da} = K \tag{3.2}$$

for some positive constant $K$. Also note that the edge $[\alpha_3, \alpha_1]$ in the local barycentric coordinates coincides with the $x$-axis and thus

$$dx = x_C d\alpha_1 \ . \tag{3.3}$$

The stretching energy is formulated as

$$E_{stretch}(\mathbf{b}) = \int_C |\mathbf{w}_a(a)|^2 da = K^2 \Delta a_i \ , \tag{3.4}$$

where $\Delta a_i = a_{i+1} - a_i$.

In order to obtain the bending energy, we observe that

$$\begin{aligned} \mathbf{w}_{aa} &= \frac{d}{da}\left(\frac{d\mathbf{w}}{ds}\cdot\frac{ds}{da}\right) \\ &= \mathbf{w}_{ss}\left(\frac{ds}{da}\right)^2 + \frac{d\mathbf{w}}{ds}\cdot\frac{d^2s}{da^2} \\ &= K^2\mathbf{w}_{ss} \end{aligned} \tag{3.5}$$

as $ds/da = K$ and $d^2s/da^2 = 0$ from (3.2). Therefore by (3.5),

$$|\mathbf{w}_{aa}|^2 = K^4|\mathbf{w}_{ss}|^2 = K^4\kappa^2 \ , \tag{3.6}$$

In order to obtain a direct relationship between the material and barycentric coordinates, we use (3.2), (2.10), and (3.3) to obtain

$$\begin{aligned} \frac{da}{d\alpha_1} &= \frac{da}{ds}\frac{ds}{dx}\frac{dx}{d\alpha_1} \\ &= \frac{1}{K}\sqrt{1+y_x^2}\, x_C \\ &= \frac{x_C}{K}\frac{(\nabla S^{\mathsf T}\mathbf{J}\mathbf{J}^{\mathsf T}\nabla S)^{1/2}}{|\mathbf{J}_2{}^{\mathsf T}\nabla S|} \ . \end{aligned} \tag{3.7}$$

Therefore by (2.11), (3.6), and (3.7), the bending energy may be expressed as

$$\begin{aligned} E_{bend}(\mathbf{b}) &= \int_{a_i}^{a_{i+1}} |\mathbf{w}_{aa}|^2 da \\ &= K^4 \int_{a_i}^{a_{i+1}} \kappa^2 da \\ &= K^4 \int_0^1 \frac{\left(\nabla S^{\mathsf T}\mathbf{J}\mathbf{P}^{\mathsf T}\mathbf{J}^{\mathsf T}\nabla^2 S\mathbf{J}\mathbf{P}\mathbf{J}^{\mathsf T}\nabla S\right)^2}{\left(\nabla S^{\mathsf T}\mathbf{J}\mathbf{J}^{\mathsf T}\nabla S\right)^3}\frac{x_C}{K}\frac{(\nabla S^{\mathsf T}\mathbf{J}\mathbf{J}^{\mathsf T}\nabla S)^{1/2}}{|\mathbf{J}_2^{\mathsf T}\nabla S|}\, d\alpha_1 \\ &= K^3 x_C \int_0^1 \frac{\left(\nabla S^{\mathsf T}\mathbf{J}\mathbf{P}^{\mathsf T}\mathbf{J}^{\mathsf T}\nabla^2 S\mathbf{J}\mathbf{P}\mathbf{J}^{\mathsf T}\nabla S\right)^2}{\left(\nabla S^{\mathsf T}\mathbf{J}\mathbf{J}^{\mathsf T}\nabla S\right)^{5/2}\left|\mathbf{J}_2^{\mathsf T}\nabla S\right|}\, d\alpha_1 \ . \end{aligned} \tag{3.8}$$

### 3.3 Simplified Elastic Strain Energy Model of A-Splines

The following approach is based on an assumption that if a polynomial function is energy-minimized, its zero contour is not very far away from an energy-minimized state. Hence, instead of optimizing the zero contour, we optimize the polynomial function, whose formulation is a lot simpler. One way to think of the hyper-energy approximating the energy of an A-spline [BX92] is by considering the energy of an approximating banded piecewise polynomial function which contains the zero contour. By continuous subdivision of the curve domain and discarding the triangles that do not contain the zero contour, the banded curve converges to the A-spline.

For a cubic A-spline, let $S_i(\alpha)$ be a BB function defined over a simplex $T_i$ and let $F_i(x, y)$ be the same function in Cartesian coordinates. The stretching energy is approximated by

$$E_{stretch}^{(i)} = \int_{T_i} [\beta_1 (F_i)_x^2 + \beta_2 (F_i)_y^2] dx dy$$

and the bending energy is approximated by

$$E_{bend}^{(i)} = \int_{T_i} [\gamma_{11}(F_i)_{xx}^2 + 2\gamma_{12}(F_i)_{xy}^2 + \gamma_{22}(F_i)_{yy}^2] dx dy \ ,$$

so the elastic energy of the piece is approximated by

$$E_i = E_{bend}^{(i)} + E_{stretch}^{(i)} \ . \tag{3.9}$$

The total energy of a piecewise spline curve is defined as

$$E = \Sigma E_i \ , \tag{3.10}$$

the sum of the energies of each piece.

# 4 Energy Optimization

We now consider how to minimize the different energy functions of the earlier section over the constrained degrees of freedom (domain vertices and control weights) of the A-spline curve.

## 4.1 Local Minimization of Total Energy

Here we take into account both the bending and stretching energy as defined in equation (3.1). The minimization is obtained locally by varying the free weights of each individual A-spline curve within its triangle.

For an A-spline, we can absorb the constants in equations (3.4) and (3.8) into new constants $\beta$ and $\gamma$, and with the help of (3.7), we can write (3.1) as

$$\begin{aligned} E_{total} &= \int_{a_i}^{a_{i+1}} (\beta + \gamma\kappa^2)\, da \\ &= K x_C \int_0^1 \frac{(\nabla S^{\mathsf{T}} \mathbf{J}\mathbf{J}^{\mathsf{T}} \nabla S)^{1/2}}{|\mathbf{J}_2{}^{\mathsf{T}} \nabla S|} \left(\beta + \gamma\kappa^2\right) d\alpha_1 \end{aligned} \tag{4.1}$$

where

$$\kappa^2 = \frac{[\nabla S^{\mathsf{T}} \mathbf{J}\mathbf{P}^{\mathsf{T}} \mathbf{J}^{\mathsf{T}} \mathbf{H}(S) \mathbf{J}\mathbf{P}\mathbf{J}^{\mathsf{T}} \nabla S]^2}{(\nabla S^{\mathsf{T}} \mathbf{J}\mathbf{J}^{\mathsf{T}} \nabla S)^3} \ . \tag{4.2}$$

An energy-minimized setting is a solution to

$$\nabla_{\mathbf{b}} E_{total} = \mathbf{0} \tag{4.3}$$

or

$$\int_{a_i}^{a_{i+1}} \gamma\kappa \nabla_{\mathbf{b}} \kappa \, da = \mathbf{0} \ .$$

We propose two simplifications to the minimization process. The first simplification restricts the freedom to just one control weight. Here the energy minimizing equation is reduced to a non-linear univariate equation.

The second one is an iterative linear optimization scheme. We generate a piecewise linear approximation of the A-spline such that the gradient over each curve piece is pretty much constant. By this simplification, we are able to reduce the minimization of the nonlinear system 4.3 to an iterative solution of linear systems.

### 4.1.1 Exact solutions

System 4.3 is in general a nonlinear system of $\mathbf{b}$. A nonlinear system is not guaranteed to be solvable. However, by restricting the freedom to one variable, we reduce to the system to a univariate non-linear system, which is easy to solve.

Let $\mathbf{b}$ be a vector function of some parameter $u$. For an A-spline curve, system 4.3 is reduced to

$$\frac{\partial E(u)}{\partial u} = 0 \ .$$

Let

$$f(u) = \frac{dE(u)}{du} = \int_{a_i}^{a_{i+1}} \gamma\kappa\kappa_u \, da$$

so that

$$f'(u) = \int_{a_i}^{a_{i+1}} \gamma(\kappa_u^2 + \kappa\kappa_{uu})\, da \ .$$

*** PROBLEM: $a$ is a function of $u$. This differentiation is invalid. *** $f(u)$ and $f'(u)$ are both continuous. We can use standard methods, such as Newton's method, to solve for the roots. Please note that the evaluations of $f(u)$ and $f'(u)$ would involve numerical integrations.

**Case study: A Quadratic A-spline**

Here we present an example of an energy-minimizing quadratic A-spline with $C^1$ continuity. First we derive the general equation for such a curve in BB form. The general quadratic spline is $S(\alpha_1,\alpha_2) = b_{200}\alpha_1^2 + 2b_{110}\alpha_1\alpha_2 + 2b_{101}\alpha_1(1-\alpha_1-\alpha_2) + b_{020}\alpha_2^2 + 2b_{011}\alpha_2(1-\alpha_1-\alpha_2) + b_{002}(1-\alpha_1-\alpha_2)^2 = 0$. However, $S(1,0) = 0 \Rightarrow b_{200} = 0$ and $S(0,0) = 0 \Rightarrow b_{002} = 0$. Furthermore, the tangent to $S(\alpha_1,\alpha_2)$ at $(1,0)$ is parallel to the line $\alpha_1+\alpha_2 = 1$. Therefore $d\alpha_2/d\alpha_1 = -(dS/d\alpha_1)/(dS/\alpha_2) = -1$, or $dS/d\alpha_1 = dS/d\alpha_2$ at that point. This now implies that $b_{110} = 0$. Also, the tangent to $S(\alpha_1,\alpha_2)$ at $(0,1)$ is parallel to the $\alpha_2$-axis. Therefore $dS/d\alpha_2 = 0$ there, and this implies that $b_{011} = 0$. We now have

$$S(\alpha_1,\alpha_2) = 2b_{101}\alpha_1(1-\alpha_1-\alpha_2) + b_{020}\alpha_2^2 = 0 \ , \tag{4.4}$$

in accordance with [BX94].

Suppose the curve $F(x,y) = 0$ passes through the points $\mathbf{p}_3 = (0,0)$ and $\mathbf{p}_1 = (1,0)$, and that we are given that the tangent lines at these points have slopes 2 and $-3$, respectively. Then the intersection of the tangent lines is $\mathbf{p}_2 = (3/5, 6/5)$. In this case (2.3) gives $(\alpha_1,\alpha_2) = ((2x-y)/2,\, 5y/6)$. We take $b_{020} = -1$ as in [BX92], and take $K = 1$, $\beta = 1$, and $\gamma = 1$. The integral we wish to minimize is this specialization of (4.1):

$$\begin{aligned}\int_{a_i}^{a_{i+1}} (\beta+\gamma\kappa^2)\, da &= Kx_C \int_0^1 \frac{(\nabla S^{\mathsf T}\mathbf{J}\mathbf{J}^{\mathsf T}\nabla S)^{1/2}}{|\mathbf{J}_2{}^{\mathsf T}\nabla S|}(\beta+\gamma\kappa^2)\, d\alpha_1 \\ &= \int_0^1 \frac{(\nabla S^{\mathsf T}\mathbf{J}\mathbf{J}^{\mathsf T}\nabla S)^{1/2}}{-\mathbf{J}_2{}^{\mathsf T}\nabla S}(1+\kappa^2)\, d\alpha_1 \ .\end{aligned} \tag{4.5}$$

The inequality constraints from [BX92] indicate that $b_{020} = -1$ and $b_{101} > 0$. Under these conditions $F_y = \mathbf{J}_2{}^{\mathsf T}\nabla S < 0$, hence the replacement of the absolute value sign by a minus sign in the above equation.

The curvature (4.2) is given by

$$\begin{aligned}\kappa^2 = {} & 180b_{101}^2\,[-2b_{101}(b_{101}-2)(\alpha_1^2+\alpha_1\alpha_2-\alpha_1) - (b_{101}-2)\alpha_2^2 + b_{101}]^2 \Big/ \\ & [29b_{101}^2\alpha_1^2 + (30b_{101}^2 - 2b_{101})\alpha_1\alpha_2 - 30b_{101}^2\alpha_1 \\ & + (9b_{101}^2 - 6b_{101} + 5)\alpha_2^2 - (18b_{101}^2 - 6b_{101})\alpha_2 + 9b_{101}^2]^3 \ ,\end{aligned}$$

and the first factor in the integrand is given by

$$\begin{aligned}\frac{(\nabla S^{\mathsf T}\mathbf{J}\mathbf{J}^{\mathsf T}\nabla S)^{1/2}}{-\mathbf{J}_2{}^{\mathsf T}\nabla S} = {} & \sqrt{5}[29b_{101}^2\alpha_1^2 + (30b_{101}^2 - 2b_{101})\alpha_1\alpha_2 - 30b_{101}^2\alpha_1 \\ & + (9b_{101}^2 - 6b_{101} + 5)\alpha_2^2 - (18b_{101}^2 - 6b_{101})\alpha_2 + 9b_{101}^2]^{1/2} \Big/ \\ & [-b_{101}\alpha_1 - (3b_{101}-5)\alpha_2 + 3b_{101}] \ .\end{aligned}$$

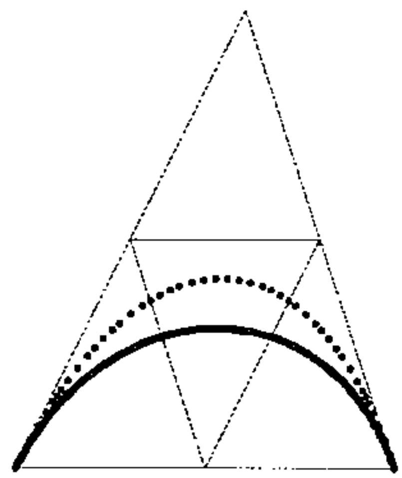


Figure 4.1: The solid line, obtained for $b_{101} = 0.391$, gives the energy-minimizing quadratic A-spline with $C^1$ continuity. The dotted line shows the curve obtained for $b_{101} = 1$.

From (4.4), we have

$$\alpha_2 = -b_{101}\alpha_1 + \sqrt{b_{101}\alpha_1[(b_{101} - 2)\alpha_1 + 2]} \ ,$$

where the root selected is chosen so that $\alpha_2 = 0$ when $\alpha_1 = 1$ and $b_{101} > 0$. Using this substitution, the function in (4.5) could not be integrated symbolically, so it was integrated numerically for several different values of $b_{101}$. The integral attained its minimum value of 6.13821 at $b_{101} = 0.391$. This gives an ellipse whose equation in Cartesian coordinates is

$$-0.782x^2 + 0.1303333xy - 0.5641111y^2 + 0.782x - 0.391y = 0 \ ,$$

or

$$7.038x^2 - 1.173xy + 5.077y^2 - 7.038x + 3.519y = 0 \ .$$

This elliptic arc is shown with the solid line in Figure 4.1. For comparison, the ellipse obtained with $b_{101} = 1$, whose Cartesian equation is $72x^2 - 12xy + 13y^2 - 72x + 36y = 0$, is shown by the dotted line in Figure 4.1.

### 4.1.2 Approximate Solutions

Another effort we make to overcome the non-linearity of the multivariate system is an iterative scheme, which starts with an initial value $\mathbf{b}_0$, repeatedly improves the setting by solving a linear system at each iteration. Instead of integrating along the curve, we take an approximation by integrating along some uniform piecewise linear subdivision of the curve. Methods to generate piecewise linear subdivisions of algebraic curves, or specifically A-splines, can be found in [WH94, BCX95].

We first discuss the case of an A-spline. Assume that at iteration $i$, the weights are given as $\mathbf{b}^{(i)}$. We compute a uniform piecewise linear subdivision of the curve $\{((\alpha_1)_j^{(i)}, (\alpha_2)_j^{(i)})\}$, $j = 0, \ldots, n$. The subdivision is fine enough so that we can assume that $h(\mathbf{b}, \alpha_1, \alpha_2) = (\nabla S^{\mathsf{T}} \mathbf{J}\mathbf{J}^{\mathsf{T}} \nabla S)^3$ in the denominator of the curvature formula 4.2 is a constant.

We substitute $\mathbf{H}(S(\mathbf{b}^{(i)}))$ by $\mathbf{H}(S(\mathbf{b}^{(i+1)}))$. In the $j^{\text{th}}$ piece, the denominator $h(\mathbf{b}, \alpha_1, \alpha_2)$ is substituted by $\overline{h}_j = (1/2)[h(\mathbf{b}^{(i)}, (\alpha_1)_j^{(i)}, (\alpha_2)_j^{(i)}) + h(\mathbf{b}^{(i)}, (\alpha_1)_{j+1}^{(i)}, (\alpha_2)_{j+1}^{(i)})]$. Hence the curvature is approximated by

$$\kappa_j^{(i)} = \frac{\left|[\nabla S(\mathbf{b}^{(i)})]^{\mathsf{T}} \mathbf{J}\mathbf{P}^{\mathsf{T}}\mathbf{J}^{\mathsf{T}} \mathbf{H}(S(\mathbf{b}^{(i+1)}))\mathbf{J}\mathbf{P}\mathbf{J}^{\mathsf{T}}[\nabla S(\mathbf{b}^{(i)})]\right|}{\overline{h}_j}$$

The energy minimizing system is

$$\sum_j \nabla_{\mathbf{b}^{(i+1)}} E^{(i)} = \mathbf{0}$$

where

$$\begin{aligned}
\nabla_{\mathbf{b}^{(i+1)}} E^{(i)} &= \int_C 2\kappa_j^{(i)} \nabla_{\mathbf{b}^{(i+1)}} \kappa_j^{(i)} da \\
&= \frac{1}{\overline{h}_j^2} \int_C \left[ 2{\mathbf{N}^{(i)}}^{\mathsf{T}} \mathbf{H}(S(\mathbf{b}^{(i+1)}))\mathbf{N}^{(i)}{\mathbf{N}^{(i)}}^{\mathsf{T}} \nabla_{\mathbf{b}^{(i+1)}} \mathbf{H}(S(\mathbf{b}^{(i+1)}))\mathbf{N}^{(i)} \right] da
\end{aligned}$$

and

$$\mathbf{N}^{(i)} = \mathbf{J}\mathbf{P}\mathbf{J}^{\mathsf{T}} \nabla S(\mathbf{b}^{(i)}) \ .$$

*** PROBLEM: $a$ is a function of $\mathbf{b}$. This differentiation is invalid. ***

Now the problem is how to symbolically integrate along an implicit curve $C : S(\alpha_1, \alpha_2) = 0$. We instead integrate along piecewise straight lines $[((\alpha_1)_j^{(i)}, (\alpha_2)_j^{(i)}), ((\alpha_1)_{j+1}^{(i)}, (\alpha_2)_{j+1}^{(i)})]$. Hence

$$\begin{aligned}
(\alpha_1)^{(i)}(u) &= (\alpha_1)_j^{(i)}(1-u) + (\alpha_1)_{j+1}^{(i)} u \\
(\alpha_2)^{(i)}(u) &= (\alpha_2)_j^{(i)}(1-u) + (\alpha_2)_{j+1}^{(i)} u
\end{aligned}$$

$$dC = \sqrt{(d\alpha_1)^2 + (d\alpha_2)^2} = \sqrt{[(\alpha_1)_{j+1}^{(i)} - (\alpha_1)_j^{(i)}]^2 + [(\alpha_2)_{j+1}^{(i)} - (\alpha_2)_j^{(i)}]^2}\, du = l_j\, du$$

The energy minimizing system is approximated as

$$\sum_j \frac{1}{\overline{h}_j^2} \int_0^1 2{\mathbf{N}^{(i)}}^{\mathsf{T}} \mathbf{H}(S(\mathbf{b}^{(i+1)}))\mathbf{N}^{(i)}{\mathbf{N}^{(i)}}^{\mathsf{T}} \nabla_{\mathbf{b}^{(i+1)}} \mathbf{H}(S(\mathbf{b}^{(i+1)}))\mathbf{N}^{(i)} l_j du = \mathbf{0} \ . \qquad (4.6)$$

Note that each integral function is a polynomial, hence the integration can be done symbolically off-line. Furthermore, the left hand side is linear in terms of $\mathbf{b}^{(i+1)}$. Hence $\mathbf{b}^{(i+1)}$ can be found by solving the linear system.

We iterate this procedure until $\|\mathbf{b}^{(i+1)} - \mathbf{b}^{(i)}\|$ is less than some $\epsilon$.

### 4.1.3 Case study: A Cubic A-spline

Here we present an example of an energy-minimizing cubic A-spline with $C^1$ continuity. We use the same boundary conditions as in the example in Section 4.1.1: the curve $F(x,y)=0$ passes through $\mathbf{p}_3=(0,0)$ and $\mathbf{p}_1=(1,0)$, with tangent lines of slopes 2 and $-3$, respectively.

The general cubic spline in BB form is

$$\begin{aligned} S(\alpha_1,\alpha_2) &= b_{300}\alpha_1^3+3b_{210}\alpha_1^2\alpha_2+3b_{201}\alpha_1^2(1-\alpha_1-\alpha_2)+3b_{120}\alpha_1\alpha_2^2 \\ &\quad +6b_{111}\alpha_1\alpha_2(1-\alpha_1-\alpha_2)+3b_{102}\alpha_1(1-\alpha_1-\alpha_2)^2+b_{030}\alpha_2^3 \\ &\quad +3b_{021}\alpha_2^2(1-\alpha_1-\alpha_2)+3b_{012}\alpha_2(1-\alpha_1-\alpha_3)^2+b_{003}(1-\alpha_1-\alpha_3)^2=0\,. \end{aligned}$$

However, we have $S(1,0)=0 \Rightarrow b_{300}=0$ and $S(0,0)=0 \Rightarrow b_{003}=0$. Furthermore, the tangent to $S(\alpha_1,\alpha_2)$ at $(1,0)$ is parallel to the line the line $\alpha_1+\alpha_2=1$. Therefore $d\alpha_2/d\alpha_1 = -(dS/d\alpha_1)/(dS/\alpha_2)=-1$, or $dS/d\alpha_1=dS/d\alpha_2$ at that point. This now implies that $b_{210}=0$. Also, the tangent to $S(\alpha_1,\alpha_2)$ at $(0,0)$ is parallel to the $\alpha_2$-axis. Therefore $dS/d\alpha_2=0$ there, and this implies that $b_{012}=0$. We now have

$$\begin{aligned} S(\alpha_1,\alpha_2) &= 3b_{201}\alpha_1^2(1-\alpha_1-\alpha_2)+3b_{120}\alpha_1\alpha_2^2+6b_{111}\alpha_1\alpha_2(1-\alpha_1-\alpha_2) \\ &\quad +3b_{102}\alpha_1(1-\alpha_1-\alpha_2)^2+3b_{021}\alpha_2^2(1-\alpha_1-\alpha_2)+b_{030}\alpha_2^3=0\ , \end{aligned} \tag{4.7}$$

in accordance with [BX94].

For the initial guess, we can take the values obtained from the example in Section 4.1.1 that correspond to the cubic coefficients here. These turn out to be

$$\begin{aligned} b_{201}=2b_{111}=b_{102}=0.2607 \\ 3b_{120}=b_{030}=3b_{021}=-1\ . \end{aligned}$$

(The value for $b_{111}$ is 1/3 of the value of $b_{101}$ in Section 4.1.1.) These values are consistent with the inequality constraints in [BX92], where it is shown that $b_{201}$ and $b_{102}$ must be positive, while $b_{030}$, $b_{120}$, and $b_{021}$ must be negative. (See Figure 2.2.)

The inequality constraints guarantee that a line drawn from the vertex $\mathbf{p}_2$ to the edge $\overline{\mathbf{p}_1\mathbf{p}_3}$ intersects the curve $S(\alpha_1,\alpha_2)=0$ exactly once. Thus we choose to divide $\overline{\mathbf{p}_1\mathbf{p}_3}$ into $n$ subintervals, and let the $((\alpha_1)_j^{(i)},(\alpha_2)_j^{(i)})$ be the intersections of $S(\alpha_1,\alpha_2)=0$ with lines drawn through $\mathbf{p}_2$ and the subdivision points of $\overline{\mathbf{p}_1\mathbf{p}_3}$. These lines have equations $n\alpha_1+j\alpha_2=j$, $j=0,\ldots,n$. Specifically, $((\alpha_1)_j^{(1)},(\alpha_2)_j^{(1)})=$

## 4.2 Local Minimization of Simplified Energy

We now consider minimizing the simplified strain energy formulation of Section 3.3. The energy $E$ of an A-spline curve is a quadratic function of the weights of $S_i(\alpha)$'s. Some of these weights are linearly related to each other in the smoothness conditions ($C^0$, $C^1$, etc. ...). Hence the minimization of $E$ is an optimization problem of a quadratic function with linear constraints. It can be solved by Lagrangian techniques. In particular, we introduce auxiliary variables $\lambda_j$'s and define

$$L(\mathbf{b},\boldsymbol{\lambda})=E(\mathbf{b})+\boldsymbol{\lambda}^{\mathsf{T}}\mathbf{h}(\mathbf{b})$$

where $\mathbf{h}(\mathbf{b})=\mathbf{0}$ are the linear constraints among the weights $\mathbf{b}$. The problem of minimizing $E$ with constraints is reduced to an unconstrained optimization problem of $L$.

It is not difficult to see that $\nabla_{\mathbf{b}}L$, the gradient of $L$ over the weights $\mathbf{b}$, consists of linear vectors in $\mathbf{b}$. Solving the linear system

$$\begin{aligned}\nabla_{\mathbf{b}}L &= \mathbf{0}\\ \nabla_{\boldsymbol{\lambda}}L &= \mathbf{0}\end{aligned}$$

we achieve the configuration that minimizes $E$.

If we include the connectivity conditions of the A-splines, which are in the form of linear inequalities, $b_k \le 0$, or $b_k \ge 0$, the optimization problem of $E$ becomes one with linear inequality constraints. In particular,

$$E = \Sigma E_i$$

with linear equality constraints

$$\mathbf{Cb} = \mathbf{0}$$

and linear inequality constraints

$$\mathbf{Sb} \le \mathbf{0}$$

where $\mathbf{S}$ is a diagonal matrix whose diagonal entry is either $-1$, 0, or 1.

For example, given a conic $F(x, y) = (1 - x - y)^2 + 2kxy, x, y \in (0, 1)$ with a free coefficient $k$, optimizing

$$E = \int\int (F_{xx}^2 + 2F_{xy}^2 + F_{yy}^2)\, dxdy = \int\int [8 + 8(k+1)^2]\, dxdy$$

yields $k = -1$, which sets $F(x, y) = 0$ to be the circle $(x-1)^2 + (y-1)^2 = 1$.

### 4.3 Global Minimization of Simplified Energy

Here we consider the minimization of the simplified energy by varying the domain endpoint vertices of a chain of A-spline curves.

In the examples below we will use a quadratic spline to interpolate points on a curve. Let $\mathbf{p}_1 = (x_1, y_1)$ and $\mathbf{p}_3 = (x_3, y_3)$ be two points on a curve $F(x, y) = 0$, and let $\mathbf{p}_2$ be the intersection of the tangent lines to the curve at $\mathbf{p}_1$ and $\mathbf{p}_3$. (We assume these tangent lines are not parallel.) Then take $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ to have the BB coordinates $(1, 0)$, $(0, 1)$, and $(0, 0)$ respectively, where we are omitting the third BB coordinate $\alpha_3 = 1 - \alpha_1 - \alpha_2$.

As in (4.4), the general $C^1$-continuous quadratic spline is

$$S(\alpha_1, \alpha_2) = 2b_{101}\alpha_1(1 - \alpha_1 - \alpha_2) + b_{020}\alpha_2^2 = 0 \ .$$

The simplified energy in (3.9) may be written explicitly, but the expressions involved are complicated and are presented in the appendix. For now we can say that

$$\begin{aligned}E &= \int_0^1 \int_0^{1-\alpha_1} [\beta_1(\mathbf{J}_1{}^\mathsf{T}\nabla S)^2 + \beta_2(\mathbf{J}_2{}^\mathsf{T}\nabla S)^2 \\ &\quad + \gamma_{11}(\mathbf{J}_1{}^\mathsf{T}\nabla^2 S\mathbf{J}_1)^2 + 2\gamma_{12}(\mathbf{J}_1{}^\mathsf{T}\nabla^2 S\mathbf{J}_2)^2 + \gamma_{22}(\mathbf{J}_2{}^\mathsf{T}\nabla^2 S\mathbf{J}_2)^2] \det(\mathbf{J}^{-1}) d\alpha_2 d\alpha_1 \\ &= k_1 b_{101}^2 + k_2 b_{101} b_{020} + k_3 b_{020}^2 \ ,\end{aligned}$$

where the $k_i$ are given by (6.1) in the appendix.

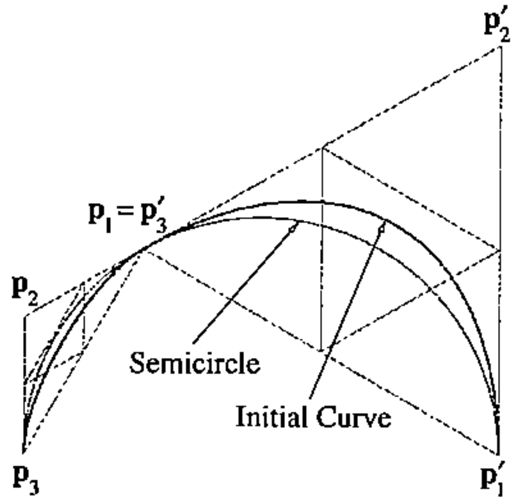



Figure 4.2: Initial configuration. The point $\mathbf{p}_1 = \mathbf{p}_3'$ is allowed to slide along the semicircle. The spline labeled "initial curve" is the minimal-energy spline obtained for $\beta =$?? if $\mathbf{p}_1 = \mathbf{p}_3'$ is at the location indicated.

The coefficients $k_1$ and $k_3$ turn out to be positive when $\Delta$ (the determinant in (2.5)), the $\beta_i$, and the $\gamma_{ij}$ are positive. Therefore the simplified energy function is convex, and obtains a global minimum at the critical point where $dE/db_{101} = 0$, which is at

$$b_{101} = -\frac{k_2}{2k_1} b_{020} \ . \tag{4.8}$$

This minimum energy is

$$\frac{4k_1k_3 - k_2^2}{4k_1} b_{020}^2 \ , \tag{4.9}$$

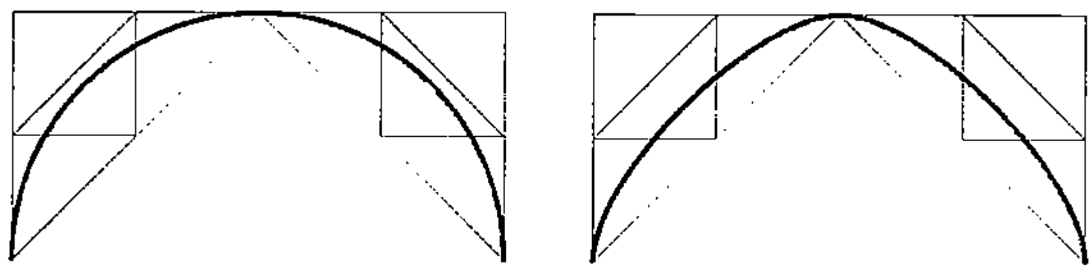
**Case Study 1:** As a simple example, suppose we have $(1,0)$ and $(-1,0)$ as two fixed points on the unit circle, and we wish to find the point $(x_0, y_0)$ on the unit upper semicircle such that the simplified energy is minimized when $\beta_1 = \beta_2 = \beta$ and $\gamma_{11} = \gamma_{12} = \gamma_{22} = \gamma$. This will require the sum of two components, the first of which has $\mathbf{p}_3 = (x_1, y_1) = (-1, 0)$ and $\mathbf{p}_1 = (x_0, y_0)$, and the second with $p_1' = (x_1', y_1') = (1, 0)$ and $p_3' = (x_0, y_0)$ (see Figure 4.2). The points $\mathbf{p}_2 = (x_2, y_2)$ and $\mathbf{p}_2' = (x_2', y_2')$ are the intersections of the tangent lines to the circle through $\mathbf{p}_1$ and $\mathbf{p}_3$ and through $\mathbf{p}_1'$ and $\mathbf{p}_3'$, respectively; these will be $(-1, (1 + x_0)/y_0)$ and $(1, (1 - x_0)/y_0)$ in the two cases.

For $\triangle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ the simplified energy is

$$\frac{[(1 + x_0)^2(3 - x_0)\beta^2 + 24(3x_0^2 - 5x_0 + 4)\beta\gamma + 144(1 - x_0)\gamma^2]\sqrt{1 - x_0^2}}{3(1 + x_0)[(2 - x_0)(1 + x_0)^2\beta + 12(x_0^2 + 1)\gamma]}$$

while the simplified energy for $\triangle \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'$ is

$$\frac{[(1 - x_0)^2(3 + x_0)\beta^2 + 24(3x_0^2 + 5x_0 + 4)\beta\gamma + 144(1 + x_0)\gamma^2]\sqrt{1 - x_0^2}}{3(1 - x_0)[(2 + x_0)(1 - x_0)^2\beta + 12(x_0^2 + 1)\gamma]} \ .$$



(a) (b)

Figure 4.3: Optimal configurations for (a): $\beta = 0$; (b): $\beta = 6\gamma$ .

The sum of these obtains its minimum value at $x_0 = 0$, at which point the energy is $(\beta^2 + 32\beta\gamma + 48\gamma^2)/(\beta + 6\gamma)$. At this point of minimum energy, $b_{101} = (\beta - 12\gamma)/[2(\beta + 6\gamma)]b_{020}$ for both pieces.

For $\Delta \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$, we have $\alpha_1 = 1 - y$ and $\alpha_2 = y - x - 1$ (from (2.3)), and the Cartesian equation is

$$(\beta + 6\gamma)x^2 - 3\beta xy + (\beta + 6\gamma)y^2 + 3\beta x - 3\beta y + 2\beta - 6\gamma = 0 \ .$$

This is an arc of a circle if $\beta = 0$, and an arc of an ellipse if $0 < \beta < 12\gamma$. The second part of the spline is the reflection of the first piece across the $y$-axis. The resulting splines for the cases where $\beta = 0$ and where $\beta = 6\gamma$ are shown in Figure 4.3.

If $\beta \geq 12\gamma$, then we find that $b_{101} \leq 0$ when $b_{020} = -1$, and this violates an inequality constraint in [BX92]. As $\beta$ increases to $12\gamma$, the A-spline for the first piece approaches the straight line segment between $(-1, 0)$ and $(0, 1)$. For $\beta \geq 12\gamma$ there is no unique unrestricted simplified energy-minimizing curve; instead one can take the elliptic arc with a greatest allowed curvature at the endpoints. The curvature at $(-1, 0)$ and $(0, 1)$ is $2(\beta + 6\gamma)/(12\gamma - \beta)$, so if there is a maximum allowable curvature of $K$, we can choose the A-spline corresponding to $\beta = 12[(K - 1)/(K + 2)]\gamma$ whenever $\beta > 12[(K - 1)/(K + 2)]\gamma$.

**Case Study 2:** For another example, we consider the problem of minimizing the total simplified energy of a closed contour with one point on each of the sides of the triangle $\Delta \mathbf{q}_1\mathbf{q}_2\mathbf{q}_3$, where $\mathbf{q}_1 = (0, 0)$, $\mathbf{q}_2 = (7, 0)$, and $\mathbf{q}_3 = (6, 5)$. Let the three points on the sides of $\Delta \mathbf{q}_1\mathbf{q}_2\mathbf{q}_3$ be denoted by $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, as in Figure 4.4. We will also impose the condition that each of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ is the same fraction along the way of their respective edges. That is, $\overline{\mathbf{q}_1\mathbf{r}_1}/\overline{\mathbf{q}_1\mathbf{q}_2} = \overline{\mathbf{q}_2\mathbf{r}_2}/\overline{\mathbf{q}_2\mathbf{q}_3} = \overline{\mathbf{q}_3\mathbf{r}_3}/\overline{\mathbf{q}_3\mathbf{q}_1} = z$ for some $z$ in $[0, 1]$. With this condition the points on the edges of $\Delta \mathbf{q}_1\mathbf{q}_2\mathbf{q}_3$ are found to be $\mathbf{r}_1 = (7z, 0)$, $\mathbf{r}_2 = (7 - z, 5z)$, and $\mathbf{r}_3 = (5 - 5z, 6 - 6z)$. We will again use a $C^1$ quadratic spline, with $\beta_1 = \beta_2 = \beta$, $\gamma_{11} = \gamma_{12} = \gamma_{22} = \gamma$, and find the value of $z$ which minimizes the total simplified energy.

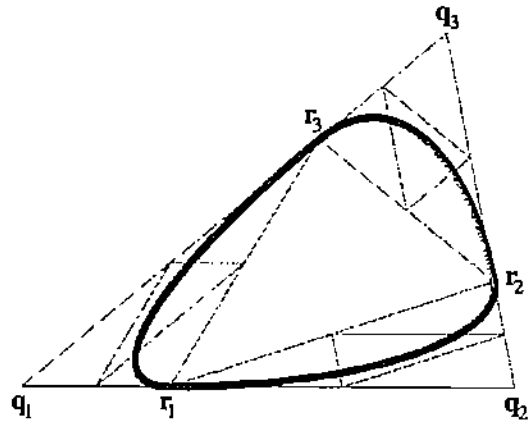



Figure 4.4: Initial configuration. The points $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ slide along the edges of $\Delta\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3$ in such a way that $\overline{\mathbf{q}_1\mathbf{r}_1}/\overline{\mathbf{q}_1\mathbf{q}_2} = \overline{\mathbf{q}_2\mathbf{r}_2}/\overline{\mathbf{q}_2\mathbf{q}_3} = \overline{\mathbf{q}_3\mathbf{r}_3}/\overline{\mathbf{q}_3\mathbf{q}_1} = z$ for some $z$ in $[0,1]$.

Consider first the triangle $\Delta\mathbf{r}_2\mathbf{q}_2\mathbf{r}_1$. Here the points $\mathbf{r}_2$, $\mathbf{q}_2$, and $\mathbf{r}_1$ take on the roles of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, so that the BB coordinates of these three points are $(1,0)$, $(0,0)$, and $(0,1)$, respectively. Thus in (4.9) we have $(x_1,y_1) = (7-z, 5z)$, $(x_2,y_2) = (7,0)$, and $(x_3,y_3) = (7z,0)$, and we obtain a minimum energy of

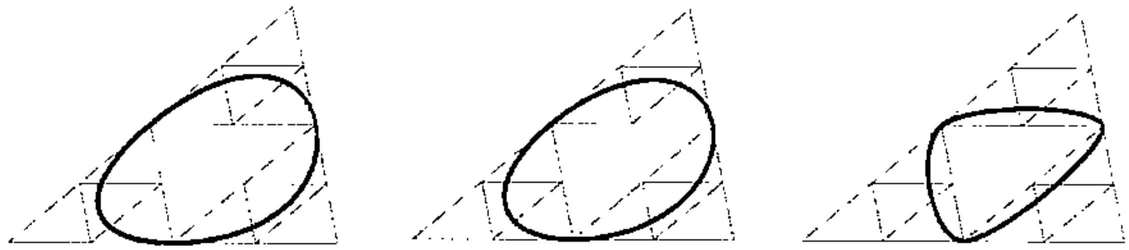
$$\begin{aligned} E_1 \;=\; & [1225z^2(1-z)^2(89z^2-112z+49)(239z^2-308z+147)\beta^2 \\ & + 24(89z^2-112z+49)(13640z^4-31535z^3+27489z^2-11319z+2401)\beta\gamma \\ & + 288(2979z^4-9408z^3+14308z^2-9604z+2401)\gamma^2]\Big/ \\ & \qquad \{20580z^3(1-z)^3[25(82z^2-105z+49)\beta+324\gamma]\}\ . \end{aligned}$$

Next consider $\Delta\mathbf{r}_3\mathbf{q}_3\mathbf{r}_2$, with $\mathbf{r}_3$, $\mathbf{q}_3$, and $\mathbf{r}_2$ take on the roles of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, respectively. Thus in (4.9) we have $(x_1,y_1) = (5-5z, 6-6z)$, $(x_2,y_2) = (6,5)$, and $(x_3,y_3) = (7-z,5z)$, and we obtain a minimum energy of

$$\begin{aligned} E_2 \;=\; & [1225z^2(1-z)^2(125z^2-90z+26)(299z^2-194z+78)\beta^2 \\ & + 24(125z^2-90z+26)(24344z^4-39273z^3+23148z^2-5174z+676)\beta\gamma \\ & + 288(3675z^4-1260z^3+3334z^2-2704z+676)\gamma^2]\Big/ \\ & \qquad \{420z^3(1-z)^3[1225(106z^2-71z+26)\beta+23364\gamma]\}\ . \end{aligned}$$

Now consider $\Delta\mathbf{r}_1\mathbf{q}_1\mathbf{r}_3$, with $\mathbf{r}_1$, $\mathbf{q}_1$, and $\mathbf{r}_3$ take on the roles of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, respectively. Thus in (4.9) we have $(x_1,y_1) = (7z,0)$, $(x_2,y_2) = (0,0)$, and $(x_3,y_3) = (5-5z, 6-6z)$, and we obtain a minimum energy of

$$E_2 \;=\; [1225z^2(1-z)^2(194z^2-206z+61)(138z^2-150z+61)\beta^2$$



(a) (b) (c)

Figure 4.5: Optimal configurations for (a): $\beta = 0$ ($b_{101} = 1$); (b): $\beta = \gamma$ ($b_{101} = 0.7857$); (c): $\beta = 10\gamma$ ($b_{101} = 0.0625$).

$$
\begin{aligned}
&+ 8(194z^2 - 206z + 61)(57740z^4 - 120640z^3 + 89274z^2 - 27694z + 3721)\beta\gamma \\
&+ 96(2594z^4 - 7828z^3 + 18798z^2 - 14884z + 3721)\gamma^2]\Big/ \\
&\quad \{6860z^3(1-z)^3[25(152z^2 - 164z + 61)\beta + 1164\gamma]\} \ .
\end{aligned}
$$

The total minimum energy is found by adding the energy of the three pieces, and setting the derivative with respect to $z$ of the sum, which is an 18th degree polynomial in $z$, equal to zero. Different answers are obtained for different values of $\beta/\gamma$:

| | $\beta = 0$ | $\beta = \gamma/10$ | $\beta = \gamma/2$ | $\beta = \gamma$ | $\beta = 2\gamma$ | $\beta = 10\gamma$ | $\gamma = 0$ |
|---|---|---|---|---|---|---|---|
| $z$ | 0.5003 | 0.5009 | 0.5018 | 0.5019 | 0.5018 | 0.5011 | 0.5006 |

Figure 4.5 shows the resulting closed contour for three of these cases, in which $\beta = 0$, $\beta = \gamma$, and $\beta = 10\gamma$.

An example of how the entire curve deforms when just one control point is moved in shown in figure 4.6.

# 5 Conclusion

Several elastic models using A-splines have been proposed, each of which has its own advantages and shortcomings. Besides the traditional energy model adapted from theory of elasticity, we give several different simplified models that take advantage of the A-spline formulation and also yield efficient computation of the minimum energy solution. A subsequent paper [] reports the use of these energy splines in image processing applications.

# 6 Appendix

Here we present the energy expressions used in Section (4.3). The simplified energy in (3.9) is

$$
E \;=\; \int_0^1 \int_0^{1-\alpha_1} [\beta_1(\mathbf{J}_1{}^{\mathsf{T}}\nabla S)^2 + \beta_2(\mathbf{J}_2{}^{\mathsf{T}}\nabla S)^2
$$

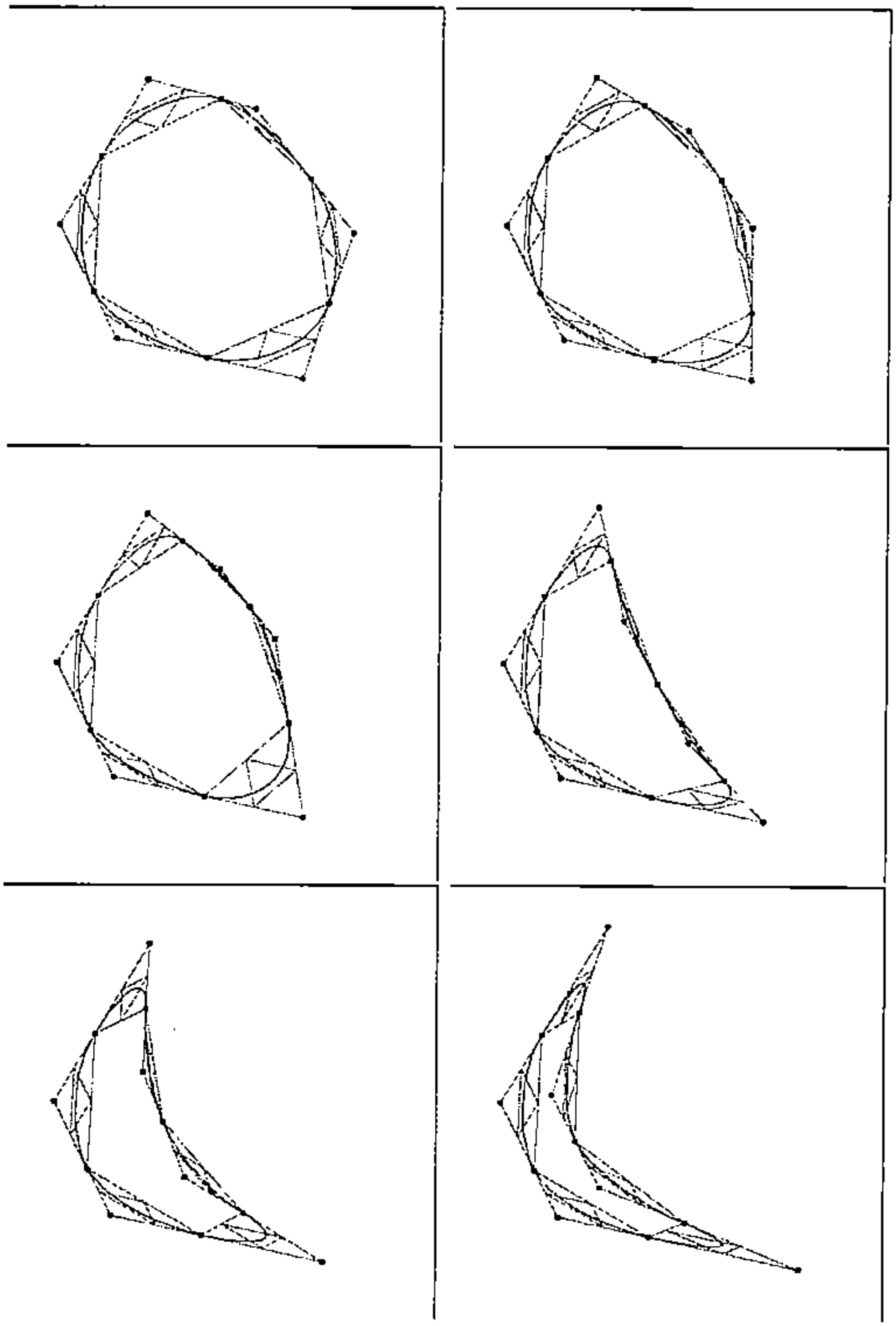


Figure 4.6: Moving one control point forces all the others to move as well

$$+\gamma_{11}(\mathbf{J_1}^\mathsf{T}\nabla^2 S\mathbf{J}_1)^2 + 2\gamma_{12}(\mathbf{J_1}^\mathsf{T}\nabla^2 S\mathbf{J}_2)^2 + \gamma_{22}(\mathbf{J_2}^\mathsf{T}\nabla^2 S\mathbf{J}_2)^2]\det(\mathbf{J}^{-1})d\alpha_2 d\alpha_1$$
$$= k_1 b_{101}^2 + k_2 b_{101} b_{020} + k_3 b_{020}^2 \ ,$$

where

$$\begin{aligned}
k_1 &= \frac{1}{3\Delta}\left[(y_1^2 - y_1y_2 - y_1y_3 + y_2^2 - y_2y_3 + y_3^2)\beta_1 + (x_1^2 - x_1x_2 - x_1x_3 + x_2^2 - x_2x_3 + x_3^2)\beta_2\right] \\
&\quad + \frac{4}{\Delta^3}\left[2(y_1-y_2)^2(y_2-y_3)^2\gamma_{11}\right. \\
&\qquad + (x_1y_2 - x_1y_3 + x_2y_1 + x_2y_3 - x_3y_1 + x_3y_2 - 2x_2y_2)^2\gamma_{12} \\
&\qquad \left. + 2(x_1-x_2)^2(x_2-x_3)^2\gamma_{22}\right] \ , \\
k_2 &= -\frac{1}{3\Delta}\left[(y_1-y_3)^2\beta_1 + (x_1-x_3)^2\beta_2\right] \\
&\quad + \frac{8}{\Delta^3}\left[(y_1-y_3)^2(y_1-y_2)(y_2-y_3)\gamma_{11}\right. \\
&\qquad + (x_1-x_3)(y_1-y_3)(x_1y_2 - x_1y_3 + x_2y_1 + x_2y_3 - x_3y_1 + x_3y_2 - 2x_2y_2)\gamma_{12} \\
&\qquad \left. + (x_1-x_3)^2(x_1-x_2)(x_2-x_3)\gamma_{22}\right] \ , \\
k_3 &= \frac{1}{3\Delta}\left[(y_1-y_3)^2\beta_1 + (x_1-x_3)^2\beta_2\right] \\
&\quad + \frac{2}{\Delta^3}\left[(y_1-y_3)^4\gamma_{11} + 2(x_1-x_3)^2(y_1-y_3)^2\gamma_{12} + (x_1-x_3)^4\gamma_{22}\right] \ ,
\end{aligned} \tag{6.1}$$

and

$$\Delta = x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2$$

as in (2.5).

As stated above in Section 4.3, the simplified energy attains its global minimum at

$$b_{101} = -\frac{k_2}{2k_1} b_{020} \ .$$

This minimum energy is

$$\frac{4k_1k_3 - k_2^2}{4k_1} b_{020}^2 \ ,$$

which equals

$$[k_{11}\beta_{11}^2 + k_{12}\beta_{11}\beta_{22} + k_{13}\beta_{11}\gamma_{11} + k_{14}\beta_{11}\gamma_{12} + k_{15}\beta_{11}\gamma_{22} + k_{22}\beta_{22}^2 + k_{23}\beta_{22}\gamma_{11} + k_{24}\beta_{22}\gamma_{12} + k_{25}\beta_{22}\gamma_{22} + k_{34}\gamma_{11}\gamma_{12} + k_{35}\gamma_{11}\gamma_{22} + k_{45}\gamma_{12}\gamma_{22}]b_{020}^2/(36\Delta^4 k_1) \ ,$$

where

$$\begin{aligned}
k_{11} &= (y_3-y_1)^2[3(y_3-y_1)^2 - 4(y_2-y_1)(y_3-y_2)]\Delta^2 \\
k_{12} &= 2[3(x_3-x_1)^2(y_3-y_1)^2 - 2(x_3-x_1)^2(y_3-y_1)(y_2-y_1) + 2(x_3-x_1)^2(y_2-y_1)^2 \\
&\quad - 2(x_3-x_1)(x_2-x_1)(y_3-y_1)^2 + 2(x_2-x_1)^2(y_3-y_1)^2]\Delta^2 \\
k_{13} &= 24(y_3-y_1)^2[4(y_2-y_1)^2(y_2-y_3)^2 - (y_3-y_1)^2(y_2-y_3)^2 + (y_3-y_1)^3(y_2-y_1)] \\
k_{14} &= 48(y_3-y_1)^2[(x_3-x_1)^2(y_3-y_1)^2 + 2(x_3-x_1)^2(y_2-y_1)^2
\end{aligned}$$

$$
\begin{aligned}
& + (x_3 - x_1)(x_2 - x_1)(y_3 - y_1)^2 - 4(x_3 - x_1)(x_2 - x_1)(y_2 - y_1)^2 \\
& + (x_2 - x_1)^2(y_3 - y_1)^2 - 4(x_2 - x_1)^2(y_3 - y_1)(y_2 - y_1) + 4(x_2 - x_1)^2(y_2 - y_1)^2] \\
k_{15} &= 24[(x_3 - x_1)^4(y_3 - y_1)^2 - (x_3 - x_1)^4(y_3 - y_1)(y_2 - y_1) + (x_3 - x_1)^4(y_2 - y_1)^2 \\
& + 2(x_3 - x_1)^3(x_2 - x_1)(y_3 - y_1)^2 + 2(x_3 - x_1)^2(x_2 - x_1)^2(y_3 - y_1)^2 \\
& - 8(x_3 - x_1)(x_2 - x_1)^3(y_3 - y_1)^2 + 4(x_2 - x_1)^4(y_3 - y_1)^2] \\
k_{22} &= (x_3 - x_1)^2[3(x_3 - x_1)^2 - 4(x_2 - x_1)(x_3 - x_2)]\Delta^2 \\
k_{23} &= 24[(x_3 - x_1)^2(y_3 - y_1)^4 + 2(x_3 - x_1)^2(y_3 - y_1)^3(y_2 - y_1) \\
& + 2(x_3 - x_1)^2(y_3 - y_1)^2(y_2 - y_1)^2 - 8(x_3 - x_1)^2(y_3 - y_1)(y_2 - y_1)^3 \\
& + 4(x_3 - x_1)^2(y_2 - y_1)^4 - (x_3 - x_1)(x_2 - x_1)(y_3 - y_1)^4 + (x_2 - x_1)^2(y_3 - y_1)^4] \\
k_{24} &= 48(x_3 - x_1)^2[(x_3 - x_1)^2(y_3 - y_1)^2 + (x_3 - x_1)^2(y_3 - y_1)(y_2 - y_1) \\
& + (x_3 - x_1)^2(y_2 - y_1)^2 - 4(x_3 - x_1)(x_2 - x_1)(y_2 - y_1)^2 + 2(x_2 - x_1)^2(y_3 - y_1)^2 \\
& - 4(x_2 - x_1)^2(y_3 - y_1)(y_2 - y_1) + 4(x_2 - x_1)^2(y_2 - y_1)^2] \\
k_{25} &= 24(x_3 - x_1)^2[4(x_2 - x_1)^2(x_2 - x_3)^2 - (x_3 - x_1)^2(x_2 - x_3)^2 + (x_3 - x_1)^3(x_2 - x_1)] \\
k_{34} &= 288(y_3 - y_1)^2(y_1 - 2y_2 + y_3)^2 \\
k_{35} &= 576[(x_3 - x_1)(y_2 - y_1) - (x_3 - x_2)(y_3 - y_1)]^2 \\
k_{45} &= 288(x_3 - x_1)^2(x_1 - 2x_2 + x_3)^2 \ .
\end{aligned}
$$

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