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On the Solution of the Convection-Diffusion Equation by Iteration*

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Abstract

The discretization of the two-dimensional convection-diffusion equation usually leads to a linear system whose matrix coefficient is block two-cyclic consistently ordered. For the solution of the resulting linear system, several efficient stationary iterative methods were proposed, among others, by Chin and Manteuffel (1988), Elman and Golub (1990), de Pillis (1991) and Eiermann, Niethammer and Varga (1992). In the present work, we propose, as an alternative, the stationary Modified Successive Overrelaxation (MSOR) method or an "equivalent" 2-step method applied to the cyclically reduced linear system. It is shown both theoretically and experimentally that the application of a "continuous" version of Manteuffel's algorithm to derive the optimal parameters produces an iterative method that is asymptotically faster than the previous methods.

Keywords: convection-diffusion equation, linear systems, iterative methods, modified successive overrelaxation method, 2-step iterative methods, Manteuffel's algorithm
Subject Classification: AMS(MOS): 65F10, 65F50; CR: 5.14
Running Title: Convection-Diffusion Equation

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1 Introduction and Preliminaries

For the numerical solution of the singularly perturbed convection-diffusion equation

$$-\varepsilon\Delta u + \alpha(x,y)u_x + \beta(x,y)u_y + \gamma(x,y)u = f(x,y) \tag{1.1}$$

on a bounded convex domain with Dirichlet boundary conditions, the discretization leads to the solution of a real linear system of equations of order n of the general form

$$Ax = b. \tag{1.2}$$

Under certain assumptions, the coefficient matrix A, in (1.2), is nonsingular and block tridiagonal, with an invertible block diagonal matrix D, and its associated block Jacobi iteration matrix

$$B := I - D^{-1}A \tag{1.3}$$

has a spectrum $\sigma(B)$ contained in a "bow tie" region $R \subseteq R_j$, $j = 1, 2$, (see [2]), where

$$R_1 := \left\{ z \in \mathbb{C} : \; |z \pm c| \le c < \frac{1}{2} \right\} \tag{1.4}$$

(see Fig. 1) or

$$R_2 := \{ z \in \mathbb{C} : \; |z \pm ic| \le c \; \text{ and } \; |\mathrm{Re}\, z| < 1 \} \tag{1.5}$$

(see Fig. 2). Note that since A is block two-cyclic consistently ordered, B in (1.3) is weakly cyclic of index 2 (see, e.g., [15], [16] or [1]) and, therefore, can be transformed by a certain permutation transformation into its normal form $\tilde{B}$ (see, e.g., [15]). More specifically,

$$(PBP^T =) \; \tilde{B} := \begin{bmatrix} O_1 & B_1 \\ B_2 & O_2 \end{bmatrix}, \tag{1.6}$$

where P is the permutation transformation matrix and O_1, O_2 are square nullmatrices of orders n_1 and n_2, respectively ($n_1 + n_2 = n$). Under the same permutation transformation, it is

$$PAP^T =: \tilde{A}, \quad PDP^T =: \widetilde{D}. \tag{1.7}$$

By setting $\tilde{x} := Px$ and $\tilde{b} := Pb$, (1.2) and (1.3) become $\tilde{A}\tilde{x} = \tilde{b}$ and $\tilde{B} := I - \widetilde{D}^{-1}\tilde{A}$, that is of exactly the same forms as before. To simplify the notation, we drop all the tildes from the tilded symbols above and so we can refer again to (1.2) and (1.3), with B having now the form of the right hand side of (1.6). In [2] the numerical solution to (1.2)-(1.3) is found by the application of the optimal block Successive Overrelaxation (SOR) method. The optimal relaxation factor $\hat{\omega}$, in both cases of (1.4) and (1.5), is determined there by a very ingenious but very complicated analysis. It is noted that the same optimal parameters
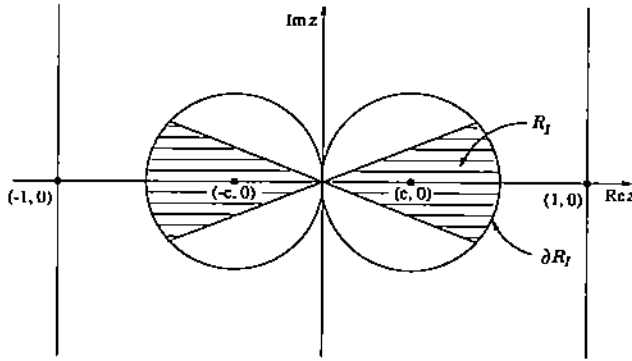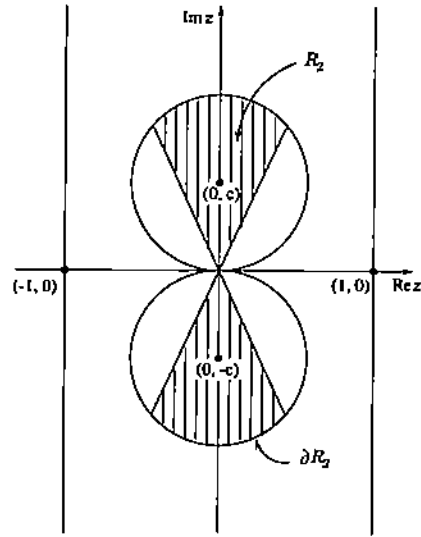
Fig. 1

Fig. 2

$\hat{\omega}$ were recovered in [6] by the application of a "continuous" version of the Young-Eidson's algorithm [17] (see also [16]).

The solution of the discretized problem (1.2) was also considered by Elman and Golub [5] who applied (convergent) block Jacobi iterations to the cyclically reduced linear system resulting from (1.2), (1.3) and (1.6) by exploiting the expressions of the nonzero entries of the Jacobi matrix which they derived explicitly. However, iterative methods analogous to the one in [2], applied either to (1.2) or to its cyclically reduced system, have been considered, to the best of our knowledge, only by de Pillis [3] and by Eiermann, Niethammer and Varga [4]. We note that in the latter two works only the spectrum case (1.4) was examined.

It is the main objective of the present work to solve the linear system (1.2) by the Modified (M)SOR iterative method (see, e.g., [16]) or, equivalently, in the Chebyshev sense (see [8]), to solve its cyclically reduced linear system by a 2-step iterative method. As will be proven, our method is asymptotically faster than any of the methods mentioned so far and can also cover a wider class of matrices A.

For the solution of a linear system similar to (1.2), (1.3) and (1.6), derived from the cubic Hermite collocation discretization method of a class of elliptic PDEs, the MSOR iterative method was successfully applied in [7]. As was shown there to determine the two parameters associated with the optimal 2-cyclic MSOR method is equivalent to determining the two parameters in the optimal Manteuffel's algorithm (see [10], [11], [12], [13]) or of a "continuous" version of it. For this it can be shown that the MSOR method applied to (1.2) is equivalent to applying a 2-step method to the equivalent to (1.2) linear system

$$(I - B^2)x = (I + B)D^{-1}b. \tag{1.8}$$

Moreover, it can be shown that the aforementioned 2-step method is equivalent to another

2-step one associated with the solution of the cyclically reduced linear system

$$(I_2 - B_2 B_1)x_2 = \bar{b}_2 + B_2 \bar{b}_1 \tag{1.9}$$

where I_1 and I_2 are unit matrices of order n_1 and n_2, respectively, and $x = [x_1^T \; x_2^T]^T$, $\bar{b} = [\bar{b}_1^T \; \bar{b}_2^T] = [(D^{-1}b)_1^T \; (D^{-1}b)_2^T]^T$, $x_1, b_1 \in \mathbb{R}^{n_1}$, $x_2, b_2 \in \mathbb{R}^{n_2}$.

The material in this work is organized as follows. In Section 2 the MSOR method and its equivalent 2-step method together with its cyclically reduced one are presented. In Section 3, Manteuffel's algorithm is briefly mentioned and its "continuous" counterpart is introduced. In Section 4 some basic elements of the boundary curves of the regions R_1^2 and R_2^2 in (1.4) and (1.5) are given. These boundary curves, denoted by $\mathcal{C}_1 \equiv \partial R_1^2$ and $\mathcal{C}_2 \equiv \partial R_2^2$, are *cardioids*. In Section 5 the optimal "continuous" Manteuffel's algorithms, first for spectra $\sigma(B) \subset R_1$ and then for spectra $\sigma(B) \subset R_2$, are developed. Finally, in Section 6, a discussion is made and numerical examples, covering various cases, in two illustrative Tables are presented that show the superiority of our method over the previous ones.

2   MSOR and Related 2-Step Methods

The MSOR method for the solution of the linear system $(I - B)x = D^{-1}b$, equivalent to (1.2)-(1.3), is defined by (see, e.g., [16])

$$
\begin{aligned}
x^{(m+1)} &= \mathcal{L}_\Omega x^{(m)} + (I - \Omega L)^{-1}\Omega D^{-1}b, \quad m = 0, 1, 2, \ldots, \\
\mathcal{L}_\Omega &:= (I - \Omega L)^{-1}(I - \Omega + \Omega U), \\
\Omega &:= \mathrm{diag}(\omega_1 I_1, \omega_2 I_2), \quad L + U =: B,
\end{aligned} \tag{2.1}
$$

where L and U are strictly lower and strictly upper triangular matrices, respectively, $x^{(0)} \in \mathbb{R}^n$ arbitrary and $\omega_1, \omega_2 \in \mathbb{R} \backslash \{0\}$ are the two relaxation factors. In [8] it was proved that in the Chebyshev sense, the method (2.1) is equivalent to the 2-step method given below

$$x^{(m+1)} = (\omega_1' I + \omega_2' B^2)x^{(m)} + (1 - \omega_1' - \omega_2')x^{(m-1)} + \omega_2'(I + B)D^{-1}b \tag{2.2}$$

where $x^{(-1)}, x^{(0)} \in \mathbb{R}^n$ are arbitrary and

$$\omega_1' = 2 - \omega_1 - \omega_2, \quad \omega_2' = \omega_1 \omega_2. \tag{2.3}$$

Since $\omega_2' \neq 0$, the iterative method (2.2) is completely consistent with (1.8).

In view of (1.6), we partition $x^{(m)}$, $m = -1, 0, 1, 2, \ldots$, in accordance with the 2-cyclic partitioning of B and split iterative scheme (2.2) into the two uncoupled 2-step methods

$$x_1^{(m+1)} = (\omega_1' I_1 + \omega_2' B_1 B_2)x_1^{(m)} + (1 - \omega_1' - \omega_2')x_1^{(m-1)} + \omega_2'(\bar{b}_1 + B_1 \bar{b}_2) \tag{2.4}$$

and

$$x_2^{(m+1)} = (\omega_1' I_2 + \omega_2' B_2 B_1)x_2^{(m)} + (1 - \omega_1' - \omega_2')x_2^{(m-1)} + \omega_2'(\bar{b}_2 + B_2 \bar{b}_1). \tag{2.5}$$

The pair of methods in (2.4) and (2.5) are completely consistent with the two linear systems to which (1.8) is equivalent. It is clear that only one of the two methods (2.4) or (2.5) needs to be applied to find a good approximation to one of the vector components of x, say x_2. Then, (2.5) is usually rearranged in the following way

$$x_2^{(m+1)} = \omega_1' x_2^{(m)} + \omega_2'(B_2(B_1 x_2^{(m)} + \bar{b}_1)) + (1 - \omega_1' - \omega_2')x_2^{(m-1)} + \omega_2'\bar{b}_2 \qquad (2.6)$$

to indicate that there are only two matrix-vector multiplications per iteration step involving the matrices B_1 and B_2. The other vector component x_1, of x, will then be found from $x_1 = \bar{b}_1 + B_1 x_2$.

To apply the best 2-step method (2.5) (or (2.6)) one has to find the best ellipse in the spirit of Manteuffel (see, e.g., [11]) that captures the spectrum $\sigma(I_2 - B_2 B_1)$. However, since $\sigma(B_2 B_1)\backslash\{0\} = \sigma(B^2)\backslash\{0\}$ it suffices to find the best ellipse that captures $\sigma(I - B^2)$. The ellipse in question is found by means of Manteuffel's algorithm ([10], [11]) in the way described in [7]. Specifically, let $\hat{a}$, $\hat{b}$ and $\hat{d}$ denote the lengths of the "real" semiaxis, the "imaginary" semiaxis, and the distance of the center from the origin $Z(0,0)$ of the best capturing ellipse. According to [7], the optimal parameters $\hat{\omega}_1'$ and $\hat{\omega}_2'$ of (2.6) will be given by

$$\hat{\omega}_1' = \frac{2(\hat{d} - 1)}{\hat{d} + (\hat{d}^2 - \hat{a}^2 + \hat{b}^2)^{1/2}}, \quad \hat{\omega}_2' = \frac{2}{\hat{d} + (\hat{d}^2 - \hat{a}^2 + \hat{b}^2)^{1/2}}, \qquad (2.7)$$

while the optimal asymptotic convergence factor by

$$\hat{\rho} = \frac{\hat{a} + \hat{b}}{\hat{d} + (\hat{d}^2 - \hat{a}^2 + \hat{b}^2)^{1/2}}. \qquad (2.8)$$

If needed, the corresponding relaxation factors for the optimal MSOR method will then be found by using (2.7) in (2.3).

3   The "Continuous" Manteuffel Algorithm

To begin with our analysis let H^+ denote the positive hull, that is the upper half of the smallest convex polygon symmetric with respect to the real axis that contains the spectrum $\sigma(I - B^2)$ in the closure of its interior. As is known (see [10], [11]) Manteuffel's algorithm distinguishes three basic cases. One of them is trivial and corresponds to the 1-point case that is when H^+ has only one vertex. In the second one, the 2-point case, when H^+ has two vertices, the elements of the unique best ellipse are found as functions of the unique real zero, lying in a specified interval, of a certain cubic or quintic polynomial. In the third case, the many-point case, when H^+ has more than two vertices, the elements of the best ellipse are those of the unique 2-point ellipse that captures H^+, if such an ellipse exists, or the

unique ellipse among all the ellipses passing through three of the vertices of H^+, capturing H^+ and corresponding to the smallest convergence factor ρ in (2.8).

To derive the "continuous" version of Manteuffel's algorithm one has to examine some limiting cases. First, the limiting 2-point case will be that where one of the two points (vertices of H^+) moves along a continuous smooth curve and tends to the other. Secondly, the limiting 3-point case will be that of an ellipse passing through three points (vertices) when one of the points moves along a curve as before and tends to one of the others. It can be very easily checked and found out analytically that the limiting 2-point best ellipse turns out to be the trivial case of the 1-point (double point, in this case) best ellipse. Also, the limiting ellipse of one that passes through three points turns out to be an ellipse that passes through two points (one is a double point) and shares with the aforementioned curve the tangent at the double point. So, one can use all the formulas in the theory developed and the algorithm given by Manteuffel. For example, in the 2-point case these formulas depend on the coordinates of the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $(x_1 < x_2)$ and specifically on the quantities

$$A = \frac{x_2 - x_1}{2}, \quad B = \frac{x_2 + x_1}{2}, \quad S = \frac{y_2 - y_1}{2}, \quad T = \frac{y_2 + y_1}{2}. \tag{3.1}$$

Under the assumption $P_1 \to P_2$ (or vice versa) it will be

$$\lim_{P_1 \to P_2} A = 0, \quad \lim_{P_1 \to P_2} B = x_1, \quad \lim_{P_1 \to P_2} S = 0, \quad \lim_{P_1 \to P_2} T = y_1, \tag{3.2}$$

and also,

$$\lim_{P_1 \to P_2} \frac{S}{A} = f'(x_1). \tag{3.3}$$

In (3.2) and (3.3), the double point is relabeled as P_1, if necessary, and $y = f(x) \in C^1$ is the equation of the curve along which one of the two points moves and tends to the other. In the 3-point case, matters are a little more complicated. Here we present very briefly one of the two cases of the limiting 3-point ellipse that passes through the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ under the assumption that i) such an ellipse exists, ii) $x_1 < x_2 < x_3$, and iii) $P_2 \to P_3$ (or vice versa). As is known the elements d, a, b of such an ellipse are given by the corresponding expressions in (4.12) of [11]. To derive the formulas for d, a, b in the limiting case, first we rewrite the three different expressions that are present in the numerators and denominators of the formulas (4.12) of [11] as follows

$$y_1^2(x_2^2 - x_3^2) + y_2^2(x_3^2 - x_1^2) + y_3^2(x_1^2 - x_2^2) = (y_2^2 - y_1^2)(x_3^2 - x_2^2) + (y_2^2 - y_3^2)(x_2^2 - x_1^2), \tag{3.4}$$

$$y_1^2(x_2 - x_3) + y_2^2(x_3 - x_1) + y_3^2(x_1 - x_2) = (y_2^2 - y_1^2)(x_3 - x_2) + (y_2^2 - y_3^2)(x_2 - x_1), \tag{3.5}$$

$$y_1^2 x_2 x_3 (x_2 - x_3) + y_2^2 x_1 x_3 (x_3 - x_1) + y_3^2 x_1 x_2 (x_1 - x_2)$$
$$= (y_2^2 x_1 - y_1^2 x_2) x_3 (x_3 - x_2) + [y_2^2(x_3 - x_2) + x_2(y_2^2 - y_3^2)] x_1 (x_2 - x_1). \tag{3.6}$$

Then, we substitute the expressions in (3.4)-(3.6) into (4.12) of [11], divide each numerator and denominator there by $x_3 - x_2$, simplify, if possible, take limits as $P_2 \to P_3$ and relabel, if necessary, to finally obtain

$$d = \frac{(y_2^2 - y_1^2)x_2 - (x_2^2 - x_1^2)y_2 f'(x_2)}{(y_2^2 - y_1^2) - 2(x_2 - x_1)y_2 f'(x_2)} ,$$

$$a^2 = d^2 - \frac{(2y_2^2 x_1 x_2 - y_1^2 x_2^2 - y_2^2 x_1^2) - 2(x_2 - x_1)x_1 x_2 y_2 f'(x_2)}{(y_2^2 - y_1^2) - 2(x_2 - x_1)y_2 f'(x_2)} , \qquad (3.7)$$

$$b^2 = a^2 \frac{(y_2^2 - y_1^2) - 2(x_2 - x_1)y_2 f'(x_2)}{(x_2 - x_1)^2} .$$

4   The Cardioids

The two cardioids C_1 and C_2 will be examined separately in Cases I and II below.

Case I: In polar coordinates the equation of the boundary ∂R_1 of the region R_1 in (1.4) (see Fig. 1) is

$$r = 2c \, |\cos\theta|, \;\; 0 < c < \frac{1}{2} , \;\; 0 \le \theta < 2\pi. \qquad (4.1)$$

Therefore, the equation of the boundary ∂R_1^2 of the region R_1^2 that will contain the spectrum $\sigma(B^2)$ will be

$$r' = 2c^2(1 + \cos\theta'), \;\; 0 < c < \frac{1}{2} , \;\; 0 \le \theta' < 2\pi. \qquad (4.2)$$

In cartesian coordinates, the equation of ∂R_1^2 will be

$$(x^2 + y^2)^2 - 4c^2 x(x^2 + y^2) - 4c^4 y^2 = 0. \qquad (4.3)$$

As is known, equation (4.2), or (4.3), is that of a cardioid C_1. Since (4.3) is of even degree with respect to y, C_1 is symmetric with respect to the real axis. It can be found out that the upper half part of C_1 has tangents parallel to the imaginary axis at the points $A(4c^2, 0)$ and $F\left(-\frac{c^2}{2}, \frac{\sqrt{3}c^2}{2}\right)$ and a tangent parallel to the real axis at the point $E\left(\frac{3c^2}{2}, \frac{3\sqrt{3}c^2}{2}\right)$. Since we are interested in the eigenvalue spectrum $\sigma(I - B^2)$ it is easy to see that this will be contained in a cardioid, that is denoted again by C_1, whose equation in cartesian coordinates will be derived from (4.3) by setting $1 - x$ for x. The equation for this new cardioid C_1 will be

$$[(1 - x)^2 + y^2]^2 - 4c^2(1 - x)[(1 - x)^2 + y^2] - 4c^4 y^2 = 0 \qquad (4.4)$$

and the characteristic points of C_1 mentioned before will now have coordinates $A(1 - 4c^2, 0)$, $E\left(1 - \frac{3c^2}{2} , \frac{3\sqrt{3}c^2}{2}\right)$ and $F\left(1 + \frac{c^2}{2} , \frac{\sqrt{3}c^2}{2}\right)$ (see Fig. 3).

Case II: The boundary ∂R_2 of the region R_2 in (1.5) has equation in polar coordinates

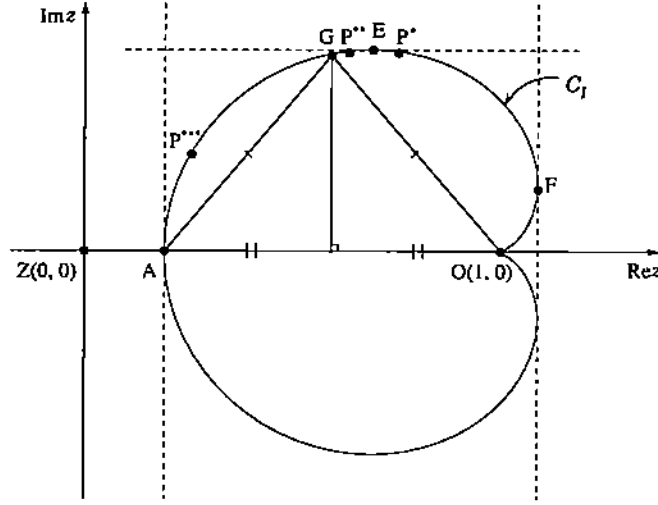$$r = 2c|\sin\theta|, \;\; 0 < c < 1, \;\; 0 \le \theta < 2\pi \qquad (4.5)$$

Fig. 3

(see Fig. 2). The boundary ∂R_2^2 of R_2^2 will have equation

$$r' = 2c^2(1 - \cos\theta'), \quad 0 < c < 1, \quad 0 \le \theta' < 2\pi. \tag{4.6}$$

In cartesian coordinates it will be

$$(x^2 + y^2)^2 + 4c^2 x(x^2 + y^2) - 4c^4 y^2 = 0. \tag{4.7}$$

The cardioid C_2, whose equation is given in (4.6) or (4.7), is symmetric with respect to the real axis. It can be found out that the images of the corresponding points A, E, F, denoted again by A, E, F, have coordinates $A(-4c^2, 0)$, $E\left(-\frac{3c^2}{2}, \frac{3\sqrt{3}c^2}{2}\right)$, $F\left(\frac{c^2}{2}, \frac{\sqrt{3}c^2}{2}\right)$, respectively. The spectrum $\sigma(I - B^2)$ is also contained in a cardioid, denoted again by C_2, with equation

$$[(1 - x)^2 + y^2]^2 + 4c^2(1 - x)[(1 - x)^2 + y^2] - 4c^4 y^2 = 0. \tag{4.8}$$

The previous characteristic points have now coordinates $A(1 + 4c^2, 0)$, $E\left(1 + \frac{3c^2}{2}, \frac{3\sqrt{3}c^2}{2}\right)$, $F\left(1 - \frac{c^2}{2}, \frac{\sqrt{3}c^2}{2}\right)$ (see Fig. 4).

5 Application of the "Continuous" Algorithm

The two cardioids C_1 and C_2 of the previous section, derived in the spectra cases (1.4) and (1.5), respectively, will be considered and studied separately. However, most of the basic
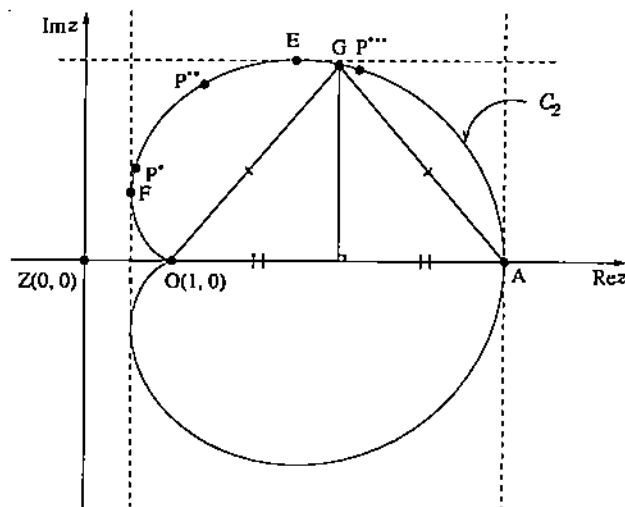
Fig. 4

results regarding the cardioid C_1 can be applied almost directly to the case of the cardioid C_2.

Case I:

The arc $\overset{\frown}{AEF}$ of the cardioid C_1 is a smooth concave curve defined on the closed interval $\left[1 - 4c^2, \ 1 + \frac{c^2}{2}\right]$. As in Section 3, let H^+ be the upper half part of the hull of $\sigma(I - B^2)$. Obviously, its vertices will be points of the arc $\overset{\frown}{AEF}$. Consider the set of all possible H^+ with a finite number of vertices that may lie anywhere on $\overset{\frown}{AEF}$. Our problem will be then that of determining the best ellipse that captures the set of all H^+. To determine it, it suffices to consider the best ellipse that captures the "worst" possible H^+ case. In mathematical terms, the problem just described can be defined as follows: *Determine the asymptotic convergence factor $\hat{r}$ that is defined via the formula*

$$\hat{r} = \sup_{\forall \ H^+ \subset C_1} \hat{\rho} \qquad (5.1)$$

where $\hat{\rho}$, given by (2.8), corresponds to the best capturing ellipse for $H^+ \subset C_1$ and which is found by Manteuffel's algorithm.

As is well known, to determine completely an ellipse $\mathcal{E}$ five independent elements of it must be known. For the best ellipse in question, or for any other ellipse that is a potential candidate for the best one, only three of its elements are to be determined. This is because the other two are already known; specifically, its center lies on the real axis and its "real" semiaxis lies on the same axis too. Let $\mathcal{F}$ denote the set of all the ellipses that are potential candidates for the best ellipse capturing the cardioid C_1 and let $\mathcal{E}$ denote any member of $\mathcal{F}$. Obviously, any $\mathcal{E}$ and C_1 cannot have more than four points in common in the upper half

plane. This is because the equations of C_1 and $\mathcal{E}$, in cartesian coordinates, are of degree four and two, respectively, and the two curves are symmetric with respect to the real axis. Also, any $\mathcal{E} \in \mathcal{F}$ that captures C_1 cannot have with C_1 four or three points in common in the upper half plane such that any one of them is a single point. For if P_1 is one single common point of $\mathcal{E}$ and C_1 the point P_1 will be a point of intersection of the two curves. This will imply that points of $\mathcal{E}$ in an arbitrarily small neighborhood of P_1 will be interior and exterior points of C_1. As a consequence, $\mathcal{E}$ will not capture C_1.

An immediate conclusion of the preceding analysis is that if $\mathcal{E} \in \mathcal{F}$ captures C_1 then, in the upper half plane, $\mathcal{E}$ and C_1 will have i) at most two points in common P_1 and P_2 ($\neq P_1$) that will be double ones (points of contact of the two curves), with P_1 and P_2 being different from A and F or ii) two points in common one of which will coincide with A and the other one will be a double point. First, we shall consider the second case and use the notation $\mathcal{E}_{P_1 P_2 P_3}$ to denote that $\mathcal{E}_{P_1 P_2 P_3} \in \mathcal{F}$ and passes through the points P_1, P_2, P_3 of the upper half plane.

Below, we state and prove a statement that constitutes one of the basic tools for the subsequence analysis.

Lemma 1: The ellipse $\mathcal{E}_{AEE}$, that is tangent to C_1 at A and E, captures C_1. (*Note:* In fact three elements, besides the two already known, are given. $\mathcal{E}_{AEE}$ passes through A and E and $\mathcal{E}_{AEE}$ is tangent to the cardioid C_1 at E. The fact that it is tangent at A as well is not a new element since A lies on the common axis of symmetry of the two curves.)

Proof: Let $\mathcal{E}_{AEE}$ be the ellipse, which passes through the points $A(1-4c^2, 0)$, $E\left(1 - \frac{3c^2}{2}, \frac{3\sqrt{3}c^2}{2}\right)$ and has "real" semiaxis $a = 1 - \frac{3c^2}{2} - (1 - 4c^2) = \frac{5c^2}{2}$, "imaginary" semiaxis $b = \frac{3\sqrt{3}c^2}{2}$ and distance of its center from the origin $d = 1 - \frac{3c^2}{2}$. As is known, a point $P(x, y) \in \bar{\mathcal{E}}_{AEE}$ if and only if

$$\frac{4\left(x - 1 + \frac{3c^2}{2}\right)^2}{25c^4} + \frac{4y^2}{27c^4} \leq 1. \tag{5.2}$$

Let $P \in C_1$. Then, from (4.2), it will be

$$x = 1 - 2c^2(1 + \cos\theta')\cos\theta', \quad y = 2c^2(1 + \cos\theta')\sin\theta'. \tag{5.3}$$

Therefore $P \in \bar{\mathcal{E}}_{AEE}$ if and only if its coordinates, given in (5.3), satisfy (5.2). Namely, if and only if

$$\frac{4\left[2c^2(1 + \cos\theta')\cos\theta' - \frac{3c^2}{2}\right]^2}{25c^4} + \frac{4c^4(1 + \cos\theta')^2\sin^2\theta'}{27c^4} \leq 1$$

or, equivalently, if and only if

$$\frac{[4(1 + \cos\theta')\cos\theta' - 3]^2}{25} + \frac{[(1 + \cos\theta')\sin\theta']^2}{27} \leq 1. \tag{5.4}$$

However, (5.4) is a valid relationship as this was proven in Thm. 8.1 of [3]. This completes the proof of the present lemma. □

Let now $B\left(1 + \frac{c^2}{2}, 0\right)$ be the projection of F onto the real axis and $C(1 + c^2, 0)$ be the other vertex of $\mathcal{E}_{AEE}$ on the real axis. Using, among others, these two points and the theory developed so far one can prove a number of statements which will eventually lead to the determination of the best ellipse that captures the cardioid C_1, in the sense already explained.

Lemma 2: Let D be any point on the "real" semiaxis with abscissa $x_D \geq 1 + \frac{c^2}{2}$. Then there exists a unique ellipse with "real" axis AD that, besides at A, is tangent to the cardioid C_1 at another point P and captures C_1.

Proof: For this, consider the family of all ellipses with "real" semiaxis $a = \frac{(AD)}{2}$. Let b denote the length of the "imaginary" semiaxis of any member of the family. Assume that b increases continuously from 0 to ∞. For $b = 0$, the member of the family in question is a degenerate ellipse, namely the double line segment AD, that intersects C_1 at two points; at A and at the point $O(1,0)$. For $b \to \infty$ the members of the family tend to a limiting ellipse that consists of the pair of the parallel to the y-axis straight lines that capture C_1 and are tangent to C_1 at A only. Therefore for $D \not\equiv B$ there will be a member of the family corresponding to the largest possible b, let it be denoted by $\bar{b}$, with $\bar{b} \in (0, \infty)$ such that for all $b \in (0, \bar{b}]$ each member of the family in question has two points in common with C_1. In view of the continuous increase of b and of the fact that for $b_1 < b_2$ the ellipse corresponding to b_1 lies entirely in the interior of the one corresponding to b_2, while their only common points are A and D, it is concluded that the ellipse corresponding to $b = \bar{b}$ is unique. □

Based on continuity arguments one can formally prove that $\bar{b}$ is a continuous function of (A, D) or of the abscissa x_D of the point D. Also, it can be proved that the coordinates of the point of contact P are continuous functions of $\bar{b}$ and therefore of x_D. Furthermore, as D moves continuously from the point B to C and then away from C the above point of contact P will move continuously along the cardioid from F to E and then from E to A, respectively.

Lemma 3: Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two distinct points on the arc $\overarc{AEF}$ of the cardioid C_1. If $x_1 < x_2$ and $P_1 \not\equiv A$, $\mathcal{E}_{AP_1P_1}$ and $\mathcal{E}_{AP_2P_2}$ intersect each other (at A and) at a point with abscissa strictly between x_1 and x_2.

Proof: From Lemma 2 it follows that $\mathcal{E}_{AP_1P_1}$ and $\mathcal{E}_{AP_2P_2}$ capture the cardioid. So, P_1 is a strictly interior point of $\mathcal{E}_{AP_2P_2}$ while P_2 is a strictly interior point of $\mathcal{E}_{AP_1P_1}$. Since $\mathcal{E}_{AP_1P_1}$ and $\mathcal{E}_{AP_2P_2}$ cannot have more than two common points in the upper half plane, they will have one more common point, besides A, satisfying the restrictions of the statement of the lemma. □

Examining now the possibility of the existence of an ellipse from the set $\mathcal{F}$, capturing C_1 and having with C_1 two points of contact, P_1, P_2, both different from A, we can prove the

following statement.

Theorem 4: There exists no ellipse $\mathcal{E}_{P_1P_1P_2P_2}$ that captures C_1 and has with it two distinct points of contact P_1 and P_2 both different from A.

Proof: Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be such that $x_1 < x_2$. Together with $\mathcal{E}_{P_1P_1P_2P_2}$ let us consider $\mathcal{E}_{AP_1P_1}$. The point A lies strictly in the interior of $\mathcal{E}_{P_1P_1P_2P_2}$. Since the two ellipses are symmetric with respect to the real axis and touch each other at P_1 it is implied that the arc $\overset{\frown}{AP_1}$ of $\mathcal{E}_{AP_1P_1}$ lies in the interior of $\mathcal{E}_{P_1P_1P_2P_2}$. So do the points of the arc of $\mathcal{E}_{AP_1P_1}$ that are beyond P_1 and are arbitrarily close to it. However, since P_2 of C_1 (and of $\mathcal{E}_{P_1P_1P_2P_2}$) lies in the interior of $\mathcal{E}_{AP_1P_1}$ it follows that $\mathcal{E}_{AP_1P_1}$ and $\mathcal{E}_{P_1P_1P_2P_2}$ will intersect each other at a point strictly between P_1 and P_2. But then $\mathcal{E}_{AP_1P_1}$ and $\mathcal{E}_{P_1P_1P_2P_2}$ will have three points (one simple and one double) in common in the upper half plane which is not possible. □

Lemma 5 : Let P be any point of the arc $\overset{\frown}{AE}$ of the cardioid C_1. The best 2-point ellipse $\widehat{\mathcal{E}}_{AP}$ captures the entire arc $\overset{\frown}{AP}$.

Proof: From the analysis of Manteuffel's algorithm (see [10], [11] or [9]), it is implied that since the abscissas and the ordinates of the points $A(x_A, y_A)$ and $P(x_P, y_P)$ satisfy $x_A < x_P$ and $0 = y_A < y_P$, the center of the best 2-point ellipse $\widehat{\mathcal{E}}_{AP}$ will be to the right of the point $\left(\frac{x_A + x_P}{2}, 0\right)$. Since $\widehat{\mathcal{E}}_{AP}$ and $\mathcal{E}_{APP}$ cannot have any other point in common, in the upper half plane, except A and P, the arc $\overset{\frown}{AP}$ of $\widehat{\mathcal{E}}_{AP}$ will be outside $\mathcal{E}_{APP}$. However, since $\mathcal{E}_{APP}$ is a capturing ellipse for C_1 and especially for the arc $\overset{\frown}{AP}$ of C_1 so will be $\widehat{\mathcal{E}}_{AP}$ for $\overset{\frown}{AP}$ of C_1. □

From the theory so far it has become clear that in order to determine the best ellipse in the spirit of Manteuffel that captures C_1 one must examine all possible 2-point best ellipses that capture C_1, if such ellipses exist. It is also clear that one of the two points in a 2-point best ellipse must always be the point A. On the other hand, the other point must be a double one, let it be P^*, if such a point exists. In the statement below, we prove the existence and uniqueness of such a point P^* and therefore the existence (and uniqueness) of the best 2-point ellipse that captures the cardioid C_1.

Theorem 6: There exists a unique point $P^* \in C_1$ such that the best 2-point ellipse $\widehat{\mathcal{E}}_{AP^*}$ is the best 2-point ellipse that covers C_1. For this ellipse there holds $\widehat{\mathcal{E}}_{AP^*} \equiv \mathcal{E}_{AP^*P^*}$.

Proof: In view of the proof of Lemma 5, for all the points P on the arc $\overset{\frown}{AE}$ of the cardioid C_1 the best 2-point ellipse $\widehat{\mathcal{E}}_{AP}$ not only intersects $\mathcal{E}_{APP}$ at A and P, but also captures the whole arcs $\overset{\frown}{AP}$ of $\mathcal{E}_{APP}$ and $\overset{\frown}{AP}$ of C_1. Since the best 2-point ellipse $\widehat{\mathcal{E}}_{AF}$ does not capture the infinite(!) arc $\overset{\frown}{AF}$ of the limiting ellipse $\mathcal{E}_{AFF}$ (that is the pair of lines parallel to the imaginary axis and tangent to the cardioid at A and F) there must exist at least a point on the arc $\overset{\frown}{EF}$ of the cardioid C_1, let it be P^*, such that $\widehat{\mathcal{E}}_{AP^*} \equiv \mathcal{E}_{AP^*P^*}$. Let P^* be the closest to E point on $\overset{\frown}{EF}$ of C_1 with this property. Let also that there exists another point
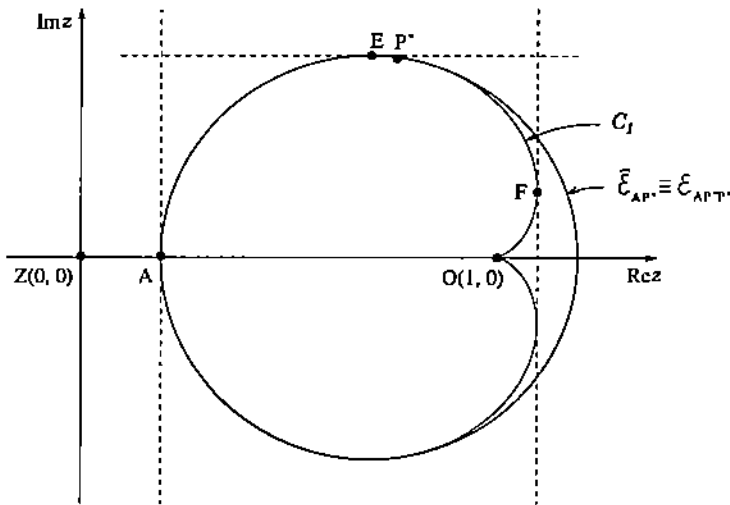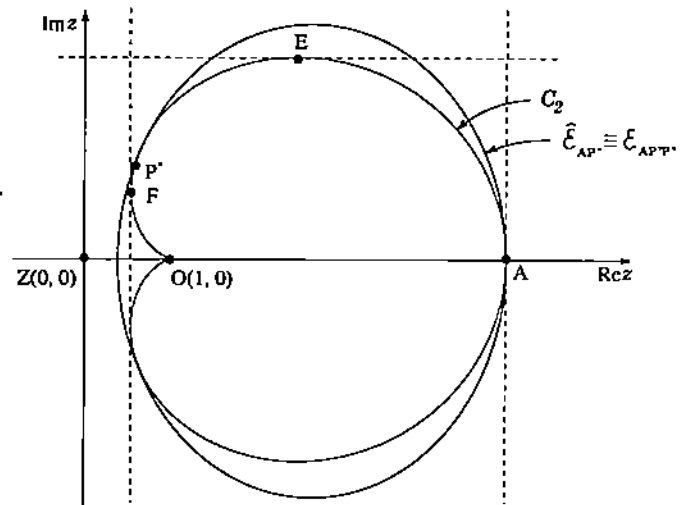
Fig. 5

Fig. 6

$P^{-\prime}$ on the arc $\overset{\frown}{P^{-}F}$ of C_1 with the same property. That is $\widehat{\mathcal{E}}_{AP^{-\prime}} \equiv \mathcal{E}_{AP^{-\prime}P^{-\prime}}$. Then, for the triad of points A, P^{-}, $P^{-\prime}$ there would be two best 2-point ellipses $\widehat{\mathcal{E}}_{AP^{-}}$ and $\widehat{\mathcal{E}}_{AP^{-\prime}}$. However, this contradicts the uniqueness of the best 2-point ellipse capturing the triad of the points in question. □

Remark: The determination of P^{-} on C_1 can be done only computationally. For this we look for the unique point P on the arc $\overset{\frown}{EF}$ on the cardioid C_1 for which the following will hold true. The value of the asymptotic convergence factor $\hat{\rho}_{AP}$ of the best 2-point ellipse $\widehat{\mathcal{E}}_{AP}$ is equal to the value of the asymptotic convergence factor $\hat{\rho}_{APP}$ of the (limiting) 3-point ellipse $\mathcal{E}_{APP}$. In Figure 5 the best ellipse has been drawn for $c = 0.45$.

From the analysis so far, in the present Case I, one can also determine the best ellipse in case the arc of the cardioid and therefore that of $\sigma(I - B^2)$ is limited to $\overset{\frown}{AP}$, where P is any point in the interior of $AEP^{-}FO$. Apparently, for any position of P on $\overset{\frown}{P^{-}FO}$ the best ellipse capturing $\overset{\frown}{AP}$, and therefore $\sigma(I - B^2)$, will be $\widehat{\mathcal{E}}_{AP^{-}}$. As P moves on the cardioid towards A, continuity arguments can show that the best 2-point ellipse for $\sigma(I - B^2)$ will be $\widehat{\mathcal{E}}_{AP}$. However, since at the point G on the arc $\overset{\frown}{AE}$ of the cardioid C_1 for which $(GA) = (GO)$ the best ellipse for $\sigma(I - B^2)$ is the 3-point ellipse $\mathcal{E}_{AGO}$, there will be a point P^{--} on $\overset{\frown}{GP^{-}}$ such that $\widehat{\mathcal{E}}_{AP^{--}} = \mathcal{E}_{AP^{--}O}$ while for any point P on $\overset{\frown}{GP^{--}}$ the best ellipse will be $\mathcal{E}_{APO}$. As P moves on from G towards A, and since for $P \equiv A$ the best ellipse for $\sigma(I - B^2)$ is $\widehat{\mathcal{E}}_{AO}$, there will be a point on $\overset{\frown}{AG}$, let it be P^{---}, such that $\widehat{\mathcal{E}}_{AP^{---}} \equiv \mathcal{E}_{AP^{---}O}$. Obviously, for $P \in \overset{\frown}{P^{---}G}$ the best ellipse will be $\mathcal{E}_{APO}$ while for $P \in \overset{\frown}{AP^{---}}$ it will be $\widehat{\mathcal{E}}_{PO}$.

Case II:

Since Manteuffel's algorithm, with real parameters, works if and only if $\sigma(I - B^2)$ is strictly to the right of the imaginary axis, it is concluded that the leftmost point of the cardioid C_2, that is $F\left(1 - \frac{c^2}{2}, \frac{\sqrt{3}c^2}{2}\right)$, must have a strictly positive abscissa. In other words, it must be $c < \sqrt{2}$. However, this restriction is weaker than the one considered in [2] ($c < 1$) and given in (1.5). As a result, the MSOR method and its "equivalent" 2-step method, we propose in this work, can handle more general classes of problems, of type (1.1), (1.2), (1.3) and (1.5), than similar methods in the literature can.

The theory developed in the Case I, with C_1 being the cardioid, holds more or less in the present case of the cardioid C_2. Some "obvious" slight changes and modifications are presented in the sequel.

Lemma 1 holds as it stands. The only difference is that the ellipse $\mathcal{E}_{AEE}$, although it is a capturing one for C_2, lies strictly to the right of the imaginary axis if and only if the abscissa of the other vertex of its "real" axis is strictly positive. Namely, when $2\left(1 + \frac{3c^2}{2}\right) - (1 + 4c^2) = 1 - c^2 > 0$ or $c < 1$. In other words we "recover" the restriction considered in [2]. In our case $c < 1$ does **not** constitute a restriction. It simply suggests that there may be a class of ellipses of the type $\mathcal{E}_{APP}$, with P strictly to the right of E, for $c < 1$, or even strictly to the right of F, for $c < \sqrt{2}$, which do not entirely lie strictly to the right of the imaginary axis. To determine the point P in question, let it be denoted by $\bar{P}(\bar{x}, \bar{y})$, we find the equation of the ellipse that shares with the cardioid C_2 the tangent at $\bar{P}$ and require that this ellipse passes through the origin $Z(0,0)$. To determine $\bar{P}(\bar{x}, \bar{y})$ for a given $c < \sqrt{2}$ we find the unique solution of a nonlinear system of three equations with three unknowns. To prove the uniqueness of the solution, we follow a reasoning similar to that used in Lemma 2. Specifically, we consider the family of all the ellipses with "real" axis ZA and imaginary semiaxis b which increases continuously from 0 to ∞. It is obvious that there exists a unique value of b, let it be $\bar{b}$, such that the corresponding ellipse and the cardioid C_2 touch each other at a unique point $\bar{P}(\bar{x}, \bar{y})$ in the upper half plane. To determine the point of contact $\bar{P}$ we consider the ellipse in question whose equation is

$$\frac{(x - \bar{d}^2)}{\bar{a}^2} + \frac{y^2}{\bar{b}^2} = 1,$$

$$\bar{d} = \bar{a} = 0.5 + 2c^2 \tag{5.5}$$

Since the coordinates of $\bar{P}$ satisfy both (4.8) and (5.5) we will have

$$[(1 - \bar{x})^2 + \bar{y}^2]^2 + 4c^2(1 - \bar{x})[(1 - \bar{x})^2 + \bar{y}^2] - 4c^4\bar{y}^2 = 0 \tag{5.6}$$

and

$$\frac{(\bar{x} - 0.5 - 2c^2)^2}{(0.5 + 2c^2)^2} + \frac{\bar{y}^2}{\bar{b}^2} = 1. \tag{5.7}$$

On the other hand, the slopes of the curves (4.8) and (5.5) at $\tilde{P}(\bar{x}, \bar{y})$ are given by

$$\frac{3c^2(1-\bar{x})^2 + c^2\bar{y}^2 + (1-\bar{x})\bar{y}^2 + (1-\bar{x})^3}{[(1-\bar{x})^2 + \bar{y}^2 + 2c^2(1-\bar{x}) - 2c^4]\bar{y}} \tag{5.8}$$

and

$$-\frac{(\bar{x} - 0.5 - 2c^2)\bar{b}^2}{(0.5 + 2c^2)^2\bar{y}} , \tag{5.9}$$

respectively. Equating the expressions in (5.8) and (5.9) gives the third equation, namely

$$\frac{3c^2(1-\bar{x})^2 + c^2\bar{y}^2 + (1-\bar{x})\bar{y}^2 + (1-\bar{x})^3}{(1-\bar{x})^2 + \bar{y}^2 + 2c^2(1-\bar{x}) - 2c^4} = -\frac{(\bar{x} - 0.5 - 2c^2)\bar{b}^2}{(0.5 + 2c^2)^2} , \tag{5.10}$$

which together with (5.6) and (5.7) constitute the system of the three equations with the three unknowns $\bar{b}$, $\bar{x}$, $\bar{y}$. This system has three real solutions. The one that corresponds to $\bar{y} > 0$ is that we seek. We simply note that in considering ellipses $\mathcal{E}_{APP}$ as potential candidates for the best (limiting) 3-point ellipse capturing C_2 all ellipses with P on the arc $\overparen{AP}$ of the cardioid must be discarded.

Lemma 2 holds the same except that the abscissa of D is now $x_D \leq 1 - \frac{c^2}{2}$

Lemma 3 is exactly the same with the obvious change of $x_1 < x_2$ to $x_1 > x_2$.

Theorem 4 and Lemma 5 are identically the same with those in the previous Case I.

From our analysis it follows directly that Theorem 6, with some obvious slight changes in its proof, together with its Remark, are still valid in the present case of the cardioid C_2.

In Figure 6 the best ellipse has been drawn for $c = 1.00$.

We conclude this part of Case II by noting that everything that was said in the previous Case I regarding the best capturing ellipse, when only part of the arc $\overparen{AEFO}$ of the cardioid C_2 constitutes the curved boundary for $\sigma(I - B^2)$ in the upper half plane, is valid.

One more point before we close this section. In case $c \geq \sqrt{2}$ our problem does possess a solution, in terms of the Manteuffel's algorithm if and only if $\sigma(I - B^2)$ lies in a part of the cardioid, which in turn, lies strictly to the right of the imaginary axis. In this case the arc of the cardioid C_2 to be considered is $\overparen{A\tilde{P}}$ (the point $\tilde{P}$ is excluded), where $\tilde{P}$ is the intersection of $\overparen{AEF}$ with the positive imaginary semiaxis. From (4.8) it can be obtained that

$$\tilde{y} = [(2c^4 - 2c^2 - 1) + 2c^3(c^2 - 2)^{1/2}]^{1/2}. \tag{5.11}$$

6 Discussion and Numerical Examples

First we try to compare theoretically the MSOR method, or rather the 2-step method (2.6), and the optimal results obtained in this work to those in the works by de Pillis [3], Chin and Manteuffel [2] (see also [6]) and Eiermann, Niethammer and Varga [4].

The two-step parallel stationary process (1.3) of [3] consists of two steps executed in parallel; the first one of them requires two matrix-vector multiplications as ours but the matrices and vectors involved are of full size n. So, the amount of work per iteration of the scheme in [3] is much more than that in our (2.6) method. Moreover, the convergence results obtained in [3] are based on the ellipse $\mathcal{E}_{AEE}$ which, as was seen, is not the optimal one and the parallel method, as analyzed there, has the disadvantage that does not always converge. In our opinion the parallel method in question can be greatly improved if the presence of all four real parameters (two complex ones) in (4.2) of [3] is fully exploited.

The SOR method in [2] can be written equivalently in the form of the 2-step method (2.6), with $\omega_1' = \omega_2' = \omega'$ (see also [4] and [14]). However, since it involves only one parameter, instead of two, it cannot be better than ours. Besides, although it covers both basic Cases I and II as well as their subcases, in the basic Case II it can only work for values of $c < 1$ (or $|\text{Re } z| < 1$), in (1.5), compared to the larger set of values of $c < \sqrt{2}$ (or $|\text{Re } z| < \sqrt{2}$) for which our method can work.

For the basic Case I, the 2-step stationary method (4.5) of [4], as was mentioned there, is marginally faster than the SOR method of [2]. It is worth pointing out that despite its disadvantages the method of [3] for values of c away from those for which it diverges it is faster that the previous two ones. The method we developed in this work is as was theoretically proved the fastest of them all. For the basic Case II our method is faster than the only other available one [2] and converges also for values of $c \in [1, \sqrt{2})$ for which the method of [2] diverges.

Finally, it should be mentioned that with the exception of the analysis in [2], (see also [6]), the analyses in [3] and [4] cover only the basic Case I or spectra of type (1.4).

To conclude this work, we present some numerical examples in two Tables that show the superiority of our method over those in [3], [2] and [4]. Table 1 is an extension of Table 1 of [4]. The entries in columns labeled [2] and [4] are the squares of the corresponding ones in [4] as they should have been given there. Table 2 presents numerical examples of the only (two) available best stationary methods. The one proposed in [2] and ours.

Before we close this section we would like to note that for nonstationary methods, one should adopt and follow the analysis presented in [12].

Table 1

c	Asymptotic Convergence Factor			
	[3]	[2]	[4]	present method
0.2	0.10850	0.11696	0.11049	0.10794
0.4	0.53965	0.55520	0.53995	0.53340
0.45	0.74938	0.75020	0.73872	0.73362
0.495	No convergence	0.97208	0.97043	0.96967

Table 2

c	Asymptotic Convergence Factor	
	[2]	present method
0.2	0.10533	0.09524
0.4	0.36411	0.32185
0.6	0.65746	0.56868
0.8	0.88836	0.76504
1.0	No convergence	0.89489
1.2	>>	0.96870
1.4	>>	0.99957

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