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Abstract

Real cubic algebraic surfaces may be described by either implicit or parametric equations. Each of these representations has strengths and weaknesses and have been used extensively in computer graphics. Applications involving both representations include the efficient computation of surface intersections, and triangulation of curved surfaces. One particularly useful representation is the rational parametrization, where the three spatial coordinates are given by rational functions of two parameters. Rational parametrizations speed up many computations, and their relatively simple structure allows one to control and avoid singularities in the parametrization. These parametrizations take on different forms for different classes of cubic surfaces. Classification of real cubic algebraic surfaces into five families for the nonsingular case is based on the configuration of twenty-seven lines on them. We provide a method of extracting all these lines and from there a rational parametrization of each of these families. The parametrizations of the real cubic surface components are constructed using a pair of real skew lines for those three families which have them, and remarkably using a complex conjugate pair of skew lines, in a fourth family. The parametrization is based on the fact that a real line generally intersects a cubic surface at three points. Points on the surface are obtained by intersecting the surface with lines that pass through points on the two skew lines. We also analyze the image of the derived rational parametrization for both real and complex parameter values, together with "base" points where the parametrizations are ill-defined.

1 Introduction

Low degree real algebraic surfaces (quadrics, cubics and quartics) play a significant role in constructing accurate computer models of physical objects and environments for purposes of simulation and prototyping[6]. While quadrics such as spheres, cones, hyperboloids and paraboloids prove sufficient for constructing restricted classes of models, cubic algebraic surface patches are sufficient to model the boundary of objects with arbitrary topology in a C^1 piecewise smooth manner [7].

Real cubic algebraic surfaces are the real zeros of a *polynomial* equation $f(x,y,z) = 0$ of degree three. In this representation the cubic surface is said to be in *implicit* form. The irreducible cubic surface which is not a cylinder of a nonsingular cubic curve, can alternatively be described explicitly by rational functions of parameters u and v:

$$x = \frac{f_1(u,v)}{f_4(u,v)},\ y = \frac{f_2(u,v)}{f_4(u,v)},\ z = \frac{f_3(u,v)}{f_4(u,v)}, \tag{1}$$

where f_i, $i = 1 \ldots 4$ are polynomials. In this case the cubic surface is said to be in *rational parametric* form.

Real cubic algebraic surfaces thus possess dual implicit-parametric representations and this property proves important for the efficiency of a number of geometric modeling and computer graphics display operations [6, 18]. For example, with dual available representations the intersection of two surfaces or surface patches reduces simply to the sampling of an algebraic curve in the planar parameter domain [4]. Similarly, point-surface patch incidence classification, a prerequisite for boolean set operations and ray casting for graphics display, is greatly simplified in the case when both the implicit and parametric representations are available [4]. Additional examples in the computer graphics domain which benefit from dual implicit-parametric representations are the rapid triangulation for curved surface display [8] and image texture mapping on curved surface patches [12].

Deriving the rational parametric form from the implicit representation of algebraic surfaces, is a process known as rational parametrization. Algorithms for the rational parametrization of cubic algebraic surfaces have been given in [2, 21], based on the classical theory of skew straight lines and rational curves on the cubic surface [9, 13, 22]. One of the main results of our current paper is to constructively address the parametrization of cubic surfaces based on the reality of the straight lines on the real cubic surface. In doing so we provide an algorithm to construct all twenty-seven straight lines (real and complex) on the real nonsingular cubic surface. We prove that the parametrizations of the real cubic surface components are constructed using a pair of real skew lines for those three families which have them, and remarkably using a complex conjugate pair of skew lines, in a fourth family. There does not appear to be a similar rational parametrization for the fifth family that covers all or almost all of the surface, so instead we use two disjoint parametrizations which involve one square root each. A rational parametrization that covers part of the surface is described in [21]. In that scheme points which lie on tangent planes through points on a real line are covered, but these points do not necessarily comprise most of the surface, and the covering is in general two-to-one instead of one-to-one. All of the parametrizations described in this paper are one-to-one, meaning that for any point on the cubic surface there can be just one set of values (u, v) which give rise to that point.

We also analyze the image of the derived rational parametrization for both real and complex parameter values, together with "base" points where the parametrizations are ill-defined. These base points cause a finite number (at most five) of lines and points, and possibly two conic sections lying on the surface, to be missed by the parametrizations. One of these conics can be attained by letting $u \to \pm\infty$ and the other with $v \to \pm\infty$ separately, or by using projective coordinates $\{u, u^*\}$ and $\{v, v^*\}$ instead of (u, v) and setting $v = 0$ and $u = 0$, respectively.

2 Preliminaries

One of the gems of classical algebraic geometry has been the theorem that twenty-seven distinct straight lines lie completely on a nonsingular cubic surface [19]. See figure 1. Schläfi's double-six notation elegantly captures the complicated and many-fold symmetry of the configurations of the twenty-seven lines [20]. He also partitions all nonsingular cubic surfaces $f(x, y, z) = 0$ into five families $F_1, \ldots, F_5$ based on the reality of the twenty-seven lines. Family F_1 contains 27 real straight lines, family F_2 contains 15 real lines, and family F_3 contains 7 real lines while families F_4 and F_5 contain 3 real lines each. What distinguishes F_4 from F_5 is that while 6 of the 12 conjugate complex line pairs of F_4 are skew (and 6 pairs are coplanar), each of the 12 conjugate pairs of complex line pairs of F_5 is coplanar. When a nonsingular cubic surface F tends to a singular cubic surface G (with an isolated double point or a double line) 12 of F's straight lines (constituting a double six) tend to 6 lines through a double point of G [22]. Hence singular cubic surfaces have only twenty-one distinct straight lines.

Alternatively a classification of cubic surfaces can be obtained from computing all 'base' points of

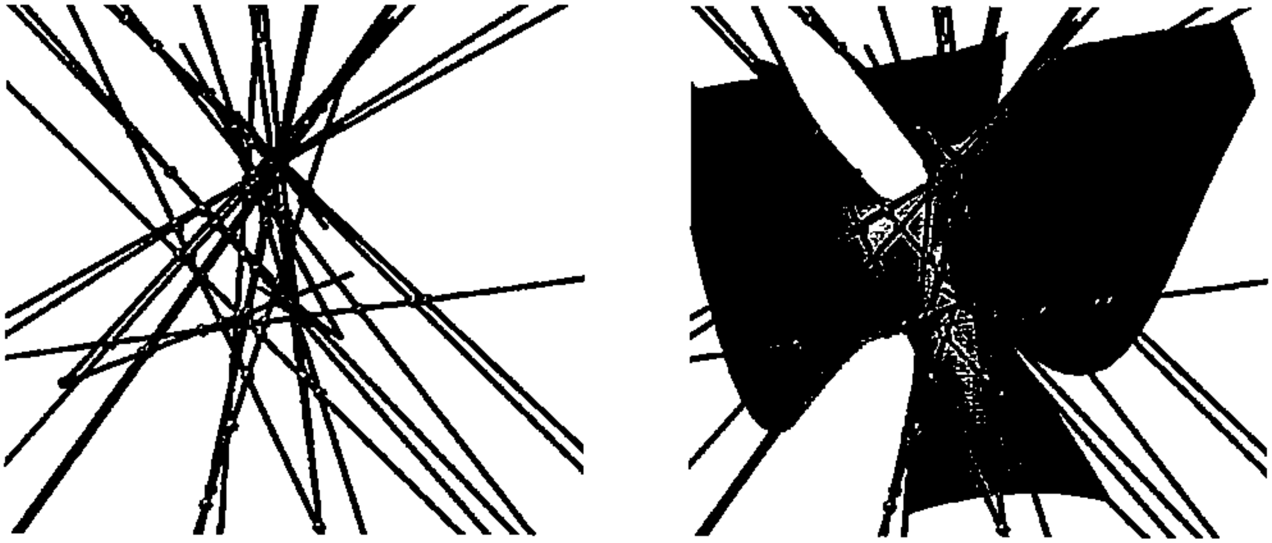

Figure 1: A configuration of twenty seven real lines of a cubic surface shown with and without the surface. Intersections of the coplanar straight lines are also shown.

its parametric representation,

$$x = \frac{f_1(u,v)}{f_4(u,v)},\ y = \frac{f_2(u,v)}{f_4(u,v)},\ z = \frac{f_3(u,v)}{f_4(u,v)},$$

Base points of a surface parametrization are those isolated parameter values which simultaneously satisfy $f_1 = f_2 = f_3 = f_4 = 0$. It is known that any nonsingular cubic surface can be expressed as a rational parametric cubic with six base points. The classification of nonsingular real cubic surfaces then follows from:

1. If all six base points are real, then all 27 lines are real, i.e. the F_1 case.

2. If two of the base points are a complex conjugate pair then 15 of the straight lines are real, i.e. the F_2 case.

3. If four of the base points are two complex conjugate pairs then 7 of the straight lines are real, i.e. the F_3 case.

4. If all base points are complex then three of the straight lines are real. In this case the three real lines are all coplanar, i.e. the F_4 and F_5 cases.

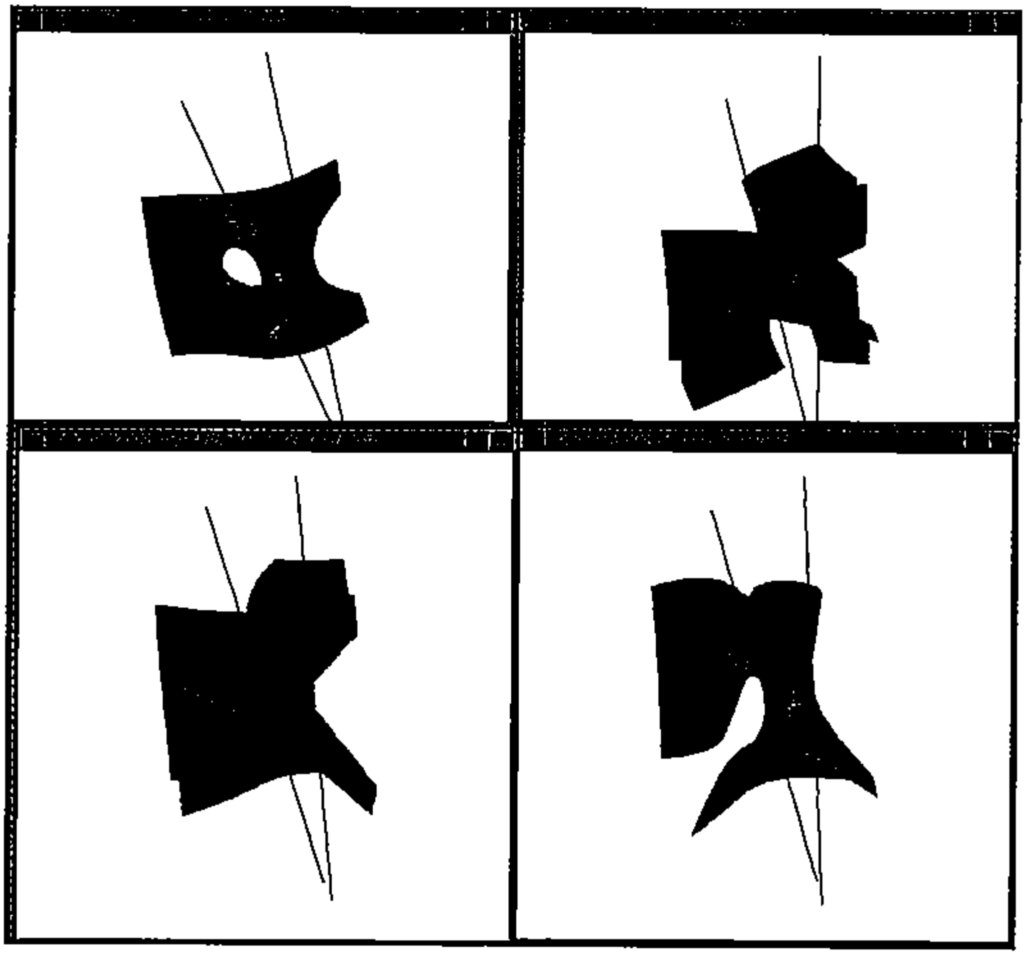

Figure 2: A cubic surface family with skew real straight lines

3 Real and Rational Points on Cubic Surfaces

We first begin by computing a simple real point (with a predefined bit precision) on a given real cubic surface $f(x,y,z) = 0$. For obvious reasons of exact calculations with bounded precision it is very desirable to choose the simple point to have rational coordinates. Mordell in his 1969 book [16] mentions that no method is known for determining whether rational points exist on a general cubic surface $f(x,y,z) = 0$, or finding all of them if any exist. We are unaware if a general criterion or method now exists or whether Mordell's conjecture below has been resolved.

The following theorems and conjecture exhibit the difficulty of this problem, and are repeated here for information.

Theorem[[16],chap 11]: All rational points on a cubic surface can be found if it contains two lines whose equations are defined by conjugate numbers of a quadratic field and in particular by rational numbers.

Theorem[[16],chap 11]: The general cubic equation (irreducible cubic and not a function of two independent variables nor a homogeneous polynomial in linear functions of its variables) has either none or an infinity of rational solutions.

Mordell Conjecture[[16],chap 11]: The cubic equation $F(X,Y,Z,W) = 0$ is solvable if and only if the congruence $F(X,Y,Z,W) \equiv 0 \pmod{p^r}$ is solvable for all primes p and integers $r > 0$ with $(X,Y,Z,W,p) = 1$.

We present a straightforward search procedure to determine a real point on $f(x,y,z) = 0$, and if lucky one with rational coordinates.

Collect the highest degree terms of $f(x,y,z)$ and call this homogeneous form $F_3(x,y,z)$. Recursively determine if $F_3(x,y,z) = 0$ has a rational point. Being homogeneous, one only needs to check for $F_3(x,y,1) = 0$ and $F_3(x,y,0) = 0$, which are both polynomials in one less variable, and hence the recursion is in dimension. Now for a univariate polynomial equation $g(x) = 0$ we use the technique of [15] to determine the existence and coordinates of a rational root. If not, one computes a real root having the desired bit precision as explained below.

Additionally, if the highest degree terms of $f(x,y,z)$ do not yield a rational point, we compute the resultant and linear subresultants of f and f_x, eliminating x to yield new polynomials $f_1(y,z)$ and $xf_2(y,z) + f_3(x,y,z)$ (see [5] for details of this computation). Recursively compute the rational points of $f_1(y,z) = 0$, using the equation $xf_2(y,z) + f_3(x,y,z) = 0$ to determine the rational x coordinate given rational y and z coordinates of the point.

In the general case, therefore, we are forced to take a real simple point on the cubic surface. We can bound the required precision of this real simple point so that the translations and resultant computations in the straight line extraction and cubic surface parametrization algorithm of the next section, are performed correctly. The lower bound of this value can be estimated as in [10] by use of the following gap theorem:

Gap Theorem ([10],p70). *Let $\mathcal{P}(d,c)$ be the class of integral polynomials of degree d and maximum coefficient magnitude c. Let $f_i(x_1,\cdots,x_n) \in \mathcal{P}(d,c)$, $i = 1,\cdots,n$ be a collection of n polynomials in n variables which has only finitely many solutions when homogenized. If $(\alpha_1,\cdots,\alpha_n)$ is a solution of the system, then for any j either $\alpha_j = 0$, or $|\alpha_j| > (3dc)^{-nd^n}$.*

4 Algebraic Reduction

Given two skew lines $\mathbf{l}_1(u) = \begin{bmatrix} x_1(u) \\ y_1(u) \\ z_1(u) \end{bmatrix}$ and $\mathbf{l}_2(v) = \begin{bmatrix} x_2(v) \\ y_2(v) \\ z_2(v) \end{bmatrix}$ on the cubic surface $f(x,y,z) = 0$, the cubic parametrization formula for a point $\mathbf{p}(u,v)$ on the surface is :

$$\mathbf{p}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} = \frac{a\mathbf{l}_1 + b\mathbf{l}_2}{a+b} = \frac{a(u,v)\mathbf{l}_1(u) + b(u,v)\mathbf{l}_2(v)}{a(u,v)+b(u,v)} \tag{2}$$

where

$$a = a(u,v) = \nabla f(\mathbf{l}_2(v)) \cdot [\mathbf{l}_1(u) - \mathbf{l}_2(v)]$$
$$b = b(u,v) = \nabla f(\mathbf{l}_1(u)) \cdot [\mathbf{l}_1(u) - \mathbf{l}_2(v)]$$

The total degree of the numerator of the parametrization formula in $\{u,v\}$ is 4 while the denominator total degree is 3. Note that if the lines are coplanar, formula (2) can only produce points on the plane of the lines, hence the search for skew lines on the cubic surface.

Following the notation of [2], a real cubic surface has an implicit representation of the form

$$f(x,y,z) = Ax^3 + By^3 + Cz^3 + Dx^2y + Ex^2z + Fxy^2 + Gy^2z + Hxz^2 + Iyz^2 + Jxyz$$
$$+ Kx^2 + Ly^2 + Mz^2 + Nxy + Oxz + Pyz + Qx + Ry + Sz + T = 0 \,.$$

Compute a simple (nonsingular) point (x_0, y_0, z_0) on the surface. We can move the simple point to the origin by a translation $x = x' + x_0$, $y = y' + y_0$, $z = z' + z_0$, producing

$$f'(x', y', z') = Q'x' + R'y' + S'z' + \dots \text{ terms of higher degree.}$$

Next, we wish to rotate the tangent plane to $f(x', y', z')$ at the origin to the plane $z'' = 0$. This can be done by the transformation

$$\begin{array}{ll} x' = x'',\ y' = y'',\ z' = (z'' - Q'x'' - R'y'')/S' & \text{if } S' \neq 0 \\ x' = x'',\ y' = (z'' - Q'x'')/R',\ z' = y'' & \text{if } S' = 0 \text{ and } R \neq 0 \\ x' = z''/Q',\ y' = x'',\ z' = y'' & \text{if } S' = 0,\ R' = 0, \text{ and } Q' \neq 0 \ . \end{array}$$

Fortunately Q', R', and S' cannot all be zero, because then the selected point (x_0, y_0, z_0) would be a singular point on the cubic surface.

The transformed surface can be put in the form

$$f''(x'', y'', z'') = z'' + [f_2(x'', y'') + f_1(x'', y'')z'' + f_0 z''^2] + [g_3(x'', y'') + g_2(x'', y'')z'' + g_1(x'', y'')z''^2 + g_0 z''^3]\ ,$$

where $f_j(x'', y'')$ and $g_j(x'', y'')$ are terms of degree j in x'' and y''. In general, this surface intersects the tangent plane $z'' = 0$ in a cubic curve with a double point at the origin (as its lowest degree terms are quadratic). This curve can be rationally parametrized as

$$\begin{aligned} x'' &= K(t) = -\frac{L''t^2 + N''t + K''}{B''t^3 + F''t^2 + D''t + A''} \\ y'' &= L(t) = tK(t) = -\frac{L''t^3 + N''t^2 + K''t}{B''t^3 + F''t^2 + D''t + A''} \\ z'' &= 0\ , \end{aligned} \tag{3}$$

where $A'', B'', \dots$ are the coefficients in $f''(x'', y'', z'')$ that are analogous to $A, B, \dots$ in $f(x, y, z)$. In the special case that the singular cubic curve is reducible (a conic and a line or three lines), a parametrization of the conic is taken instead.

We transform the surface again to bring a general point on the parametric curve specified by t to the origin by the translation

$$\begin{aligned} x'' &= \bar{x} + K(t) \\ y'' &= \bar{y} + L(t) \\ z'' &= \bar{z}\ . \end{aligned}$$

The cubic surface can now be expressed by

$$\bar{f}(\overline{x},\overline{y},\overline{z}) = \overline{Q}(t)\overline{x} + \overline{R}(t)\overline{y} + \overline{S}(t)\overline{z} + \cdots \text{ terms of higher degree } .$$

We make the tangent plane of the surface at the origin coincide with the plane $\hat{z} = 0$ by applying the transformation

$$\begin{aligned}
\overline{x} &= \hat{x} \\
\overline{y} &= \hat{y} \\
\overline{z} &= -\frac{\overline{Q}(t)}{\overline{S}(t)}\hat{x} - \frac{\overline{R}(t)}{\overline{S}(t)}\hat{y} + \frac{1}{\overline{S}(t)}\hat{z} \ .
\end{aligned}$$

The equation of the surface now has the form

$$f(\hat{x},\hat{y},\hat{z}) = \hat{z} + [\hat{f}_2(\hat{x},\hat{y}) + \hat{f}_1(\hat{x},\hat{y})\hat{z} + \hat{f}_0\hat{z}^2] + [\hat{g}_3(\hat{x},\hat{y}) + \hat{g}_2(\hat{x},\hat{y})\hat{z} + \hat{g}_1(\hat{x},\hat{y})\hat{z}^2 + \hat{g}_0\hat{z}^3] \ .$$

The intersection of this surface with $\hat{z} = 0$ gives

$$\hat{f}_2(\hat{x},\hat{y}) + \hat{g}_3(\hat{x},\hat{y}) = 0 \ . \tag{4}$$

Recall that $\hat{x}$ and $\hat{y}$, and hence $\hat{f}_2$ and $\hat{g}_3$, are functions of t. As shown in [2], equation (4) is reducible, and hence contains a linear factor, for those values of t for which $\hat{f}_2(\hat{x},\hat{y})$ and $\hat{g}_3(\hat{x},\hat{y})$ have a linear or quadratic factor in common. These factors correspond to lines on the cubic surface, and our goal is to find the values of t which produce these lines.

The values of t may be obtained by taking the resultant of $\hat{f}_2(\hat{x},\hat{y},t)$ and $\hat{g}_3(\hat{x},\hat{y},t)$ by eliminating either $\hat{x}$ or $\hat{y}$. Since $\hat{f}_2$ and $\hat{g}_3$ are homogeneous in $\{\hat{x},\hat{y}\}$ it does not matter with respect to which variable the resultant is taken[23]; the result will have the other variable raised to the sixth power as a factor. Apart from the factor of $\hat{x}^6$ or $\hat{y}^6$, the resultant consists of an 81st degree polynomial $P_{81}(t)$ in t. At first glance it would appear that there could be 81 values of t for which a line on the cubic surface is produced, but this is not the case:

Theorem 1: The polynomial $P_{81}(t)$ obtained by taking the resultant of $\hat{f}_2$ and $\hat{g}_3$ factors as $P_{81}(t) = P_{27}(t)[P_3(t)]^6[P_6(t)]^6$, where $P_3(t) = B''t^3 + F''t^2 + D''t + A''$, the denominator of $K(t)$ and $L(t)$, and $P_6(t)$ is the numerator of $\overline{S}(t)$ $\quad (P_6(t) = \overline{S}(t)[P_3(t)^2])$.

Sketch of proof: This proof was performed through the use of the symbolic manipulation program Maple [11]. When expanded out in full, $P_{81}(t)$ contains hundreds of thousands of terms, so a direct approach was not possible. Instead, $P_{81}(t)$ was shown to be divisible by both $[P_3(t)]^6$ and $[P_6(t)]^6$.

When $\hat{f}_2$ and $\hat{g}_3$ were expressed in terms of the numerators of $\overline{Q}(t)$, $\overline{R}(t)$, and $\overline{S}(t)$, it was possible to take the resultant without overflowing the memory capabilities of the machine. The resultant could be factored, and $[P_6(t)]^6$ was found to be one of the factors.

The factor $[P_3(t)]^6$ proved to be more difficult to obtain. After the factor $[P_6(t)]^6$ was removed, the remaining factor was split into several pieces, according to which powers of $\overline{Q}(t)$, $\overline{R}(t)$, and $\overline{S}(t)$ they contained. These pieces were each divided by $[P_3(t)]^6$, and the remainders taken. The remainders were expressed as certain polynomials times various powers of $P_3(t)$, as in $a_0(t)+a_1(t)P_3(t)+a_2(t)[P_3(t)]^2+a_3(t)[P_3(t)]^3+a_4(t)[P_3(t)]^4+a_5(t)[P_3(t)]^5$. We were able to show that $a_0(t)$ is in fact divisible by $P_3(t)$. Then we could show that $a_0(t)/P_3(t)+a_1(t)$ is also divisible by $P_3(t)$, and so on up the line until we could show the whole remaining factor is divisible by $[P_3(t)]^6$. Details are given in the appendix.

The solutions of $P_{27}(t)=0$ correspond to the 27 lines on the cubic surface. A method of partial classification is suggested by considering the number of real roots of $P_{27}(t)$: if it has 27, 15, or 7 real roots the cubic surface is F_1, F_2, or F_3, respectively, and if $P_{27}(t)=0$ has three real roots the surface can be either F_4 or F_5. However, this is not quite accurate. In exceptional cases, $P_{27}(t)$ may have a double root at $t=t_0$, which corresponds to $\hat{f}_2$ and $\hat{g}_3$ sharing a quadratic factor. If this quadratic factor is reducible over the reals, the double root corresponds to two (coplanar) real lines; if the quadratic factor has no real roots it corresponds to two coplanar complex conjugate lines.

Theorem 2: *Simple real roots of $P_{27}(t)=0$ correspond to real lines on the surface.*

Proof: Let t_0 be a simple real root of $P_{27}(t)=0$. Since $P_{27}(t)$ is a factor of the resultant of $\hat{f}_2$ and $\hat{g}_3$ obtained by eliminating $\hat{x}$ or $\hat{y}$, $\hat{f}_2(\hat{x},\hat{y},t_0)$ and $\hat{g}_3(\hat{x},\hat{y},t_0)$ must have a linear or quadratic factor in common. If $\hat{f}_2(\hat{x},\hat{y},t_0)$ and $\hat{g}_3(\hat{x},\hat{y},t_0)$ have just a linear factor in common, then that factor is of the form $c_1\hat{x}+c_2\hat{y}$ where c_1 and c_2 are real constants since all the coefficients of $\hat{f}_2(\hat{x},\hat{y},t_0)$ and $\hat{g}_3(\hat{x},\hat{y},t_0)$ are real and $\hat{f}_2(\hat{x},\hat{y},t_0)$ and $\hat{g}_3(\hat{x},\hat{y},t_0)$ are homogeneous in $\hat{x}$ and $\hat{y}$. In this case the real line $c_1\hat{x}+c_2\hat{y}=0$ lies on the surface.

If $\hat{f}_2(\hat{x},\hat{y},t_0)$ and $\hat{g}_3(\hat{x},\hat{y},t_0)$ have a quadratic factor in common, then that factor is of the form $c_1\hat{x}^2+c_2\hat{x}\hat{y}+c_3\hat{y}^2$. We will show that if this is the case, then $P_{27}(t)$ has at least a double root at $t=t_0$. This will be sufficient to prove that simple roots of $P_{27}(t)$ can only correspond to common linear factors of $\hat{f}_2(\hat{x},\hat{y},t_0)$ and $\hat{g}_3(\hat{x},\hat{y},t_0)$, and hence real lines on the cubic surface.

If we write $\hat{f}_2(\hat{x},\hat{y},t)=Q_1(t)\hat{x}^2+Q_2(t)\hat{x}\hat{y}+Q_3(t)\hat{y}^2$ and $\hat{g}_3(\hat{x},\hat{y},t)=Q_4(t)\hat{x}^3+Q_5(t)\hat{x}^2\hat{y}+Q_6(t)\hat{x}\hat{y}^2+$

$Q_7(t)\hat{y}^3$, then the resultant of $\hat{f}_2(\hat{x},\hat{y},t)$ and $\hat{g}_3(\hat{x},\hat{y},t)$ obtained by eliminating $\hat{x}$ is

$$R(\hat{f}_2,\hat{g}_3) = \begin{vmatrix} Q_1(t) & Q_2(t) & Q_3(t) & 0 & 0 \\ 0 & Q_1(t) & Q_2(t) & Q_3(t) & 0 \\ 0 & 0 & Q_1(t) & Q_2(t) & Q_3(t) \\ Q_4(t) & Q_5(t) & Q_6(t) & Q_7(t) & 0 \\ 0 & Q_4(t) & Q_5(t) & Q_6(t) & Q_7(t) \end{vmatrix} \hat{y}^6 \ . \tag{5}$$

We need to show that if $\hat{f}_2(\hat{x},\hat{y},t)$ and $\hat{g}_3(\hat{x},\hat{y},t)$ have a quadratic factor in common when $t = t_0$, then $R(\hat{f}_2,\hat{g}_3)/\hat{y}^6$ has a double root at $t = t_0$. This is equivalent to showing that $R(\hat{f}_2(t_0),\hat{g}_3(t_0)) = 0$ and $(d/dt)[R(\hat{f}_2(t_0),\hat{g}_3(t_0))] = 0$. If $\hat{f}_2(\hat{x},\hat{y},t_0)$ and $\hat{g}_3(\hat{x},\hat{y},t_0)$ have a quadratic factor in common, then $\hat{g}_3(t_0) = k(c_1\hat{x} - c_2\hat{y})\hat{f}_2(t_0)$ for some real constants k, c_1, and c_2. Thus $Q_4(t_0) = kc_1Q_1(t_0)$, $Q_5(t_0) = k[c_1Q_2(t_0) - c_2Q_1(t_0)]$, $Q_6(t_0) = k[c_1Q_3(t_0) - c_2Q_2(t_0)]$, and $Q_7(t_0) = -kc_2Q_3(t)$. Making these substitutions in (5), we find that indeed both $R(\hat{f}_2(t_0),\hat{g}_3(t_0)) = 0$ and $(d/dt)[R(\hat{f}_2(t_0),\hat{g}_3(t_0))] = 0$. ∎

To summarize, the simple real roots of $P_{27}(t) = 0$ correspond to real lines on the cubic surface. Double real roots may correspond to either real or complex lines, depending on whether the quadratic factor $\hat{f}_2(\hat{x},\hat{y},t)$ and $\hat{g}_3(\hat{x},\hat{y},t)$ have in common is reducible or not over the reals. Higher order roots indicate some type of singularity. Complex roots can only correspond to complex lines in nonsingular cases. If t_0, a complex root of $P_{27}(t) = 0$, corresponded to a real line $c_1\hat{x} - c_2\hat{y}$ on the surface, then $\overline{t_0}$ would correspond to the same line, as a real line is its own complex conjugate. Thus one real line would be leading to two distinct values for t_0.

When the cubic surface is of class F_1, F_2, or F_3, it contains at least two real skew lines, and the parametrization in [2] is used. Having obtained skew lines $l_1(u) = [x_1(u) \quad y_1(u) \quad z_1(u)]$ and $l_2(v) = [x_1(v) \ \ y_1(v) \ \ z_1(v)]$, we consider the net of lines passing through a point on each. This is given by

$$\frac{z - z_1}{x - x_1} = \frac{z_2 - z_1}{x_2 - x_1} \qquad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \ .$$

Solving these for y and z in terms of x, and substituting into the cubic surface $f(x,y,z) = 0$ gives a cubic equation in x with coefficients in u and v, say $G(x,u,v) = 0$. Since $x = x_1$ and $x = x_2$ satisfy this equation, $G(x,u,v)$ is divisible by $x - x_1$ and $x - x_2$, and we have that

$$H(u,v,x) = \frac{G(x,u,v)}{[x - x_1(u)][x - x_2(v)]} \tag{6}$$

is a linear polynomial in x. This is solved for x as a rational function of u and v. Rational functions for y and z are obtained analogously.

The parametrization (1) is then computed as in (2):

$$(x,\ y,\ z) = (x(u,v),\ y(u,v),\ z(u,v)) = (f_1(u,v)/f_4(u,v),\ \ f_2(u,v)/f_4(u,v),\ f_3(u,v)/f_4(u,v))$$

where

$$\begin{aligned} f_1(u,v) &= a(u,v)x_1(u) + b(u,v)x_2(v) \\ f_2(u,v) &= a(u,v)y_1(u) + b(u,v)y_2(v) \\ f_3(u,v) &= a(u,v)z_1(u) + b(u,v)z_2(v) \\ f_4(u,v) &= a(u,v) + b(u,v)\ , \end{aligned} \tag{7}$$

with

$$a(u,v) = \nabla f(\mathrm{l}_2(v)) \cdot [\mathrm{l}_1(u) - \mathrm{l}_2(v)]\ , \qquad b(u,v) = \nabla f(\mathrm{l}_1(u)) \cdot [\mathrm{l}_1(u) - \mathrm{l}_2(v)]$$

In this notation $-f_1(u,v)$ and $f_4(u,v)$ are the coefficients of x^0 and x^1, respectively, in $H(u,v,x)$. The symbolic manipulation program Maple was used to verify that the expressions $f_1(u,v)/f_4(u,v)$, $f_2(u,v)/f_4(u,v)$, and $f_3(u,v)/f_4(u,v)$ do simplify to x, y, and z respectively.

Using floating-point arithmetic, it may be the case that some terms with very small coefficients appear in $f_1(u,v)$, $f_2(u,v)$, $f_3(u,v)$, and $f_4(u,v)$ when the coefficients should in fact be zero. Specifically, these are the terms containing u^3, v^3, u^4, v^4, u^3v and uv^3 in f_1, f_2, and f_3, and terms containing u^3 and v^3 in f_4. These coefficients were shown to be zero using Maple, so in the algorithm they are subtracted off in case they appear in the construction of f_1, f_2, f_3, and f_4.

5 Parametrizations without Real Skew Lines

When the cubic surface is of class F_4 or F_5 it does not contain any pair of real skew lines. In the F_4 case we derive a parametrization using complex conjugate skew lines, and in the F_5 case we obtain a parametrization by parametrizing conic sections which are the further intersections of the cubic surface with planes through a real line on the surface.

5.1 The F_4 Case

In this case there are 12 pairs of complex conjugate lines. For 6 of these pairs, the two lines intersect (at a real point). In the other 6 pairs, the two lines are skew. Let one pair of complex conjugate skew lines be given by $(x_1(u+vi),\ y_1(u+vi),\ z_1(u+vi))$ and $(x_1(u-vi),\ y_1(u-vi),\ z_1(u-vi))$. Here x_1, y_1, and z_1 are (linear) complex functions of a complex variable, and x_2, y_2, z_2 may be considered to be the complex conjugates of x_1, y_1, z_1. Also the real parameters u and v are unrestricted. Then the parametrization is again given by (7). Even though the quantities x_i, y_i, and z_i are complex, the expressions for $x(u,v)$, $y(u,v)$, and $z(u,v)$ turn out to be real when x_2, y_2, and z_2 are the complex conjugates of x_1, y_1, and z_1. The symbolic manipulation program Maple was used to verify that the quantities $f_1(u,v)/i$, $f_2(u,v)/i$, $f_3(u,v)/i$ and $f_4(u,v)/i$ are all real when (x_1,y_1,z_1) and (x_2,y_2,z_2) are complex conjugates.

Using floating-point arithmetic, it may be the case that some terms with very small coefficients appear in $f_1(u,v)$, $f_2(u,v)$, and $f_3(u,v)$ when the coefficients should in fact be zero. Specifically, these are the terms containing u^3v and uv^3. These coefficients were shown to be zero using Maple, so in the algorithm they are subtracted off in case they appear in the construction of f_1, f_2, and f_3.

Theorem 3: The algorithm provides a valid parametrization of an F_4 cubic surface when u and v are related as follows: u is unrestricted (both real and imaginary parts), and v is the complex conjugate of u. Each real point on the F_4 surface, except for those corresponding to base points of the parametrization, is obtained for exactly one complex value of u.

Lemma: Given two skew complex conjugate lines $\mathrm{l}_1(u) = (A+Bi,\ C+Di,\ E+Fi)+(G+Hi,\ I+Ji,\ K+Li)u$ and $\overline{\mathrm{l}_1}(v) = (A-Bi,\ C-Di,\ E-Fi)+(G-Hi,\ I-Ji,\ K-Li)v$, then for an arbitrary real point $\mathbf{p} = (x,y,z)$, there exists a unique complex value u_0 such that the points $\mathbf{p}$, $\mathrm{l}_1(u_0)$, and $\overline{\mathrm{l}_1(u_0)}$ are collinear.

Proof of Lemma: The points $\mathbf{p}$, $\mathrm{l}_1(u_0)$, and $\overline{\mathrm{l}_1(u_0)}$ will be collinear if and only if the vectors $\mathbf{p}-\mathrm{l}_1(u_0)$ and $\mathbf{p}-\overline{\mathrm{l}_1(u_0)}$ are parallel. Setting the cross product of these two vectors equal to zero and splitting u_0 into real and imaginary parts as a_0+b_0i, we find that there is a solution when

$$a_0 = \frac{M_1M_3+M_2M_4}{M_1^2+M_2^2} \qquad \text{and} \qquad b_0 = \frac{M_2M_3-M_1M_4}{M_1^2+M_2^2}\ , \tag{8}$$

and

$$M_1 = \begin{vmatrix} B & G & H \\ D & I & J \\ F & K & L \end{vmatrix} \qquad M_2 = \begin{vmatrix} x-A & G & H \\ y-C & I & J \\ z-E & K & L \end{vmatrix}$$

$$M_3 = \begin{vmatrix} B & x-A & H \\ D & y-C & J \\ F & z-E & L \end{vmatrix} \qquad M_4 = \begin{vmatrix} B & G & x-A \\ D & I & y-C \\ F & K & z-E \end{vmatrix}$$

The denominators $M_1^2 + M_2^2$ are positive because M_1 is nonzero exactly when $\mathbf{l}_1$ does not contain a real point. $\mathbf{l}_1$ contains a real point if and only if the vectors $[B\ \ D\ \ F]^T$, $[G\ \ I\ \ K]^T$, and $[H\ \ J\ \ L]^T$ are linearly dependent, and this is equivalent to $M_1 = 0$.

There certainly cannot be two distinct complex values u_1 and u_2 such that $\mathbf{p}$, $\mathbf{l}_1(u_1)$, and $\overline{\mathbf{l}_1(u_1)}$ are collinear and also $\mathbf{p}$, $\mathbf{l}_1(u_2)$, and $\overline{\mathbf{l}_1(u_2)}$ are collinear as that would imply $\mathbf{l}_1(u_1)$, $\mathbf{l}_1(u_2)$, $\overline{\mathbf{l}_1(u_1)}$ and $\overline{\mathbf{l}_1(u_2)}$ are coplanar, which is impossible as $\mathbf{l}_1$ and $\overline{\mathbf{l}_1}$ are skew. ∎

Proof of Theorem 3: Given an arbitrary real point (x_0, y_0, z_0) on the cubic surface, Equation (8) can be used to obtain a specific parameter value $u_0 = (a_0, b_0)$. This value of (a_0, b_0), when inserted into the parametrization (7), gives back (x_0, y_0, z_0), unless (a_0, b_0) happens to make the fractions in (7) $0/0$, which means that (a_0, b_0) is a base point of the parameter map. ∎

As will be shown in Section 6, there are five base points in this F_4 parametrization, with one of them being real. The points on the cubic surface which may be missed include one real line, which corresponds to the real base point. The other base points correspond to two pair of complex conjugates lines. For each pair, if the two lines are coplanar, and thus have a real point in common, that point is also missed in the parametrization. Skew complex lines corresponding to base points result in no missed real surface points.

It may seem odd that a real line may be missed by this parametrization, but in fact the real line does intersect the two skew complex conjugate lines. Here an extended notion of a real line is used: a line may be of the form $\mathbf{p} = \mathbf{d}u$ where $\mathbf{p}$ is a real 3D point and $\mathbf{d}$ is a real 3D vector, but in the context here we have to allow u to take on all complex values. With this understanding it is possible for an apparent real line to intersect both complex conjugate skew lines in complex points, and when it does, the points of intersection are complex conjugates. All points on this real line map into the same (a_0, b_0).

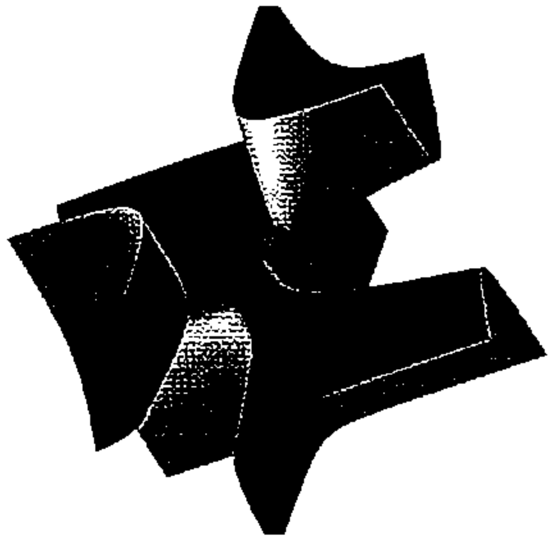

Figure 3: An Example of an F_5 cubic surface

5.2 The F_5 Case

When the cubic surface is of class F_5 (example shown in figure 3) it does not have any complex conjugate skew lines. One could attempt to use one real line and one complex line, or two non-conjugate complex skew lines, and proceed as before. However, there is no simple way to describe the values the parameters u and v may take on. In the F_1, F_2, and F_3 cases, u and v were unrestricted real parameters. In the F_4 case, when we let $u = \Re(u) + \Im(u)i$ and $v = \Re(v) + \Im(v)i$, we obtained a parametrization in which $\Re(u)$ and $\Im(u)$ are unrestricted, and then $\Re(v) = \Re(u)$ and $\Im(v) = -\Im(u)$. If we try the same idea with one real and one complex line, or two complex lines which are not conjugates, and let $\Re(u)$ and $\Im(u)$ be unrestricted, then $\Re(v)$ and $\Im(v)$ are complicated functions of $\Re(u)$ and $\Im(u)$, typically seventh degree polynomials.

In [21], a rational parametrization based on tangent planes at points lying on a real line is given. However, in general this only parametrizes part of the cubic surface. Points on the surface which do not lie on any tangent plane through a point on the chosen real line are missed, and these may account for a substantial portion of the surface. Since our goal is to parametrize the entire surface we instead parametrize the surface by parametrizing planes through one of the real lines on the surface, and then by parametrizing the conic sections which are the further intersections of these planes with the cubic surface. The parametrization of the conics will be that of [3]. One cost of parametrizing the whole surface is that we now have to use a square root in the parametrization. Another drawback of this parametrization is that there are typically two values of (u, v) corresponding to points on the cubic surface, instead of the one-to-one map resulting when both curves used in the parametrization are line, as in the F_1 through F_4 cases. Also, we have to use two distinct parametrizations; one which works when the conics are ellipses and the other for hyperbolas.

The procedure for finding the parametrization starts out like the ones for the F_1 through F_4 cases. In this case three coplanar real lines and 24 complex lines are determined, and the complex lines are found to come in 12 coplanar conjugate pairs. Since the methods of the other cases involving skew lines do not work here, one of the real lines is chosen to be mapped into the x-axis and the plane of the three real lines is mapped into the xy-plane. Specifically, suppose a real line $\mathbf{l}$ is given by $\mathbf{l}(u) = (A + Bu, C + Du, E + Fu)$ and that the normal to the plane is given by $\mathbf{N} = (N_1, N_2, N_3)$. $\mathbf{N}$ is obtained by taking the cross product of the (unit) direction vectors of two of the real lines, or by taking any unit vector perpendicular to the real lines if they are all parallel. Next, let $\mathbf{B} = (B_1, B_2, B_3)$ be the cross product of the direction vector of $\mathbf{l}$ with $\mathbf{N}$. We move a point on $\mathbf{l}$ to the origin by the translation $x = x' + A$, $y = y' + C$, $z = z' + E$, and then apply the transformation

$$\begin{aligned} x' &= (B_2N_3 - B_3N_2)x'' + (FN_2 - DN_3)y'' + (DB_3 - FB_2)z'' \\ y' &= (B_3N_1 - B_1N_3)x'' + (BN_3 - FN_1)y'' + (FB_1 - BB_3)z'' \\ z' &= (B_1N_2 - B_2N_1)x'' + (DN_1 - BN_2)y'' + (BB_2 - DB_1)z'' \ . \end{aligned} \tag{9}$$

This brings $\mathbf{l}$ to the x'' axis and the plane of the real lines to $z'' = 0$.

Planes through the x''-axis can be parametrized by $z'' = uy''$ for real values of u. All planes through the x''-axis are obtained except for $z'' = 0$, the plane containing the three real lines already found. The cubic surface now has an equation of the form $f''(x'', y'', z'') = 0$, and satisfies $f''(x'', 0, 0) = 0$. If we now make the substitution $z'' = uy''$ into $f''(x'', y'', z'')$, we obtain and equation that factors as $y''g''(x'', y'') = 0$, where $g''(x'', y'')$ is a quadratic in x'' and y''. The factor of y'' indicates that the line $y'' = 0$ is in the intersection of the cubic surface and the plane $z'' = uy''$ for any real u. The conic section $g(x'', y'') = 0$ is parametrized as in [3]: Let $g(x'', y'') = ax''^2 + by''^2 + cx''y'' + dx'' + ey'' + f$, and the discriminant $k = c^2 - 4ab$. The quantities a through f are polynomials in u.

If $k < 0$, the conic is an ellipse, and is parametrized by

$$x'' = \frac{[af(ce - 2bd) - d(t_2 + t_3)]v^2 + [df(ce - 2bd) - 2ft_3]v + f^2(ce - 2bd)}{a(t_1 + t_3)v^2 - df(c^2 - 4ab)v + f(t_1 - t_3)}$$

$$y'' = \frac{f(c^2-4ab)(av^2+dv+f)}{a(t_1+t_3)v^2 - df(c^2-4ab)v + f(t_1-t_3)},$$

where

$$t_1 = ae^2 + bd^2 - cde\ ,\ \ t_2 = t_1 + f(c^2-4ab)\ ,\ \ t_3 = \sqrt{t_1 t_2}\ .$$

This gives real points only when the terms t_1 and t_2 have the same sign or are zero. If t_1 and t_2 have opposite sign, $g(x'', y'') = 0$ has no real points, and geometrically this means that the plane $z'' = uy''$ intersects the cubic surface only in the x''-axis. Thus values of u should be restricted to those that give non-negative values for $t_1 t_2$. Upon back substitution using $z'' = uy''$ and (9), in the final parametrization x, y, and z are given by quotients of functions of the form $Q_1(u,v) + Q_2(u,v)\sqrt{Q_9(u)}$, where $Q_1(u,v)$ is of degree six in u and two in v, $Q_2(u,v)$ is of degree one in u and two in v, and $Q_3(u)$ is of degree nine in u alone. Due to the use of floating-point arithmetic, a nonzero coefficient for u^{10} may appear in $Q_3(u)$, and this is subtracted off in case it does show up.

If $k \geq 0$, the conic is a hyperbola or parabola, and is parametrized by

$$\begin{aligned} x'' &= \frac{a(c+\sqrt{c^2-4ab})v^2 + 2aev + f(c-\sqrt{c^2-4ab})}{2a\sqrt{c^2-4ab}\,v + 2ae - cd + d\sqrt{c^2-4ab}} \\ y'' &= \frac{-2a(av^2+dv+f)}{2a\sqrt{c^2-4ab}\,v + 2ae - cd + d\sqrt{c^2-4ab}}\ . \end{aligned}$$

Here real values are given for all u and v for which the denominators are nonzero. In the final parametrization x, y, and z are given by quotients of functions of the form $[Q_1(u,v) + Q_2(u,v)\sqrt{Q_3(u)}]$ $/[Q_4(u,v) + Q_5(u,v)\sqrt{Q_3(u)}]$, where $Q_1(u,v)$ is of degree three in u and two in v, $Q_2(u,v)$ is of degree one in u and two in v, $Q_3(u)$ is of degree four in u alone, $Q_4(u,v)$ is of degree three in u and one in v, and $Q_5(v)$ is of degree one in each of u and v.

6 Classification and Straight Lines from Parametric Equations

We also consider the question of deriving a classification and generating the straight lines of the cubic surface given its rational parametric equations (equation (1) above):

$$x = \frac{f_1(u,v)}{f_4(u,v)},\ y = \frac{f_2(u,v)}{f_4(u,v)},\ z = \frac{f_3(u,v)}{f_4(u,v)},$$

Note that given an arbitrary parametrization, the fact that it belongs to a cubic surface can be computed by determining the parametrization base points and multiplicities.

The computation of real base points which are the simultaneous zeros of $f_1 = f_2 = f_3 = f_4 = 0$, are obtained by first computing the real zeros of $f_1 = f_2 = 0$ using resultants and subresultants, via the method of birational maps [5] and then keeping those zeros which also satisfy $f_3 = f_4 = 0$. The classification follows from the reality of the base points, as detailed in the preliminaries section.

Having determined the base points, the straight lines on the cubic surface are then determined by the image of these points and combinations of them. In general there can be six real base points for cubic surfaces. The image of each of the six base points under the parametrization map yields a straight line on the surface. Next the fifteen pairs of base points define lines in the u, v parameter space, whose images under the parametrization map also yield straight lines. Finally the six different conics in the u, v parameter space which pass through distinct sets of five base points, also yield straight line images under the parametrization map. See Bajaj and Royappa [8] for techniques to find parametric representations of the straight lines which are images of these base points. The question of determining parametric representations of the straight lines which are the images of parameter lines or parameter conics is for now, open.

Normally a cubic surface parametrization has six base points, but in the case of our parametrizations for the F_1, F_2, F_3, and F_4 surfaces, the number of base points is reduced to five. This happens because the degree of the parametrization is sufficiently small: neither u nor v appears to a power higher than the second. Consider the intersection of the parametrized surface with a line in 3-space. Let the line be given as the intersection of two planes $a_i x + b_i y + c_i z + d_i = 0$ for $i = 1, 2$. Then when the substitutions $x = f_1(u,v)/f_4(u,v)$, $y = f_2(u,v)/f_4(u,v)$, $z = f_3(u,v)/f_4(u,v)$ are made into the equations of the lines, we obtain polynomials of degree two in each of u and v. When resultants of these polynomials are taken to eliminate either u or v, univariate polynomials of degree eight are obtained. This indicates that there could be as many as eight intersection points of the line with the surface. However, a cubic surface will intersect the line in only three (possibly complex) points, counting multiplicity and solutions at infinity. The difference between these two results (eight and three) is the number of base points. A cubic parametrization would have led to nine possible intersection points when considering the algebraic equations, and hence six, the difference of nine and three, is the number of base points for such a parametrization.

Let $\mathbf{l}_1$ and $\mathbf{l}_2$ be the two skew lines used in the parametrization, whether they be real or complex. The base points (u, v) correspond to lines on the cubic surface which intersect both $\mathbf{l}_1$ and $\mathbf{l}_2$. Real base points correspond to real lines and complex base points correspond to complex lines. One of the many useful results on nonsingular cubic surfaces is that given any two (real or complex) skew lines on the surface, there are exactly five lines that intersect both [22]. For an F_1 surface, the five transversal lines, and the base points, are all real. Thus those five real lines are missed by the parametrization (1). For an F_2 surface, three of the base points are real and the other two form a complex conjugate pair. The parametrization (1) consequently misses the three real lines incident to both $\mathbf{l}_1$ and $\mathbf{l}_2$. In addition, if the two transversal complex conjugate lines are coplanar and have a real intersection point, that point is also missed. For both F_3 and F_4 surfaces, one base point is real and the other four form two conjugate pairs. In each of these cases there is one real line through both $\mathbf{l}_1$ and $\mathbf{l}_2$, and that line is missed. Again, if a pair of transversal complex conjugate lines is coplanar, their real intersection point is missed, so there may be two such isolated points for F_3 and F_4. As will be demonstrated in the example below, the missing points on the surface can be approached as (u, v) approaches the corresponding base point in an appropriate manner.

In addition to the transversal lines, two conic sections are also missed in the parametrization of the F_1, F_2, and F_3 surfaces. One conic is obtained as follows: take the intersection of the plane containing $\mathbf{l}_1(u)$ and perpendicular to $\mathbf{l}_2(v)$ with the cubic surface. This intersection consists of $\mathbf{l}_1$ plus a conic. It turns out that the value of v at which $\mathbf{l}_2$ intersects this plane tends to $\pm\infty$. Thus points on the conic are not obtained for finite values of v, even though the line $\mathbf{l}_1$ does turn out to be reachable. The other missing conic is found by interchanging the roles of $\mathbf{l}_1$ and $\mathbf{l}_2$. These two conics lie on parallel planes, and may be obtained if the parametrization uses projective coordinates as indicated in the example below.

7 Conclusion and Future Research

We have presented a method of extracting real straight lines and from there a rational parametrization of each of four families of nonsingular cubic surfaces. The parametrizations of the real cubic surface components are constructed using a pair of real skew lines for those three families which have them, and remarkably using a complex conjugate pair of skew lines, in a fourth family. In each of these, the entire real surface is covered except for one, three, or five lines which intersect both skew lines, one or two isolated points, and two conic sections. The missing conics can be recovered through the use of projective instead of real coordinates. For the last family, in which two real skew lines do not exist, in order to cover the whole surface we had to use two separate parametrizations, each involving a square root. Fortunately many graphics applications, such as the triangulation of a real surface, will involve only the classes of cubics which do contain real skew lines. These real skew lines will correspond to non-intersecting edges of the tetrahedra. Open problems remain in computing the images of curves containing the real base points in the parameter plane. All figures of the cubic surfaces shown in this paper were made using the GANITH toolkit [8].

An additional associated line of future research is in computing invariants for cubic surfaces based on its straight lines. In Computer Vision, as pointed out by Holt-Netravali [14] and Mundy-Zisserman [17], it is essential to derive properties of curves and surfaces which are invariant to perspective projection and to be able to compute these invariants reliably from perspective image intensity data. In connection with FFT (= First Fundamental Theorem of Invariant Theory), referring to Abhyankar [1] and Mundy-Zisserman [17] for details, we attempt to calculate complete systems of symbolic invariants of cubic surfaces. In doing these calculations, it is important to know all the relations between a set of invariants which is the content of SFT (= Second Fundamental Theorem of Invariant Theory).

Turning to our specific situation, we may derive projective invariants of a cubic surface from simultaneous invariants of the 27 lines on it. Namely, by taking the coefficients of two planes through a line in 3-space we get a 2×4 matrix whose 2×2 minors are the six Grassmann coordinates of the line. Thus we get a 27×6 matrix; its 6×6 minors are invariants and pure covariants as well as dot products between them. This is the FFT of vector invariants. In this paper, since we have derived an effective classification of cubic surfaces based on the line configurations, we can now derive these invariants (symbolically). Details of this procedure are left to a subsequent paper.

Appendix A: Examples

In this appendix we provide examples of the parametrization of F_1 and F_4 cubic surfaces.

An F_1 Surface

The F_1 surface is given by the implicit equation

$$f(x,y,z) = 16x^3 - 10y^3 - 156z^3 + 3x^2y + 101x^2z - 38xy^2 + 72y^2z + 39xz^2 - 74yz^2 - 81xyz$$
$$- 32x^2 + 20y^2 + 475z^2 + 81xy - 17xz + 81yz - 480z = 0 \ .$$

Two skew lines on this surface are $\mathbf{l}_1(u) = (u+2, -u, -u)$ and $\mathbf{l}_2(v) = (1, v-4, v/3)$. See figure 4. With these lines, we obtain the parametrization $(x(u,v), y(u,v), z(u,v)) = (f_1(u,v)/f_4(u,v), f_2(u,v)$

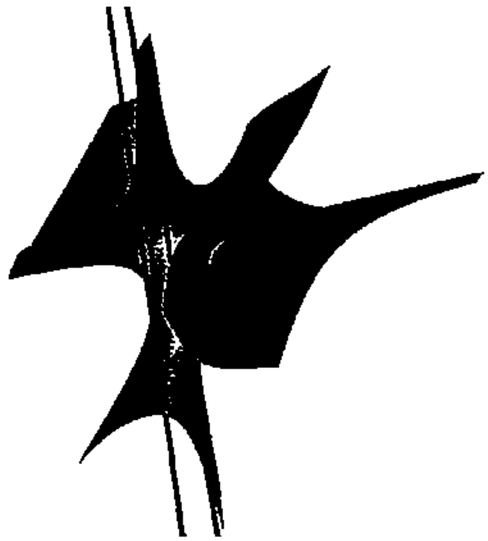

Figure 4: An F_1 cubic surface with two skew real straight lines.

$/f_4(u,v), f_3(u,v)/f_4(u,v))$ where

$$
\begin{aligned}
f_1 &= 185u^2v^2 - 2391u^2v + 4122u^2 + 467uv^2 - 7671uv + 18630u + 194v^2 - 4860v + 14400 \\
f_2 &= 55u^2v^2 - 849u^2v + 3438u^2 + 233uv^2 - 2145uv + 5346u + 618v^2 - 5892v + 13680 \\
f_3 &= -105u^2v^2 + 1791u^2v - 6642u^2 + 13uv^2 + 1551uv - 8910u + 206v^2 - 1140v \\
f_4 &= 240u^2v - 2520u^2 + 185uv^2 - 2301uv + 3078u + 97v^2 - 2121v + 5490 \ .
\end{aligned}
\tag{10}
$$

The five base points, where $f_1 = f_2 = f_3 = f_4 = 0$, are $(u,v) = (-1, 9/2), (-5/4, 5), (-12, 114/11)$, $(-37/29, 81/16)$, and $(-29/15, 156/23)$. These correspond to the lines $(1,\ w+2,\ -w)$, $(w,\ -w+2,\ 5/3\,w)$, $(w+256/47,\ -62/121\,w+192/47,\ -94/121\,w)$, $(w-25/191,\ -99/128\,w+370/191,\ 191/128\,w)$, and $(w-615/113,\ 293/322\,w-348/113,\ 113/322\,w)$, respectively. As an example of what is meant by this correspondence, consider an arbitrary point (x,y,z) in 3-space. The values of u_0 and v_0 for which the points (x,y,z), $(u_0+2, -u_0, -u_0)$ and $(1, v_0-4, v_0/3)$ are collinear are given by

$$
(u_0,\ v_0) = \left(\frac{-4x+y-3z+8}{2x-y+3z-6},\ \frac{3(4x-y+5z-8)}{2x-y+3z-4}\right) \ . \tag{11}
$$

When $(x,y,z) = (1, w+2, -w)$ is plugged into this expression, we obtain $(u_0,\ v_0) = (-1, 9/2)$. Since this is a base point, however, plugging this into (10) yields 0/0 for x, y, and z.

It is evident from (11) that a point (x,y,z) on the cubic surface will be missed when a denominator is zero while the corresponding numerator is not. In this example these points lie on the planes E_1, given by $2x - y + 3z = 6$, and E_2, given by $2x - y + 3z = 4$. E_1 contains $\mathbf{l}_2$ and is perpendicular to $\mathbf{l}_1$, while E_2 contains $\mathbf{l}_1$ and is perpendicular to $\mathbf{l}_2$. Note that E_1 and E_2 are parallel.

The intersection of E_1 with the cubic surface consists of the line $\mathbf{l}_2$ and a conic section. Solving for z from the equation for E_1 and substituting that into $f(x, y, z) = 0$ yields $(x-1)(55x^2 - 185y^2 - 130xy - 519x + 1281y + 1386) = 0$. When both $x = 1$ and $2x - y + 3z = 6$, the numerator for u_0 in (11) is zero as well as the denominator. The result of this is that points on $\mathbf{l}_2$ are reached by (10). Substituting in $u = 1$ gives $u_0 = -1$, and substituting $u = -1$ into (10) gives $(x, y, z) = (1, -(55v - 327)/(11v - 3), -(55v - 327)/(11v - 3))$, which does in fact give $\mathbf{l}_2$ (except for the point (1, 35/31, 9/31), obtained when $v = 9/2$, a base point. The f_i/f_4 approach different values as $(u, v) \to (-1, 9/2)$ in different directions.)

On the other hand, the points on the conic cannot be obtained by this parametrization, but can only be approached by letting $u \to \pm\infty$. In fact, the conic can be parametrized by letting $u \to \pm\infty$ in (10). The result is

$$(x, y, z) = \left(\frac{185v^2 - 2391v + 4122}{120(2v - 21)}, \frac{55v^2 - 849v + 3438}{120(2v - 21)}, \frac{-35v^2 + 597v - 2214}{40(2v - 21)}\right) .$$

Another way of handling this is to use projective coordinates. We can replace u by u/u^* and v by v/v^* in (10) and clear denominators, yielding polynomials that are biquadratic in $\{u, u^*\}$ and $\{v, v^*\}$. The variables would then range over the Cartesian cross product of two one-dimensional projective spaces. Points on the above conic are attained for $u^* = 0$ and $v^* = 1$. Naturally a symmetric argument holds for the intersection of plane E_2 with the cubic surface, and points on the corresponding conic are attained when $u^* = 1$ and $v^* = 0$.

The projective space approach still will not resolve the problems with the base points. However, every point on the lines that are missed can be approached as (u, v) approaches the corresponding base point in the appropriate manner. For example, consider the transversal line $(1, w + 2, -w)$ from above, with base point $(-1, 9/2)$. If we let (u, v) approach $(-1, 9/2)$ along the line given by $(u, v) = (-1 + m, 9/2 - [3(289w + 111)]/[16(31w + 9)]m)$, then $(x, y, z) \to (1, w + 2, -w)$ as $m \to 0$. This gets us every point on $(1, w + 2, -w)$ except for $w = -9/31$, but that point is approached if $(u, v) \to (-1, 9/2)$ along the line $u = -1$.

An F_4 Surface

Figure 5: An F_4 cubic surface

The F_4 surface (shown in figure 5) is given by the implicit equation

$$f(x,y,z) = 1696x^3 - 1196y^3 + 881z^3 - 2984x^2y - 62x^2z + 2424xy^2 + 1174y^2z - 913xz^2 - 781yz^2 \\ + 450xyz - 1802x^2 + 443y^2 - 1217z^2 + 1786xz + 266xy - 1596yz + 1696z = 0 \ .$$

Two skew complex conjugate lines on this surface are $\mathrm{l}_1(u) = ((1-i)u+1+i,\ (-1+2i)u+2-i,\ (-2-3i)u+3+2i)$ and $\mathrm{l}_2(u) = ((1+i)u+1-i,\ (-1-2i)u+2+i,\ (-2+3i)u+3-2i)$. With these lines, we obtain the parametrization $(x(u,v), y(u,v), z(u,v)) = (f_1(u,v)/f_4(u,v), f_2(u,v)/f_4(u,v), f_3(u,v)/f_4(u,v))$ where

$$\begin{aligned}
f_1 &= 68358u^4 - 69411u^3 + 136716u^2v^2 + 42607u^2v - 22381u^2 - 69411uv^2 \\
&\quad - 39230uv + 43253u + 68358v^4 + 42607v^3 - 5775v^2 + 8221v - 11755 \\
f_2 &= -68958u^4 + 284194u^3 - 137916u^2v^2 + 4441u^2v - 366491u^2 + 284194uv^2 \\
&\quad + 11300uv + 193570u - 68958v^4 + 4441v^3 - 124361v^2 - 8901v - 36677 \qquad (12) \\
f_3 &= -133716u^4 + 417667u^3 - 267432u^2v^2 - 37422u^2v - 466042u^2 + 417667uv^2 \\
&\quad + 58622uv + 224171u - 133716v^4 - 37422v^3 - 164742v^2 - 22866v - 39654 \\
f_4 &= 2(33879u^3 + 300u^2v - 62530u^2 + 33879uv^2 + 3994uv \\
&\quad + 38739u + 300v^3 - 22624v^2 - 2804v - 8072) \ .
\end{aligned}$$

The real base point is $(u,v) = (2/3, -1/6)$, which corresponds to the line $(w+1,\ 3w+1/6,\ 2w+1/6)$. The four complex base points, $(0.67336 \pm 0.02735i, -0.07294 \pm 0.11195i)$ and $(0.69678 \pm 0.02251i, -0.05028 \mp 0.13900i)$ correspond to the pairs of skew complex conjugate lines $(w + 0.16675 \mp 0.18781i, (0.93864 - 0.59824i)w + 0.72700 \mp 0.06977i, (0.55461 \mp 0.58502i) * w)$ and $(w + 1.45840 + 0.89959i, (1.26868 + 1.30057i)w + 0.09568 + 0.80755i, (0.31897 + 1.11820i)w)$, respectively. Since these complex conjugate lines are skew, no isolated real points are missed by the F_4 parametrization here. Also, since the lines l_1 and l_2 are complex, there are no real conics missed that lie in the planes containing one of these lines and perpendicular to the other, as was the case in the F_1 example. Indeed, if we let u and/or v approach $\pm\infty$ in (12), all three of (x,y,z) become infinite. Because of this property it may be desirable to use the skew complex-line parametrization in the other cases in which it may be used, namely the F_2 and F_3 surfaces.

Appendix B

Theorem 1: The polynomial $P_{81}(t)$ obtained by taking the resultant of $\hat{f}_2$ and $\hat{g}_3$ factors as $P_{81}(t) = P_{27}(t)[P_3(t)]^6[P_6(t)]^6$, where $P_3(t) = B''t^3 + F''t^2 + D''t + A''$, the denominator of $K(t)$ and $L(t)$, and $P_6(t)$ is the numerator of $\overline{S}(t)$ $\quad (P_6(t) = \overline{S}(t)[P_3(t)^2])$.

Proof: This proof was performed through the use of Maple. When expanded out in full, $P_{81}(t)$ contains hundreds of thousands of terms, so a direct approach was not possible. Instead, $P_{81}(t)$ was shown to be divisible by both $[P_3(t)]^6$ and $[P_6(t)]^6$.

The quantities $\hat{f}_2$ and $\hat{g}_3$ were expressed in terms of the numerators of $\overline{Q}(t)$, $\overline{R}(t)$, and $\overline{S}(t)$, and the numerator and denominator of $K(t)$. Let $K(t) = P_2(t)/P_3(t)$, where

$$\begin{aligned} P_2(t) &= -(Lt^2 + Nt + K) \\ P_3(t) &= Bt^3 + Ft^2 + Dt + A \end{aligned} \tag{13}$$

(For brevity in this appendix we drop the double primes on the coefficients A'' through P'' of $f(x'', y'', z'')$.) Then we have

$$\begin{aligned} \overline{Q}(t) &= \frac{[(Ft^2 + 2Dt + 3A)P_2(t) + (Nt + 2K)P_3(t)]P_2(t)}{[P_3(t)]^2} = \frac{Q^*}{P_3^2} \\ \overline{R}(t) &= \frac{[(3Bt^2 + 2Ft + D)P_2(t) + (2Lt + N)P_3(t)]P_2(t)}{[P_3(t)]^2} = \frac{R^*}{P_3^2} \\ \overline{S}(t) &= \frac{(Gt^2 + Jt + E)[P_2(t)]^2 + (Pt + O)P_2(t)P_3(t) + S[P_3(t)]^2}{[P_3(t)]^2} = \frac{S^*}{P_3^2}\ . \end{aligned} \tag{14}$$

Then we obtain

$$
\begin{aligned}
\hat{f}_2 \;=\; & \{[(It+H)P_2+MP_3]Q^{*2}-[(Jt+2E)P_2+oP_3]Q^*S^*+[(Dt+3A)P_2+KP_3]S^{*2}\}\hat{x}^2 \\
& +\{[(2It+2H)P_2+2MP_3]Q^*R^*-[(2Gt+J)P_2+PP_3]Q^*S^* \\
& \quad -[(Jt+2E)P_2+OP_3]R^*S^*+[(2Ft+2D)P_2+NP_3]S^{*2}\}\hat{x}\hat{y} \\
& +\{[(It+H)P_2+MP_3]R^{*2}-[(2Gt+J)P_2+PP_3]R^*S^*+[(3Bt+F)P_2+LP_3]S^{*2}\}\hat{y}^2 \ ,
\end{aligned}
$$

$$
\begin{aligned}
\hat{g}_3 \;=\; & (-CQ^{*3}+HQ^{*2}S^*-EQ^*S^{*2}+AS^{*3})\hat{x}^3 \\
& +(-3CQ^{*2}R^*+IQ^{*2}S^*+2HQ^*R^*S^*-JQ^*S^{*2}-ER^*S^{*2}+DS^{*3})\hat{x}^2\hat{y} \\
& +(-3CQ^*R^{*2}+2IQ^*R^*S^*-GQ^*S^{*2}+HR^{*2}S^*-JR^*S^{*2}+FS^{*3})\hat{x}\hat{y}^2 \\
& +(-CR^{*3}+IR^{*2}S^*-GR^*S^{*2}+BS^{*3})\hat{y}^3 \ .
\end{aligned}
$$

With this representation it was possible to take the resultant of $\hat{f}_2$ and $\hat{g}_3$ with respect to $\hat{x}$ without overflowing the memory capabilities of the machine. The resultant could be factored, and $[P_6(t)]^6$ was found to be one of the factors.

The factor $[P_3(t)]^6$ proved to be more difficult to obtain. After the factor $[P_6(t)]^6$ was removed from the resultant, the substitution $Q^* = P_2^2P_3 - tR^*$ was used to eliminate Q^* from the remaining factor. This remaining factor was split into 28 terms as follows:

$$
\begin{aligned}
& A_1R^{*6}+A_2R^{*5}S^*+A_3R^{*5}+A_4R^{*4}S^{*2}+A_5R^{*4}S^*+A_6R^{*4}+A_7R^{*3}S^{*3} \\
& +A_8R^{*3}S^{*2}+A_9R^{*3}S^*+A_{10}R^{*3}+A_{11}R^{*2}S^{*4}+A_{12}R^{*2}S^{*3}+A_{13}R^{*2}S^{*2}+A_{14}R^{*2}S^* \\
& +A_{15}R^{*2}+A_{16}R^*S^{*5}+A_{17}R^*S^{*4}+A_{18}R^*S^{*3}+A_{19}R^*S^{*2}+A_{20}R^*S^*+A_{21}R^* \\
& +A_{22}S^{*6}+A_{23}S^{*5}+A_{24}S^{*4}+A_{25}S^{*3}+A_{26}S^{*2}+A_{27}S^*+A_{28}
\end{aligned}
\tag{15}
$$

The coefficients A_i are functions of A through P, P_2, and P_3, and range from 76 terms in the case of A_{22} to 1674 terms for A_5. Thus these coefficients must be omitted here for space considerations. Next these substitutions were made:

$$
\begin{aligned}
R^* &= M_2P_2^2+N_2P_2P_3 \\
S^* &= M_3P_2^2+N_3P_2P_3+SP_3^2 \ .
\end{aligned}
\tag{16}
$$

Later on these substitutions will be made:

$$
\begin{aligned}
M_2 &= 3Bt^2+2Ft+D \quad & M_3 &= Gt^2+Jt+E \\
N_2 &= 2Lt+N & N_3 &= Pt+O
\end{aligned}
\tag{17}
$$

so that the system (16, 17) agrees with the definitions of (13, 14). The reason behind these substitutions is to express the resultant in terms of P_3 as much as possible so as to be able to more readily determine what powers of P_3 divide into the coefficients A_i.

Upon making the substitutions in (16), each of the terms $A_i R^{*j} S^{*k}$ becomes a term B_i, where the B_i are functions of A through P, P_2, and P_3. The number of terms in the B_i ranges from 140 for B_{28} to 48960 for B_7. Each B_i can be regarded as a polynomial in P_3. The highest power of P_3 appearing in any term is P_3^{15}, in B_{28}. Since we are trying to show that $\sum_{i=1}^{28} B_i$ is divisible by P_3^6, we need only consider the terms of the B_i which do not contain any power of P_3 greater than or equal to six. That is,

$$\text{if } B_i = \sum_{i=0}^{15} b_i P_3^i \ ,$$
$$\text{let } C_i = \sum_{i=0}^{5} b_i P_3^i \ .$$

It turns out that each of the C_i is divisible by P_2^{10}, so let

$$D_i = C_i / P_2^{10} \ .$$

We now make the substitutions

$$A = P_3 - Bt^3 - Ft^2 - Dt$$
$$K = -P_2 - Lt^2 - Nt$$

into the terms D_i to produce more terms E_i. The latter are now functions of $B, C, \ldots, J, L, M, \ldots, P, P_2$, and P_3. Each of the E_i turns out to be divisible by P_3^2. As was the case with the B_i, we remove powers of P_3 greater than or equal to six from the E_i. When we do that, all of the resulting terms are divisible by P_2. Thus,

$$\text{if } E_i = \sum_{i=2}^{8} e_i P_3^i \ ,$$
$$\text{let } F_i = (\sum_{i=2}^{5} b_i P_3^i)/P_2 \ .$$

(the highest power of P_3 appearing in the E_i is 8, in seven of the E_i.)

The sum of all the terms of the F_i is 61170. Since this is less than 2^{16}, all of the F_i can be added together in Maple to obtain one large expression. This can be expressed as a polynomial in P_2 and P_3 as follows:

$$(G_1P_2^4 + G_2P_2^3 + G_3P_2^2 + G_4P_2 + G_5)P_3^5 + (G_6P_2^4 + G_7P_2^3 + G_8P_2^2 + G_9P_2)P_3^4$$
$$+ (G_{10}P_2^4 + G_{11}P_2^3 + G_{12}P_2^2)P_3^3 + (G_{13}P_2^4 + G_{14}P_2^3)P_3^2 \ .$$

Through the use of Maple we were able to show that each of the four terms enclosed in parentheses in (18) vanish. The fourth term, $(G_{13}P_2^4+G_{14}P_2^3)$, was shown to be zero by making the three substitutions of (17), namely $N_2 = 2Lt+N$, $N_3 = Pt+O$, and (after simplifying) $M_3 = Gt^2+Jt+E$, then determining that the result was divisible by $M_2 - 3Bt^2 - 2Ft - D$. The same procedure worked for the third term in parentheses in (18), $(G_{10}P_2^4 + G_{11}P_2^3 + G_{12}P_2^2)$, and for these combinations: $(G_6P_2^4 + G_7P_2^3)$, $G_8P_2^2$, G_9P_2, $(G_1P_2^4 + G_2P_2^3 + G_3P_2^2)$, G_4P_2, and G_5. Thus the expression in (18) vanishes, and since this is the remainder of the resultant (15) upon division by P_3^6, we conclude that the entire expression (15) is divisible by P_3^6. ∎

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