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1992

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Report Number:

92-060

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CSD-TR-92-060
September 1992

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Technical Report CSD-TR-92-060
CAPO Report CER-92-26
September 1992

Abstract

To solve the real nonsingular linear system $Ax = b$ (1) on parallel and vector machines, we consider multisplitting methods, m-step preconditioners and m-step additive preconditioners, generalizing some of the results and methods developed in previous related works. In particular we generalize the method and the corresponding convergence results in [14], and determine suitable relaxed m-step preconditioners ([1], [6]) treating also the problem of minimizing the related condition number, with respect to the relaxation (extrapolation) parameter involved, in various cases. We also generalize the theory for determining suitable m-step additive preconditioners [2] and finally we solve completely the problem of determining the optimum SOR-additive iterative method [2] for 2-cyclic positive definite matrices.

Key words and phrases: multisplitting methods, m-step preconditioners, extrapolation method, successive overrelaxation (SOR) method.

AMS (MOS) Subject Classifications: 65F10. CR categories: 5.14.

[‡]Research supported in part by NSF grant CCR-8619817 and by AFOSR grant 8810243.
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1 Introduction

For solving the large nonsingular linear system of equations

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n,n}$, $b \in \mathbb{R}^n$, parallel iterative methods, called multisplitting methods, were introduced in [12]. According to [12], given a multisplitting of A

$$A = M_k - N_k, \qquad \det(M_k) \neq 0, \qquad k = 1(1)p, \tag{1.2}$$

the corresponding multisplitting method is defined by

$$x^{(m+1)} = \sum_{k=1}^{p} D_k M_k^{-1} N_k x^{(m)} + \sum_{k=1}^{p} D_k M_k^{-1} b, \qquad m = 0, 1, 2, \ldots, \tag{1.3}$$

where D_k is a diagonal matrix, with $D_k \geq 0$, $k = 1(1)p$, and $\sum_{k=1}^{p} D_k = I$. Setting

$$H = \sum_{k=1}^{p} D_k M_k^{-1} N_k \qquad \text{and} \qquad G = \sum_{k=1}^{p} D_k M_k^{-1}, \tag{1.4}$$

(1.3) takes the form

$$x^{(m+1)} = H\, x^{(m)} + c, \qquad m = 0, 1, 2, \ldots, \tag{1.5}$$

where $c = Gb$. Moreover we have

$$H = I - GA. \tag{1.6}$$

According to [18], Thm. 2.6, p. 68, (1.5) is consistent with (1.1). Furthermore (1.5) is completely consistent with (1.1) iff G is nonsingular. From now on we assume that (1.5) is completely consistent with (1.1); hence it is obvious that (1.5) can be obtained using the splitting

$$A = G^{-1} - G^{-1}H. \tag{1.7}$$

It is well known that (1.5) converges to $A^{-1}b$ for any starting vector $x^{(0)}$ iff $\rho(H) < 1$, where $\rho(\cdot)$ denotes spectral radius. Convergence results of (1.5), under various assumptions, can be found in the literature (see, e.g., [4], [5], [7], [8], [11], [12], [14], [16], [17]).

In [1], [6] for the linear system (1.1), where A is positive definite (cf. [18], p. 21) a splitting $A = M - N$, $\det(M) \neq 0$, is considered, where M is positive definite and $\rho(M^{-1}N) < 1$, and the associated preconditioning matrix or m-step preconditioner is defined by

$$M_m = M(I + G + G^2 + \ldots + G^{m-1})^{-1}, \qquad m > 1, \tag{1.8}$$

where $G = M^{-1}N$. If $A \approx M$, then M_m is an improved approximation to A and is used instead of M for accelerating the rate of convergence of Chebyshev and Conjugate Gradient methods. Also in

[2] for the same purpose m-step additive preconditioners are defined, which are connected with the multisplitting method (1.5) for $p = 2$ and $D_1 = D_2 = \frac{1}{2}I$. In particular, in [2] the SOR-additive preconditioner is defined and an optimal value ω_{opt} for the parameter ω of the 2-cyclic SOR-additive iterative method is also determined.

In the present paper we give in Section 2 two theorems concerning the convergence of the method (1.5), when: (i) A in (1.1) satisfies $A^{-1} \geq 0$ and (1.2) are weak regular splittings (cf. [3]) and (ii) A is positive definite and (1.2) are P-regular splittings (see [13]). Also in Section 2 we generalize the two-splitting method (method of the arithmetic mean) treated in [14] and prove some theorems which generalize Thms 1, 2, 3 in [14]. In Section 3 we give a method for finding a suitable m-step preconditioner M_m, $m \geq 1$, for system (1.1). The given preconditioner contains a parameter ω and we determine in more than half of the cases the optimal value of ω so that the condition number of $M_m^{-1}A$ is minimized. We also generalize the procedure given in [2] for defining m-step additive preconditioners and prove a theorem giving sufficient conditions for determining suitable additive preconditioners. Finally, in Section 4 we completely solve the problem of determining the optimal ω of the SOR-additive iterative method studied in [2]. As we show the theoretical analysis in [2] concerning this problem was not complete.

2 Convergence Results

We consider the linear system (1.1) and the multisplitting method (1.5). Then we obtain the following results which are useful in the sequel (see also Thm 1 (a), (b) in [12] and Thm 1 and Cor 1 in [17]).

Theorem 2.1

If in (1.1) $A^{-1} \geq 0$ and (1.2) are weak regular splittings of A, then (1.7) is also a weak regular splitting of A; hence (1.5) converges ($\rho(H) < 1$).

Proof

It follows from Thm 1 and Cor 1 in [17]. □

Theorem 2.2

If A in (1.1) is positive definite, (1.2) are P-regular splittings of A and $D_k = a_k I$ ($a_k \geq 0$, $\sum_{k=1}^{p} a_k = 1$), then (1.7) is also a P-regular splitting of A; hence (1.5) converges.

Proof

From the hypothesis M_k is nonsingular and $M_k + N_k$ is positive real (see [18], Thm 2.9, p. 24), i.e., $M_k + N_k + (M_k + N_k)^T$ is positive definite or equivalently $M_k + M_k^T - A$, $k = 1(1)p$, is positive definite (C^T denotes the transpose of C). Since A is positive definite, according to [18], Thm 5.3, p. 79, it suffices to show that

$$M + M^T - A = \frac{1}{2}[M + N + (M+N)^T] \tag{2.1}$$

is positive definite, where $M = G^{-1}$, $N = G^{-1}H$ ($A = M - N$), or equivalently that

$$M^{-1}(M + M^T - A)M^{-T} = M^{-T} + M^{-1} - M^{-1}AM^{-T} =: Q \tag{2.2}$$

is positive definite. Thus we have

$$\begin{aligned} Q &= \textstyle\sum_{k=1}^{p} a_k(M_k^{-T} + M_k^{-1} - M_k^{-1}AM_k^{-T}) + \sum_{k=1}^{p} a_k M_k^{-1}AM_k^{-T} \\ &- \left(\textstyle\sum_{k=1}^{p} a_k M_k^{-1}\right) A \left(\textstyle\sum_{k=1}^{p} a_k M_k^{-T}\right). \end{aligned}$$

The matrix $S_1 \equiv \sum_{k=1}^{p} a_k(M_k^{-T} + M_k^{-1} - M_k^{-1}AM_k^{-T}) = \sum_{k=1}^{p} a_k M_k^{-1}(M_k + M_k^T - A)M_k^{-T}$ is positive definite, since $a_k \geq 0$ and $M_k^{-1}(M_k + M_k^T - A)M_k^{-T}$, $k = 1(1)p$, is positive definite. Moreover, for the symmetric matrix $S_2 \equiv Q - S_1$ we have

$$\begin{aligned} S_2 &= \left(\textstyle\sum_{j=1}^{p} a_j\right)\left(\textstyle\sum_{k=1}^{p} a_k M_k^{-1}AM_k^{-T}\right) - \textstyle\sum_{k,j=1}^{p} a_k a_j M_k^{-1}AM_j^{-T} \\ &= \textstyle\sum_{k,j=1}^{p} a_k a_j M_k^{-1}AM_k^{-T} - \sum_{k,j=1}^{p} a_k a_j M_k^{-1}AM_j^{-T} \\ &= \textstyle\sum_{k,j=1}^{p} a_k a_j [M_k^{-1}AM_k^{-T} - M_k^{-1}AM_j^{-T}]. \end{aligned}$$

Hence

$$\begin{aligned} 2S_2 &= S_2 + S_2^T \\ &= \textstyle\sum_{k,j=1}^{p} a_k a_j (M_k^{-1}AM_k^{-T} - M_k^{-1}AM_j^{-T}) + \sum_{k,j=1}^{p} a_k a_j (M_k^{-1}AM_k^{-T} - M_j^{-1}AM_k^{-T}) \\ &= \textstyle\sum_{k,j=1}^{p} a_k a_j (M_k^{-1}AM_k^{-T} - M_k^{-1}AM_j^{-T} + M_j^{-1}AM_j^{-T} - M_j^{-1}AM_k^{-T}) \\ &= \textstyle\sum_{k,j=1}^{p} a_k a_j \left[(M_k^{-1} - M_j^{-1})A(M_k^{-1} - M_j^{-1})^T\right]. \end{aligned}$$

S_2, as a sum of nonnegative definite matrices, is nonnegative definite. This implies that Q is positive definite and that $A = G^{-1} - G^{-1}H$ is a P-regular splitting of A; hence $\rho(H) < 1$. □

Remarks

i) As one can see the proof in Theorem 2.2 parallels that of Thm 1(b) in [12]. However, it is based on a simpler (equivalent) theorem than that in [12]. This makes the corresponding expressions for

S_1 and S_2 be simpler and easier to handle. ii) Note that S_2 may be nonnegative definite iff all M_j, $j = 1(1)p$, share a common eigenvalue-eigenvector pair.

In the following a generalization, in various directions, of the method of the arithmetic mean of [14] is suggested. Consider the splittings of A

$$A = M_k - N_k, \qquad \det(M_k) \neq 0, \qquad k = 1(1)2q, \tag{2.3}$$

where

$$M_k = \frac{1}{\omega} D + W_k - L, \quad N_k = (\frac{1}{\omega} - 1)\, D + W_k + U, \quad k = 1(1)q, \tag{2.4}$$

and

$$M_k = \frac{1}{\omega} D + W_k - U, \quad N_k = (\frac{1}{\omega} - 1) D + W_k + L, \quad k = q + 1(1)2q. \tag{2.5}$$

In (2.4), (2.5) W_k is a diagonal matrix, $W_k > 0$, $k = 1(1)2q$, and ω a real positive parameter. For the corresponding multisplitting method (1.5), where $p = 2q$ and M_k is given by (2.4), (2.5), $k = 1(1)2q$, we prove the theorems below, which generalize Thms 1, 2, 3 in [14]. We simply mention that in [14], $p = 2$, $\omega = 1$, $W_k = \rho W (\rho > 0,\ W > 0)$, and $D_1 = D_2 = \frac{1}{2} I$.

Theorem 2.3

If A in (1.1) is an irreducibly diagonally dominant L-matrix ([15], p. 23 and [18], p. 42), then the multisplitting method (1.5), where $p = 2q$, M_k is given by (2.4), (2.5), $k = 1(1)2q$, and $0 < \omega \leq 1$, converges.

Proof

The matrix M_k is nonsingular, since $D > 0$, $W_k > 0$ and $\omega > 0$, $k = 1(1)2q$. According to the hypothesis (see [15], Cor 1, p. 85) A is a nonsingular M-matrix with $A^{-1} > 0$. Obviously M_k is a strictly diagonally dominant L-matrix, $k = 1(1)2q$; hence M_k is an M-matrix and therefore $M_k^{-1} \geq 0$, $k = 1(1)2q$. We also have $N_k \geq 0$, $k = 1(1)2q$. Consequently, (2.3) are regular splittings of A and hence weak regular splittings of A. Now, by Thm 2.1 we have $\rho(H) < 1$. □

Theorem 2.4

Let A in (1.1) be a positive real matrix. Then the multisplitting method (1.5), where $p = 2q$, M_k is given by (2.4), (2.5) with $\omega = 1$ and $W_k = \rho_k I$, $k = 1(1)2q$, $D_k = a_k I$ and

$$\rho_k > \begin{cases} \max\{0, -\frac{\mu_m}{\lambda_m}\} & \text{for} \quad k = 1(1)q \\ \max\{0, -\frac{\nu_m}{\lambda_m}\} & \text{for} \quad k = q + 1(1)2q, \end{cases} \tag{2.6}$$

where λ_m is the smallest eigenvalue of $A + A^T$ and μ_m, ν_m are the smallest eigenvalues of the matrices $(D - L)(D - L)^T - UU^T$ and $(D - U)(D - U)^T - LL^T$, respectively, converges.

Proof

Since A is positive real, we have that A is nonsingular, $B \equiv A + A^T$ is positive definite and $D > 0$. Consequently M_k is nonsingular, $k = 1(1)2q$, since $\rho_k > 0$. Moreover we have $\lambda_m > 0$. The matrices $C_1 \equiv (D-L)(D-L)^T - UU^T$ and $C_2 \equiv (D-U)(D-U)^T - LL^T$ are symmetric and for any $z \in I\!R^n$, $z \neq 0$, we have

$$\frac{z^T(\rho_k B + C_1)z}{z^T z} \geq \rho_k \lambda_m + \mu_m, \quad \frac{z^T(\rho_k B + C_2)z}{z^T z} \geq \rho_k \lambda_m + \nu_m. \tag{2.7}$$

Because of (2.6), (2.7) implies that the matrices $\rho_k B + C_1$, $k = 1(1)q$, and $\rho_k B + C_2$, $k = q+1(1)2q$, are positive definite. Setting $G_k = M_k^{-1} N_k$, $k = 1(1)2q$, it can be shown that

$$\rho_k B + C_1 = M_k(I - G_k G_k^T) M_k^T, \quad k = 1(1)q \tag{2.8}$$

and

$$\rho_k B + C_2 = M_k(I - G_k G_k^T) M_k^T, \quad k = q + 1(1)2q. \tag{2.9}$$

From (2.8), (2.9) we have that $I - G_k G_k^T$, $k = 1(1)2q$, are positive definite; hence the eigenvalues of $G_k G_k^T$ belong to $[0,1)$, $k = 1(1)2q$. Thus we obtain $||G_k||_2 = [\rho(G_k G_k^T)]^{1/2} < 1$, $k = 1(1)2q$, and

$$||H||_2 = ||\sum_{k=1}^{2q} a_k G_k||_2 \leq \sum_{k=1}^{2q} a_k ||G_k||_2 < \sum_{k=1}^{2q} a_k = 1,$$

implying that the method converges. □

Theorem 2.5

If A in (1.1) is a positive definite matrix, then the multisplitting method (1.5), where $p = 2q$, M_k is given by (2.4), (2.5), $D_k = a_k I$ and $0 < \omega < 2$, converges.

Proof

In this case we have $U = L^T$ and $A = D - L - L^T$, $D > 0$. The splittings (2.4), (2.5) are P-regular splittings, since M_k is nonsingular and $M_k + N_k + (M_k + N_k)^T = 2(M_k + M_k^T - A) = 2[(\frac{2-\omega}{\omega})D + 2W_k]$, $k = 1(1)2q$. Thus by Thm 2.2 we obtain the desired result. □

3 m-Step Preconditioners

We consider the linear system (1.1), where A is positive definite. If

$$A = M - N, \quad \det(M) \neq 0, \tag{3.1}$$

then using the iterative method

$$Mx^{(m+1)} = Nx^{(m)} + b, \qquad m = 0, 1, 2, \ldots,$$

we solve in every iteration a linear system of the form

$$My = c. \tag{3.2}$$

It is known that M is chosen so that it approximates A as well as possible ($A \approx M$) and $\rho(G) < 1$, where $G = M^{-1}N$. Choosing a positive definite M ($A \approx M$) with $\rho(G) < 1$, we can find improved approximations to A using the Neumann expansion (see e.g., [1], [2], [6])

$$A^{-1} = (I - G)^{-1}M^{-1} = (I + G + G^2 + \ldots)M^{-1}. \tag{3.3}$$

Thus we have

$$A \approx M_m = M(I + G + G^2 + \ldots + G^{m-1})^{-1}, \quad m \geq 1. \tag{3.4}$$

It can be shown (see Thm 3.1 of [6]), that under the above assumptions M_m is also positive definite and therefore M_m^{-1} is usually used to accelerate convergence of the Conjugate Gradient method. The matrix M_m is the preconditioning matrix or m-step preconditioner. One comment here: In Thm 1 of [1], it was proved that for m odd the hypothesis "A and M are positive definite" is sufficient for M_m to be positive definite. However, this hypothesis does not guarantee that M_m will be a better than M approximation to A, since then

$$M_m^{-1}N_m = M_m^{-1}(M_m - A) = I - (I + G + \ldots + G^{m-1})(I - G) = G^m.$$

Therefore the condition $P(G) < 1$ should be included in our assumptions for all m (odd or even).

Taking into consideration the theory mentioned previously (see also [10]), in order to find suitable m-step preconditioners for (1.1), we can work as follows: We choose some positive definite matrix M and write $A = M - N$. Then $G = M^{-1}N$ has real eigenvalues λ_i, $i = 1(1)n$, such that $\lambda_i < 1$, $i = 1(1)n$. Suppose that λ_i are ordered as $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < 1$. We consider now the splitting

$$A = \hat{M} - \hat{N}, \tag{3.5}$$

where $\hat{M} = \frac{1}{\omega}M$. As is known the splitting (3.5) defines the extrapolated method based on the original splitting. Obviously $\hat{M}$ is positive definite for $\omega > 0$ and it is $\rho(\hat{M}^{-1}\hat{N}) < 1$ iff $0 < \omega < \frac{2}{1-\lambda_1}$. Hence an m-step preconditioner, which is positive definite and approximates A well, is given by

$$\hat{M}_m = \hat{M}(I + \hat{G} + \hat{G}^2 + \ldots + \hat{G}^{m-1})^{-1}, \quad m \geq 1, \tag{3.6}$$

where $\hat{G} = \hat{M}^{-1}\hat{N}$ and $\omega \in (0, \frac{2}{1-\lambda_1})$. Certainly $\hat{M}_m$ depends on ω and the problem as how to choose ω for a fixed m, so that the condition number $k(\hat{M}_m^{-1}A)$ of $\hat{M}_m^{-1}A$ is as small as possible, arises. It is easy to show that

$$I - \hat{M}_m^{-1}A = \hat{G}^m = [(1-\omega)I + \omega G]^m; \tag{3.7}$$

hence

$$k(\hat{M}_m^{-1}A) = \frac{\max_i \mu_i^{(m)}}{\min_i \mu_i^{(m)}}, \tag{3.8}$$

where $\mu_i^{(m)}$, $i = 1(1)n$, are the eigenvalues of $\hat{M}_m^{-1}A$. We note that the eigenvalues of $\hat{G}$ are ordered as

$$-1 < 1-\omega+\omega\lambda_1 \leq 1-\omega+\omega\lambda_2 \leq \ldots \leq 1-\omega+\omega\lambda_n < 1. \tag{3.9}$$

Because of (3.7) we have

$$k(\hat{M}_m^{-1}A) = \frac{\max_i\{1-[1-\omega+\omega\lambda_i]^m\}}{\min_i\{1-[1-\omega+\omega\lambda_i]^m\}}, \quad m \geq 1. \tag{3.10}$$

It can be shown, as in [1], that

$$k(\hat{M}_m^{-1}A) = \begin{cases} \frac{1-(1-\omega+\omega\lambda_1)^m}{1-(1-\omega+\omega\lambda_n)^m}, & \text{if } m \text{ is odd,} \\ \\ \frac{1-[\min_i |1-\omega+\omega\lambda_i|]^m}{1-[\max_i |1-\omega+\omega\lambda_i|]^m}, & \text{if } m \text{ is even,} \end{cases} \tag{3.11}$$

where $\omega \in (0, \dfrac{2}{1-\lambda_1})$.

The problem of finding $\min_{\omega} k(\hat{M}_m^{-1}A)$ seems to be not an easy one in the general m odd case. In the sequel we solve first this problem for $m = 1$ (trivial case), $m = 3$ and for any even $m \geq 2$. The results are given in Thms 3.1 and 3.3. In these theorems it is assumed that $\lambda_1 < \lambda_n$, for if $\lambda_1 = \lambda_n$, then $k(\hat{M}_m^{-1}A) = 1$ for all m and all permissible values of ω.

Theorem 3.1

The condition number $k_m = k_m(\omega)$ of $\hat{M}_m^{-1}A$, given by (3.11), for $m = 1$ is independent of ω and is given by $k_1 = \frac{\nu_1}{\nu_n}$, while for $m = 3$, is minimized with respect to ω for

$$\omega = \omega_{opt} = \frac{\nu_1 + \nu_n - \sqrt{\nu_1^2 + \nu_n^2 - \nu_1\nu_n}}{\nu_1\nu_n}, \tag{3.12}$$

where $\nu_1 = 1 - \lambda_1$, $\nu_n = 1 - \lambda_n$.

Proof

For $m = 1$ the result is trivially obtained. For $m = 3$ it can be shown after some manipulation that $sign\ \left(\frac{\partial k_3(\omega)}{\partial \omega}\right) = sign\ (\phi_3(\omega))$, where

$$\phi_3(\omega) = -\nu_1\nu_n\omega^2 + 2(\nu_1+\nu_n)\omega - 3. \tag{3.13}$$

The two roots of $\phi_3(\omega)$ are real and are given by

$$\rho_1 = \frac{\nu_1+\nu_n+\sqrt{\nu_1^2+\nu_n^2-\nu_1\nu_n}}{\nu_1\nu_n}, \qquad \rho_2 = \frac{\nu_1+\nu_n-\sqrt{\nu_1^2+\nu_n^2-\nu_1\nu_n}}{\nu_1\nu_n}. \tag{3.14}$$

It can be proved that $0 < \rho_2 < \frac{2}{\nu_n} < \rho_1$. Moreover $\frac{\partial k_3}{\partial \omega} < 0$ if $0 < \omega < \rho_2$ while $\frac{\partial k_3}{\partial \omega} > 0$ if $\rho_2 < \omega < \dfrac{2}{\nu_n}$. Hence $\min_{\omega} k_3(\omega) = k_3(\rho_2)$ and our assertion follows. □

Remarks:

(i) For $m = 1$ the extrapolation parameter (damping factor) ω was used in conjuction with the Jacobi iteration matrix in [10]. Thm 3.1 effectively shows that if ω is kept fixed during the iterations **no** improvement over the original preconditioner should be expected! (ii) For odd $m \geq 5$ the function $\phi_m(\omega)$ is a polynomial of degree $m-1$ whose sign determination as ω varies in $(0, \dfrac{2}{\nu_n})$ seems not an easy problem to study. This is what makes the whole problem difficult to solve.

To derive the optimal results for even $m \geq 2$ first we introduce the notation "$a \sim b$" to denote that the expressions a and b are of the same sign and then state and prove the lemma below, a basic key to the proof of one of our main results.

Lemma 3.1:

For any even $m \geq 2$ the function

$$\phi_m \equiv \phi_m(x) := \frac{x^{m-1}-x^m}{1-x^m}, \qquad x \in (-1,1) \tag{3.15}$$

is a strictly increasing function of x in $(-1,1]$.

Proof

Differentiating (3.15) with respect to x we obtain

$$\frac{\partial \phi_m}{\partial x} \sim (m-1) - mx + x^m = (m-1)(1-x) - x(1-x^{m-1}). \tag{3.16}$$

If $x \in (-1,0]$, the rightmost expression in (3.16) is positive since $1-x > 0$, $-x \geq 0$ and $1-x^{m-1} > 0$, implying that ϕ_m strictly increases in $(-1,0]$. For $x \in [0,1)$ let

$$z \equiv z(x) := (m-1) - mx + x^m, \qquad x \in [0,1). \tag{3.17}$$

Then on differentiation we take

$$\frac{\partial z}{\partial x} = -m(1 - x^{m-1}) < 0$$

and therefore $z(x)$ strictly decreases in [0,1) with $\lim_{x \to 1^-} z(x) = 0$, and $z(0) = m - 1 > 0$. Hence $z(x)$ takes on positive values only and by virtue of (3.17) and (3.16) so does $\frac{\partial \phi_m}{\partial x}$. Consequently ϕ_m strictly increases in [0,1). □

In the sequel we state and prove two theorems that solve the problem of determining the optimal extrapolation parameter for all even $m \geq 2$.

Theorem 3.2:

Let the eigenvalues λ_i, $i = 1(1)n$, of $\hat{G}$ in (3.7) satisfy

$$-1 < -\lambda_n = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < 1, \qquad (\lambda_1 \leq 0 \leq \lambda_n). \tag{3.18}$$

Then the condition number $k_m = k_m(\omega)$ of $\hat{M}_m^{-1}A$, given by (3.11) for even $m \geq 2$, is minimized with respect to $\omega \in (0, \frac{2}{1-\lambda_1})$ for

$$\omega_{opt} = 1. \tag{3.19}$$

Proof

Let λ_i and λ_{i+1}, $i \in \{1, 2, \ldots, n-1\}$ be the absolutely smallest nonpositive and nonnegative eigenvalues of G, respectively. Two cases are distinguished depending on the sign of $\lambda_i + \lambda_{i+1}$.

Case I: Let $\lambda_{i+1} + \lambda_i < 0$. (The subcase $\lambda_{i+1} + \lambda_i = 0$ can be trivially examined after the analysis is complete.) We subdivide the interval for ω, $(0, \frac{2}{1-\lambda_1})$, into a number of (at most $2n+1$) subintervals. For continuity arguments to apply all of them are taken to be closed, except the first and the last ones. The subdivision points are

$$\frac{1}{1-\lambda_1}, \frac{2}{2-\lambda_1-\lambda_2}, \frac{1}{1-\lambda_2}, \frac{2}{2-\lambda_2-\lambda_3}, \ldots, \frac{1}{1-\lambda_i}, \frac{2}{2-\lambda_i-\lambda_{i+1}}, 1, \frac{1}{1-\lambda_{i+1}}, \frac{2}{2-\lambda_{i+1}-\lambda_{i+2}}, \ldots.$$

The last point is either $\frac{1}{1-\lambda_j}$ for some $j \in \{i+1, i+2, \ldots, n\}$ iff $\frac{1}{1-\lambda_j} < \frac{2}{1-\lambda_1} \leq \frac{2}{1-\lambda_j-\lambda_{j+1}}$ or $\frac{2}{2-\lambda_{j-1}-\lambda_j}$ for some $j \in \{i+2, i+3, \ldots, n\}$ iff $\frac{2}{2-\lambda_{j-1}-\lambda_j} < \frac{2}{1-\lambda_1} \leq \frac{1}{1-\lambda_j}$. Let $I_1, I_2, I_3, \ldots,$ $I_{2i}, I_{2i+1}, I_{2i+2}, \ldots$ be the successive subintervals of $(0, \frac{2}{1-\lambda_1})$ defined by these points. Let also

$$\lambda_k(\omega) := 1 - \omega + \omega\lambda_k, \qquad k = 1(1)n. \tag{3.20}$$

As can be readily checked, the ordering of the eigenvalues $\lambda_k(\omega)$ of $\hat{G} \equiv G_\omega$ is the same as that of the λ_k's in (3.18). We then claim that: "$k_m = k_m(\omega)$ is a strictly decreasing function of ω in each subinterval I_ℓ, $\ell = 1(1)2i+1$, and a strictly increasing one in each I_ℓ, $\ell \geq 2i+2$". The proof of our claim will prove (3.19). For this we shall distinguish four cases: (a) $\omega \in I_\ell$, $\ell = 2(2)2i$, (b) $\omega \in I_\ell$,

$\ell = 1(2)2i+1$, (c) $\omega \in I_\ell$, $\ell = 2i+2,\, 2i+4, \ldots$, and (d) $\omega \in I_\ell$, $\ell = 2i+3,\, 2i+5, \ldots$. In case (a), $\omega \in [\frac{1}{1-\lambda_k}, \frac{2}{2-\lambda_k-\lambda_{k+1}}]$, $k = \ell/2$. It can be readily checked that $\lambda_k(\omega)$ and $\lambda_{k+1}(\omega)$ are, respectively, the absolutely smallest nonpositive and nonnegative eigenvalues of G_ω with $0 \le -\lambda_k(\omega) \le \lambda_{k+1}(\omega)$. On the other hand $0 \le -\lambda_1(\omega) \le \lambda_n(\omega)$. So, $k_m(\omega)$ will be given by the expression

$$k_m(\omega) = \frac{1-\lambda_k^m(\omega)}{1-\lambda_n^m(\omega)}. \tag{3.21}$$

Since m is even, and both $\lambda_k(\omega)$ and $\lambda_n(\omega)$ strictly decrease with ω increasing it is concluded that the numerator and the denominator of the expression in (3.21) decreases and increases, respectively, making $k_m(\omega)$ be a strictly decreasing function of $\omega \in I_\ell$. In case (b), $\omega \in [\frac{2}{2-\lambda_{k-1}-\lambda_k}, \frac{1}{1-\lambda_k}]$, $k = \dfrac{\ell+1}{2}$. (I_1 is open on the left with bound 0 and I_{2i+1} is closed on the right with bound 1.) Now $-\lambda_{k-1}(\omega) \ge \lambda_k(\omega) \ge 0$, so that $k_m(\omega)$ will be given again by (3.21). However, this time both terms of the fraction strictly increase with ω. Thus, differentiating with respect to ω one obtains

$$\begin{array}{rclcl} \frac{\partial k_m}{\partial \omega} & \sim & (1-\lambda_n^m(\omega))(1-\lambda_k)\lambda_k^{m-1}(\omega) & - & (1-\lambda_k^m(\omega))(1-\lambda_n)\lambda_n^{m-1}(\omega) \\ & \sim & \frac{\lambda_k^{m-1}(\omega)(1-\lambda_k(\omega))}{1-\lambda_k^m(\omega)} - \frac{\lambda_n^{m-1}(\omega)(1-\lambda_n(\omega))}{1-\lambda_n^m(\omega)} & = & \phi_m(\lambda_k(\omega)) - \phi_m(\lambda_n(\omega)), \end{array} \tag{3.22}$$

because of $\omega(1-\lambda_j) = 1-\lambda_j(\omega)$, $j = k, n$, and in view of (3.15). Since ω varies in $I_{2k-1} \subset (0,1]$ and $\lambda_k(\omega) \le \lambda_n(\omega)$ Lemma 3.1 applies, implying that $\frac{\partial k_m}{\partial \omega} \le 0$, with equality concerning limiting cases only. Therefore $k_m(\omega)$ strictly decreases in I_{2k-1}. In case (c), where I_ℓ, $\ell = 2i+2, 2i+4, \ldots$, is of the general type $[\frac{2}{2-\lambda_{k-1}-\lambda_k}, \frac{1}{1-\lambda_k}]$, $k = \ell/2$, except the first and maybe the last interval, we have a similar situation to that of case (a). This time $k_m(\omega)$ is given by the expression

$$k_m(\omega) = \frac{1-\lambda_k^m(\omega)}{1-\lambda_1^m(\omega)}. \tag{3.23}$$

Since $\lambda_k(\omega) \ge 0 \ge \lambda_1(\omega)$ and both $\lambda_k(\omega)$ and $\lambda_1(\omega)$ decrease with ω increasing, $k_m(\omega)$ strictly increases with ω. In case (d) we have a similar situation to that in case (b). The interval I_ℓ, $\ell = 2i+3, 2i+5, \ldots$, is of the general type $[\frac{1}{1-\lambda_k}, \frac{2}{2-\lambda_k-\lambda_{k+1}}]$, $k = (\ell-1)/2$, except maybe the last one, and k_m is given by (3.23), where this time $0 \ge \lambda_k(\omega) \ge \lambda_1(\omega)$, so both terms of the fraction in (3.23) decrease with ω increasing. On differentiation we have a series of relationships similar to those in (3.22) but this time

$$\frac{\partial k_m}{\partial \omega} \sim \phi_m(\lambda_k(\omega)) - \phi_m(\lambda_1(\omega)). \tag{3.24}$$

Based now on Lemma 3.1 we have again the desired result, namely $k_m(\omega)$ strictly increases on I_ℓ. Summarizing the conclusions of cases (a)–(d) leads to (3.19).

Case II: In case $\lambda_{i+1} + \lambda_i > 0$ we work in a similar way as in Case I. This time $1 \in [\frac{1}{1-\lambda_i}, \frac{2}{2-\lambda_i-\lambda_{i+1}})$ and we have $2i$ subintervals to the left and at most $2(n-i)+1$ ones to the right of 1. The function $k_m(\omega)$ behaves in exactly the same way as before in the subintervals which are to the left and to

$k_m(\omega)$ behaves in exactly the same way as before in the subintervals which are to the left and to the right of 1 as is readily checked and consequently we arrive at exactly the same conclusion. This completes the proof of our theorem. □

Suppose now that the eigenvalues of G in (3.7) satisfy

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < 1, \tag{3.25}$$

that is without the further assumption $\lambda_n = -\lambda_1$ of Thm 3.2. Suppose also that we extrapolate G using any parameter $\omega \in (0, \frac{2}{1-\lambda_1})$. The answer now to the question "What is the value of ω_{opt} in this case?" can be given almost immediately. Having in mind the fact that "The extrapolation with a parameter ω_2 of an extrapolation with parameter ω_1 is also an extrapolation with parameter $\omega = \omega_2\omega_1$", which can be easily checked (see also [9]), leads us to writing ω as $\omega = \omega_2\omega_1$, where $\omega_1 = \frac{2}{2-\lambda_1-\lambda_n}$. The eigenvalues $\lambda_i' = 1 - \omega_1 + \omega_1\lambda_i$, $i = 1(1)n$, of G_{ω_1} satisfy

$$-1 < -\lambda_n' = \lambda_1' \leq \lambda_2' \leq \ldots \leq \lambda_n' < 1, \; (\lambda_1' \leq 0 \leq \lambda_n'), \tag{3.26}$$

that is all the assumptions of Thm 3.2. So, extrapolation of G_{ω_1} becomes optimal iff $\omega_2 = 1$. Thus we have just proved:

Theorem 3.3:

Let the eigenvalues of G in (3.7) satisfy (3.25). Then the condition number $k_m = k_m(\omega)$ of $\hat{M}_m^{-1}A$, given by (3.11) for even $m \geq 2$, is minimized with respect to $\omega \in (0, \frac{2}{1-\lambda_1})$ for

$$\omega_{opt} = \frac{2}{2 - \lambda_1 - \lambda_n}. \tag{3.27}$$

As an immediate corollary we have

Corollary 3.1:

If A is real symmetric positive definite and point (or block) 2-cyclic consistently ordered and M, in the splitting $A = M - N$, is the diagonal (or the block diagonal part corresponding to the block partitioning) of A, then the condition number $k_m = k_m(\omega)$ of $\hat{M}_m^{-1}A$, given by (3.11) for even $m \geq 2$, is minimized for $\omega_{opt} = 1$.

Note: If the only information available on the spectrum of G is its spectral radius $\rho(G) = \lambda_n < 1$, then ω_{opt} should be taken to be 1.

We close this section by noting that the idea in [2] for defining m-step additive preconditioners of (1.1), where A is positive definite, can be generalized. For this we consider the multisplitting

$$A = P_k - Q_k, \qquad \det(P_k) \neq 0, \qquad k = 1(1)p, \tag{3.28}$$

and the iteration matrix H of the corresponding multisplitting method (1.5) with $D_k = a_kI$, $k = 1(1)p$. Setting

$$G_k = P_k^{-1}Q_k, \qquad M^{-1} = \sum_{i=1}^{p} a_i P_i, \tag{3.29}$$

then

$$H = \sum_{i=1}^{p} a_i G_i \tag{3.30}$$

and the m-step additive preconditioner is defined by

$$M_m = M(I + H + H^2 + \ldots + H^{m-1})^{-1}, \qquad m \geq 1, \tag{3.31}$$

provided that M_m is positive definite (and $A \approx M_m$). We note that the m-step additive preconditioner is an m-step preconditioner (see (3.4)) related to the splitting defining a multisplitting method. Certainly, if M is positive definite and $\rho(H) < 1$, then M_m is also positive definite and $A \approx M_m$. In the following Theorem we give sufficient conditions for M_m to be positive definite.

Theorem 3.4

Let A in (1.1) be positive definite and

$$A = P_k - Q_k, \qquad k = 1(1)2q, \tag{3.32}$$

where

$$P_{q+i} = P_i^T, \qquad i = 1(1)q. \tag{3.33}$$

If the splittings (3.32) for $k = 1(1)q$ are P-regular splittings of A, then the m-step additive preconditioner (3.31), where

$$M = \left(\sum_{i=1}^{2q} a_i P_i^{-1}\right)^{-1}, \quad a_i = \frac{1}{2q}, \quad i = 1(1)2q, \quad H = \sum_{i=1}^{2q} a_i G_i, \quad G_i = P_i^{-1}Q_i, \tag{3.34}$$

is positive definite.

Proof

Since (3.32) for $k = 1(1)q$ are P-regular splittings and (3.33) holds, it follows that (3.32) for $k = q+1(1)2q$ are also P-regular splittings of A. Thus we have that $P_k+Q_k+(P_k+Q_k)^T = 2(P_k+P_k^T-A)$ is positive definite, $k = 1(1)2q$. Consequently $P_k + P_k^T$ is positive definite, $k = 1(1)2q$. Moreover, using (3.33), we find

$$M^{-1} = \sum_{i=1}^{2q} a_i P_i^{-1} = \frac{1}{2q}\sum_{i=1}^{q}(P_i^{-1} + P_{q+i}^{-1}) = \frac{1}{2q}\sum_{i=1}^{q}[(P_i^{-1})^T(P_i^T + P_i)P_i^{-1}]. \tag{3.35}$$

Since $P_i + P_i^T$ is positive definite, $i = 1(1)q$, and M^{-1} is a sum of positive definite matrices, M^{-1} and hence M is positive definite. Moreover it is $\rho(H) < 1$ by Thm 2.2. Now, using Thm 3.1 of [6] we obtain the desired result. □

4 Optimum SOR-Additive Iterative Method

We again consider system (1.1), where

$$A = D - L - L^T \tag{4.1}$$

and A is positive definite. Given the splittings $A = P_k - Q_k$, $k = 1, 2$, with

$$P_1 = \frac{1}{\omega}(D - \omega L), \qquad P_2 = P_1^T = \frac{1}{\omega}(D - \omega L^T) \tag{4.2}$$

and $\omega \neq 0$ a real parameter, it can be shown that $A = P_1 - Q_1$ is a P-regular splitting of A, if $0 < \omega < 2$. Hence Thm 3.4 for $q = 1$ (see also Thm 2.2) implies that the SOR two-splitting method or SOR-additive method [2]

$$x^{(m+1)} = Hx^{(m)} + c, \qquad m = 0, 1, 2, \ldots, \tag{4.3}$$

where

$$H = H(\omega) = \frac{1}{2}(G_1 + G_2), \quad c = \frac{1}{2}(P_1^{-1} + P_2^{-1})b, \quad G_i = P_i^{-1}Q_i, \quad i = 1, 2, \tag{4.4}$$

converges. Under the further assumption that A has the 2-cyclic form

$$A = \begin{bmatrix} D_1 & -X \\ -X^T & D_2 \end{bmatrix} \tag{4.5}$$

(D_1, D_2 are diagonal matrices), it was proved in [2] that if λ is an eigenvalue of H, then

$$\lambda = \frac{1}{2}[\omega^2\mu^2 + \omega(2 - \omega)\mu + 2(1 - \omega)], \tag{4.6}$$

where μ is an eigenvalue of the Jacobi iteration matrix $J = I - D^{-1}A$ for A. It is noted that J has real eigenvalues, which occur in $\pm$ pairs and $\rho(J) < 1$. Moreover it was shown in [2] that $\min_{0<\omega<2} \rho(H(\omega)) = \rho(H(\omega_{opt}))$, where

$$\omega_{opt} = \frac{\mu_m - \frac{3}{2} + \sqrt{3 - 2\mu_m^2}}{\frac{1}{4} + \mu_m - \mu_m^2}, \qquad \mu_m = \rho(J). \tag{4.7}$$

We observe first that $\lim_{\mu_m \to 0^+} \mu = 0$ for all the eigenvalues μ of J and from (4.7) we obtain $\lim_{\mu_m \to 0^+} \lambda = 1 - \omega$, which means that the optimum ω satisfied $\lim_{\mu_m \to 0^+} \omega_{opt} = 1$. On the other hand, (4.7) for $\mu_m = 0$ gives

$$\omega_{opt} = \frac{-\frac{3}{2}+\sqrt{3}}{\frac{1}{4}} (\approx 0.9282) \neq 1. \tag{4.8}$$

This observation suggests that the theoretical determination of the optimum value of ω must be reconsidered. In what follows we give the complete solution to this problem and the results are contained in the following theorem.

Theorem 4.1

If A in (1.1) is positive definite, $A = D - L - L^T$ and has the form (4.5), then the optimum value ω_{opt} for ω $(0 < \omega < 2)$ of the SOR-additive method defined by (4.3) is given by

$$\omega_{opt} = \begin{cases} \frac{1-\sqrt{1-2\mu_m^2}}{\mu_m^2}, & \text{if} \quad 0 < \mu_m \leq \frac{1}{\sqrt{6}} \\ \frac{\mu_m - \frac{3}{2} + \sqrt{3-2\mu_m^2}}{\frac{1}{4}+\mu_m+\mu_m^2}, & \text{if} \quad \frac{1}{\sqrt{6}} \leq \mu_m < 1 \end{cases} \tag{4.9}$$

where $\mu_m = \rho(J)$ and $J = I - D^{-1}A$.

Proof

The problem we solve is: Find

$$\min_{\omega} \max_{\mu} |\lambda|, \tag{4.10}$$

where λ is given by (4.6), $0 < \omega < 2$, $\mu \in [-\mu_m, \mu_m]$ and $\mu_m < 1$. For this we have that $\frac{\partial \lambda}{\partial \mu} = 0$ iff $\mu = \frac{\omega - 2}{2\omega} \equiv \mu^*$. Moreover,

$$\mu^* \in [-\mu_m, \mu_m] \quad \text{iff} \quad \omega^* \equiv \frac{2}{1+2\mu_m} \leq \omega < 2. \tag{4.11}$$

With $\lambda = \lambda(\mu)$ we find

$$A = A(\omega) \equiv |\lambda(\mu_m)| = \frac{1}{2}|\omega^2\mu_m^2 + \omega(2-\omega)\mu_m + 2(1-\omega)|, \tag{4.12}$$

$$B = B(\omega) \equiv |\lambda(-\mu_m)| = \frac{1}{2}|\omega^2\mu_m^2 - \omega(2-\omega)\mu_m + 2(1-\omega)|, \tag{4.13}$$

$$C = C(\omega) \equiv |\lambda(\mu^*)| = \begin{cases} \frac{1}{8}(\omega^2 + 4\omega - 4), & \text{if} \quad 2(\sqrt{2}-1) \leq \omega < 2 \\ \frac{1}{8}(4 - 4\omega - \omega^2), & \text{if} \quad 0 < \omega \leq 2(\sqrt{2}-1) \end{cases}. \tag{4.14}$$

Hence

$$\max_{\mu} |\lambda| = \max\{A, B, C\}. \tag{4.15}$$

It can be proved that

(i) If $0 < \mu_m < \frac{\sqrt{2}}{2}$ and $0 < \omega \le \omega_1 \equiv \frac{1-\sqrt{1-2\mu_m^2}}{\mu_m^2}$ or $\frac{\sqrt{2}}{2} \le \mu_m < 1$ and $0 < \omega < 2$, then

$$B \le A = \frac{1}{2}[\omega^2\mu_m^2 + \omega(2-\omega)\mu_m + 2(1-\omega)].$$

(ii) If $0 < \mu_m < \frac{\sqrt{2}}{2}$ and $\omega_1 \le \omega < 2$, then

$$A \le B = \frac{1}{2}[\omega(2-\omega)\mu_m - \omega^2\mu_m^2 - 2(1-\omega)].$$

Thus, we distinguish the following cases:

Case I: $\frac{\sqrt{2}}{2} \le \mu_m < 1$. Then it can be shown that $\omega^* \le 2(\sqrt{2}-1)$ and

$$\max\{A,B,C\} = \begin{cases} A \text{ if } 0 < \omega \le \rho_2 \\ C \text{ if } \rho_2 \le \omega < 2, \end{cases} \tag{4.16}$$

where

$$\rho_2 = \frac{\mu_m - \frac{3}{2} + \sqrt{3 - 2\mu_m^2}}{\frac{1}{4} + \mu_m - \mu_m^2}. \tag{4.17}$$

Now, we find that $\frac{\partial A}{\partial \omega} < 0$ and $\frac{\partial C}{\partial \omega} > 0$, implying $\min_\omega A = A(\rho_2)$ and $\min_\omega C = C(\rho_2) = A(\rho_2)$. Hence we obtain $\omega_{opt} = \rho_2$ and $\min_\omega \max_\mu |\lambda| = A(\rho_2) = C(\rho_2) = \frac{1}{8}(\rho_2^2 + 4\rho_2 - 4)$.

Case II: $0 < \mu_m < \frac{\sqrt{2}}{2}$. Then it can be shown that:

(i) If $0 < \mu_m \le \frac{1}{\sqrt{6}}$, then $2(\sqrt{2}-1) < \omega_1 \le \omega^*$.

(ii) If $\frac{1}{\sqrt{6}} \le \mu_m < \frac{\sqrt{2}}{2}$, then $2(\sqrt{2}-1) < \omega^* \le \omega_1$.

Therefore we must distinguish the following subcases:

Case IIa: $0 < \mu_m \le \frac{1}{\sqrt{6}}$. Then we find

$$\max\{A,B,C\} = \begin{cases} A & \text{if} \quad 0 < \omega \le \omega_1 \\ B & \text{if} \quad \omega_1 \le \omega \le \omega^* \\ C & \text{if} \quad \omega^* \le \omega \le 2 \end{cases} \tag{4.18}$$

and

$$\min_\omega A(\omega) = A(\omega_1), \quad \min_\omega B(\omega) = B(\omega_1), \quad \min_\omega C(\omega) = C(\omega^*) = B(\omega^*) \geq B(\omega_1).$$

Hence we have $\omega_{opt} = \omega_1$ and $\min_\omega \max_\mu |\lambda| = A(\omega_1) = B(\omega_1)$.

Case IIb: $\frac{1}{\sqrt{6}} < \mu_m < \frac{\sqrt{2}}{2}$. Then it can be proved that

$$0 < 2(\sqrt{2} - 1) < \omega^* < \rho_2 < \omega_1 < 2 \tag{4.19}$$

and

$$\max\{A, B, C\} = \begin{cases} A & \text{if} \quad 0 < \omega \leq \rho_2 \\ C & \text{if} \quad \rho_2 \leq \omega < 2. \end{cases} \tag{4.20}$$

As in Case I we find that $\omega_{opt} = \rho_2$ and $\min_\omega \max_\mu |\lambda| = A(\rho_2) = C(\rho_2) = \frac{1}{8}(\rho_2^2 + 4\rho_2 - 4)$. Combining the above results of Cases I, IIa, IIb we obtain (4.8).

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