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**OBTAINING BOUNDARIES WITH RESPECT:
A SIMPLE APPROACH TO PERFORMING
SET OPERATIONS ON POLYHEDRA**

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Obtaining Boundaries with Respect:
A simple approach to
Performing Set Operations on Polyhedra

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Abstract

A regularized set operation on two solids can be separated into four steps: partition the faces of the boundaries of the two solids to impose respect, obtain an eight-way classification of the faces, create a solid according to the set operation, and reduce the representation to its minimal form. Of these four steps, the first step is the most difficult. This paper presents and proves correct a general approach for imposing respect on two boundary representations. The approach is based on a data-driven, binary form of decomposition.

Categories and Subject Descriptors: I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—*Geometric algorithms*.

General Terms: Algorithms, Design

Additional Key Words and Phrases: nonregular decomposition, boolean set operations, polyhedra, nonmanifolds.

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1 Introduction

To date, many representations have been devised for modeling solids. Of these, the boundary representation technique is most widely used. A boundary representation (BRep) models a solid by explicitly encoding the bounding surfaces of the solid as a collection of vertices, edges and faces. Such BReps are frequently modified using regularized union, \cup^* , intersection, \cap^* , and difference, $-^*$. The operators provide a conceptually simple method to construct complex solids by combining simpler ones.

The basic approach for performing set operations on two boundary representations can be separated into four distinct steps:

1. Impose *respect* on the faces; that is, partition every face of one solid, A , by the boundary of the other solid, B , so that each of the resulting faces in its entirety is either inside, outside or on the boundary of B .
2. Classify the faces; that is, identify which faces of one solid are inside the other solid, which faces are outside the other solid, and which faces lie on the boundary of the other solid.
3. Depending on the set operation at hand, assemble the appropriate faces.
4. Topologically reduce the result to a minimal boundary representation.

In this paper, we focus on solving the first step of the regularized set operations and present a new method that does this; that is, a method that imposes respect; and we prove its correctness. The motivation for this approach is to utilize spacial decomposition methods to directly manipulate boundary representation models without resorting to some intermediate hierarchical data structures. The presented solution has several appealing properties:

- The approach is purely face-based. An implementation needs only to keep track of faces partitions. This greatly simplifies or entirely eliminates the use of intermediate data structures.
- The approach handles all solids uniformly without making nonmanifolds the exceptions.
- The eight-way classification allows the simultaneous creation of the union, the intersection and the difference without the need to copy and reclassify the original boundaries for each operation in turn.
- The approach allows both regular and the nonregular decomposition methods to be applied. A hybrid method that utilizes both methods improves efficiency.

Section 2 defines solids and their boundaries. Section 3 gives a detailed outline of the four steps for performing the set operations. The first step and its algorithm are then given in detail in Section 4. Section 5 discusses the merits of the algorithm,

2 The Modeling Domain

A commonly used class of solids is the class of compact 3-D manifolds with boundary that are planar polyhedra. Solids in this class are three-dimensional objects that have a finite interior and a boundary that is a closed two-manifold[Req80]. This class of solids is not closed under regularized Boolean operations because performing a set operation on two solids with two-manifold boundaries may not result in a solid with a two-manifold boundary[Req77, RV77, TR80]. So, we work with a domain which also includes nonmanifolds. We call this larger class of solids the modeling domain \mathcal{M} (see Definition 1). Let \mathcal{H} be the class of all half-spaces bounded by a plane. That is, each half-space H is defined as the point set

$$H = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz + d \leq 0\}$$

for some numbers a, b, c, d . We consider all finite intersections of half-spaces in \mathcal{H} such that the resulting point set is compact. Then define the class \mathcal{C} of simple convex polyhedra. The class \mathcal{M} is now defined as the set of polyhedra obtained from convex polyhedra by a finite number of regularized Boolean operations, \cup^* , \cap^* , and $-^*$. Note that this class includes nonmanifold objects as considered by Weiler [Wei86] and by Hoffman, Hopcroft and Karasick [HHK87].

Definition 1 *The class of solids \mathcal{M} is defined as follows:*

1. *If $S \in \mathcal{C}$ then $S \in \mathcal{M}$.*
2. *If S_1 and S_2 are in \mathcal{M} and $\langle \text{op} \rangle^*$ is one of the regularized Booleans you consider, then $S_1 \langle \text{op} \rangle^* S_2 \in \mathcal{M}$.*
3. *Nothing else is in \mathcal{M} .*

The boundary of any solid in \mathcal{M} can be partitioned into a set of faces, edges and vertices. The faces can be defined in terms of maximal faces which are uniquely given by the half-spaces of \mathcal{H} that form the solids of \mathcal{M} :

Definition 2 *A maximal face of S at H , where $S \in \mathcal{M}$ and $H \in \mathcal{H}$, is*

$$f_{\max}(H, S) = i^\circ \left(r^\circ \left((S \cap^* H) \cap bS \right) \right),$$

where r° and i° are the regularization and the interior operators in the relative topology of bH , and bS is the boundary of S .

Each maximal face consists of one or more connected components called the *maximally connected faces*. A *minimal boundary description*, σ_{\min} , is a triple consisting of a set of the maximally connected faces, a set of maximally connected edges, and a set of vertices defined as follows:

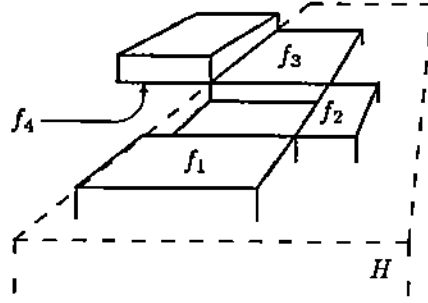


Figure 1: The maximal face determined by the half-space H is $f_{\max} = f_1 \cup f_2 \cup f_3$. Face f_4 is the maximal face for half-space c^*H .

Definition 3 Let $S \in \mathcal{M}$. Then σ_{\min} of S is the triple (V, E, F) where V , E , and F are the set of vertices, edges, and faces respectively, for which:

$$F(\sigma_{\min}) = \{f_{\text{cmax}} \mid (\exists H \in \mathcal{H}) (f_{\text{cmax}} \subseteq f_{\max}(H, S)) \\ \text{and } f_{\text{cmax}} \text{ is maximally connected}\} \quad (4)$$

$$E(\sigma_{\min}) = \{e_{\text{cmax}} \mid (\exists f_1, f_2 \in F(\sigma_{\min})) e_{\text{cmax}} \subseteq i^\circ(r^\circ f_1 \cap r^\circ f_2) \\ \text{and } e_{\text{cmax}} \text{ is maximally connected}\} \quad (5)$$

$$V(\sigma_{\min}) = \{v \mid (\exists e \in E(\sigma_{\min})) v \in b^\circ e\} \quad (6)$$

The maximal faces and the maximally connected faces have definitions similar to the definitions of C-faces and M-faces given by Silva[Sil81]. The difference is that Silva assumes faces that are closed sets, but for us the faces of $F(\sigma_{\min})$, as well as the edges of $E(\sigma_{\min})$, are open sets. Therefore, the faces, edges and vertices are pairwise disjoint point sets. As an illustration, Figure 1 shows a closed half-space H whose boundary contains the four faces f_1, \dots, f_4 . Of these four faces in bH , only f_1, f_2 and f_3 have the same orientation as H , and so the maximal face f_{\max} in H is $f_1 \cup f_2 \cup f_3$. Within this maximal face there exist three maximally connected and open faces, namely f_1, f_2 and f_3 .

All valid boundary descriptions of solid S in the modeling domain \mathcal{M} comprise the set $\mathcal{B}(S)$.

Definition 7 Let $S \in \mathcal{M}$. Then $\mathcal{B}(S) = \{\sigma \mid \sigma = (V, E, F)\}$ is the set of all valid boundary descriptions such that

$$F(\sigma) = \{f \mid f \text{ is a maximally connected face } f_{\text{cmax}} \text{ in } F(\sigma_{\min}), \\ \text{or } f \text{ is the interior of the intersection of } f_{\text{cmax}} \text{ and a convex region}\}$$

$$E(\sigma) = \{e \mid (\exists f_1, f_2 \in F(\sigma)) (e \subseteq i^\circ(r^\circ f_1 \cap r^\circ f_2))\}$$

$$V(\sigma) = \{v \mid (\exists e \in E(\sigma)) v \in b^\circ e\}$$

and where

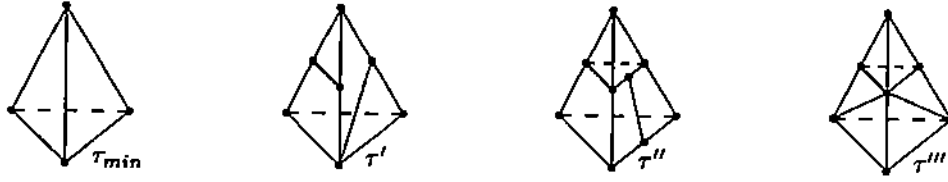


Figure 2: Several different arrangements of faces, edges and vertices of a tetrahedral solid.

1. $F(\sigma)$, $E(\sigma)$ and $V(\sigma)$ are finite sets,
2. each set $F(\sigma)$, $E(\sigma)$ and $V(\sigma)$ is pairwise disjoint,
3. the union of all the faces, edges and vertices form the boundary of S :

$$\partial S = V(\sigma) \cup \bigcup E(\sigma) \cup \bigcup F(\sigma).$$

Consider the various boundary descriptions of a specific tetrahedral solid T . Figure 2 shows graphically four different descriptions of the boundary of T . Each face set, along with the corresponding edges and the vertices, is one of the valid boundary descriptions in

$$B(T) = \{\tau_{min}, \tau', \tau'', \tau''', \dots\}.$$

In the following, lower case Greek letters, with exceptions to V , E , and F , denote the actual solids. Furthermore, lower case Roman letters are arbitrary variables, which are defined appropriately.

3 Outline of the Algorithm for the Boolean Operations

Consider the spatial locations occupied by two solids S and T , where $S = T$ is allowed. Given $\sigma \in \mathcal{B}(S)$ and $\tau \in \mathcal{B}(T)$, we ask whether the boundaries described by σ and τ penetrate or touch each other, and if so, where? If they do, the penetrating faces are subdivided so that they do not penetrate each other. In consequence, the construction of the result is simplified. Recall the four basic steps needed for computing the set operations given on Page 1.

- We will say that a boundary description *respects* a solid if each face of the boundary description is homogeneous in relation to the other solid. A face of $F(\sigma)$ is *homogeneous* in relation to solid T if the face is entirely inside T , outside T , or on the boundary of T . That is, no face of $F(\sigma)$ is both inside and outside, or partially on T .

In the first step, σ and τ are used to derive new boundary descriptions $\sigma' \in \mathcal{B}(S)$ and $\tau' \in \mathcal{B}(T)$ that respect each other's solids. The function that performs this is

$$\text{Respect}(\sigma, \tau) \rightarrow [\sigma', \tau'].$$

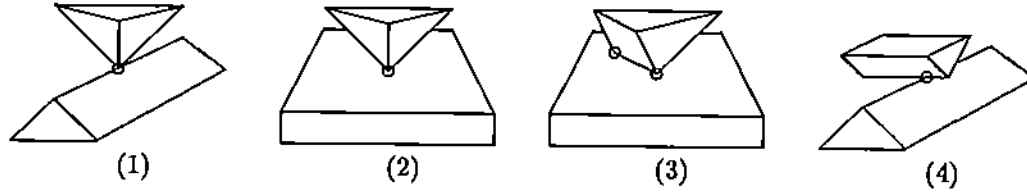


Figure 3: All allowed singularities: vertex-on-edge, vertex-on-face, edge-on-face, edge-on-edge

When σ respects T , for example, the faces, and the edges of σ are homogeneous almost everywhere in relation to T . Here “almost everywhere” means that singularities such as those shown in Figure 3 are allowed. Thus, a face or an edge is homogeneous with a few allowed exceptions.

The following property establishes one of the four conditions necessary for a face to be homogeneous (almost everywhere) in relation to another solid:

Property 8 *Let $\sigma \in B(S)$, and $\tau_{min} \in B(T)$ where $S, T \in \mathcal{M}$. Then a face $f \in F(\sigma)$ is homogeneous (almost everywhere) in relation to T if one and only one of the following relations holds true:*

$$\begin{aligned}
 fINT & \text{ iff } f \subset \left(T - \bigcup F(\tau_{min}) \right) \\
 fOUTT & \text{ iff } f \subset \left(c^*T - \bigcup F(\tau_{min}) \right) \\
 fWITHT & \text{ iff } (\exists f' \in F(\tau_{min})) (f \subseteq k^\circ f') \text{ and } (\forall p \in (f \cap f')) (N_S(p) = N_T(p)) \\
 fANTIT & \text{ iff } (\exists f' \in F(\tau_{min})) (f \subseteq k^\circ f') \text{ and } (\forall p \in (f \cap f')) (N_{e^*S}(p) = N_T(p));
 \end{aligned}$$

$N_X(p)$ is the regularized neighborhood of point p with respect to the solid X ; it describes the local region around p .

Relation $fINT$ holds when the face is inside T with the exception of some boundary points of T —namely, some vertices or some edges. Relation $fOUTT$ holds when the face f is outside T with the exception of some boundary points of T . If only manifolds were being considered, then the conditions could simply be stated as $f \subset iT$ and $f \subset cT$. Since however, nonmanifolds are part of the modeling domain \mathcal{M} , the singularities are tolerated.

Relations $fWITHT$ or $fANTIT$ hold when the face lies completely on the boundary of T —with some exceptions. The two relations distinguish between the two solids touching along the face, or overlapping each other along the face. Refer to Figure 3.

So, if exactly one of the relations hold for each face of a boundary description and some solid, then the boundary description is said to respect the solid. Given that σ respects



Figure 4: (a) shows two solid touching. (b) shows two solids overlapping.

T does not imply that τ respects S . From Property 8, we can say that $\sigma \in \mathcal{B}(S)$ and $\tau \in \mathcal{B}(T)$ respect each other if and only if σ respects T and τ respects S .

- In the second step of the Boolean operation, the faces of one boundary description are classified in relation to the other solid. The face classification process partitions the faces, $F(x)$, of each boundary description, x , in relation to solid y into four classification sets $F_{cy}(x)$, $F_{iy}(x)$, $F_{wy}(x)$, and $F_{ay}(x)$. The notation $F_R(x)$ is chosen to resemble the notation used for a fragment (defined later) where R is a region containing a subset of $F(x)$.

In the first step, Respect returned σ' and τ' . Because σ' respects T and τ' respects S , each face of one solid along with the other solid belongs to exactly one of the four relations of Property 8. Specifically, the Relations (1)-(4) of Property 8 correspond to the relations that hold for faces of the four classification sets. That is, the eight-way classification of σ and τ —that respect each other's solids—is given by

$$\text{Classify}([\sigma, \tau]) = [F_{cT}(\sigma), F_{iT}(\sigma), F_{wT}(\sigma), F_{aT}(\sigma), \\ F_{cS}(\tau), F_{iS}(\tau), F_{wS}(\tau), F_{aS}(\tau)],$$

where

$$\begin{aligned} F_{iT}(\sigma) &= \{f \in F(\sigma) \mid f_{INT}\} & F_{iS}(\tau) &= \{f \in F(\tau) \mid f_{INS}\} \\ F_{cT}(\sigma) &= \{f \in F(\sigma) \mid f_{OUTT}\} & F_{cS}(\tau) &= \{f \in F(\tau) \mid f_{OUTS}\} \\ F_{wT}(\sigma) &= \{f \in F(\sigma) \mid f_{WITHT}\} & F_{wS}(\tau) &= \{f \in F(\tau) \mid f_{WITHS}\} \\ F_{aT}(\sigma) &= \{f \in F(\sigma) \mid f_{ANTIT}\} & F_{aS}(\tau) &= \{f \in F(\tau) \mid f_{ANTIS}\}. \end{aligned}$$

Thus,

$$\begin{aligned} F(\sigma') &= F_{cT}(\sigma') \cup F_{iT}(\sigma') \cup F_{wT}(\sigma') \cup F_{aT}(\sigma'), \\ F(\tau') &= F_{cS}(\tau') \cup F_{iS}(\tau') \cup F_{wS}(\tau') \cup F_{aS}(\tau'). \end{aligned}$$

These sets roughly correspond to Tilove's classification sets $XOUTS$, and $XINS$, and the two cases for $XONS$. They can be thought of as the *in*, *out*, *with* and *anti* sets.

As an example of face classification, Figure 5 shows the union of two cubes, and face classification of each face. The six faces of each cube are made to respect the other cube by splitting the top, the front, the right, and the back faces of the lower cube, and splitting

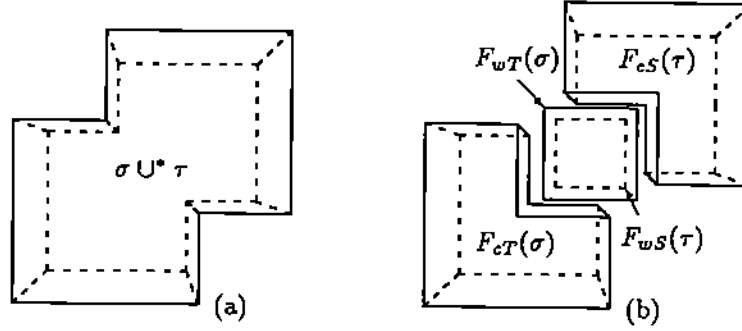


Figure 5: (a) shows the union of two boxes, σ and τ . The faces of the classification sets $F_{cS}(\tau)$, $F_{cT}(\sigma)$, $F_{wS}(\tau)$ and $F_{wT}(\sigma)$ needed for the union are shown in (b).

	σ				τ			
	F_{cT}	F_{iT}	F_{wT}	F_{aT}	F_{cS}	F_{iS}	F_{wS}	F_{aS}
$\sigma \cup \tau$	\oplus		\oplus		\oplus			
$\sigma \cap \tau$		\oplus				\oplus	\oplus	
$\sigma - \tau$	\oplus			\oplus		\ominus		
$\tau - \sigma$		\ominus			\oplus			\oplus

Table 1: The relevant sets of faces for each set operation are indicated by \oplus and \ominus . \ominus signifies the complementation.

the left, the bottom, the front and the back faces of the upper cube. In consequence, each cube will have ten homogeneous faces. Of these, six faces are outside, two faces are on, and two faces are inside the other cube. The two *on* faces belong to F_{wy} sets.

- Once the eight-way face classification is done, we construct a boundary description of the result of a regularized Boolean operation $\langle \text{op} \rangle^*$, $\zeta = \sigma' \langle \text{op} \rangle^* \tau'$, namely,

$$\text{Create} \left(\langle \text{op} \rangle, [F_{cT}(\sigma'), \dots, F_{aS}(\tau')] \right) \rightarrow \zeta,$$

where, $\zeta \in \mathcal{B}(C)$ of some solid $C = S \langle \text{op} \rangle^* T$. ζ describes the boundary of C and consists of the faces in either $F(\sigma')$ or $F(\tau')$. The faces of $F(\zeta)$ are all the faces of exactly three of the eight classification sets of $\text{Classify}([\sigma', \tau'])$. Which three depends only on the set operation, as shown in Table 1. The appropriate face sets are indicated by the symbols \oplus and \ominus .

When constructing the BRep ζ , the face normals of certain faces must be complemented. For the difference operation, $\zeta = \sigma - \tau$, the faces comprise the set $F_{iS}(\tau')$ or $F_{iT}(\sigma')$. A complemented face covers the same area, but the solid which is on one side of the face

is on the other side of its complement. In the table, the sets with faces that need to be complemented are indicated by the symbol \ominus .

- The created boundary description ζ resulting from Create is fragmented. That is, ζ is not necessarily minimal. In the fourth step of the set operation algorithm, function Reduce maps ζ to a minimal boundary description ζ_{\min} :

$$\text{Reduce}(\zeta) \rightarrow \zeta_{\min}.$$

This mapping is called topological reduction. Coplanar faces that have at least one common edge are merged into a single maximally connected face, and adjacent collinear edges are merged into a single maximally connected edge.

It is now possible to express the composition of functions to compute a set operation. Given $\sigma \in \mathcal{B}(S)$ and $\tau \in \mathcal{B}(T)$, along with a binary set operation $\langle \text{op} \rangle$, the result of applying the set operation to σ and τ is

$$\sigma \langle \text{op} \rangle \tau = \text{Reduce} \left(\text{Create} \left(\langle \text{op} \rangle, \text{Classify} \left(\text{Respect}(\sigma, \tau) \right) \right) \right).$$

4 Obtaining Boundaries with Respect

Establishing respect requires the partitioning of existing faces and edges, and the introduction of new edges and vertices. Whenever a face does not satisfy one of the four relations of Property 8, the face must be subdivided into two or more faces, so that each face does satisfy one of the four relations. Instead of viewing the problem as one of partitioning two boundary descriptions in relation to each other, it can be viewed differently as a problem of partitioning two boundary descriptions in relation to common regions of space. This way, a face of one boundary description is compared only to some common region of space and is not directly compared to the other solid.

Regions, as used here, are point sets that have an interior, a boundary and an exterior, however, unlike solids, regions need not be closed sets. A *region*, R , is the intersection of a finite number of open or closed half-spaces with planar boundaries. The boundary of R is $bR = rR - iR$, and so, portions of the boundary of R do not necessarily belong to R .

If, for some $\sigma \in \mathcal{B}(S)$ and some region R , each face of $F(\sigma)$ is either completely in the region or completely outside the region, then σ *respects* R .

Definition 9 σ respects a region R if $(\forall f \in F(\sigma)) (f \subseteq R \text{ or } f \subseteq cR)$.

If σ respects R , a *fragment* is the set of all faces of $F(\sigma)$ that lie in R , written $F_R(\sigma)$, where

$$F_R(\sigma) \subseteq F_{\mathcal{E}^3}(\sigma) = F(\sigma).$$

Observe that a face of $F(\sigma)$ lying on bR may or may not be in the fragment $F_R(\sigma)$, since R need be neither open nor closed. Similarly, since a face is an open set, the edges and the vertices of a face lying in R may or may not be in R . Formally, we define fragments as follows:

Definition 10 For a nonempty region R and a σ that respects R , the fragment of σ in R is

$$F_R(\sigma) = \{f \in F(\sigma) \mid f \subseteq R\}.$$

It is not hard to see that the faces of $F_R(\sigma)$ and the faces of $F(\tau) - F_R(\tau)$ are pairwise disjoint. That is, the faces of $F(\sigma)$ within R cannot intersect any of the faces of $F(\tau)$ that are not in R , and vice-versa. In consequence, an algorithm for imposing respect can be based on the divide-and-conquer paradigm [Ben80]. A partitioning of space into the two regions R and cR , divides the problem into two smaller subproblems.

Consider a finite set of n convex regions $\mathcal{R} = \{R_1, \dots, R_n\}$, that are pairwise disjoint and together cover \mathbb{E}^3 along with some σ and τ that both respect each of the n regions. The n regions of \mathcal{R} partition the faces into n fragments,

$$F(x) = F_{R_1}(x) \cup F_{R_2}(x) \cup \dots \cup F_{R_n}(x),$$

for x either σ or τ . The problem of imposing respect on σ and τ consists of n subproblems of imposing respect on each of the two fragments of the n regions independently.

We wish to specify an algorithm that obtains respect by splitting regions and fragments. In particular, the algorithm should produce a sequence

$$(\sigma_1, \tau_1, \mathcal{R}_1), (\sigma_2, \tau_2, \mathcal{R}_2), \dots, (\sigma_n, \tau_n, \mathcal{R}_n), \dots \quad (11)$$

in which the n th triple is known to contain σ_n and τ_n that respect each other, and where the i th triple, for $1 \leq i < n$, does not. To map the i th triple to the $(i + 1)$ triple, three operations are needed:

1. Select some region $R \in \mathcal{R}_i$ for decomposition,
2. select some splitting plane \mathcal{P} that intersects R , and
3. partition R and subdivide the faces of $F_R(\sigma_i)$ and $F_R(\tau_i)$ by \mathcal{P} , producing σ_{i+1} and τ_{i+1} .

The operations are now as follows:

- First, we select a region. Any region of \mathcal{R}_i which contains a nonhomogeneous face needs to be further decomposed and must be eventually selected. The order in which such regions are selected is arbitrary.

The ideal action is to select a region that has nonhomogeneous faces and not to select a region that has homogeneous faces. However, distinguishing between such regions is computationally expensive, and so it is not done. Instead, a region is selected unless it is plainly obvious that all the faces it contains are homogeneous. The following function checks four different conditions to determine the existence of respect within a given region

R :

$$FRel(F_R(\sigma), F_R(\tau)) = \begin{cases} 1 & \text{if } F_R(\sigma) = \emptyset \text{ and } F_R(\tau) = \emptyset \\ 2 & \text{if } F_R(\sigma) = \emptyset \text{ and } F_R(\tau) \neq \emptyset \\ 3 & \text{if } F_R(\tau) = \emptyset \text{ and } F_R(\sigma) \neq \emptyset \\ 4 & \text{if } F_R(\sigma) = \{f\}, F_R(\tau) = \{f'\} \text{ and} \\ & \quad k \circ f = k \circ f' = k \circ R \\ 0 & \text{otherwise} \end{cases}$$

FRel checks the number of faces of both fragments in the region. Conditions one through three hold when R is nonplanar and one or both of the fragments is empty. Condition four holds when R is planar and contains two coplanar and equal faces, one face from each solid. A zero value indicates that based on the four conditions, the two fragments are not known to contain homogeneous faces.

- Next, we select a splitting plane for the chosen region. Any violations of respect occur along the boundaries of either solid. Subdividing the faces of one solid by a plane that contains a face of the other solid is a step in the right direction, and suggests a selection strategy in which a face of either solid determines the splitting plane.

Because regions can be either planar or nonplanar, the selection strategy must account for the region planarity. For a nonplanar region, a splitting plane should contain one of the faces. For a planar region, a splitting plane should be perpendicular to one of the faces and pass through one of the edges. This in effect is analogous to cutting polygons in 2D by splitting lines.

To achieve respect with a finite number of cuts requires a careful selection of a face. Some faces in a region take priority over other faces. For example, faces that lie on the boundary of a nonplanar region should not be chosen as long as other faces exist in the interior of the region. Doing so would result in ill-formed regions that violate the correctness of this method (this will be demonstrated). This shows that only a subset of the faces constitutes a set of candidate faces from which a single candidate face can be chosen. Consequently, four issues have to be considered in devising a splitting plane selection strategy:

1. Given a fragment, what are its candidate faces?
2. Given that both fragments have nonempty sets of candidate faces, which fragment should contribute a candidate face?
3. Given all the candidate faces, which is the most desirable face?
4. How should the splitting plane be oriented in relation to the candidate face?

To address these issues, we must first establish the relationship between a face and its containing region, and an edge and its containing region. We define five face-region relations Δ_1 through Δ_5 , and two edge-region relations Λ_1 and Λ_2 . The relations are

$f\Delta_1R$	The face is <i>inside</i> the nonplanar region R —although the edges of the face lying on the convex hull of the face may lie on the boundary of the region.
$f\Delta_2R$	The face lies on the border of the region, and the region is <i>below</i> the face.
$f\Delta_3R$	The face lies on the border of the region, and the region is <i>above</i> the face.
$f\Delta_4R$	The face is in but not equal to the planar region; i.e., $f \subset R$.
$f\Delta_5R$	The face is equal to the planar region; i.e., $k^\circ f = k^\circ R$.
$e\Lambda_1R$	The edge is inside the region; i.e., $e \subset iR$.
$e\Lambda_2R$	The edge lies on the boundary of R ; i.e., $e \subseteq b^\circ R$. Note that $f\Delta_5R$ if and only if $\forall e \in E(f) e\Lambda_2R$, where $E(f)$ are the edges adjacent to f .

Table 2: The face-region and edge-region relations.

defined in Table 2. The five face-region relations are pairwise disjoint, and so for any region R and any face f in R , $f\Delta_iR$ and $f\Delta_jR$ imply that $i = j$. The same holds for the two edge-region relations. Let the *face-region index* and the *edge-region index* be the subscripts of Δ_i and Λ_i respectively. Using the face-region indices, the faces in a region can be grouped into five sets. Similarly, the edges can be grouped into two sets based on the edge-region indices. The value of $I_R(\sigma, \tau)$ is then the smallest face-region index of any face in that region, where

$$I_R(\sigma, \tau) = \min \left\{ i \mid f\Delta_iR \text{ for } f \in (F_R(\sigma) \cup F_R(\tau)) \right\}.$$

The subset of all the faces in R having the smallest face-region index is called the *set of candidate faces*, $C_R(\sigma, \tau)$, defined as:

$$\left\{ f \in (F_R(\sigma) \cup F_R(\tau)) \mid f\Delta_{I_R(\sigma, \tau)}R \right\}. \quad (12)$$

With the set of candidate faces, the splitting plane selection function can now be stated. The oriented splitting plane selected to split a region R containing the fragments $F_R(\sigma)$ and $F_R(\tau)$, not both empty, is

$$\text{Choose}(F_R(\sigma), F_R(\tau)) = \begin{cases} \text{Plane}(f) & \text{if } f\Delta_1R \text{ or } f\Delta_2R \\ -\text{Plane}(f) & \text{if } f\Delta_3R \\ \text{Perp}(f, e) & \text{if } f\Delta_4R \text{ and } e\Lambda_1R, \text{ for } e \in E(f), \end{cases} \quad (13)$$

where

1. $f \in C_R(\sigma, \tau)$ is the candidate face,
2. $\text{Plane}(f)$ is the plane containing f , and oriented so that the plane's normal vector points away from the solid, and
3. $\text{Perp}(f, e)$ is a plane perpendicular to $\text{Plane}(f)$ where

- (a) e is some edge of $E(f)$, the set of edges adjacent to face f , and
- (b) e does not lie on a boundary of R .

The orientation of the plane $\text{Perp}(f, e)$ is arbitrary.

This finishes the discussion on two of the four issues, namely, what are the candidate faces and how should the splitting plane be oriented in relation to the selected candidate face. These two issues address correctness. The other two issues, namely, what fragment to choose a face from and what is the best candidate face, address efficiency. In this paper, only the correctness issue is dealt with. Details pertaining to efficiency can be found in Vanecek's thesis[Van89].

- Finally, we partition the selected region and subdivide the faces of the region by the selected splitting plane.

Given the plane \mathcal{P} as the tuple (a, b, c, d) , where the components are real,

$$\mathcal{P}_> = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz + d > 0\}$$

is the *open half-space above* \mathcal{P} , and $\mathcal{P}_<$ is the *open half-space below* \mathcal{P} . Analogously, $\mathcal{P}_\geq = \mathcal{P}_> \cup \mathcal{P}$ and $\mathcal{P}_\leq = \mathcal{P}_< \cup \mathcal{P}$ are the appropriate closed half-spaces.

A plane \mathcal{P} that intersects a region R can partition R into regions $R \cap \mathcal{P}_\geq$, and $R \cap \mathcal{P}_<$, referred to as the *region on or above*, and the *region below* \mathcal{P} , and labeled R_\geq and $R_<$ respectively.

The subdividing is performed by a function

$$\text{Cut}(x, R, \mathcal{P}) \rightarrow x', \tag{14}$$

where x is a boundary description that respect the region R , and \mathcal{P} is the selected splitting plane intersecting R . x' is the resulting boundary description that in addition to respecting R also respects the regions $R_< = R \cap \mathcal{P}_<$ and $R_\geq = R \cap \mathcal{P}_\geq$. That is,

$$F_R(x') = F_{R_<}(x') \cup F_{R_\geq}(x').$$

The faces of $F(x')$ are those of $F(x)$, except for the faces of R which are cut by \mathcal{P} (a face f of R that crosses \mathcal{P} transversely is indicated by $f \boxtimes \mathcal{P} = (f \cap \mathcal{P} \text{ and } f \not\subset \mathcal{P})$), namely:

$$\begin{aligned} & (F(x) - F_R(x)) \cup \\ & \{f \in F_R(x) \mid f \cap \mathcal{P} = \emptyset\} \cup \\ & \{f \in F_R(x) \mid f \subset \mathcal{P}\} \cup \\ & \{f' \subset (f - \mathcal{P}) \mid f \in F_R(x), f \boxtimes \mathcal{P}, \text{ and } f' \text{ is maximally connected}\}. \end{aligned} \tag{15}$$

In the fragment $F_R(x')$, none of the faces cross \mathcal{P} . The three sets indicated in Eq. (15) are the faces of $F_R(x)$ that do not cross \mathcal{P} , the faces that lie in \mathcal{P} , and the faces that

result from subdividing the faces that cross \mathcal{P} . Each face that crosses \mathcal{P} results in two or more new subfaces in $F_R(x')$.

In addition to changing the face set of \mathbf{x} , the edge and the vertex sets are also changed. The edges that cross \mathcal{P} and that are adjacent to the subdivided faces get cut and new edges are created from the portions of the faces that lie on \mathcal{P} . Let $E_R(x)$ be the subset of the edge set $E(x)$ with edges that are adjacent to the faces of $F(x)$. The new edge set $E(x')$ is

$$\begin{aligned} & (E(x) - E_R(x)) \cup \\ & \{e \in E_R(x) \mid e \cap \mathcal{P} = \emptyset \text{ or } e \subset \mathcal{P}\} \cup \\ & \{e' \subset (e - \mathcal{P}) \mid e \in E_R(x), E \cap \mathcal{P} \neq \emptyset, e \not\subset \mathcal{P} \text{ and } e' \text{ is maximally connected}\} \cup \\ & \{e \subset (f \cap \mathcal{P}) \mid f \in F_R(x), f \not\subset \mathcal{P}, \text{ and } e \text{ is maximally connected}\} \end{aligned}$$

The vertices of $V(x')$ consist of the original vertices of $V(x)$ and the vertices created by subdividing the edges of $E_R(x)$, namely,

$$V(x) \cup \{e \cap \mathcal{P} \mid e \in E_R(x) \text{ and } e \not\subset \mathcal{P}\}.$$

With the three operations, the i th triple of Sequence (11) can be mapped to the $(i+1)$ triple by a function h , as follows:

$$h(\langle \sigma, \tau, \mathcal{R} \rangle) = \begin{cases} \langle \sigma, \tau, \mathcal{R} \rangle & \text{if } (\forall R \in \mathcal{R}) (\text{FRel}(F_R(\sigma), F_R(\tau)) \neq 0) \\ \langle \sigma', \tau', \mathcal{R}' \rangle & \text{otherwise} \end{cases} \quad (16)$$

where

$$\begin{aligned} R & \in \{R' \in \mathcal{R} \mid \text{FRel}(F_R(\sigma), F_R(\tau)) = 0\}, \\ \mathcal{P} & = \text{Choose}(F_R(\sigma), F_R(\tau)), \\ \sigma' & = \text{Cut}(\sigma, R, \mathcal{P}), \\ \tau' & = \text{Cut}(\tau, R, \mathcal{P}), \text{ and} \\ \mathcal{R}' & = (\mathcal{R} - \{R\}) \cup \{R_<, R_\geq\}, \text{ s.t. } R_< = R \cap \mathcal{P}_< \text{ and } R_\geq = R \cap \mathcal{P}_\geq. \end{aligned}$$

In term of the Function h , Respect is defined as the pair $[\sigma_n, \tau_n]$ corresponding to the triple $\langle \sigma_n, \tau_n, \mathcal{R}_n \rangle$, with the smallest integer n for which all regions of \mathcal{R}_n contain fragments that have a nonzero FRel value. That is,

$$\text{Respect}(\sigma, \tau) = \min_{n>0} \left\{ [\sigma_n, \tau_n] \mid \langle \sigma_n, \tau_n, \mathcal{R}_n \rangle = h^n(\langle \sigma, \tau, \{\mathbb{E}^3\} \rangle) \text{ and } (\forall R \in \mathcal{R}_n) (\text{FRel}(F_R(\sigma_n), F_R(\tau_n)) > 0) \right\},$$

where $h^n(y)$ denotes $n-1$ compositions of h .

We will now show that for all σ and τ there is an n at which σ_n and τ_n respect each other.

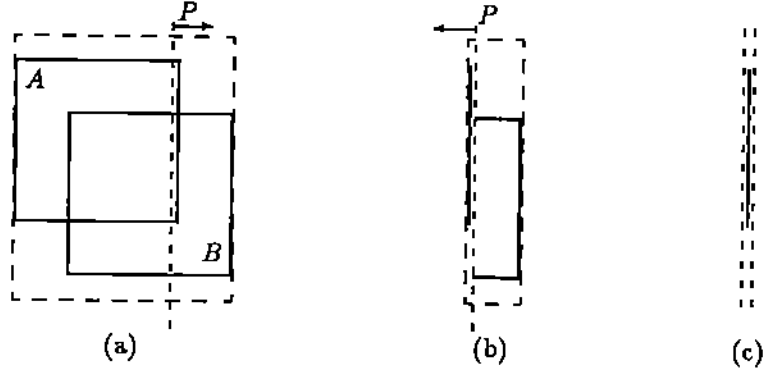


Figure 6: An example of an ill constructed region. (a) shows two solids A and B in a region marked by dashes and cut by a plane P . (b) shows the region on and above P . (c) shows a planar region lying both inside and outside B and containing the right face of solid A .

What needs to be shown first is that for any $R \in \mathcal{R}_i$ constructed by h , $F_R(\sigma) = \emptyset$ implies that $R \subseteq iS$ or $R \subset cS$ almost everywhere. Given any region R that is not necessarily constructed by h ,

$$(F_R(\sigma) = \emptyset) \Rightarrow (\forall f \in F(\sigma)) (f \cap R = \emptyset),$$

namely, an empty fragment means that none of the faces of $F(\sigma)$ lie in R . However, $F_R(\sigma) = \emptyset$ implies nothing about the relation of R and iS . One can contrive an R such that σ respects R , $F_R(\sigma) = \emptyset$, and yet R lies both inside and outside S (recall that the faces are open sets). Such an ill-formed region is constructed in Figure 6. The figure shows the projections of two blocks A and B , and two cuts necessary to create the desired region (shown in Figure 6(c)). The ill-formed region is a planar region that resulted from choosing splitting planes other than those given by Choose.

Theorem 17 *If $h^i((\sigma, \tau, \{\mathbb{E}^3\})) = (\sigma_i, \tau_i, \mathcal{R}_i)$ for any $i > 1$, then for any $R \in \mathcal{R}_i$,*

$$\begin{aligned} F_R(\sigma_i) = \emptyset &\Rightarrow \left(R \subset S - \bigcup F(\sigma_i) \right) \text{ or } \left(R \subset c^*S - \bigcup F(\sigma_i) \right), \text{ and} \\ F_R(\tau_i) = \emptyset &\Rightarrow \left(R \subset T - \bigcup F(\tau_i) \right) \text{ or } \left(R \subset c^*T - \bigcup F(\tau_i) \right). \end{aligned}$$

Theorem 17 states that a region of \mathcal{R} that is constructed by h cannot be both inside and outside a given solid if the solid's fragment in the region is empty. Once it is clear that an empty fragment indicates that the region is entirely inside or outside a given solid, it is easy to show the relation of the faces to the solids.

Corollary 18 *If $h^i((\sigma, \tau, \{\mathbb{E}^3\})) = (\sigma_i, \tau_i, \mathcal{R}_i)$ for any $i > 1$, then for any $R \in \mathcal{R}_i$,*

1. if $F_R(\tau) = \emptyset$ and $F_R(\sigma) \neq \emptyset$ then $(\forall f \in F_R(\sigma)) f_{INT}$ or $(\forall f \in F_R(\sigma)) f_{OUTT}$,
2. if $F_R(\sigma) = \emptyset$ and $F_R(\tau) \neq \emptyset$ then $(\forall f \in F_R(\tau)) f_{INS}$ or $(\forall f \in F_R(\tau)) f_{OUTS}$.

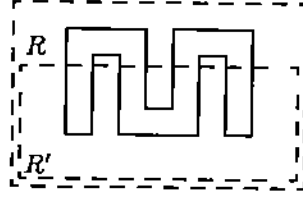


Figure 7: A child region R' that contains more faces than the parent region R .

For solids with coplanar and touching faces, a region containing the touching faces cannot be completely decomposed into regions satisfying Corollary 18. Instead, h produces a planar and convex region that contains two of the touching faces.

Lemma 19 *If $h^i(\langle \sigma, \tau, \{\mathbf{E}^3\} \rangle) = \langle \sigma_i, \tau_i, \mathcal{R}_i \rangle$ for any $i > 1$, then for any $R \in \mathcal{R}_i$, if $F_R(\sigma) = \{f\}$, $F_R(\tau) = \{f'\}$ and $k^\circ f = k^\circ f'$ then*

$$(f \text{ WITH } T \text{ and } f' \text{ WITH } S) \text{ or } (f \text{ ANTI } T \text{ and } f' \text{ ANTI } S).$$

It remains to be shown that h converges, namely that,

$$(\exists n > 0) (\forall i \geq n) (h^i(x) = h^{i+1}(x)).$$

At first thought, it appears that the size of a candidate-face set $C_{R'}(\sigma_{i+1}, \tau_{i+1})$ can be larger than the candidate-face set $C_R(\sigma_i, \tau_i)$ of its parent region $R \supseteq R'$. This suggests that the child region R' can have more choices of splitting planes than the parent region R , and that fragmentation increases not only the total number of faces but also the number of possible splitting planes. However, this is not the case. $C_{R'}(\sigma_{i+1}, \tau_{i+1})$ may contain several faces that belong to the same maximal face. The number of candidate splitting planes obtainable from R' is bounded by the number of maximal faces that cross R' and not by the number of faces in the region. The actual number of the unique planes obtainable from the candidate face set is the same as the number of maximal faces that intersect the region, and is

$$N_R(\sigma, \tau) = \sum_{f_{max}} \begin{cases} 1 & \text{if } (\exists f \in F) f \subseteq f_{max} \text{ and } f \Delta_{I_R(\sigma, \tau)} R \\ 0 & \text{otherwise,} \end{cases}$$

where $F = F_R(\sigma) \cup F_R(\tau)$ and the sum is over all maximal faces f_{max} of S and T that also intersect R .

In the case that R is a planar region, the splitting planes are taken to be perpendicular to the region and are determined from the edges in the region rather than from the faces. As such every face is a candidate face. The actual number of splitting planes is the number of maximal edges that intersect the interior of the region. Thus, for planar regions,

$$N_R(\sigma, \tau) = \sum_{e_{max}} \begin{cases} 1 & \text{if } (\exists e \in E) e \subseteq e_{max} \text{ and } e \subset i^\circ R \\ 0 & \text{otherwise,} \end{cases}$$

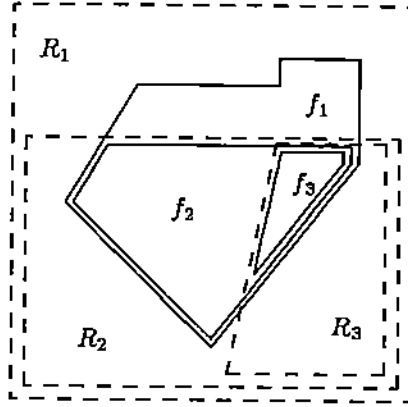


Figure 8: Fragmentation of a face f_1 , where $f_3 \subset f_2 \subset f_1$.

where $E = E_R(\sigma) \cup E_R(\tau)$ and the sum is over all maximal edges e_{max} of S and T that also intersect the interior of R .

It is important to note that a face with a face-region index i resulting from the subdivision of a face with a face-region index j implies that $j \leq i$. This is portrayed in Figure 8 which shows three regions R_1 , R_2 , and R_3 , such that $R_3 \subset R_2 \subset R_1$, and three faces f_1, f_2, f_3 . Given that that faces have $f_1 \Delta_i R_1$, $f_2 \Delta_j R_2$, and $f_3 \Delta_k R_3$, their face-region indices are nondecreasing, $i \leq j \leq k$.

Lemma 20 *Let $f \in F_R(\sigma_k)$ and $f' \in F_{R'}(\sigma_{k+1})$ where $R \supseteq R'$ and $f \supseteq f'$, and where $f \Delta_i R$ and $f' \Delta_j R'$. Then $i \leq j$.*

We can now show that as more and more regions are created, the amount of work reduces in that the face-region indices increase, as the number of possible splitting planes decreases.

Theorem 21 *Given the sequence $(\sigma_1, \tau_1, \mathcal{R}_1), (\sigma_2, \tau_2, \mathcal{R}_2), \dots$ then for all $i \geq 1$ and for all $R \in \mathcal{R}_i$, one of the following two conditions hold:*

1. $R \in \mathcal{R}_{i+1}$, $I_R(\sigma_i, \tau_i) = I_R(\sigma_{i+1}, \tau_{i+1})$, and $N_R(\sigma_i, \tau_i) = N_R(\sigma_{i+1}, \tau_{i+1})$; or
2. $R \notin \mathcal{R}_{i+1}$ and $\exists R_1, R_2 \in \mathcal{R}_{i+1}$ where $R = R_1 \cup R_2$, such that for $x = 1, 2$ exactly one of the following conditions hold:
 - (a) $F_{R_x}(\sigma_{i+1}) = \emptyset$ and $F_{R_x}(\tau_{i+1}) = \emptyset$;
 - (b) $(I_R(\sigma_i, \tau_i) = I_{R_x}(\sigma_{i+1}, \tau_{i+1})$ and $N_R(\sigma_i, \tau_i) > N_{R_x}(\sigma_{i+1}, \tau_{i+1})$); or
 - (c) $I_R(\sigma_i, \tau_i) < I_{R_x}(\sigma_{i+1}, \tau_{i+1})$.

Each region in \mathcal{R}_i either appears in \mathcal{R}_{i+1} in which case nothing changed (condition (1)), or is split (condition (2)). If the region R is split into R_1 and R_2 , three possibilities arise for each R_x , where $x = 1, 2$:

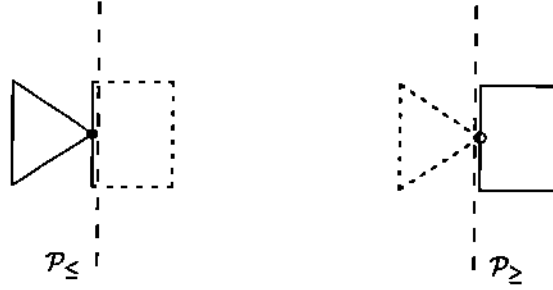


Figure 9: Both figures show a cone and a block. In figure (a), the left face of the block is merged with the apex of the cone. In figure (b), the left face is not.

- 2a) R_x contains no faces. Thus, $R_x \in \mathcal{R}_j$, for all subsequent j .
- 2b) The smallest face-region index in R_x remains the same as in R . In this case, the number of possible splitting planes diminished from that of R .
- 2c) The smallest face-region index in the R_x is higher than in the R_x .

This suggests that at some point, the splitting of regions must stop as the decomposition process runs out of splitting planes and all regions become homogeneous. It follows, then, that from any σ and τ , we can derive some σ_n , and τ_n that respect each other.

Theorem 22 Given $\sigma \in \mathcal{B}(S)$ and $\tau \in \mathcal{B}(T)$ for $S, T \in \mathcal{M}$,

$$(\exists n > 0) \left((\sigma_n, \tau_n, \mathcal{R}_n) = h^n(\langle \sigma, \tau, \{\mathbf{E}^3\} \rangle) \text{ and } (\forall R \in \mathcal{R}_n) (\text{FRel}(F_R(\sigma_n), F_R(\tau_n)) > 0) \right).$$

The proof follows directly from Theorem 21.

5 Conclusion

This paper presented the set operations as a four step problem, and focused on the first step, namely, imposing respect. Respect is achieved by a method that can be described as a two-way, input-directed, spatial decomposition method with four appealing properties:

1. The method is purely face-based. Only face sets need to be manipulated. This is a direct consequence of partitioning a region into two subregions, called a two-way decomposition method, rather than into three regions, called a three-way decomposition method.

The use of the two-way decomposition method provides only the weak form of respect for which the obtained boundary descriptions contain faces that are homogeneous almost everywhere in relation to the other solid. Thus, it cannot be consistently determined if an edge or a vertex that touches the other solid is going to be incorporated into the touching face or not. This is illustrated graphically in Figures 9(a) and (b). The two figures show

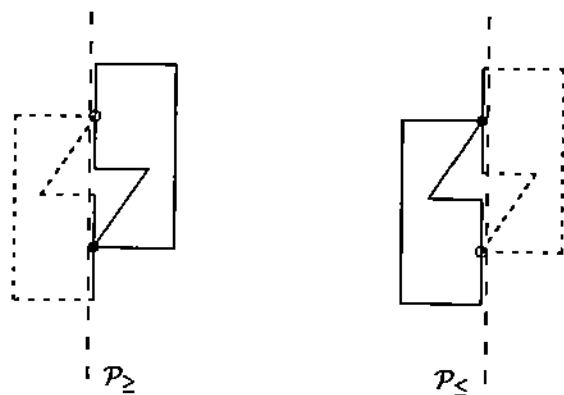


Figure 10: Solid circles indicate where a vertex is merged into a face. An open circle indicates where a vertex is not merged.

a cone touching a block. In (a), the splitting plane causes the left face of the block to lie in the same region as the cone, and so the left face of the block is subsequently made homogeneous in relation to the cone. On the other hand, Figure (b) shows that a split can cause the block and the cone to separate so that the left face of the block does not merge with the apex of the cone.

The example of Figure 9 suggests that a more informed splitting plane selection strategy that properly chooses a splitting plane can produce the strong form of respect. This, however, is not possible. Consider the same type of example shown in Figure 10. Both orientations of the splitting plane prevent one of the two faces lying in the splitting plane to merge with the touching vertex. The open circles mark the vertices that are not merged into the touching face.

As a consequence, the two-way decomposition method necessarily results in the weak form of respect, and requires a post-processing step to merge the four singular cases illustrated in Figure 3. This post-processing step can be done efficiently by using the plane-sweep method[PS85].

2. With the use of a nonmanifold boundary representation such as the fedge-based data structure [Van89], all solids are handled uniformly.
3. The eight-way classification allows the nondestructive construction of $A \cup^* B$, $A \cap^* B$, $A -^* B$, and $B -^* A$ simultaneously without reclassifying for each operation.
4. The splitting plane selection strategy used in Choose can be augmented with a simple regular-decomposition method. Initial cuts can be selected so that a minimal rectilinear region enclosing both solids is successively cut in half without affecting the correctness of the method. This is illustrated in Figure 11. Note that by itself, regular-decomposition

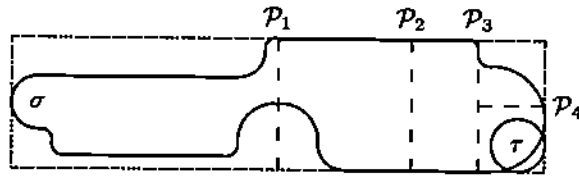


Figure 11: The first four cuts performed by a regular decomposition method.

is not sufficient for imposing respect. Nevertheless, mixing regular-decomposition with the input-directed method in general reduces the number of generated regions. Paterson and Yao demonstrated that while a binary partition that restricts every splitting plane to contain a face can be quadratic in size a binary partition without the restriction may exist that is only linear in size [PY89, Example II].

A solid modeler using the presented method has been implemented in Common Lisp and runs on a Texas Instrument's Explorers and on Symbolics. It uses the fedge-based data structure to represent the boundaries of solids and uses set operations to create complex solids from few parameterized primitives.

Appendix: Proofs

Proof: [of Theorem 17] The proof is given only for σ_i since the proof for the second implication (i.e., τ_i) is identical.

Suppose by way of contradiction that the implication is false, and let $F = \bigcup F(\sigma_i)$. Then

$$\begin{aligned}
 & \neg(F_R(\sigma_i) = \emptyset \Rightarrow (R \subset S - F) \vee (R \subset c^*S - F)) \\
 \Rightarrow & \neg(F_R(\sigma_i) \neq \emptyset \vee (R \subset S - F) \vee (R \subset c^*S - F)) \\
 \Rightarrow & F_R(\sigma_i) = \emptyset \wedge (R \not\subset S - F) \wedge (R \not\subset c^*S - F) \\
 \Rightarrow & F \cap R = \emptyset \wedge (R \cap (cS \cup F)) \neq \emptyset \wedge (R \cap (iS \cup F)) \neq \emptyset \\
 \Rightarrow & F \cap R = \emptyset \wedge (R \cap cS) \neq \emptyset \wedge (R \cap iS) \neq \emptyset.
 \end{aligned}$$

That is, R does not contain any of the faces, but it is simultaneously inside and outside S . This can hold only if R crosses the boundary of S (i.e., $R \cap bS \neq \emptyset$). However, since $F \cap R = \emptyset$, R can cross bS only through edges or vertices. Now, since R is a convex region, it can only cross bS at only one vertex or at only one edge. If R crosses bS at a vertex, R must be a subset of a line passing through the point coincident with the vertex. If R crosses bS at an edge, R must be a subset of a plane passing through the line containing that edge.

So suppose without loss of generality that $R \notin \mathcal{R}_{i-1}$. Then there must exist some region $R' \in \mathcal{R}_{i-1}$ for which

- $\text{FRel}(F_{R'}(\sigma_{i-1}), F_{R'}(\tau_{i-1})) = 0$; and
- $R = R' \cap \mathcal{P}_<$ or $R = R' \cap \mathcal{P}_\geq$, where $\mathcal{P} = \text{Choose}([F_{R'}(\sigma_{i-1}), F_{R'}(\tau_{i-1})])$.

Now consider the two possible ways that R can cross bS :

1. R is a subset of a line. Clearly, R' is a subset of a plane (i.e., a planar region) since R' contains faces. Furthermore, at least one face in R' contains an edge e for which $e \Lambda_1 R'$, since otherwise, every face f would be $f \Delta_5 R'$, and R' would no longer be a candidate for decomposition. Since R' is planar, to get R to be a subset of a line means that $R = R' \cap SP_\geq$, and $\mathcal{P} = \text{Perp}(f, e)$, where e lies on the border of R' , namely, $e \Lambda_2 R'$. But this is contradicts the condition of Choose, and so R cannot be a subset of a line.
2. R is planar. Without loss of generality, assume that R' is non-planar. Clearly, R is the result of $R' \cap \mathcal{P}_\geq$, and since R' is non-planar than the face -region index of each face in R' is less than four. To get the desired region R , the candidate face f in R' must have the face-region index equal to two

or three, and there must be other faces lying in the interior of R' (i.e., with a face-region index of one). However, this contradicts the definition of Choose which uses a candidate face from the candidate face set consisting of faces with the minimal face-region index.

The above two cases (based on the premise that the implication of the theorem is false) derive contradictions thereby showing that the implications of the theorem are true. \square

Proof: [of Corollary 18] From Theorem 17,

$$F_R(\tau) = \emptyset \Rightarrow \left(R \subset T - \bigcup F(\tau) \right) \text{ or } \left(R \subset c^*T - \bigcup F(\tau) \right).$$

Since $F_R(\sigma) \neq \emptyset$, then all its faces must be contained entirely in R . So using Property 8, consider in turn, the two disjuncts of the above consequent:

$$\begin{aligned} \forall f \in F_R(\sigma)(f \subseteq R) &\Rightarrow \forall f \in F_R(\sigma) \left(f \subset T - \bigcup F(\tau) \right) \\ &\Rightarrow \forall f \in F_R(\sigma) f \text{ IN } \tau, \text{ or} \\ \forall f \in F_R(\sigma)(f \subseteq R) &\Rightarrow \forall f \in F_R(\sigma) \left(f \subset c^*T - \bigcup F(\tau) \right) \\ &\Rightarrow \forall f \in F_R(\sigma) f \text{ OUT } \tau, \end{aligned}$$

which completes the proof. \square

Proof: [of Lemma 19] Assume the antecedent. Since $k^\circ f = k^\circ f'$, then $f \subseteq k^\circ f'$, and $f' \subseteq k^\circ f$, which means that

$$\forall p \in f \cap f' (N_S(p) = N_T(p)), \text{ or} \quad (23)$$

$$\forall p \in f \cap f' (N_{c^*S}(p) = N_T(p)). \quad (24)$$

Now, if Eq. (23) holds, then $f \text{ WITH } \tau$ and $f' \text{ WITH } \sigma$. Otherwise, if Eq. (24) holds, then $f \text{ ANTI } \tau$ and $f' \text{ ANTI } \sigma$. \square

Proof: [of Lemma 20] Since $x_{k+1} = \text{Cut}(x_k, R, \mathcal{P})$, $F_{R'}(x_{k+1})$ is either $F_R(x_{k+1}) \cap \mathcal{P}_<$ or $F_R(x_{k+1}) \cap \mathcal{P}_\geq$.

Assume the former. From Definition 14, the region on or above \mathcal{P} consists of the faces of three sets, that are correspondingly the faces that do not cross \mathcal{P} , that lie in \mathcal{P} , and that result from faces crossing \mathcal{P} . Of these, only the faces lying on \mathcal{P} change their face-region index when they become part of R_\geq . The others, not lying on \mathcal{P} are not affected, and so their face-region index does not change. Therefore, $i = j$. For the faces that do lie on \mathcal{P} , they migrated from the interior to the boundary. Therefore, $i < j$.

Assume the latter. The same line of reasoning follows as the former case with the exception that there are no faces on \mathcal{P} . Therefore, each face of the region below \mathcal{P} maintains the same face-region index. \square

Proof: [of Theorem 21] Proof by strong induction on the regions of \mathcal{R}_i .

Base Case: For $i = 1$, $\mathcal{R}_1 = \{\mathbb{E}^3\}$. So, $\mathbb{E}^3 \notin \mathcal{R}_2$, and $\mathbb{E}^3 = R_{<} \cup R_{\geq}$ for some $R_{<}, R_{\geq} \in \mathcal{R}_2$. Now, the candidate face $f \in C_{\mathbb{E}^3}(\sigma_1, \tau_1)$ has $f \Delta_1 \mathbb{E}^3$ and lies on \mathcal{P} . This means that f , as well as all the other faces lying on \mathcal{P} , fall in R_{\geq} . Now assume, without loss of generality, that the candidate face f is in $F(\sigma_2)$. It needs to be shown that condition two holds for both regions $R_{<}$ and R_{\geq} .

First, consider $R_{<}$. Since S is a solid (with finite volume), $S \cap R_{<} \neq \emptyset$, which means that $F_{R_{<}}(\sigma_2) \neq \emptyset$. According to Lemma 20 $I_{\mathbb{E}^3}(\sigma_1, \tau_1) = I_{R_{<}}(\sigma_2, \tau_2)$. However, f is in R_{\geq} , so the candidate-face set for $R_{<}$ is smaller by at least one choice of splitting plane. Therefore, condition two holds for $R_{<}$.

Second, consider R_{\geq} . Since $f \Delta_2 R_{\geq}$, then according to Lemma 20, either $I_{\mathbb{E}^3}(\sigma_1, \tau_1) = I_{R_{\geq}}(\sigma_2, \tau_2) = 1$ and so there are some faces that are not contained in \mathcal{P} which means that the candidate-face set for R_{\geq} is smaller by at least one choice of splitting plane, or the face f (or other faces on \mathcal{P}) is the only face in R_{\geq} and so $I_{\mathbb{E}^3}(\sigma_1, \tau_1) = 1 < I_{R_{\geq}}(\sigma_2, \tau_2) = 3$. Therefore, condition two holds for R_{\geq} .

Inductive Hypothesis: Assume that for some k , $k > 1$, and for all j , $1 \leq j < k$, the two conditions of the theorem hold.

Inductive Step: Given the sequence of triples $\{(\sigma_k, \tau_k, \mathcal{R}_k)\}_{i=1}^k$, the next triple in the sequence is defined by choosing some region $R \in \mathcal{R}_k$ for which $\text{FRel}(F_R(\sigma_k), F_R(\tau_k)) = 0$, selecting some splitting plane \mathcal{P} by Choose, and creating the regions $R_{<} = R \cap \mathcal{P}_{<}$ and $R_{\geq} = R \cap \mathcal{P}_{\geq}$ along with the appropriate fragments.

If there is no region R for which FRel is zero, nothing changes. That is, $\mathcal{R}_{k+1} = \mathcal{R}_k$, and by the induction hypothesis condition one holds for all $R \in \mathcal{R}_k$.

So suppose that there is some region $R \in \mathcal{R}_k$ for which FRel is zero.

All the other regions $R' \in (\mathcal{R}_k - R)$ appear unchanged in \mathcal{R}_{k+1} so condition one holds for each region R' .

Since R is split into $R_{<}$ and R_{\geq} , $R \notin \mathcal{R}_{k+1}$, so it remains to be shown that condition two holds for R .

First, consider the region below \mathcal{P} , namely $R_{<}$, and take each of the three conditions in turn:

- (a) Since the candidate face lies on \mathcal{P} , it belongs to R_{\geq} and not $R_{<}$. If there all other faces lie in R_{\geq} , the region $R_{<}$ is void of any faces.

- (b) Given that the region $R_{<}$ contains some faces, Lemma 20 states that each face must have the same face-region index as its parent face in R . Thus, $I_R(\sigma_i, \tau_i) = I_{R_{<}}(\sigma_{i+1}, \tau_{i+1})$. Furthermore, since all the faces of R that lie on \mathcal{P} belong to R_{\geq} , the maximally connected faces on \mathcal{P} do not intersect $R_{<}$. Thus, $N_R(\sigma_i, \tau_i) > N_{R_{<}}(\sigma_{i+1}, \tau_{i+1})$.
- (c) Lemma 20 shows that this condition cannot occur.

Second, consider the region on and above \mathcal{P} , namely R_{\geq} , and take each of the three conditions in turn:

- (a) Since the candidate face of R lies on \mathcal{P} , the face belongs to R_{\geq} , and so the region contains at least one face. Therefore, this condition cannot occur.
- (b) For this condition to hold, not all the faces with the smallest face-region index (i.e., faces of $C_R(\sigma_k, \tau_k)$) can be coplanar. Since if not, than no matter what face is selected,

$$C_{R_{\geq}}(\sigma_{k+1}, \tau_{k+1}) \subset C_R(\sigma_k, \tau_k) \text{ and } C_{R_{\geq}}(\sigma_{k+1}, \tau_{k+1}) \neq \emptyset.$$

- (c) For this condition to hold, all the faces in R with the smallest face-region index (i.e., faces of $C_R(\sigma_k, \tau_k)$) are coplanar and lie on \mathcal{P} . All the other faces (i.e., those not in $C_R(\sigma_k, \tau_k)$) must have a larger face-region index. The faces lying on \mathcal{P} all increase their face-region index in R_{\geq} . Therefore, all the faces of R_{\geq} have a face-region index larger than $I_R(\sigma_k, \tau_k)$.

This proves that the two conditions hold for all $i \geq 1$ and for all $R \in \mathcal{R}_i$. \square

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