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Computer Sciences Department
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Technical Report CSD-TR-838
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We present a new characterization of the lowest degree, implicitly defined, real algebraic surfaces, which smoothly contain any given number of rational space curves, of arbitrary degree. The characterization is constructive, yielding efficient algorithms for generating families of such algebraic surfaces. Smooth containment is similar to C^1-continuity interpolation, and is a generalization of Hermite interpolation applied to fitting curves through point data, equating derivatives at those points. In this paper, we deal with containment and matching of "normals" (vectors orthogonal to tangents) along with entire span of the space curves.

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Hermite Interpolation of Rational Space Curves using Real Algebraic Surfaces
(*Preliminary Draft*)

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Abstract

We present a new characterization of the lowest degree, implicitly defined, real algebraic surfaces, which smoothly contain any given number of rational space curves, of arbitrary degree. The characterization is constructive, yielding efficient algorithms for generating families of such algebraic surfaces. Smooth containment is similar to C^1-continuity interpolation, and is a generalization of Hermite interpolation applied to fitting curves through point data, equating derivatives at those points. In this paper, we deal with containment and matching of "normals" (vectors orthogonal to tangents) along the entire span of the space curves.

1 Introduction

Importance: In the course of developing a geometric modeling system for the construction of accurate computer models of solid physical objects [1], we have developed a technique of automatically generating real interpolation surfaces of low degree, which yields a piecewise, tangent-plane-continuous mesh of algebraic surface patches for the boundary of these physical objects. Modeled physical objects with algebraic surfaces of the lowest degree, lends itself to faster computations in manipulating these models for tasks such as computer-aided design and computer graphics.

Why algebraic surfaces ? A real, algebraic surface S is implicitly defined by a single polynomial equation $f(x, y, z) = 0$, where coefficients of f are over the real numbers $\mathbb{R}$. Manipulating polynomials, as opposed to arbitrary analytic functions, is computationally more efficient and as we show here, these surfaces lend themselves very well to the complex problem of Hermite interpolation. Most prior approaches to interpolation and surface fitting, have focused on the parametric representation of surfaces [2, 11, 15]. Contrary to opinion and as we exhibit here, implicitly defined surfaces are quite appropriate for interpolation. Additionally, while all algebraic surfaces can be

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represented implicitly, only a subset of them have the alternate parametric representation, with x, y and z given explicitly as rational functions of two parameters.

The Problem: A real, rational algebraic space curve $\mathbf{r}(t)$ is represented by the triple $(x = G_1(t), y = G_2(t), z = G_3(t))$, where G_1, G_2 and G_3 are rational functions in t, again with coefficients over the real numbers $\mathbb{R}$. We assume that the curve is smooth and only singly defined under the parameterization map, i.e., each triple of values for (x, y, z), is mapped to a single value of t.

We consider the problem of constructing a real algebraic surface S, which *smoothly* interpolates n given rational space curves $\mathbf{r}_i(t)$, $i = 1 \ldots n$, having prespecified "normal" directions $\mathbf{n}_i(t)$, along each of the curves. A rational space curve $\mathbf{r}(t)$ has an infinity of "normal" directions $\mathbf{n}(t)$ at each point of the curve. These are all vectors orthogonal to the tangent vector $\mathbf{r}'(t)$, that is, $\mathbf{r}'(t) \cdot \mathbf{n}(t) = 0$. Also, we assume that $\mathbf{n}(t)$ is not zero identically, a phenomenon that may occur at singular curves. By *smoothly* interpolates we mean that S contains each of the curves $\mathbf{r}_i$ and furthermore along the entire span of the $\mathbf{r}_i$'s, has its gradient in the same direction as the normal directions $\mathbf{n}_i$. This is a generalization of Hermite interpolation, applied to fitting curves through point data, and equating derivatives at those points. We are interested in constructing real and nonsingular algebraic surfaces of the lowest possible degree.

Related Work: Sarraga in [11] presents techniques for constructing a C^1-continuous surface of rectangular Bézier (parametric) surface patches, interpolating a net of cubic Bézier curves. Other approaches to parametric surface fitting and transfinite interpolation are also mentioned in that paper, as well as in [15]. Least squares fitting of algebraic surfaces through given data points, is shown in [9]. Meshing of given algebraic surface patches using control techniques of joining Bézier polyhedrons is shown in [12]. Surface blending consisting of "rounding" and "filleting" surfaces (smoothing the intersection of two primary surfaces), a special case of Hermite interpolation, has been considered for polyhedral models in [4] and for algebraic surfaces in [5, 6, 7, 13, 14, 15].

Results: We show in Sections 3 and 4 that generalized Hermite interpolation of rational curves with algebraic surfaces satisfies a linear system. As applications of the characterization of Hermite interpolated, real algebraic surfaces, we can show, for example that:

- Two space lines with constant-direction normals can be Hermite interpolated with a real quadric if and only if the lines are parallel or intersect at a point, and the normals are not orthogonal to the plane containing them. The real quadric is a "cylinder" when the lines are parallel and a "cone" when the lines intersect.

- Two skewed lines with constant-direction normals cannot be Hermite interpolated with real quadrics. The only real quadratic surface which satisfies both containment and tangency conditions reduces into two planes. But there exists a 5 parameter family of cubic surfaces which Hermite interpolates them.

- Two parallel or intersecting space lines with linearly-varying normals can be blended with a 5 parameter family of cubic surfaces while two skewed lines with linearly-varying normals in the space can be Hermite interpolated with a 3 parameter family of cubic surfaces.

Lines in space with constant-direction normals, occur naturally as edges of polyhedra, with the Hermite interpolating surfaces being used to "smooth" planar faces containing those edges. Lines with linearly-varying normals occur on real quadric and cubic surfaces. See for example Table 1 in Appendix. Similar results to the ones above, can also be derived for Hermite interpolation of conics and space cubic curves. Since these rational curves lie on quadrics, cubic surfaces and higher degree algebraic surfaces, our method gives a powerful way of automatically, generating low degree "blending" and "meshing" surfaces with tangent continuity at intersections.

2 Preliminaries

For any multivariate polynomial f, partial derivatives are written by subscripting, for example, $f_x = \partial f/\partial x$, $f_{xy} = \partial^2 f/(\partial x \partial y)$, and so on. Since we consider algebraic curves and surfaces, we have $f_{xy} = f_{yx}$ etc. Vectors and vector functions are denoted by bold letters. The *inner product* of vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted $\mathbf{a} \cdot \mathbf{b}$. The *length* of the vector $\mathbf{a}$ is $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

The *gradient* of $f(x, y, z)$ is the vector $\nabla f = (f_x, f_y, f_z)$. A point $p = (x_0, y_0, z_0)$ is a *simple* point of f if the gradient of f at p is not null; otherwise the point is *singular*. An algebraic surface is non-singular or smooth if all its points are simple. A point p of the space curve $\mathbf{r}(t)$ is *singular* if either of the conditions below are satisfied :

- The vector $\mathbf{r}'$ at p is zero.
- There exist two distinct values of t satisfying $\mathbf{r}(t) = p$.

The Frenet-Serret formulae ([3], p. 107) are:

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n}, \qquad \frac{d\mathbf{b}}{ds} = -T\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = T\mathbf{b} - \kappa \mathbf{t},$$

where s is arc length, $\mathbf{t}$ is the unit tangent, $\mathbf{n}$ is the principle normal, $\mathbf{b}$ is the binormal, $\kappa = 1/\rho$ is curvature, and $T = 1/\tau$ is torsion. The vectors $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ form an orthonormal triad with

$$\mathbf{n} = \mathbf{b} \times \mathbf{t}.$$

The "normal" direction $\mathbf{n}_i$ of a given smooth rational curve $\mathbf{r}_i$ is thus some prespecified linear combination of the principal normal $\mathbf{n}$ and binormal $\mathbf{b}$, defined at each point along the curve.

3 Hermite Interpolation of One Rational Space Curve

We now characterize the real algebraic surfaces $f(x, y, z)$ of degree n which *smoothly contains* a given rational space curve $\mathbf{r}(t) = (G_1(t), G_2(t), G_3(t))$ of degree d with a normal $\mathbf{n}(t) = (n_x(t), n_y(t), n_z(t))$ of degree m. This method transforms constraints into a homogeneous linear system of equations whose variables are coefficients of the surface $f(x, y, z)$ to be determined.

Definition 3.1 *Let* $\mathbf{r}(t)$ *be a rational space curve with a normal* $\mathbf{n}(t)$*. A real algebraic surface* $f(x,y,z)$ *is said to Hermite Interpolate* $\mathbf{r}(t)$ *if*
(1) $f(x,y,z)$ *contains* $\mathbf{r}(t)$. **(containment condition)**
(2) $\nabla f(\mathbf{r}(t))$ *is not identically zero and is linearly dependent on* $\mathbf{n}(t)$ *for all* t. **(tangency condition)**[1]

3.1 The Method

Algorithm 3.1 (Generation of a linear system of equations)

- *INPUT: A rational curve* $\mathbf{r}(t) = (G_1(t), G_2(t), G_3(t))$ *of degree* d *with a normal* $\mathbf{n}(t) = (n_x(t), n_y(t), n_z(t))$ *of degree* m.
- *OUTPUT: A real algebraic surface* $f(x,y,z)$ *of degree* n*, which Hermite Interpolates* $\mathbf{r}(t)$*, if any.*

Let $f(x,y,z)$ *be a generic polynomial in* $x, y,$ *and* z *of degree* n *whose coefficients are unknowns. A surface* $f(x,y,z)$ *of degree* n *has* $c = \binom{n+3}{3}$ *coefficients of which* $c-1$ *are independent. A normalization of* f *can be achieved in a variety of ways, see for e.g., [9].*

1. **(Containment condition)** *Substitute* $\mathbf{r}(t)$ *into* $f(x,y,z)$*. Then, the resulting formula is a rational polynomial of* t *of degree, at most,* $nd+1$*. Since each coefficient of the numerator of that polynomial must be zero, we obtain at most* $nd+1$ *linear equations in which the variables are the coefficients of* $f(x,y,z)$*.*
2. **(Tangency condition)**
   (a) *Compute* $\nabla f(x,y,z) = (f_x, f_y, f_z)$ *and* $\mathbf{r}'(t) = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$.
   (b) i. *If* $\frac{dx}{dt} \neq 0$*, use the equation* $f_y(\mathbf{r}(t)) \cdot n_z(t) - n_y(t) \cdot f_z(\mathbf{r}(t)) = 0$.
   ii. *If* $\frac{dy}{dt} \neq 0$*, use the equation* $f_x(\mathbf{r}(t)) \cdot n_z(t) - n_x(t) \cdot f_z(\mathbf{r}(t)) = 0$.
   iii. *If* $\frac{dz}{dt} \neq 0$*, use the equation* $f_x(\mathbf{r}(t)) \cdot n_y(t) - n_x(t) \cdot f_y(\mathbf{r}(t)) = 0$.
   In each case, the numerator of the equation is set to zero. This yields additionally at most, $(n-1)d + m + 1$ *linear equations in the coefficients of the surface* $f(x,y,z)$*.*
3. *In total we obtain a homogeneous linear system composed of ,at most,* $(2n-1)d+m+2$ *linear equations. Let* k *be the rank of this system.*
4. *If* $c \leq k$*, there does not exist a nontrivial solution, except for degenerate cases. Let* n_0 *be such that* $\binom{n_0+3}{3} \leq k$ *while* $\binom{n_0+4}{3} > k$*. Then surfaces of degree* $n \leq n_0$ *cannot Hermite interpolate the given rational curve. Hence, choose a surface of degree* $n \geq n_0 + 1$*. Then* $c > k$ *and one can solve the linear system, where the solution for* k *coefficients can be expressed in terms of*

[1] For the time being, we exclude surfaces which are singular along the curve.

the remaining $c-k$ symbolic coefficients. This yields a $c-k-1$-parameter family of surfaces $f(x,y,z)$ of degree n.

5. *Consider the conditions that $f(x,y,z)$ is singular along $\mathbf{r}(t)$. This yields at most $3(n-1)d+3$ linear constraint equations on the $c-k$ symbolic coefficients. Choose values for the coefficients which do not satisfy these constraint equations. If such choice is impossible, repeat steps 1. to 5. for a larger n.*

3.2 Proof of Correctness

In the following lemma, we assert that a surface $f(x,y,z)$ obtained successfully from Algorithm 3.1 Hermite interpolates a given rational space curve $\mathbf{r}(t)$ with a normal $\mathbf{n}(t)$. Furthermore, it is proven that if there exists such a surface, Algorithm 3.1 always finds it.

Lemma 3.1 *For a given rational space curve $\mathbf{r}(t)$ with a normal $\mathbf{n}(t)$ a surface $f(x,y,z)$ of degree n, if any, returned from Algorithm 3.1 Hermite interpolates $\mathbf{r}(t)$.*

Proof : Let $\nabla f(x,y,z) = (f_x, f_y, f_z)$, $\mathbf{r}'(t) = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$, and $\mathbf{n}(t) = (n_x(t), n_y(t), n_z(t))$. Since the coefficients of $f(x,y,z)$ satisfies the homogeneous linear equations from the *containment condition* of Algorithm 3.1, $f(x,y,z)$ contains $\mathbf{r}(t)$.

Now, let's prove that $\nabla f(\mathbf{r}(t)) = \alpha(t)\cdot\mathbf{n}(t)$ for some nonzero $\alpha(t)$. In the step *2.(b)* of Algorithm 3.1, assume without loss of generality that $\frac{dx}{dt} \neq 0$. Since $f(\mathbf{r}(t)) = 0$, we , by chain rule, get an equation :

$$f_x(\mathbf{r}(t))\cdot\frac{dx}{dt} + f_y(\mathbf{r}(t))\cdot\frac{dy}{dt} + f_z(\mathbf{r}(t))\cdot\frac{dz}{dt} = 0 \tag{1}$$

From the definition of a normal of a space curve,

$$n_x(t)\cdot\frac{dx}{dt} + n_y(t)\cdot\frac{dy}{dt} + n_z(t)\cdot\frac{dz}{dt} = 0 \tag{2}$$

¿From the step *2.(b)i* of Algorithm 3.1, we have an equation :

$$f_y(\mathbf{r}(t))\cdot n_z(t) - n_y(t)\cdot f_z(\mathbf{r}(t)) = 0 \tag{3}$$

It is impossible that both $n_y(t)$ and $n_z(t)$ are identically zero for the following reason. Suppose $n_y(t) = n_z(t) = 0$. Then this and equation (2) imply $n_x(t)\cdot\frac{dx}{dt} = 0$. Since $\frac{dx}{dt} \neq 0$, $n_x(t)$ becomes 0. This is contradiction to the general assumption that $\mathbf{n}(t)$ is not identically zero. Hence, at least, one of $n_y(t)$ and $n_z(t)$ must be nonzero. Without loss of generality, let $n_y(t) \neq 0$. Also, let $\alpha(t) = \frac{f_y(\mathbf{r}(t))}{n_y(t)}$. Then,

$$f_y(\mathbf{r}(t)) = \alpha(t)\cdot n_y(t) \tag{4}$$

By substituting equation (4) into equation (3), we get $\alpha(t)\cdot n_y(t)\cdot n_z(t) - n_y(t)\cdot f_z(\mathbf{r}(t)) = n_y(t)\cdot(\alpha(t)\cdot n_z(t) - f_z(\mathbf{r}(t))) = 0$. Since $n_y(t) \neq 0$,

$$f_z(\mathbf{r}(t)) = \alpha(t)\cdot n_z(t) \tag{5}$$

From equations (1),(4), and (5),

$$f_x(\mathbf{r}(t)) \cdot \frac{dx}{dt} + \alpha(t) \cdot n_y(t) \cdot \frac{dy}{dt} + \alpha(t) \cdot n_z(t) \cdot \frac{dz}{dt} = 0 \tag{6}$$

Here, we see that $\alpha(t) \neq 0$. Otherwise, $\nabla f(\mathbf{r}(t)) = (0,0,0)$ and this leads to the contradiction to the fact that $f(x,y,z)$ is not singular along $\mathbf{r}(t)$. By multiplying $\alpha(t)$ to equation (2) and subtracting equation (6) from it, we get an equation :

$$f_x(\mathbf{r}(t)) \cdot \frac{dx}{dt} = \alpha(t) \cdot n_x(t) \cdot \frac{dx}{dt} \tag{7}$$

Since $\frac{dx}{dt} \neq 0$,

$$f_x(\mathbf{r}(t)) = \alpha(t) \cdot n_x(t) \tag{8}$$

Equations (4), (5), and (8) imply that $\nabla f(\mathbf{r}(t)) = \alpha(t) \cdot \mathbf{n}(t)$. Hence, the tangency condition is met. ♠

Lemma 3.2 *For a given rational space curve* $\mathbf{r}(t)$ *with a normal* $\mathbf{n}(t)$*, Algorithm 3.1 always finds a surface* $f(x,y,z)$ *of degree* n *which Hermite interpolates* $\mathbf{r}(t)$*, if there exists such a surface.*

Proof : If there is such a surface, its coefficients satisfies the linear system generated by Algorithm 3.1. Hence, it is trivial to see that Algorithm 3.1 always succeeds to find such a surface. ♠

Lemma 3.1 and Lemma 3.2 together imply the following theorem.

Theorem 3.1 *For a given rational space curve* $\mathbf{r}(t)$ *with a normal* $\mathbf{n}(t)$*, Algorithm 3.1 returns a surface* $f(x,y,z)$ *of degree* n *which Hermite interpolates* $\mathbf{r}(t)$ *if and only if there exists such a surface.*

Example 1: [The intersection of a sphere $x^2+y^2+z^2-1=0$ with the plane $z=0$]

Let $\mathbf{r}(t) = (\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, 0)$, and $\mathbf{n}(t) = (\frac{4t}{1+t^2}, \frac{2-2t^2}{1+t^2}, 0)$. To find a surface of degree 2 which Hermite interpolates $\mathbf{r}(t)$, we let $f(x,y,z) = ax^2+by^2+cz^2+dxy+eyz+fzx+gx+hy+iz+j$. From the containment condition, we get 5 equations :

$$\begin{aligned} j-h+b &= 0 \\ 2g-2d &= 0 \\ 2j-2b+4a &= 0 \\ 2g+2d &= 0 \\ j+h+b &= 0 \end{aligned}$$

From the tangency condition, we also get 5 equations :

$$-2i+2e = 0$$

$$
\begin{aligned}
-4f &= 0 \\
-4e &= 0 \\
4f &= 0 \\
2i + 2e &= 0
\end{aligned}
$$

The homogeneous linear system composed of the above 10 equations has rank 8. Its solution is $a = -j, b = -j, d = e = f = g = h = i = 0$. Hence, a Hermite interpolating surface is $f(x, y, z) = -jx^2 - jy^2 + cz^2 + j = 0$. Note that $f(x, y, z)$ is reducible when $j = 0$. So, we can assume that j is not 0. Dividing $f(x, y, z)$ by $-j$ results in $f(x, y, z) = x^2 + y^2 + c'z^2 - 1 = 0$ which is controlled by one parameter c'. We can make $f(x, y, z)$ contain a point, say, $(1, 0, 1)$. That is, $f(1, 0, 1) = 1 + c' - 1 = 0$. So, for example, the circular cylinder $f(x, y, z) = x^2 + y^2 - 1 = 0$ is an appropriate interpolating surface.

4 Hermite Interpolation of Two or More Rational Space Curves

Definition 4.1 *Let $\mathbf{r}_1(t), \ldots, \mathbf{r}_l(t)$ be rational space curves with normals $\mathbf{n}_1(t), \ldots, \mathbf{n}_l(t)$, respectively. An irreducible surface $f(x, y, z)$ is said to Hermite interpolate all the $\mathbf{r}_i(t)$, $i = 1, \ldots, l$, if*
(1) $f(x, y, z)$ contains all $\mathbf{r}_i(t)$
(2) For all i, $\nabla f(\mathbf{r}_i(t))$ is not identically zero, and is linearly dependent on $\mathbf{n}_i(t)$ for all t.

4.1 The Method

The following algorithm shows how to come up with a Hermite interpolating surface if any.

Algorithm 4.1 (Simultaneously interpolating rational space curves)

- *INPUT : Rational space curves $\mathbf{r}_i(t)$, of respective degrees d_i, with normals $\mathbf{n}_i(t)$, of respective degrees m_i, $i = 1, \ldots, l$.*
- *OUTPUT : A surface $f(x, y, z)$ of degree n ,if any, which Hermite interpolates the $\mathbf{r}_i(t)$, $i = 1, \ldots, l$.*

Let $f(x, y, z)$ be a generic polynomial in x, y, and z of degree n whose coefficients are unknowns.

1. *For each i, apply the first two steps of Algorithm 3.1 to $\mathbf{r}_i(t)$ and $\mathbf{n}_i(t)$.*
2. *There are totally , at most, $\sum_{i=1}^{l}[(2n-1)d_i + m_i] + 2l$ linear equations in the coefficients of $f(x, y, z)$, generated in step 1. Let c be $\binom{n+3}{3}$ and k be the rank of the entire homogeneous linear system . If $c \leq k$ then there can only exist degenerate cases. Let n_0 be such that*

$\binom{n_0+3}{3} \le k$ *while* $\binom{n_0+4}{3} > k$. *Then surfaces of degree* $n \le n_0$ *cannot Hermite interpolate the given rational curve. Hence, choose a surface of degree* $n \ge n_0 + 1$. *Then for all* i, $c > k$. *Solve the linear system. This yields a* $c - k - 1$*-parameter family of surfaces* $f(x, y, z)$.

3. *Check if* $f(x, y, z)$ *is singular along any of the* $\mathbf{r}_i(t)$. *If yes, repeat steps* 1. *to* 3. *for a larger* n.

4. *Examine if* $f(x, y, z)$ *is reducible. If yes, repeat steps* 1. *to* 4. *for a larger* n.

Example: Let $\mathbf{r}_1(t) = (\frac{2t}{1+t^2}, \frac{-6t}{1+t^2}, \frac{1-t^2}{1+t^2})$, $\mathbf{n}_1(t) = (\frac{4t}{1+t^2}, 0, \frac{2-2t^2}{1+t^2})$ and $\mathbf{r}_2(t) = (\frac{-6t}{1+t^2}, \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$, $\mathbf{n}_2(t) = (0, \frac{4t}{1+t^2}, \frac{2-2t^2}{1+t^2})$ be two space curves with normals. Note that $\mathbf{r}_1(t)$ is an intersection of a cylinder $x^2+z^2-1 = 0$ with a plane $3x+y = 0$, and $\mathbf{r}_2(t)$ is an intersection of a cylinder $y^2+z^2-1 = 0$ with a plane $x + 3y = 0$. Hence, a surface which Hermite interpolates $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ will blend the cylinders along the curves. Warren in [14], shows that there exists such a surface of degree 2. Using our method, we construct a surface of degree 2 which Hermite interpolates the rational curves. Let $f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + iz + j$. Applying the first 2 steps of algorithm 4.1 produces 16 equations of rank 9. Its solution is $a = b = j/8, c = -j, d = 3j/4$. Hence, we get a degree 2 surface $f(x, y, z) = jx^2/8 + jy^2/8 - jz^2 + 3jxy/4 + j = 0$. Since j can not vanish, the equation can be divided by $j/8$, resulting in $f(x, y, z) = x^2 + y^2 - 8z^2 + 6xy + 8 = 0$. This is a hyperboloid of one sheet, and passes the tests in the last 2 steps of algorithm 4.1.

4.2 Applications

The simultaneous Hermite interpolation of two or more space curves can be used to find a "blending" surface of two given primary surfaces, along prespecified tangency curves. There is freedom in the choice of rational curves on the primary surfaces to be blended, so that the blending surface has good properties. Being of low degree is a very desirable property of blending surfaces.

4.2.1 Interpolating Two Lines in Space

In this section, we consider some special cases of Hermite interpolation of rational curves. We restrict the space curve $\mathbf{r}(t)$ to be of the form $\mathbf{r}(t) = (\alpha_1 \cdot t + \beta_1, \alpha_2 \cdot t + \beta_2, \alpha_3 \cdot t + \beta_3)$. That is, we consider Hermite interpolation of space lines with normals. Applications to this special case arise, for e.g., in smoothing a rectangular or a triangular face of a polyhedra. As we see below, a quadric surface always suffices.

Constant-Direction Normals

The simplest case is when lines have constant-direction normals $\mathbf{n}(t) = (\kappa, \lambda, \mu)$. We can prove that two space lines with constant-direction normals can be blended with a quadric when and only when either those lines are parallel or they intersect at a point.

Theorem 4.1 *Two parallel space lines with constant-direction normals can be Hermite interpolated* *with quadratic surfaces if and only if neither of the normals are orthogonal to the plane which contains both parallel lines.*

Proof : ($\Longleftarrow$) Without loss of generality, we can assume that $\mathbf{r}_1(t) = (t, 0, 0)$, $\mathbf{n}_1(t) = (0, 1, \mu_1)$, $\mathbf{r}_2(t) = (t, \alpha, 0)$, and $\mathbf{n}_2(t) = (0, 1, \mu_2)$. Note that y component of $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ can be assumed to be 1 because neither of the normals are perpendicular to the x-y plane. Clearly, $\alpha \neq 0$. Let $f(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + iz + j$. Then $f(\mathbf{r}_1(t)) = at^2 + gt + j$, $f(\mathbf{r}_2(t)) = at^2 + (g + \alpha d)t + j + \alpha h + \alpha^2 b$, $\nabla f(\mathbf{r}_1(t)) = (2at + g, dt + h, ft + i)$, and $\nabla f(\mathbf{r}_2(t)) = (2at + g + \alpha d, dt + h + 2\alpha b, ft + i + 2\alpha e)$. Using Algorithm 4.1, we get the linear system

$$
\begin{aligned}
a &= 0 \\
g &= 0 \\
j &= 0 \\
g + \alpha d &= 0 \\
j + \alpha h + \alpha^2 b &= 0 \\
f - \mu_1 d &= 0 \\
i - \mu_1 h &= 0 \\
f - \mu_2 d &= 0 \\
i + 2\alpha e - \mu_2 h - 2\mu_2 \alpha b &= 0
\end{aligned}
$$

Solving this linear system results in the quadratic surface $f(x,y,z) = by^2 + cz^2 + \frac{1}{2}b(\mu_1 + \mu_2)yz - \alpha by - \alpha\mu_1 bz$. Since $f(x,y,z)$ is reducible when $b = 0$, we choose b to be non zero. Dividing $f(x,y,z)$ out by b results in $f(x,y,z) = y^2 + c'z^2 + \frac{1}{2}(\mu_1 + \mu_2)yz - \alpha y - \alpha\mu_1 z$. It is trivial to see $f(x,y,z)$ is not singular along any of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. Now, by choosing c' appropriately, we get an irreducible quadric $f(x,y,z)$ which Hermite interpolates $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. We note that the real quadrics which satisfy the above properties are all "cylinders".

($\Longrightarrow$) Suppose that one of the two normals are orthogonal to the x-y plane. That is, the y component of either $\mathbf{n}_1(t)$ or $\mathbf{n}_2(t)$ is 0. Using Algorithm 4.1, we can show that we always get a quadric which reduces into two real planes (one of which is the x-y plane). ♠

Theorem 4.2 *Two intersecting space lines with constant-direction normals can be Hermite interpolated with quadratic surfaces if and only if neither of the normals are orthogonal to the plane which contains both intersecting lines.*

Proof : ($\Longleftarrow$) Without loss of generality, we can assume that $\mathbf{r}_1(t) = (t, 0, 0)$, $\mathbf{n}_1(t) = (0, 1, \mu_1)$, $\mathbf{r}_2(t) = (\alpha t, t, 0)$, and $\mathbf{n}_2(t) = (1, -\alpha, \mu_2)$. Note that y component of $\mathbf{n}_1(t)$ and x component of $\mathbf{n}_2(t)$ can be assumed to be 1, because neither of the normals are perpendicular to the x-y plane.

Let $f(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + iz + j$. Then $f(\mathbf{r}_1(t)) = at^2 + gt + j$, $f(\mathbf{r}_2(t)) = (\alpha d + b + \alpha^2 a)t^2 + (h + \alpha g)t + j$, $\nabla f(\mathbf{r}_1(t)) = (2at + g, dt + h, ft + i)$, and $\nabla f(\mathbf{r}_2(t)) =$

$(dt + 2a\alpha t + g, \alpha dt + 2bt + h, \alpha ft + et + i)$. Using Algorithm 4.1, we get the linear system

$$\begin{aligned} a &= 0 \\ g &= 0 \\ j &= 0 \\ \alpha d + b + \alpha^2 a &= 0 \\ h + \alpha g &= 0 \\ f - \mu_1 d &= 0 \\ i - \mu_1 h &= 0 \\ e + \alpha f - \mu_2 d &= 0 \end{aligned}$$

Solving this linear system results in the quadratic surface $f(x,y,z) = -\alpha dy^2 + cz^2 + dxy + (\mu_2 - \alpha\mu_1)dyz + \mu_1 dzx$. Since f(x,y,z) is reducible when $d = 0$, we choose d to be non zero. Dividing $f(x,y,z)$ out by d results in $f(x,y,z) = -\alpha y^2 + c'z^2 + xy + (\mu_2 - \alpha\mu_1)yz + \mu_1 zx$. It is trivial to see $f(x,y,z)$ is not singular along any of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. Now, by choosing c' appropriately, we get an irreducible quadric $f(x,y,z)$ which Hermite interpolates $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. Since the real irreducible quadrics which satisfy the above properties need to have a point singularity, the Hermite interpolant in this case can only be one of the real quadric "cones".

($\Longrightarrow$) Using Algorithm 4.1, we can show that we always get a quadric which reduces into two planes when either of the normals are orthogonal to the x-y plane. Details are omitted. ♠

We can prove, using Algorithm 4.1, that two skewed lines with constant normals cannot be Hermite interpolated with quadrics. The only quadratic surface which satisfies both containment and tangency condition reduces into two planes. But there are a five parameter family of cubic surfaces which Hermite interpolate them.

Theorem 4.3 *Two skewed lines with constant-direction normals in the space can be Hermite interpolated with a cubic surface.*

Sketch of proof : Let's define, wlg., two general skewed lines with constant-direction normals as follows : $\mathbf{r}_1(t) = (t,0,0)$, $\mathbf{n}_1(t) = (0,\mu_2,\mu_3)$, and $\mathbf{r}_2(t) = (\alpha t, t, \beta)$, $\mathbf{n}_2(t) = (\lambda_1, -\alpha\lambda_1, \lambda_3)$. Algorithm 4.1 generates a linear system consisted of 14 equations whose rank is 14. Since a cubic surface is specified by 19 independent parameters, we obtain a Hermite interpolated cubic surface $f(x,y,z)$ which is controllable by 5 parameters. By choosing appropriate values of parameters, we can find a suitable cubic surface which passes the last two tests of Algorithm 4.1. ♠

Linearly Varying Normals

Theorem 4.4 *Two parallel space lines with linearly-varying normals in the space can be Hermite interpolated with a cubic surface.*

Sketch of proof : Let's define, wlg., two general lines with linearly-varying normals as follows : $\mathbf{r}_1(t) = (t,0,0)$, $\mathbf{n}_1(t) = (0, \beta_1 t + \beta_2, \kappa_1 t + \kappa_2)$, and $\mathbf{r}_2(t) = (t,\alpha,0)$, $\mathbf{n}_2(t) = (0, \beta_3 t + \beta_4, \kappa_3 t + \kappa_4)$.

Algorithm 4.1 generates a linear system consists of 16 equations whose rank is 14. Since a cubic surface is specified by 19 independent parameters, we obtain a Hermite interpolated cubic surface $f(x,y,z)$ which is controllable by 5 parameters. By choosing appropriate values of parameters, we can find a suitable cubic surface which passes the last two tests of Algorithm 4.1. ♠

Theorem 4.5 *Two intersecting space lines with linear-varying normals in the space can be Hermite interpolated with a cubic surface.*

Sketch of proof : Let's define, wlg., two general lines with linearly-varying normals as follows : $\mathbf{r}_1(t) = (t,0,0)$, $\mathbf{n}_1(t) = (0, \beta_1 t + \beta_2, \kappa_1 t + \kappa_2)$, and $\mathbf{r}_2(t) = (\alpha t, t, 0)$, $\mathbf{n}_2(t) = (\alpha_3 t + \alpha_4, -\alpha\alpha_3 t - \alpha\alpha_4, \kappa_3 t + \kappa_4)$. Algorithm 4.1 generates a linear system consisted of 16 equations whose rank is 14. Since a cubic surface is specified by 19 independent parameters, we obtain a Hermite interpolated cubic surface $f(x,y,z)$ which is controllable by 5 parameters. By choosing appropriate values of parameters, we can find a suitable cubic surface which passes the last two tests of Algorithm 4.1. ♠

Theorem 4.6 *Two skewed lines with linearly-varying normals in the space can be Hermite interpolated with a cubic surface.*

Sketch of proof : Let's define, wlg., two general lines with linearly-varying normals as follows : $\mathbf{r}_1(t) = (t,0,0)$, $\mathbf{n}_1(t) = (0, \beta_1 t + \beta_2, \kappa_1 t + \kappa_2)$, and $\mathbf{r}_2(t) = (\alpha t, t, \beta)$, $\mathbf{n}_2(t) = (\alpha_3 t + \alpha_4, -\alpha\alpha_3 t - \alpha\alpha_4, \kappa_3 t + \kappa_4)$. Algorithm 4.1 generates a linear system consisted of 16 equations whose rank is 16. Since a cubic surface is specified by 19 independent parameters, we obtain a Hermite interpolated cubic surface $f(x,y,z)$ which is controllable by 3 parameters. By choosing appropriate values of parameters, we can find a suitable cubic surface which passes the last two tests of Algorithm 4.1. ♠

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A Table 1.: Hermite Interpolating Conics with a Real Quadric

The table shows various conics (L : line, P : parabola, H : hyperbola, E : ellipse) with different normal directions, which can be Hermite interpolated by a single real quadric.

Real Quadrics	Constant-direction Normal	Linearly-varying Normal	Quadratically-varying Normal
Ellipsoid			E
Hyperboloid of one sheet		L	E,H
Hyperboloid of two sheets			E,H
Hyperbolic Paraboloid		L,P	H
Elliptic Paraboloid		P	E
Cone	L	L	E,H
Elliptic Cylinder	L		E
Hyperbolic Cylinder	L		H
Parabolic Cylinder	L	P	
2 Planes	L,P,H,E		