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ABSTRACT

Two images of a 4-points object which undergoes 3D rotation, translation without knowing its correspondence are given. The problems are (i) How to match the corresponding elements in the two images due to the movement of the object? Can all the possible mapping be found ? (ii) What underlying motions and associated depth components of these points could account for the two images ? (iii) What is the structure of the object? This paper presents a method which addresses all these issues in the same framework. The method reduces a 4-points problem into a set of testable conditions and a 3-points problem. This forms the basis for deriving all possible interpretations and relates the correspondence and motion problem together. Examples are provided to illustrate each step of the method. Several applications including "Structure from motion"[2], and "Perception of structure from motion"[7][8] are also described.

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1. Introduction

The correspondence problem is a fundamental issue in computer vision. One direction of research in image sequence analysis [2][3][4][9] often assumes the correspondence of the elements among frames has beeen established. The difficulty of research in stereopsis also lies in the correspondence problem.

This paper discusses the correspondence problem of four points in 3D space. The problem is formulated as follows: Consider an object consisting of four points in 3D space. Let the projections of these four points into image plane be observables. One can rotate, translate the object and observe the effect on the projections of the four points in the image plane. Clearly, there are 24 possible mappings between these two sets of four points. Some of the mappings could not be accounted for but some, which will be called as admissible mappings, could be attributed to rigid motion. The problems are: What are the admissible mappings ? and, What are the structures of object and underlying motion. Notice that there are no attributes associated with any of these points.

2. Problem Statements

Figures 1a and 1b depict two views of four points undergoing rigid motion. The task is to find out admissible mappings, motions, and structure of an object in 3D space. It seems that there is no systematic way of doing this. Further, it is awkard to solve a system of nonlinear equations in R^6 (see next section) for each possible mapping.

In this study, motion which rotates about the optical axis will be excluded. Such motions, called degenerate, can be detected [1] while the structure of an object can not be inferred since there is no multiframe information at all. It is easy to realize that one would not claim to have a sequence of images by rotating a 2D-picture. This type of degenerate motion has an effect which could not distinguish coplanar or noncoplanar points.

A general version of this problem can be seen in [1]. Suppose two views of a n-points object is observed. What are the possible admissible mappings which can be attributed to some underlying motion and relative position of these points ? In [1], we show how to reduce a n-points problem to several 4-points problems and assume that the mapping of four points objects has been established. In this study, we explore the problem of correspondence of four points in detail.

3. Method

Assume that two sets of four points in the image plane are given. To explore the problem, we first hypothesize that a mapping between the four points has been designated. Pursuing thereon, a computational method is developed to determine the motion(s) which underly the movement of four points and to determine their relative positions. In addition, several compatible conditions are also developed to check if a mapping (correspondence of points) can be admissible or not. Furthermore we show that, for each mapping, four-points problem is equivalent to several testable conditions and a three-points problem.

Let the 3D coordinates of four points be denoted by O, A_1, A_2, A_3 and O, B_1, B_2, B_3 respectively in the two scenes. Notice that translation is adjusted to zero and rotational axis is adjusted to pass through one of the four points. The relative positions of these points are referred to with respect to O and the observables are the first two components of the space coordinates. Write depth component of A_i and B_i to be s_i and t_i respectively; A = $[A_1, A_2, A_3]$ and B = $[B_1, B_2, B_3]$. Obviously, there must exist some 3D rotation R such that the following relation holds since the designated mapping is assumed to be a correct one.

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ s_1 & s_2 & s_3 \end{bmatrix} = R \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ t_1 & t_2 & t_3 \end{bmatrix} = R\,\mathrm{B} \qquad \text{(A)}$$

(A) implies that Au = RBu for all u $\in R^3$ and thus $||Au|| = ||R\,Bu||$. If u is chosen to be $[1\ 0\ 0]^t$ or $[0\ 1\ 0]^t$ or $[0\ 0\ 1]^t$ then one obtains three constraints that the length of A_i's remain the same before and after motion. If u is chosen to be $[1\ 1\ 0]^t$ or $[1\ 0\ 1]^t$ or $[0\ 1\ 1]^t$ then one obtains three constraints that the inner product of any two vectors remains the same before and after motion. The following six equations denoted by (B1-B6) represent the constraints just mentioned.

$$\alpha_1^2 + s_1^2 = \beta_1^2 + t_1^2 \qquad \text{(B1)}$$

$$\alpha_2{}^2 + s_2^2 = \beta_2{}^2 + t_2^2 \tag{B2}$$

$$\alpha_3{}^2 + s_3^2 = \beta_3{}^2 + t_3^2 \tag{B3}$$

$$\alpha_{12} + s_1 s_2 = \beta_{12} + t_1 t_2 \tag{B4}$$

$$\alpha_{13} + s_1 s_3 = \beta_{13} + t_1 t_3 \tag{B5}$$

$$\alpha_{23} + s_2 s_3 = \beta_{23} + t_2 t_3 \tag{B6}$$

where $\alpha_i^2 = a_{i1}^2 + a_{i2}^2$; $\alpha_{ij} = a_{i1}a_{j1} + a_{i2}a_{j2}$; $\beta_i^2 = b_{i1}^2 + b_{i2}^2$; $\beta_{ij} = b_{i1}b_{j1} + b_{i2}b_{j2}$.

Here, I would point out that if there exist (α β) such that

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a_{31} \\ a_{32} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_{31} \\ b_{32} \end{bmatrix} \tag{C}$$

Then these four points are coplanar under the assumption that degenerate motion is excluded [1]. If this case happens then the third columns of A and B are redudant and linearly dependent on the first two columns. The situation that (C) occurs will be dealt with at the end of this section.

It can be seen that the existence of a solution for (B1-B6) is a necessary and sufficient condition for the existence of a rotation to account for the correspondence. The proof is simple: Rewrite these six equations into $A^t A = B^t B$. Thus the relation $(B^{-1})^t\ A^t\ A\ B^{-1} = I$ which implies $(A\ B^{-1})^t\ (A\ B^{-1}) = I$ holds. From [6], one knows that $A\ B^{-1} = \boldsymbol{R}$ for some 3D rotation $\boldsymbol{R}$.

Next, the above six equations can further be reduced into another six equations denoted by (D1-D6) as below:

$$\delta_{11}\, s_1{}^2 + \delta_{12}\, s_1 s_2 + \delta_{13}\, s_2{}^2 = F_1 \tag{D1}$$

$$\delta_{21}\, s_1{}^2 + \delta_{22}\, s_1 s_3 + \delta_{23}\, s_3{}^2 = F_2 \tag{D2}$$

$$\delta_{31}\, s_2{}^2 + \delta_{32}\, s_2 s_3 + \delta_{33}\, s_3{}^2 = F_3 \tag{D3}$$

$$\alpha_1{}^2 + s_1^2 = \beta_1{}^2 + t_1^2 \tag{D4}$$

$$\alpha_2{}^2 + s_2^2 = \beta_2{}^2 + t_2^2 \tag{D5}$$

$$\alpha_3{}^2 + s_3^2 = \beta_3{}^2 + t_3^2 \tag{D6}$$

where $\delta_{11} = \alpha_2^2 - \beta_2^2$; $\delta_{12} = 2(\beta_{12} - \alpha_{12})$; $\delta_{13} = \alpha_1^2 - \beta_1^2$; $F_1 = (\delta_{12}^2 - 4 \delta_{11} \delta_{13})/4$. Other δ_{ij}'s can be written similarly but not listed here. Another three equations similar to (D1)(D2)(D3) in t_i , $1 \le i \le 3$, are required to make the new system of equations equivalent to (B1-B6).

We will study how to solve for s_i's from (D1)(D2)(D3). If s_i's can be computed*, then it becomes

very easy to derive t_i's or reject the solutions from (B1-B6). However, it is still difficult and cumbersome to solve for a system of three quadratic equations in R^3. Apparently, one has to at least solve an eighth-order polynomial in a variable if a brute force approach is used. Before we introduce another two observations which lead to a simple and efficient computational algorithm, some compatibility conditions for a mapping to be admissible will be presented. These compatibility conditions are straightforward. Since each one of the three equations are quadratic, conditions which make conics degenerate into empty set exist. Three of them are listed below and will be referred to compatibiltiy conditions when examples are described.

Condition (1): Assume $F_1 < 0$. If $\delta_{11} < 0$ or $\delta_{13} < 0$ then there is no solution.

The proof is simple. Arrange the left hand side to be the sum of two square terms or reference any book discussing conics. Using the same reasoning, one can write down another two conditions.

Condition (2): Assume $F_2 < 0$. If $\delta_{21} < 0$ or $\delta_{23} < 0$ then there is no solution.

Condition (3): Assume $F_3 < 0$. If $\delta_{31} < 0$ or $\delta_{33} < 0$ then there is no solution.

---

*First one can eliminate s_3^2 and s_3 from equations (D2)(D3) to obtain a quadratic equation in s_1 and s_2, Second one can solve two quadratic equations in R^2 (note not in R^3). Lastly, with solution of s_i's one can derive t_i's or reject the solution from system of equations (B1-B6). The footnote however would not work (they are dependent) and is used as a reminder.

Based on these three necessary conditions, one can check whether the hypothesized mapping can be admissible or not. Even though these three conditions are satisfied, it does not guarantee existence of solution for equations (B1-B6). Other conditions are needed.

Example 1: The following data represents a hypothesized mapping (see Figure 2) between A_i's and B_i's

$$O = (0\ 0),\ A_1 = (4\ 0),\ A_2 = (1\ 1).$$
$$O = (0\ 0),\ B_1 = (5\ 0),\ B_2 = (1\ 2).$$

We have $\alpha_1^2 = 16$, $\beta_1^2 = 25$,$\alpha_2^2 = 2$, $\beta_2^2 = 5$,$\alpha_{12} = 4$, $\beta_{12} = 5$. Thus one obtains $\delta_{11} = -3$, $\delta_{12} = 2$, $\delta_{13} = -9$, F_1 =−26. According to compatibility condition (1), the mapping is not admissible.

We now develop the first observation. Let $\overline{A}_i$ and $\overline{B}_i$ be the first two components of A_i and B_i respectively. Let $\overline{B}_3 = a\ \overline{B}_1 + b\ \overline{B}_2$. Since A = RB and Au = RBu for all u $\in R^3$ hold. We will choose u to be $[a\ b\ -1]'$ then we have $a\ A_1 + b\ A_2 - A_3 = R$ $[0\ 0\ {*d}]'$. Obviously, we have $a\ A_1 + b\ A_2 - A_3 = {*d}\ r_3$ where r_3 is the third column of R. It can be proved that $*d$ is nonzero if these four points are noncoplanr[1] and the motion is nondegenerate. Thus we know that the first two components of r_3 up to a scalar since $\overline{A}_i$'s are observables. Using the same technique, we can obtain the first two components of the last row of R by *interchanging the roles of the two frames.* In fact, one could already find out many properties about the motion based on this information [1], but we will pursue another route. For convenience, we write R as follows:

$$\begin{bmatrix} * & * & a1 \\ * & * & a2 \\ b1 & b2 & r_{33} \end{bmatrix}$$

Note that $(b1\ b2)$ or $(a1\ a2)$ is determined up to a sign and unknown scalar, and the magnitude of $(b1\ b2)$ and $(a1\ a2)$ must be the same.

The second observation is to choose u = $[u_1, u_2, u_3]^t$ such that u is perpendicular to the last row of A and the last row of B. If there is a motion underlying such mapping (correspondence), then u must exist. Using Au = $\boldsymbol{R}$Bu, we have $[*m\ \ *n\ \ 0]^t = \boldsymbol{R}$ $[*p\ \ *q\ \ 0]^t$ where *m, *n, *p, *q are unknown and will be derived next (u is still unknown). This means that the dot product of $(b1\ b2\ r_{33})$, the last row of $\boldsymbol{R}$, and $(*p\ \ *q\ \ 0)$ should be zero. Therefore $*p = b2$, $*q = -b1$; or $*p = -b2$, $*q = b1$. Using the same technique, one can derive that $*m = a2$, $*n = -a1$; or $*m = -a2$, $*n = a1$. The scale is not important here as long as the magnitudes of $(b1\ b2)$ and $(a1\ a2)$ are kept the same. Now we have four constraints for u_1, u_2, u_3.

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} *m \\ *n \\ *p \\ *q \end{bmatrix} \tag{E}$$

Although there are four cases for the right hand side of (E), only two needs to be explored because the other two are simply the negative of these two cases. We will write them down for easy reading.

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a2 \\ -a1 \\ b2 \\ -b1 \end{bmatrix} \text{ or } = \begin{bmatrix} -a2 \\ a1 \\ b2 \\ -b1 \end{bmatrix}$$

Obviously, the existence of solution for this linear system is a precondition to have solution for the original problem. Furthermore the rank of this 4 by 3 matrix is 3. Suppose the rank is two instead of three, then the last column can be written as a linear combination of the first two columns which essentially satisfies the coplanarity condition (C) and violates the assumption. Thus only one solution at most can exist for each case. If no solution (i.e the system of equations are not consistent) exists for either one of the cases, then the hypothesized mapping is definitely not admissible. This condition will be referred to as the U-condition in the example. Thus, $(u_1\ u_2\ u_3)$ can be derived; and $u_1\, s_1 + u_2\, s_3 + u_3\, s_3 = 0\, ;\, u_1\, t_1 + u_2\, t_2 + u_3\, t_3 = 0.$

With this condition, we can now derive another compatibility condition to check if the hypothesized mapping is consistent or not. Square these two relations, and one obtains

$$u_1^2\, s_1^2 + u_2^2\, s_2^2 + u_3^2\, s_3^2 + 2\, u_1\, u_2\, s_1\, s_2 + 2\, u_1\, u_3\, s_1\, s_3 + 2\, u_2\, u_3\, s_2\, s_3 = 0$$

$$u_1^2\, t_1^2 + u_2^2\, t_2^2 + u_3^2\, t_3^2 + 2\, u_1\, u_2\, t_1\, t_2 + 2\, u_1\, u_3\, t_1\, t_3 + 2\, u_2\, u_3\, t_2\, t_3 = 0$$

Thus a consistency condition can be derived as:

Condition (4): $\sum_{i=1}^{i=3} u_i^2\, (\, \alpha_i^2 - \beta_i^2\,) + 2 \sum_{i,j=1;i<j}^{i,j=3} u_i\, u_j\, (\, \alpha_{ij} - \beta_{ij}) = 0$

The relation $u_1\, s_1 + u_2\, s_2 + u_3\, s_3 = 0$ and $u_1\, t_1 + u_2\, t_2 + u_3\, t_3 = 0$ provide a convenient way (see previous footnoot) of solving the system of three nonlinear equations (D1)(D2)(D3) in R^3. Notice that this relation is not independent to the equations in (B1-B6) which was already shown to be equivalent to $A = \boldsymbol{R}\ B$. Indeed, it says that if one adds these two relations into (B1-B6), the enlarged system must be consistent. For convenience of discussion and without loss of generality, u_3 of u is normalized to -1.

$$\alpha_1{}^2 + s_1^2 = \beta_1{}^2 + t_1^2 \tag{p1}$$

$$\alpha_2{}^2 + s_2^2 = \beta_2{}^2 + t_2^2 \tag{p2}$$

$$\alpha_3{}^2 + s_3^2 = \beta_3{}^2 + t_3^2 \tag{p3}$$

$$\alpha_{12} + s_1 s_2 = \beta_{12} + t_1 t_2 \tag{p4}$$

$$\alpha_{13} + s_1 s_3 = \beta_{13} + t_1 t_3 \tag{p5}$$

$$\alpha_{23} + s_2 s_3 = \beta_{23} + t_2 t_3 \tag{p6}$$

$$u_1\, s_1 + u_2\, s_2 = s_3 \tag{p7}$$

$$u_1\, t_1 + u_2\, t_2 = t_3 \tag{p8}$$

Examining the enlarged system, one could find that (p3)(p5)(p6) can be replaced by

$$u_1^2\, (\alpha_1^2 - \beta_1^2) + 2\, u_1\, u_2\, (\alpha_{12} - \beta_{12}) + u_2^2\, (\alpha_2^2 - \beta_2^2) = \alpha_3^2 - \beta_3^2. \tag{p3'}$$

$$u_1\, (\alpha_1^2 - \beta_1^2) + u_2\, (\alpha_{12} - \beta_{12}) = \alpha_{13} - \beta_{13}. \tag{p5'}$$

$$u_1\,(\alpha_{12} - \beta_{12}) + u_2\,(\alpha_2^2 - \beta_2^2) = \alpha_{23} - \beta_{23}. \tag{p6'}$$

It is now clear how the relation facilitates our approach to answer the orginal question. They give us some consistency conditions (U-condition, condition (4), (p3'), (p5'), (p6')) and reduce a four-points problem into a three-points problem. Actually, these two observations lead one to realize that a four-points problem is equivalent to both testable conditions and a three-points problem. For a three-points problem, one can use equations (D1)(D4)(D5). Thus, we have

$$\delta_{11}\, s_1^2 + \delta_{12}\, s_1 s_2 + \delta_{13}\, s_2^2 = F_1 \tag{1}$$

$$\alpha_1^2 + s_1^2 = \beta_1^2 + t_1^2 \tag{2}$$

$$\alpha_2^2 + s_2^2 = \beta_2^2 + t_2^2 \tag{3}$$

Deriving solutions becomes an easy task now. (2) and (3) require that s_1^2 must be greater than $\beta_1^2 - \alpha_1^2$ and s_2^2 must be greater than $\beta_2^2 - \alpha_2^2$. Thus, the intersections of these two regions and conic represented by (1) are all the possible solutions for s_1, s_2 for the three-points problem.

$$\begin{aligned} \delta_{11}\, s_1^2 + \delta_{12}\, s_1 s_2 + \delta_{13}\, s_2^2 &= F_1 \\ s_1^2 &\geq \alpha_1^2 - \beta_1^2 \\ s_2^2 &\geq \alpha_2^2 - \beta_2^2 \end{aligned} \tag{F}$$

Next, one could use (B1)(B2) to find out t_1 and t_2. Since there could have two values of t_1, t_2, one needs to check (B4) to choose the correct pair of t_1, t_2 and s_3, t_3 follows easily from the relation.

Now we discuss the situation of (C) where coplanarity condition occurs. The condition (C) can be written as $A_3 = \alpha\, A_1 + \beta\, A_2$ and $B_3 = \alpha\, B_1 + \beta\, B_2$. It is thus easy to see that (B3)(B5)(B6) can be derived from (B1)(B2)(B4). Thus the situation is exactly the same as the above system (F). In fact, the compatibility condition, U-condition, and consistency condition are not needed in this case. The original problem is itself a three-points problem.

In the case of planar patch, the task itself is a three-points problem. One of the nice property about the planar patch is that if we know these four points are coplanar, then an algorithm can be developed to find the mapping easily (or perceptually if you prefer). Figure 1b is in fact generated from planar patch, the reader is encouraged to guess what the mapping should be before he proceeds. An algorithm could easily be suggested by Lemma 1 although it is not written down explictly.

Lemma 1: Let Γ be a coplanar patch in space as depicted in Figure 1b. The intersection point M remains inside the convex hull spanned by A,B,C,D.

Proof: Since M is inside the convex hull before the motion, then

$$M = k_1 A + k_2 B + k_3 C + k_4 D$$

for some k_i such that $k_1 + k_2 + k_3 + k_4 = 1$. Apply rotation $\boldsymbol{R}$, we have

$$\boldsymbol{R} M = k_1 \boldsymbol{R} A + k_2 \boldsymbol{R} B + k_3 \boldsymbol{R} C + k_4 \boldsymbol{R} D$$

Therefore, the intersection remains in the convex hull. Furthermore, the ratios k_1, k_2, k_3, k_4 remain the same. In particular, $M = a\ A + b\ B$; $M = c\ C + d\ D$ for some a,b,c,d. This suggests that we can use a,b,c,d as an index to decide the correspondence. Q.E.D..

In the general case (the knowledge about planar patch is not given), the above algorithm can be used to decide if there is a coplanar interpretation or not. Of course, this does not guarrantee the existence, we need to check the feasibility of three-points problem, since the fourth point is redudant. The reader is now advised to connect the line between the opposite corners as Figure 5. It is clear that what the correspondence should be.

4. Simulations and Applications

Two examples and three applications are described in this section. The input of the first example consists of two views of four noncoplanar points with correspondence established. We use this example to illustrate each step described in the theory. The second example uses the same two views of the first example without *priori* knowledge of correspondence. A complete simulation of this four-points problem is presented. For each of 24 possible mappings, compatibility conditions, U-condition and consistency conditions are examined. If a mapping passes all the conditions, then a solution is derived.

The first application is to deal with "Structure From Motion" studied by Ullman [2]. He showed that "Three different views of four noncoplanar points can uniquely determine the structure uniquely". We show how to apply the theory to this problem. The second application is to show that "Increase of observable points will not narrow down the number of solutions (motion)". The third application is related to a recent paper "Perception of structure from motion" [7][8].

Example 2: The tilt and the slant of rotational axis are both 30 degrees and the rotational angle is also 30 degrees. The coordinates of points before and after transformation are given below. The left hand side represents those before motion and the right hand side represents those after motion. Note that the translation is adjusted to zero and one of the points, O, is chosen as reference and fixed point. Only the first two components are observable to the method described above.

$$
\begin{aligned}
O &= (0.0\ \ 0.0\ \ 0.0), & O &= (0.00\ \ 0.00\ \ 0.00) \\
B_1 &= (4.0\ \ 2.0\ \ 3.0), & A_1 &= (3.253\ \ 2.976\ \ 3.091). \\
B_2 &= (2.0\ \ 3.0\ \ 5.0), & A_2 &= (1.402\ \ 2.580\ \ 5.419) \\
B_3 &= (6.0\ \ 5.0\ \ 3.0), & A_3 &= (3.780\ \ 6.494\ \ 3.678)
\end{aligned}
$$

The rotational matrix can be computed according to [5] as follows:

$$
\begin{bmatrix}
0.8911 & -0.4185 & 0.1752 \\
0.4475 & 0.8743 & -0.1875 \\
-0.0747 & 0.2455 & 0.9665
\end{bmatrix}
$$

According to the formula described above, the first two components of the last column is proportional to $[\bar{A}_1\ \bar{A}_2]\ [\bar{B}_1\ \bar{B}_2]^{-1}\ \bar{B}_3 - \bar{A}_3$. Thus, one obtains $(0.8762\ \ -0.9374)^t$ which differs from the true vector by a scalar 5. Using the same technique, one obtains $[\bar{B}_1\ \bar{B}_2]\ [\bar{A}_1\ \bar{A}_2]^{-1}\ \bar{A}_3 - \bar{B}_3 = (-0.7086\ \ 2.3270)^t$ for the first two components of the last row. It is clear that it differs by a scalar 9.48. Next, one has to adjust the magnitude of these two vectors so that they have the same magnitude and then call them $(a1\ a2)$ and $(b1\ b2)$. Now one examines the determinants of the following two matrix. In order to have a solution, at least one of them must be zero. In this case, the determinant of (B.2) is zero and the determinant of (B.1) is not zero. Therefore the solution of u is uniquely determined up to a scalar.

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & a2 \\ a_{12} & a_{22} & a_{32} & -a1 \\ b_{11} & b_{21} & b_{31} & b2 \\ b_{12} & b_{22} & b_{32} & -b1 \end{bmatrix} \tag{B.1}$$

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & a2 \\ a_{12} & a_{22} & a_{32} & -a1 \\ b_{11} & b_{21} & b_{31} & b2 \\ b_{12} & b_{22} & b_{32} & -b1 \end{bmatrix} \tag{B.2}$$

One obtains the solution $u = (2.133\ \ -1.761\ \ 0.801)$ as opposed to the accurate $u = (1.940\ \ -1.601\ \ 0.7284)$. For reader's convenience, we will write equations (D1)(D2)(D3) down:

$$4.3715\ s_1^2 - 3.509\ s_1\, s_2 + 0.5574\ s_2^2 = 0.6417 \tag{q1}$$

$$4.5315 s_1^2 - 4.744\ s_1\, s_3 + 0.5574\ s_3^2 = 3.1013 \tag{q2}$$

$$4.5315 s_2^2 - 9.871\ s_2\, s_3 + 4.3715\ s_3^2 = 4.553 \tag{q3}$$

Now, if one tries to use the new relation $2.133\ s_1 - 1.761\ s_2 + 0.801\ s_3 = 0$ with (q2) or (q3) or both, then one would not obtain any new conic (see previous footnote). In the following, we compute α_i's, β_i's, α_{ij}, β_{ij}.

$$\beta_1^2 = 20.0,\ \ \beta_2^2 = 13.0,\ \ \beta_3^2 = 61;\ \ \alpha_1^2 = 19.442,\ \ \alpha_2^2 = 8.628,\ \ \alpha_3^2 = 56.46;$$

$$\beta_{12} = 14.0,\ \ \beta_{13} = 34,\ \ \beta_{23} = 27;\ \ \alpha_{12} = 12.24,\ \ \alpha_{13} = 31.62,\ \ \alpha_{23} = 22.06;$$

Now we will normalize u_3 to -1 as discussed before. One computes the left hand side and right hand side of (p3') (p5') and (p6') as follows:

$$\begin{aligned} 4.531532 &= 4.531544 \\ 2.372189 &= 2.3772183 \\ 4.93595 &= 4.955951 \end{aligned}$$

Now, the problem is reduced to the following three equations:

$$4.3715\, s_1^2 - 3.509\, s_1 s_2 + 0.5574\, s_2^2 = 0.6417$$

$$20.0 + s_1^2 = 19.44 + t_1^2$$

$$13.0 + s_2^2 = 8.62 + t_2^2$$

From the last two equations, there is no restriction for s_1 and s_2. The only requirement is the first equation. We show a couple of solutions other than those we started:

$$s_1 = 0;\ \ s_2 = 1.072;\ \ s_3 = 2.357;$$

$$t_1 = 0.748;\ \ t_2 = 2.351;\ \ t_3 = 3.177$$

Thus we have

$$\begin{bmatrix} 3.253 & 1.402 & 3.780 \\ 2.976 & 2.580 & 6.494 \\ 0.748 & 2.351 & 3.177 \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} 4.0 & 2.0 & 6.0 \\ 2.0 & 3.0 & 5.0 \\ 0.0 & 1.072 & 2.357 \end{bmatrix}$$

Another solution could be

$$s_1 = -1.0;\ \ s_2 = -4.941;\ \ s_3 = -8.199;$$

$$t_1 = 1.248;\ \ t_2 = 5.365;\ \ t_3 = 8.471$$

Thus we have

$$\begin{bmatrix} 3.253 & 1.402 & 3.780 \\ 2.976 & 2.580 & 6.494 \\ 1.248 & 5.365 & 8.471 \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} 4.0 & 2.0 & 6.0 \\ 2.0 & 3.0 & 5.0 \\ -1.0 & -4.941 & -8.199 \end{bmatrix}$$

Example 3: We use the data created in example 2. The tilt and the slant of rotational axis are both 30 degrees and the rotational angle is 30 degrees. O, B_1, B_2, B_3 are chosen as below and O, A_1, A_2, A_3 are the corresponnding points attributed to the rotation. This mapping is an admissible one.

$$\begin{array}{ll} O = (0.0\ \ 0.0\ \ 0.0), & O = (0.00\ \ 0.00\ \ 0.00) \\ B_1 = (4.0\ \ 2.0\ \ 3.0), & A_1 = (3.253\ \ 2.976\ \ 3.091). \\ B_2 = (2.0\ \ 3.0\ \ 5.0), & A_2 = (1.402\ \ 2.580\ \ 5.419) \\ B_3 = (6.0\ \ 5.0\ \ 3.0), & A_3 = (3.780\ \ 6.494\ \ 3.678) \end{array}$$

Table 1 shows that 24 mappings all pass three compatibility conditions. Eight mappings fail on U-condition and only 2 mappings pass the consistency conditions. In a word, only two mappings are admissible. One is what we have already known. The other one is to map O to A_3, B_1 to A_2, B_2 to A_1, and B_3 to O. The solution we obtain is as below: (One needs to adjust the relative positions of these points since A_3 is now the reference point in the second frame).

$$\begin{bmatrix} -2.378 & -0.527 & -3.780 \\ -3.91 & -3.518 & -6.494 \\ -1.743 & 0.59 & 3.69 \end{bmatrix} = R \begin{bmatrix} 4.0 & 2.0 & 6.0 \\ 2.0 & 3.0 & 5.0 \\ 2.0 & 0.0 & -2.999 \end{bmatrix}$$

To see the validity of the solution, readers only need to check whether the length and the inner product remain invariant.

Next we present an application of this theory to "structure from motion" introduced in [2]. Ullman shows that the structure of a 4-points object can be uniquely determined if three different views are given. The correspondence is assumed in his analysis. We will follow its assumption although there are easy ways to check whether the correspondence in these three views is possibly valid or not. I will not elaborate further here.

Application 1: (Structure From Motion)

This example uses the data in Example 2 where one starts with O, B_1, B_2, B_3. The second frame is generated by rotating 30 degrees about axis with tilt 30 degrees and slant 30 degrees. One further obtains the third frame by rotating 45 degrees about the axis with tilt 45 degrees and slant 20 degrees. Two equations can be derived as below. From the first and second frame, one obtains $u_1 s_1 + u_2 s_2 + u_3 s_3 = 0$. which is already shown before. From the first and the third frame, one obtains $v_1 s_1 + v_2 s_2 + v_3 s_3 = 0$.

$$1.63\, s_1 - 1.90\, s_2 + 1.54\, s_3 = 0$$

$$1.94\, s_1 - 1.60\, s_2 + 0.72\, s_3 = 0$$

One can then obtain $s_1 = 1.09$, $s_2 = 1.81$, $s_3 = 1.07$ up to a scalar by taking the cross product of u and v. Substituting these into equation (q1), one derive 2.58 as the scalar. Thus $s_1 = 2.8$, $s_2 = 4.6$, $s_3 = 2.7$ as opposed to $s_1 = 3.0$, $s_2 = 5.0$, $s_3 = 3.0$.

Application 2: (Five or more points in two views)

Now consider five points in two views. We claim that the fifth point has no role in pinning down the number of the solution (motion). Suppose we have two motions which can account for the four points. We will show that these two motions can also account for correspondence of the fifth point by adjusting the depth of the fifth point. Suppose R_1 can account for the fifth point. Our task is to show that R_2 can also account for the fifth point by adjusting its depth component. As before, I shall use notation O, B_1, B_2, B_3 with the fifth point denoted as D. We know that

$$D = (\bar{D}\ s_d)^t = \alpha B_1 + \beta B_2 + \gamma B_3 \quad \text{and} \quad R_1 D = \alpha R_1 B_1 + \beta R_1 B_2 + \gamma R_1 B_3$$

Examining the first two components, we get

$$\bar{D} = \alpha \bar{B}_1 + \beta \bar{B}_2 + \gamma \bar{B}_3$$

Examining the first two components of $R_2\,(\bar{D}\ \ s_d)^t$, one obtains $R_2^*\,\bar{D} + s_d\, l_1$ where R_2^* is the principal 2×2 minor of R_2; and l_1 is the first two components of the third column of R_2. Our goal is to see if we can choose a s_d so that $R_2^*\,\bar{D} + s_d\, l_1$ becomes $\overline{R_1 D}$. Clearly, one has

$$\begin{aligned} R_2^*\,\bar{D} + s_d\, l_1 &= \alpha R_2^*\,\bar{B}_1 + \beta R_2^*\,\bar{B}_2 + \gamma R_2^*\,\bar{B}_3 + s_d\, l_1 \\ &= \alpha\,(\overline{R_2 B_1} - s_1\, l_1) + \beta\,(\overline{R_2 B_2} - s_2\, l_1) + \gamma\,(\overline{R_2 B_3} - s_3\, l_1) + s_d\, l_1 \\ &= \alpha\,\overline{R_2 B_1} + \beta\,\overline{R_2 B_2} + \gamma\,\overline{R_2 B_3} + (s_d - \alpha s_1 - \beta s_2 - \gamma s_3)\, l_1 \\ &= \alpha\,\overline{R_1 B_1} + \beta\,\overline{R_1 B_2} + \gamma\,\overline{R_1 B_3} + (s_d - \alpha s_1 - \beta s_2 - \gamma s_3)\, l_1 \end{aligned}$$

Obviously s_d can be chosen such that the coefficient of the last term of the above equation is zero. Thus R_2 can also account for the correspondence of the fifth point. Here

s_1, s_2, s_3 are the depths of B_1, B_2, B_3.

Application 3: (Perception of structure from motion)

Recently, a paper entitled "Perception of Structure from Motion" [7][8] discusses lower bounds in relation to the structure from motion problem. This problem was first treated in [2] where three views of four noncoplanar points can uniquely determine the structure (relative depth) of these four points. In [7][8], the authors go one step further to investigate the lower bounds issue. The following are two quoted paragraphs:

> We prove that two orthographic projections of four noncoplanar points admit only four interpretations (up to a reflection) of structure. This forms the basis for an algorithm to recover structure from motion ...see Abstract of [7][8].

> **Theorem 2:** Two orthographic projections of four rigidly linked noncoplanar points are compatible with at most four interpretations. (see [7][8], section 4, page 6)

Here, we would like to point out that the result (unfortunately) is wrong. In the following, a counterexample with five solutions (the reflection is not counted) is presented. Other solutions in fact could be given, but five solutions are sufficient to invalidate their result. In fact, example 2 would serve the purpose.

The following are four solutions where column vectors of the matrix on the right hand side represent space coordinates in the first scene; and column vectors of the matrix on the left hand side represent space coordinates in the second scene due to some motion.

$$\begin{bmatrix} 3.253 & 1.402 & 3.780 \\ 2.976 & 2.580 & 6.494 \\ 0.748 & 2.351 & 3.177 \end{bmatrix} = R \begin{bmatrix} 4.0 & 2.0 & 6.0 \\ 2.0 & 3.0 & 5.0 \\ 0.0 & 1.072 & 2.357 \end{bmatrix} \qquad \text{(sol.1)}$$

$$\begin{bmatrix} 3.253 & 1.402 & 3.780 \\ 2.976 & 2.580 & 6.494 \\ 1.248 & 5.365 & 8.471 \end{bmatrix} = R \begin{bmatrix} 4.0 & 2.0 & 6.0 \\ 2.0 & 3.0 & 5.0 \\ -1.0 & -4.941 & -8.199 \end{bmatrix} \qquad \text{(sol.2)}$$

$$\begin{bmatrix} 3.253 & 1.402 & 3.780 \\ 2.976 & 2.580 & 6.494 \\ 5.055 & 8.732 & 5.735 \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} 4.0 & 2.0 & 6.0 \\ 2.0 & 3.0 & 5.0 \\ -5.0 & -8.478 & -5.32 \end{bmatrix} \qquad \text{(sol.3)}$$

$$\begin{bmatrix} 3.253 & 1.402 & 3.780 \\ 2.976 & 2.580 & 6.494 \\ 0.8405 & 2.09 & 2.356 \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} 4.0 & 2.0 & 6.0 \\ 2.0 & 3.0 & 5.0 \\ 0.383 & 0.0 & -1.020 \end{bmatrix} \qquad \text{(sol.4)}$$

The above four solutions and the original one which we started already make five solutions. To check these solutions, the readers are advised to examine the invariant of the length of each vector, and the invariant of the inner product of each two vectors.

5. Discussion and Conclusion

Given two views of four points, how many interpretations could possibly exist? Naturally one would decompose the task into two phases: Correspondence Problem and Recovery (motion, structure) Problem. To the best knowledge of author, almost all studies would assume the mapping (correspondence) has been established. However, it is not clear what would happen to their individual algorithms should the mapping be wrong. As for the correspondence problem, most of the studies would rely on attributes associated with point or patch and use the best match as criterion for correspondence.

In this paper, a theory which addresses the correspondence problem and recovery problem in the same framework is presented. The method reduces a four-points problem* into a set of testable conditions - including three compatibility conditions (1)(2)(3), U-condition, and four consistency conditions (4)(p3')(p5')(p6') - and a three-points problem. If a mapping passes all these testable conditions, then the four-points problem becomes a three-points problem. This forms the basis for deriving all possible solutions and relates the correspondence and recovery problem together.

Examples are used to illustrate each step of the theory. Several applications including "Structure from Motion"[2], and "Perception of structure from motion"[7][8] are also described. It is hoped that a similar theory can be found in the case of perspective projection.

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* If these four points are coplanar and the mapping is correct, then all these testable conditions automatically holds. In fact, it is a three-points problem in itself.

References

1. Lee C.H. (1986), "On Correspondence, Motion, Scale, and Structure of Two Views of A scene" TR-591 Department of Computer Sciences, Purdue University.

2. Ullman, S. (1979), "The Interpretation of Visual Motion", MIT press.

3. Tsai, R. Y. and T.S. Huang (1984), "Uniqueness and Estimation of Three-Dimensional Motion Parameters of Rigid Objects with Curved Surfaces", *IEEE Trans. PAMI 6,* 13-27.

4. Roach, J.W. and J.K. Aggarwal, "Determining the movement of objects from a sequence of images", *IEEE Trans. PAMI 2,*

5. Rogers D.F. and J.A. Adams (1976), Mathematical Elements for Computer Graphics, McGraw-Hill, New York.

6. Bellman, R. (1960), Introduction to Matrix Analysis, McGraw-Hill, New York.

7. Aloimonos J. and Amit B. (1985), "Perception of Structure from Motion: Lower Bounds Issues", Technical Report 158, Department of Computer Science, The University of Rochestor.

8. Aloimonos J. (1986), "Perception of Structure from Motion", Conference of Computer Vision and Pattern Recognition, IEEE Computer Society, June 25-27, Miami, Florida.

9. Nagel H.H. and B. Neuman (1981), "On 3-D reconstruction from two perspective views", *Proc. IJCAI 81,* Vol II.

Mapping	Compatibility	U-Condition	Consistency
$0, A_1, A_2, A_3$	ok	ok	ok
$0, A_1, A_3, A_2$	ok	-	-
$0, A_2, A_1, A_3$	ok	-	-
$0, A_2, A_3, A_1$	ok	-	-
$0, A_3, A_1, A_2$	ok	ok	-
$0, A_3, A_2, A_1$	ok	ok	-
$A_1, 0, A_2, A_3$	ok	-	-
$A_1, 0, A_3, A_2$	ok	ok	-
$A_1, A_2, 0, A_3$	ok	ok	-
A_1, A_2, A_3, O	ok	ok	-
$A_1, A_3, 0, A_2$	ok	ok	-
A_1, A_3, A_2, O	ok	ok	-
$A_2, 0, A_1, A_3$	ok	-	-
$A_2, 0, A_3, A_1$	ok	ok	-
$A_2, A_1, 0, A_3$	ok	ok	-
$A_2, A_1, A_3, 0$	ok	ok	-
$A_2, A_3, 0, A_1$	ok	-	-
A_2, A_3, A_1, O	ok	-	-
$A_3, 0, A_1, A_2$	ok	ok	-
$A_3, 0, A_2, A_1$	ok	ok	-
$A_3, A_1, 0, A_2$	ok	-	-
$A_3, A_1, A_2, 0$	ok	ok	-
$A_3, A_2, 0, A_1$	ok	ok	-
A_3, A_2, A_1, O	ok	ok	ok

OK = Success
– = Failure

Table 1: Example 2

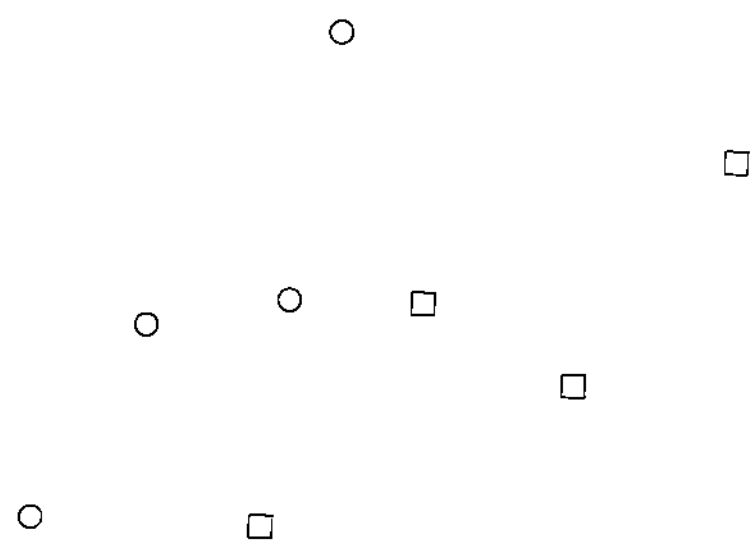

Figure 1a

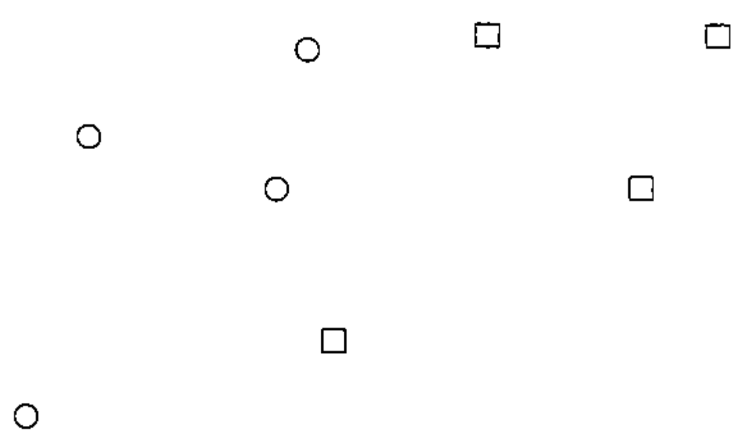

Figure 1b

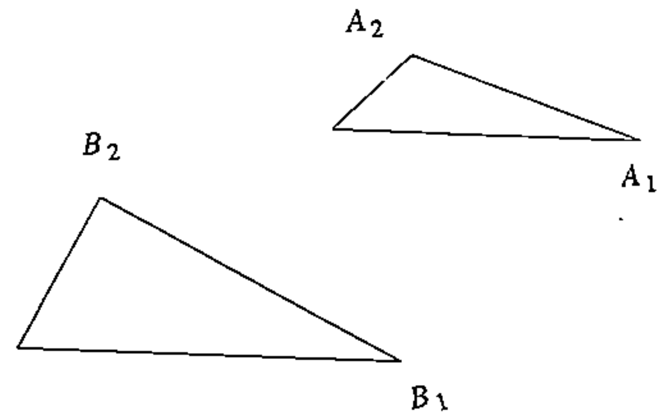

Figure 2

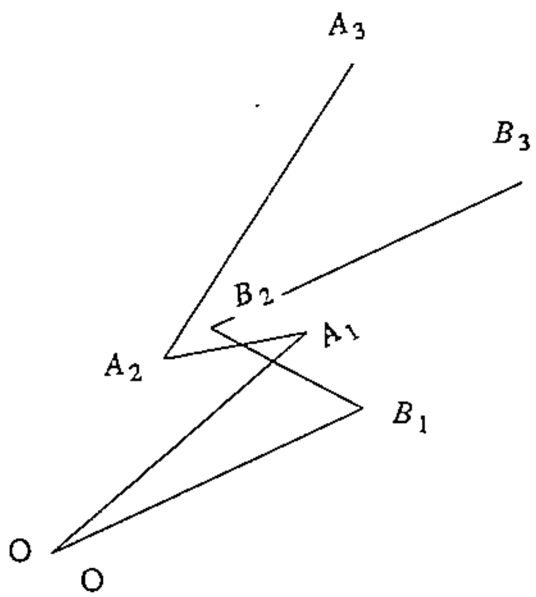

Figure 3

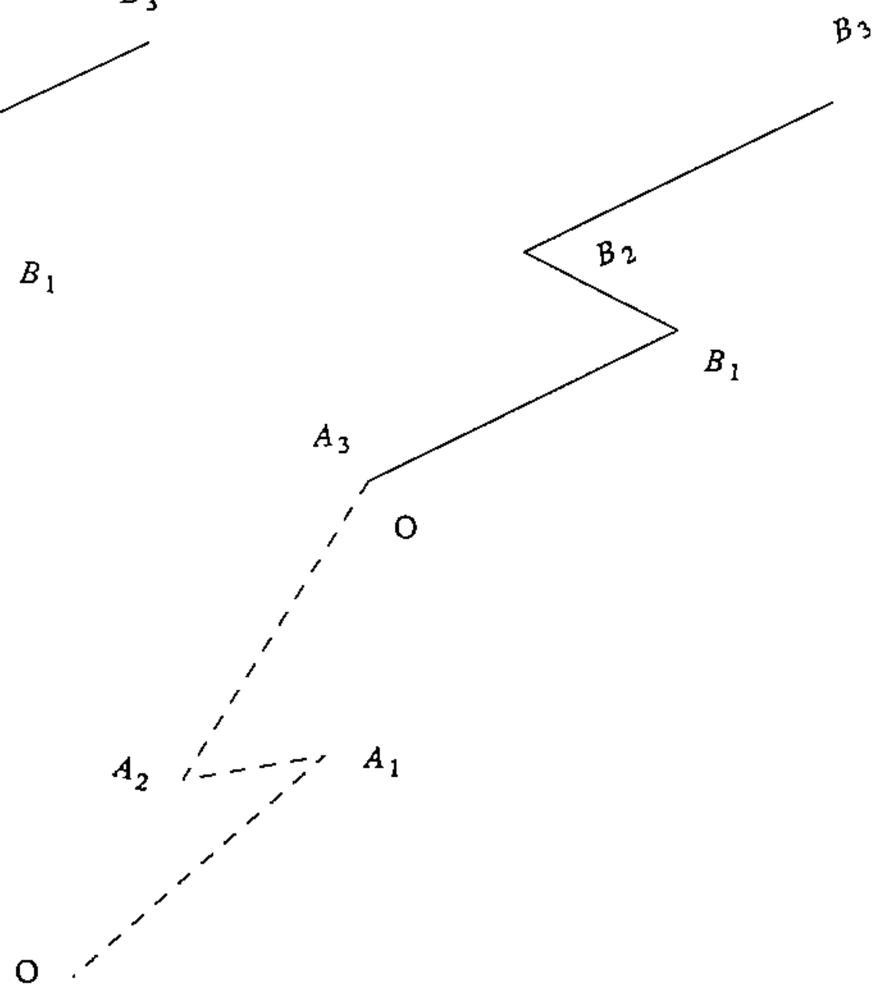

Figure 4

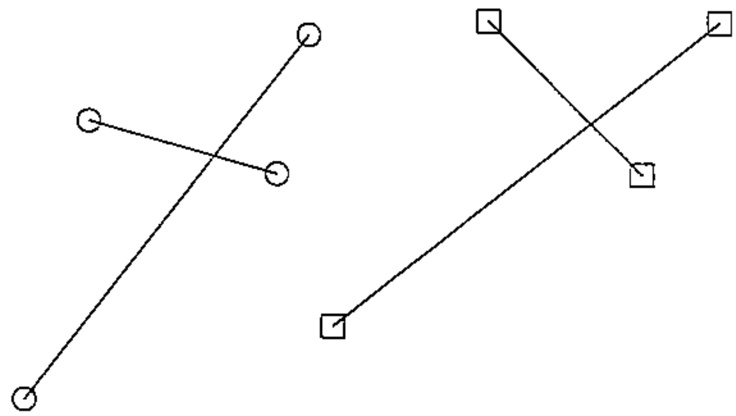

Figure 5