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Abstract. This paper examines flow balance, a basic assumption used in the operational analysis of queues and other discrete-state systems. Violation of this assumption can lead to large errors in estimates of state occupancies and average performance measures. However, if the state occupancies of a state sequence are approximated using a subsequence, then the maximum and average errors are of the order of the proportion of the state sequence discarded.

1. INTRODUCTION

The behavior of many systems can be represented by a state sequence over a finite or infinite time period. The state occupancies are the proportions of time the states are occupied in the sequence. Formulas relating the state occupancies to the parameters of the system are derived under simplifying assumptions about the state sequence. For queueing systems, the most common assumptions are flow balance and homogeneity.

For example, the behavior of a queueing network is represented by a sequence of values of the vector $\mathbf{n}(t) = (n_1(t), \ldots, n_K(t))$ that lists the number of jobs at each device at time t. Under the assumptions of flow balance and homogeneity, the occupancy $p(\mathbf{n})$ of any state $\mathbf{n}$ is easily computed from the total mean time demands for each device. Other performance metrics, such as throughput and response time, can be easily computed from the $p(\mathbf{n})$.

One of the goals of operational analysis has been to characterize the errors in formulas for performance quantities when the assumptions do not hold. The primary focus of error analyses has been the sensitivity of queueing formulas to violations in the homogeneity assumptions [1,6,7]. It has been commonly asserted that the error arising from the flow balance assumption approaches zero as the length of the state sequence over a finite state set approaches infinity.

Surprisingly, this assertion is not necessarily true. It is possible for arbitrarily large errors to exist between the actual state occupancies and estimates computed from formulas derived on the assumption of flow balance.

In contrast, relative errors will be bounded if the state occupancies of a maximal flow balanced subsequence are used as approximations for the state occupancies of the entire sequence. In this case, the absolute error cannot exceed the proportion of the state sequence falling outside the flow balanced subsequence.

This paper establishes these claims by studying errors between actual state occupancies and estimates derived on the assumption of flow balance. Bounds on absolute, relative, and average errors are derived and shown by example to be attainable. The main results are: 1) errors may be large if the state sequence in which the parameters are measured is not flow balanced, and 2) errors will be small if the parameters are measured using a significant flow balanced subsequence. The conclusion is that the common technique of removing end effects to obtain flow balanced observations of systems before measuring parameters introduces little error.

Derivations of all numbered equations are outlined in the Appendix; full details are given in [2].

2. NOTATION

Consider a state sequence

$$s_1, s_2, \ldots, s_K \;\; (s_{K+1})$$

in which each s_i is one of the integers $1, 2, \ldots, N$. The state s_{K+1} is not part of the sequence; it is recorded (in parentheses) so that an exit transition can be defined for every state in the sequence. A state sequence represents data that could be collected by sampling the system at $K+1$ arbitrary times or by observing the system continuously and recording the state at each change.

The operational notation for a state sequence is listed in Table 1. We will be interested in the relationship between the one-step transition matrix $\mathbf{Q} = [q_{ij}]$ and the occupancy vector $\mathbf{p} = [p_i]$. A one-step transition frequency, q_{ij}, is the proportion of occurrences of state i followed immediately by an occurrence of state j. A state occupancy, p_i, is the proportion of occurrences of state i. The matrix $\mathbf{Q}$ will be regarded

Table 1: Operational notation for a state sequence.

Symbol	Definition	Description
K		Length of state sequence
N		Number of unique states observed
C_{ij}		Number of one-step transitions from i to j
C_i		Number of exits from state i $\left[C_i = \sum_{j=1}^{N} C_{ij} \right]$
A_i		Number of entries into state i $\left[A_i = \sum_{j=1}^{N} C_{ji} \right]$
q_{ij}	C_{ij} / C_i	Proportion of exits from state i that immediately enter state j $\left[\sum_{j=1}^{N} q_{ij} = 1 \right]$
p_i	C_i / K	Proportion of total transitions occurring from state i $\left[\sum_{i=1}^{N} p_i = 1 \right]$
$\mathbf{Q}$	$[q_{ij}]$	One-step transition matrix
$\mathbf{p}$	$[p_i]$	State occupancy vector

as the parameters in terms of which the occupancy vector $\mathbf{p}$ must be expressed.

The physical interpretation of the vector $\mathbf{p}$ depends on the experiment used to obtain $\mathbf{Q}$. If the state sequence contains samples taken at arbitrary times, the relation between the p_i and the actual state occupancy times of the system is unknown. If all state transitions are observed, p_i can be interpreted as the proportion of all transitions occurring from state i. If the mean holding times in each state are known, the relation between the p_i and the time the system was in state i is easily computed.

(Details appear in the Appendix.)

The following sections study ways to produce an estimate $\hat{\mathbf{p}} = [\hat{p}_i]$ of the actual state occupancy vector $\mathbf{p}$ of a state sequence. Table 2 defines several measures of the error between $\mathbf{p}$ and $\hat{\mathbf{p}}$. The bounds shown in this table are derived without making any assumptions about the state sequence. (See Appendix.) A bound on the sum, E, of the error magnitudes also serves as a bound on the maximum absolute error, the average absolute error, and the weighted mean relative error.

The first part of this paper (Sections 3 and 4) assumes nothing about the system from which the state sequence was observed. In Section 3, the state occupancy vector is approximated by assuming the state sequence is flow balanced and solving the state balance equations. In Section 4, the state occupancy vector is approximated by the state occupancy vector of a flow balanced subsequence. The second part of this paper (Section 5) restricts attention to systems whose states are recurrent; in such systems every state is revisited within a bounded time.

3. APPROXIMATIONS USING STATE BALANCE EQUATIONS

A state sequence is flow balanced if the number of entries into each state is equal to the number of exits from that state; equivalently, s_1 and s_{K+1} are the same state. For any flow balanced state sequence, the state occupancy vector $\mathbf{p}$ satisfies the system of linear equations

$$\mathbf{p}\,\mathbf{Q} = \mathbf{p} \quad . \tag{3.1}$$

These equations are not linearly independent; given $\mathbf{Q}$, we can compute $\mathbf{p}$ by replacing any equation by the normalizing condition ($p_1 + \ldots + p_N = 1$) and solving the resulting system.

Table 2: Error measures.

Name	Definition	Bound
Maximum absolute	$\max_i \lvert p_i - \hat{p}_i \rvert$	$E/2 \le 1$
Average absolute	$\frac{1}{N} \sum_{i=1}^{N} \lvert p_i - \hat{p}_i \rvert = E/N$	$2/N$
Maximum relative	$\max_i \frac{\lvert p_i - \hat{p}_i \rvert}{p_i}$	$K-1$
Average relative	$\frac{1}{N} \sum_{i=1}^{N} \frac{\lvert p_i - \hat{p}_i \rvert}{p_i}$	$K-1$
Weighted mean relative	$\sum_{i=1}^{N} p_i \frac{\lvert p_i - \hat{p}_i \rvert}{p_i} = E$	2

$$\text{where } E = \sum_{i=1}^{N} \lvert p_i - \hat{p}_i \rvert$$

If a state sequence is not flow balanced, there exists one state ($i = s_{K+1}$) for which $A_i = C_i + 1$ and one state ($i = s_1$) for which $A_i = C_i - 1$. For all other states we still have $A_i = C_i$. Define $d_i = A_i - C_i$. Then $\mathbf{d} = [d_i]$ is a row vector in which all but two elements are zero. For any state sequence, the state occupancy vector $\mathbf{p}$ satisfies the system of linear equations

$$\mathbf{p}\,\mathbf{Q} = \mathbf{p} + \frac{1}{K}\,\mathbf{d} \quad . \tag{3.2}$$

Augmenting this system with the normalizing condition produces a linear system whose unique solution is the state occupancy vector $\mathbf{p}$.

Suppose flow balance is assumed when analyzing a state sequence that is not flow balanced. This means that the normalized solution to (3.1) is used as an approximation of the solution to (3.2). How much error will result?

The following example shows that the errors in Table 2 can be within $1/K$ of their bounds.

EXAMPLE. Consider the following state sequence of length $K = n_1 + n_2 + n_3$:

$$1^{n_1}\ 2^{n_2}\ 3^{n_3}\ (3) \quad .$$

The superscripts denote repetitions of a state. For this state sequence,

$$\mathbf{Q} = \begin{pmatrix} \frac{n_1-1}{n_1} & \frac{1}{n_1} & 0 \\ 0 & \frac{n_2-1}{n_2} & \frac{1}{n_2} \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

The actual state occupancy vector is $\mathbf{p} = (\frac{n_1}{K}, \frac{n_2}{K}, \frac{n_3}{K})$. The solution estimated from (3.1) is $\hat{\mathbf{p}} = (0, 0, 1)$. The vector of absolute errors is $(\frac{n_1}{K}, \frac{n_2}{K}, -\frac{K-n_3}{K})$; the vector of relative errors is $(1, 1, -\frac{K-n_3}{n_3})$. The error measures are maximized when $n_3 = 1$. In this case, state 3 has the largest absolute error of $\frac{K-1}{K}$ and the largest relative error of $K-1$; the average absolute error is $\frac{2}{3}\frac{K-1}{K}$ and the weighted mean relative error is $2\frac{K-1}{K}$.

Equations (3.1) and (3.2) differ only in the terms $\pm\frac{1}{K}$ associated with the initial and final states. It has been conjectured [5] that if the initial and final states are visited often, then the terms $\pm\frac{1}{K}$ are small compared to the occupancies of these states, and the solutions of (3.1) and (3.2) nearly the same. The previous example shows this conjecture is false. Suppose that $n_1 = n_3 = aK$ for some constant a; no matter what the value of K, $p_1 = p_3 = a$ and the largest absolute error is $1-a$. In other words, as K becomes large, the terms $\frac{1}{K}\mathbf{d}$ vanish from (3.2) and yet the largest absolute error remains close to its maximum.

The conclusion is that violation of the flow balance assumption can lead to large errors in the estimate of the occupancy vector. This statement is true even if the initial and final states occur frequently.

4. APPROXIMATIONS USING SUBSEQUENCES

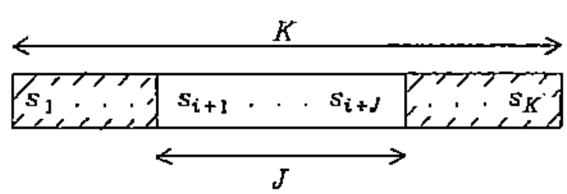
Another way to approximate the state occupancy vector of an arbitrary state sequence is to select some flow balanced subsequence, solve for its state occupancy vector, and use the result as an estimate of the state occupancy vector of the entire sequence. In this section we will derive bounds on the errors in this type of approximation. If a state sequence has no flow balanced subsequence, then every state is distinct and we know $p_i = 1/K$ for all states i.

Table 3 summarizes the necessary notation. The state occupancy vector for the entire sequence is $\mathbf{p} = (p_1, \ldots, p_N)$ and for the subsequence it is $\hat{\mathbf{p}} = (\hat{p}_1, \ldots, \hat{p}_N)$. The occupancy vector $\hat{\mathbf{p}}$ satisfies the linear system $\hat{\mathbf{p}}\hat{\mathbf{Q}} = \hat{\mathbf{p}}$, where $\hat{\mathbf{Q}}$ is the one-step transition matrix for the subsequence. Note, $\hat{p}_i$ may be zero if the subsequence contains no occurrences of state i.

The diagram below shows a typical state sequence and subsequence. The shaded areas are the states outside the subsequence; these states comprise $\frac{K-J}{K}$ of the entire sequence.

Table 3: Notation for subsequence analysis.

Symbol	Definition	Description
K		Length of state sequence
J		Length of subsequence
N		Number of unique states observed
n_i		Number of occurrences of state i in state sequence
n_i'		Number of occurrences of state i in subsequence
n_i''		Number of occurrences of state i outside subsequence
p_i	n_i / K	Proportion of occurrences of state i in sequence
$\hat{p}_i$	n_i' / J	Proportion of occurrences of state i in subsequence
$\mathbf{p}$	$[p_i]$	State occupancy vector for state sequence
$\hat{\mathbf{p}}$	$[\hat{p}_i]$	State occupancy vector for subsequence



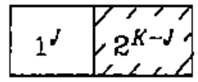
4.1 Absolute Errors

The largest absolute error magnitude in any element of $\hat{\mathbf{p}}$ is bounded by the proportion of the state sequence that is not used. That is,

$$\max_i |p_i - \hat{p}_i| \leq \frac{K-J}{K} \quad . \tag{4.1}$$

An example shows that this bound can be attained.

EXAMPLE. Consider the following state sequence:



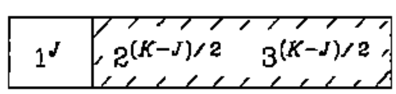
The state occupancy vector for the entire sequence is $\mathbf{p} = (\frac{J}{K}, \frac{K-J}{K})$, whereas the approximation using the subsequence is $\hat{\mathbf{p}} = (1, 0)$. The vector of absolute errors is $(\frac{J-K}{K}, \frac{K-J}{K})$. Each absolute error has magnitude equal to the bound.

While the error in some p_i may be as large as the bound in (4.1), the errors in all the p_i cannot be that large (except when $N=2$). The average absolute error magnitude is bounded by

$$\frac{1}{N} \sum_{i=1}^{N} |p_i - \hat{p}_i| \le \frac{2}{N} \frac{K-J}{K} \quad . \tag{4.2}$$

Usually both $\frac{2}{N}$ and $\frac{K-J}{K}$ will be much less than 1. Their product may easily be an order of magnitude smaller than either of the terms. An example shows that this bound can be attained.

EXAMPLE. Consider the following state sequence having three different states:



The exact solution is $\mathbf{p} = (\frac{J}{K}, \frac{K-J}{2K}, \frac{K-J}{2K})$ and the approximate solution is $\hat{\mathbf{p}} = (1, 0, 0)$. The vector of absolute errors is $(\frac{J-K}{K}, \frac{K-J}{2K}, \frac{K-J}{2K})$ and the mean absolute error magnitude is $\frac{2}{3} \frac{K-J}{K}$.

4.2 Relative Errors

The largest relative error magnitude in any element of $\hat{\mathbf{p}}$ is bounded by

$$\max_i \frac{|p_i - \hat{p}_i|}{p_i} \le \max\left[\frac{K-J}{J}, 1\right] \quad . \tag{4.3}$$

This bound can be attained by state sequences of any length. The relative error for a state not represented in the subsequence ($n_i'' = n_i$) is always 1. The relative error for a state occurring only in the subsequence ($n_i' = n_i$) is always $-\frac{K-J}{J}$.

The mean relative error magnitude is bounded by

$$\frac{1}{N} \sum_{i=1}^{N} \frac{|p_i - \hat{p}_i|}{p_i} \le 2\,\frac{K-J}{N} \quad . \tag{4.4}$$

Since the mean error is bounded by the maximum error, the tighter of the bounds in (4.3) and (4.4) can be used.

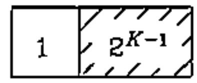
The weighted mean relative error gives more significance to errors for states that occur frequently. The weighted mean relative error is bounded by twice the proportion of the state sequence that is not used. That is,

$$\sum_{i=1}^{N} p_i \frac{|p_i - \hat{p}_i|}{p_i} \le 2\,\frac{K-J}{K} \quad . \tag{4.5}$$

This error bound is N times the mean absolute error bound in (4.2).

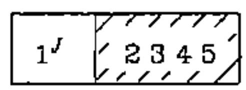
The following example shows that while both types of average errors may be large, the weighted mean can be much less than the mean.

EXAMPLE. Consider the state sequence



The relative errors in states 1 and 2 are $K-1$ and 1, respectively. The mean relative error is $\frac{K}{2}$; the weighted mean relative error is $2\frac{K-1}{K}$. This shows that the bound in (4.5) can be attained.

EXAMPLE. Even if most of the state sequence is used, the mean relative error can be within $\frac{1}{N}$ of the maximum relative error. Consider the following state sequence:



If $J \geq 4$, states 2, 3, 4, and 5 all have the largest relative error of 1. The mean relative error magnitude of $\frac{4}{5}\frac{J+1}{J}$ is greater than $\frac{4}{5}$ for all J. The weighted mean error of $\frac{8}{K}$ approaches zero as J increases.

Application of the bounds in this section is illustrated by the following example. Suppose we observe a state sequence of length $K = 1000$ and use a subsequence of length $J = 900$ to approximate the state occupancy vector. The largest absolute error for any state will be no greater than 10%. If we know that there are $N = 50$ different states in the sequence, then the mean absolute error will be no larger than 0.4%. The largest relative error and the mean relative error are both bounded by 100%. The weighted mean relative error is bounded by 20%.

5. STATE SEQUENCES WITH RECURRENT STATES

The worst cases illustrated in Sections 3 and 4 were caused by states occurring only once or many times consecutively. In reality, observed states often recur regularly. We offer an operational definition of "recurrent states" and show that the worst case errors are smaller for sequences of states of systems of recurrent states.

We will say that the states of a given system are recurrent if there exists an upper bound L on the maximum distance between consecutive occurrences of state i. This is equivalent to saying that every subsequence of length L contains at least one occurrence of every state. In many cases, an estimate of L may be known from some characteristic of the underlying system.

This definition implies that, for a given system, there exists a lower bound $p = 1/L$ on all the state occupancies p_i that can be observed in state sequences of that system. Because the property that all $p_i \geq p$ does not rule out the occurrences of a state being all in a single run, it is not equivalent to the definition of recurrent states.

For systems of recurrent states a bound on total absolute error for the balance-equation approximation is

$$E \leq 2\left(1 - \frac{1}{L}\right) \quad . \tag{5.1}$$

(We believe this bound can be tightened.) A bound on the total absolute error for the subsequence approximation is

$$E \leq \frac{2L}{K} \quad . \tag{5.2}$$

This bound shows that, for any system whose states are recurrent and any given error tolerance, there exists a sufficiently long observation that the error from the flow balance assumption will be less than the given tolerance.

6. CONCLUSIONS

If the transition matrix $\mathbf{Q}$ of a flow-imbalanced state sequence is used to solve the balance equations $\mathbf{pQ} = \mathbf{p}$, large errors may occur in the resulting estimates of the state occupancies $\mathbf{p}$. But if a flow balanced subsequence is used to approximate the state occupancies of the entire sequence, most errors are of the order of the proportion of the state sequence discarded. The conclusion is that the subsequence approximation (from section 3) is more robust and accurate than the balance-equations approximation (from section 4).

If the observed state sequence comes from a system whose states are recurrent, the errors are smaller than for unconstrained sequences. The errors induced by the subsequence approximation tend to zero as the length of the observation period increases for such systems. (We conjecture that this statement is true for the balance-equations approximation as well, but have not yet obtained a proof.)

The assumption that the approximating subsequence is flow balanced is not necessary. It is only necessary to assume that an exact solution for the subsequence has been obtain by any method. In general, the error of the solution of the subsequence must be added to the errors of our bounds. Therefore, these results can apply to any situation in which a subset of available data is used to approximate performance quantities.

The subsequence approximation appears commonly in simulation and measurement, where "end effects" due to jobs in progress at the start and end of the observation period are discarded. The performance quantities of the resulting subset of the data are used to approximate the performance quantities of the original observation period. Our results show that this technique is robust and will not introduce much error.

The principle of the subsequence approximation is also used in the theory of nearly completely decomposable systems [3,4]. If a subsystem interacts weakly with its environment, the steady state behavior of the subsystem will be a good approximation of the subsystem behavior between interactions with the environment. In our terminology, the flow balanced subsequence corresponds to a portion of the state sequence between interactions. Near complete decomposability assures that the time constants of the subsystem are short and, hence, each state of the subsystem will be observed in a short time. Thus the amount of the sequence between interactions that must be discarded to obtain a flow balanced subsequence is small and the error introduced by assuming flow balance for the full interval between interactions is small. We have not yet explored how to exploit the assumption of decomposability to partition the transition matrix **Q** and tighten the error bounds.

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Appendix

This appendix outlines the derivations of all numbered equations in the text.

Table 2 Bounds The bounds on the average absolute error and weighted mean error follow from the fact

$$\sum_{i=1}^{N} |p_i - \hat{p}_i| \le \sum_{i=1}^{N} (p_i + \hat{p}_i) = 2 \quad .$$

The bounds on the maximum and average relative errors follows from the fact that $p_i \ge 1/K$ for all i.

Define $e_i = p_i - \hat{p}_i$. Let P denote the states for which $e_i \ge 0$ and M the states for which $e_i < 0$. Now, $\sum_{i=1}^{N} e_i = 0$ implies

$$\sum_{i \in P} |e_i| = \sum_{i \in M} |e_i|$$

Then,

$$E = \sum_{i \in P} |e_i| + \sum_{i \in M} |e_i| = 2 \sum_{i \in P} |e_i| = 2 \sum_{i \in M} |e_i| \quad .$$

Since the largest $|e_i|$ is contained in one of these two sums, $\max_i |e_i| \le E/2$. Equality holds if either sum contains one term.

Section 2 The fraction of time state i is occupied is related to p_i as follows. Let T_i denote the total time state i is occupied in the original system. The mean holding time in state i is $h_i = T_i/n_i = T_i/p_i K$. The fraction of time state i is occupied is $f_i = T_i/T$. But $T = \sum_{j=1}^{N} T_j = \sum_{j=1}^{N} h_j p_j K$. Therefore,

$$f_i = h_i p_i \Big/ \sum_{j=1}^{N} h_j p_j \quad .$$

If the holding times are constant, all $f_i = p_i$.

Eqns. (3.1) and (3.2) Applying the relationships from Table 1 to $A_i = C_i + d_i$ gives

$$\sum_{j=1}^{N} \frac{C_{ji}}{C_j} \frac{C_j}{K} = \sum_{j=1}^{N} \frac{C_{ij}}{C_i} \frac{C_i}{K} + \frac{1}{K} d_i$$

or

$$\sum_{j=1}^{N} q_{ji}\, p_j = p_i + \frac{1}{K} d_i$$

for all states i. Expressing these equations in matrix form gives (3.2). For flow balanced sequences, all $d_i = 0$, and the system reduces to (3.1).

Eqns. (4.1) - (4.5) For every state i,

$$\frac{n_i}{K} \le \frac{n_i'}{K} + \frac{K-J}{K} \le \frac{n_i'}{J} + \frac{K-J}{K} .$$

Therefore, if $p_i \ge \hat{p}_i$ then $p_i - \hat{p}_i \le (K-J)/K$. Also, $n_i \ge n_i'$ implies $p_i \ge \hat{p}_i \, J/K$. If $p_i < \hat{p}_i$ then

$$\hat{p}_i - p_i \le \hat{p}_i - \frac{J}{K}\hat{p}_i \le \frac{K-J}{K} .$$

These two cases imply (4.1).

The relative error magnitude for state i is bounded by

$$\frac{|p_i - \hat{p}_i|}{p_i} \le \frac{n_i'}{n_i}\,\frac{K-J}{J} + \frac{n_i''}{n_i} .$$

If $(K-J)/J \le 1$, this expression is maximized when $n_i' = 0$ and $n_i'' = n_i$. If $(K-J)/J > 1$, the maximum occurs when $n_i' = n_i$ and $n_i'' = 0$. This gives (4.3).

The weighted relative error is

$$\sum_{i=1}^{N} p_i \,\frac{|p_i - \hat{p}_i|}{p_i} \le \sum_{i=1}^{N} \left[\frac{n_i'}{K}\,\frac{K-J}{J} + \frac{n_i''}{K} \right] .$$

This simplifies to (4.5). Multiplying each side by $1/N$ gives (4.2).

The bound on the mean relative error magnitude is obtained by applying $n_i \ge 1$ to the relative error bound for state i:

$$\frac{1}{N}\sum_{i=1}^{N} \frac{|p_i - \hat{p}_i|}{p_i} \le \frac{1}{N}\sum_{i=1}^{N} \left[n_i'\,\frac{K-J}{J} + n_i'' \right] .$$

This reduces to (4.4).

Eqs. (5.1) and (5.2) The total absolute error satisfies

$$E = \sum |p_i - \hat{p}_i| \le \sum |p_i - p| + \sum |\hat{p}_i - p| .$$

Given that all $p_i \ge p$, the first sum evaluates to $1-Np$. If all the $\hat{p}_i \ge p$ as well, then $E \le 2(1-Np)$. However, as many as $(N-1)$ of the $\hat{p}_i$ can be smaller than p, in which case the second sum can be as large as $1-p+(N-1)p$. Therefore the bound on E is $2(1-p)$. With $p=1/L$, relation (5.1) is obtained.

The bound of (4.2) says $E \le 2(K-J)/K$. For a system of recurrent states, at most L states need be discarded to find a flow balanced subsequence. Hence $K-J \le L$ and $E \le 2L/K$.