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Ronald F. Boisvert

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Ronald F. Boisvert

Purdue University
Department of Computer Science
West Lafayette, Indiana 47907

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Abstract

An analysis of two one-parameter families of fourth order finite difference methods of the HODIE type is presented for the constant coefficient Helmholtz equation $\nabla^2 u + Fu = G$ on a rectangle. It is shown that the choice of the parameter, which specifies a set of auxiliary points in the HODIE scheme, significantly affects the accuracy attained by the method. Extensions are made to the case of variable F. A sixth order method for the constant coefficients case is stated.

1. Introduction

We consider the numerical solution to the Dirichlet problem on a rectangle for the Helmholtz equation

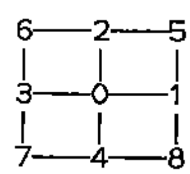
$$\nabla^2 u + Fu = G$$

where $F \le 0$, by the use of high order difference methods of the HODIE type introduced by Lynch and Rice in [3]. These methods appear promising since they retain the simplicity of $O(h^2)$ finite difference techniques while attaining the high orders possible with the finite element approach.

A nine-point HODIE discretization of the operator $Lu=G$ at the point (x_0,y_0) takes the form

$$L_h U = (1/h^2)\ \Sigma\ \alpha_i U_i = \Sigma\ \beta_j G_j = I_h G$$

where U_i is the estimate of $u(x_i,y_i)$, $G_j = G(x_j,y_j)$, and h is the grid spacing. The (x_i,y_i) are the nine grid points adjacent to (x_0,y_0) labelled as



and the points (x_j,y_j) are additional points, called auxiliary points, which are located near (x_0,y_0). The coefficients α_i and β_j are chosen to make the discretization exact for the space $\mathbb{P}_n$ of polynomials of degree at most n, that is,

$$L_h s - I_h Ls = 0 \quad \text{for all } s \in \mathbb{P}_n$$

together with a normalization equation. In [3], Lynch and

Rice show that for $n \le 7$ there exist auxiliary points for which the HODIE discretization of the Helmholtz equation is exact on $\mathbb{P}_n$ and has order of accuracy $O(h^{n-1})$ for $u \in C^{n+1}$.

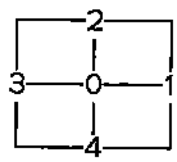
One of the key ideas of the HODIE method is that high order is attained by the addition and placement of auxiliary points and not by the use of mesh points more than one grid line away from (x_0, y_0). In this way the linear system for U is no more complex than that of the usual nine-point star discretization. It has the same block tridiagonal structure and the α coefficients are O(h) perturbations of the usual ones. The coefficients α_i and β_j must be determined by solving a linear system (possibly one system per grid point), but the number of equations in this system is fixed independant of h and so as $h \to 0$ the time to solve the system of difference equations dominates.

2. Families of Fourth Order Methods for Constant F

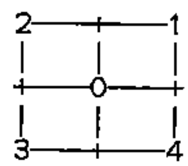
In the case of fourth order methods for the Helmholtz equation with constant F, the system defining the α_i's and β_j's is particularly simple and can be solved explicitly, the coefficients being the same independant of the location of (x_0, y_0). In order that $L_h s = I_h L s$ for all $s \in \mathbb{P}_5$, 22 conditions must be satisfied, one for each element of a basis for $\mathbb{P}_6$ plus a normalization. Since there are nine

α's, we need at least 13 β's, some of which might be zero in special cases.

We can determine that all but five β's are zero by taking advantage of symmetry. Hence we consider only those discretizations with five auxiliary points placed symmetrically with respect to both grid lines passing through (x_0,y_0). We label these as



or



Set P Set Q

Without loss of generality we assume that $x_0=0$ and $y_0=0$. We define two sets of auxiliary points,

$$P(h,\xi) = \{(0,0),(\xi h,0),(0,\xi h),(-\xi h,0),(0,-\xi h)\}$$

$$Q(h,\xi) = \{(0,0),(\xi h,\xi h),(-\xi h,\xi h),(-\xi h,-\xi h),(\xi h,-\xi h)\}$$

where $0<\xi\leq 1$ is a parameter. Finally, we focus our attention on those discretizations with the symmetry constraints $\alpha_1=\alpha_2=\alpha_3=\alpha_4$, $\alpha_5=\alpha_6=\alpha_7=\alpha_8$ and $\beta_1=\beta_2=\beta_3=\beta_4$. To indicate the simplifications produced by these assumptions we state a lemma whose proof is easy.

<u>Lemma 2–1</u>

Let $L_h u = 1/(6h^2)(\alpha_0 u_0 + \alpha_1(u_1+u_2+u_3+u_4) + \alpha_5(u_5+u_6+u_7+u_8))$ and suppose $I_h G = \beta_0 G_0 + \beta_1(G_1+G_2+G_3+G_4)$ where the four auxiliary points are chosen as either the set $P(h,\xi)$ or $Q(h,\xi)$. Then the discretization $L_h u = I_h G$ is exact for all monomials $x^i y^j$ with at least one of i and j odd.

Theorem 2-1(a)

Let the discretization $L_h u = I_h G$ be defined as in Lemma 2-1 by the coefficients

$$\alpha_0 = -20 + (6-\xi^2)Fh^2 - \tfrac{1}{2}(1-\xi^2)F^2h^4$$

$$\alpha_1=\alpha_2=\alpha_3=\alpha_4 = 4$$

$$\alpha_5=\alpha_6=\alpha_7=\alpha_8 = 1 + \tfrac{1}{4}\xi^2 Fh^2$$

$$\beta_0 = (6\xi^2-1)/(6\xi^2) - (1/12)(1-\xi^2)Fh^2$$

$$\beta_1=\beta_2=\beta_3=\beta_4 = 1/(24\xi^2)$$

and the auxiliary point set $Q(h,\xi)$. Then $L_h u = I_h G$ is exact for all $u \in \mathbb{P}_5$.

Theorem 2-1(b)

Let the discretization $L_h u = I_h G$ be defined as in Lemma 2-1 by the coefficients

$$\alpha_0 = -20 + 2(3-\xi^2)Fh^2 - \tfrac{1}{2}(1-\xi^2)F^2h^4$$

$$\alpha_1=\alpha_2=\alpha_3=\alpha_4 = 4 + \tfrac{1}{2}\xi^2 Fh^2$$

$$\alpha_5=\alpha_6=\alpha_7=\alpha_8 = 1$$

$$\beta_0 = (3\xi^2-1)/(3\xi^2) - (1/12)(1-\xi^2)Fh^2$$

$$\beta_1=\beta_2=\beta_3=\beta_4 = 1/(12\xi^2)$$

and the auxiliary point set $P(h,\xi)$. Then $L_h u = I_h G$ is exact for all $u \in \mathbb{P}_5$.

Proof :

By Lemma 2-1 we need only satisfy $L_h u = I_h L u$ for the functions 1, x^2-h^2, y^2-h^2, $x^2(x^2-h^2)$, $y^2(y^2-h^2)$ and

$(x^2-h^2)(y^2-h^2)$. Since the pairs x^2-h^2, y^2-h^2 and $x^2(x^2-h^2)$, $y^2(y^2-h^2)$ and their partial derivatives are linearly dependant on each of the given point sets we can further reduce these to the basis functions

$$1,\ x^2-h^2,\ x^2(x^2-h^2),\ (x^2-h^2)(y^2-h^2)$$

In addition, one normalization equation is needed. In this way we get a system of five equations which are non-singular for all $0<\xi\leq 1$ in the unknowns α_0, α_1, α_5, β_0, and β_1. In each case, the solution of the system leads to the coefficients given above. ∎

The discretizations with $\xi=1$ are especially appealing since, in these cases, the auxiliary points are also grid points and thus no more than one evaluation of the right side G per grid point is needed. However, in some cases a HODIE discretization with $0<\xi<1$ can produce an error small enough to warrant its use in spite of an increase in the number of evaluations of G.

Some stencils of particular interest are displayed below. The notation $L^P_{h,\xi}U = I^P_{h,\xi}G$ denotes the HODIE discretization with grid spacing h and auxiliary point set $P(h,\xi)$. Note that the stencils for $I^P_{h,\xi}$ and $I^Q_{h,\xi}$ are given at half-grid points.

Auxiliary point set P(h,1)

$$L^{P}_{h,1} = 1/(6h^2)\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} + F/12\begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix}$$

$$I^{P}_{h,1} = (1/12)\begin{bmatrix} & & 1 & & \\ & & & & \\ 1 & & 8 & & 1 \\ & & & & \\ & & 1 & & \end{bmatrix}$$

Auxiliary point set P(h,½)

$$L^{P}_{h,½} = 1/(6h^2)\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} + F/48\begin{bmatrix} & 1 & \\ 1 & 44 & 1 \\ & 1 & \end{bmatrix} - F^2h^2/16$$

$$I^{P}_{h,½} = (1/3)\begin{bmatrix} & & & & \\ & & 1 & & \\ & 1 & -1 & 1 & \\ & & 1 & & \\ & & & & \end{bmatrix} - Fh^2/16$$

Auxiliary point set Q(h,1)

$$L^{Q}_{h,1} = 1/(6h^2)\begin{array}{|c|c|c|}\hline 1 & 4 & 1 \\ \hline 4 & -20 & 4 \\ \hline 1 & 4 & 1 \\ \hline \end{array} + F/24\begin{array}{|c|c|c|}\hline 1 & & 1 \\ \hline & 20 & \\ \hline 1 & & 1 \\ \hline \end{array}$$

$$I^{Q}_{h,1} = (1/24)\begin{array}{|c|c|c|c|c|}\hline 1 & & & & 1 \\ \hline & & & & \\ \hline & & 20 & & \\ \hline & & & & \\ \hline 1 & & & & 1 \\ \hline \end{array}$$

Auxiliary point set Q(h,½)

$$L^{Q}_{h,1/2} = 1/(6h^2)\begin{array}{|c|c|c|}\hline 1 & 4 & 1 \\ \hline 4 & -20 & 4 \\ \hline 1 & 4 & 1 \\ \hline \end{array} + F/96\begin{array}{|c|c|c|}\hline 1 & & 1 \\ \hline & 92 & \\ \hline 1 & & 1 \\ \hline \end{array} - F^2h^2/16$$

$$I^{Q}_{h,1/2} = (1/6)\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline & 1 & & 1 & \\ \hline & & 2 & & \\ \hline & 1 & & 1 & \\ \hline & & & & \\ \hline \end{array} - Fh^2/16$$

The linear systems generated by these discretizations are $O(h^2)$ perturbations of the usual nine-point discretization of the Laplacian and thus they possess a number of properties which permit the use of efficient equation solution techniques. We summarize these properties in the following theorem.

Theorem 2-2

Let $-A$ be the matrix generated using any of the HODIE schemes $P(h,\xi)$ or $Q(h,\xi)$ for the Dirichlet problem for the Helmholtz equation $\nabla^2 u + Fu = G$ with constant $F \le 0$ on a rectangle. We assume that the natural ordering of equations and unknowns is used. Then, for h sufficiently small, the matrix A is

a) real and symmetric,

b) irreducibly diagonally dominant,

c) positive definite, and

d) of monotone type.

Proof :

Part a is obvious. For b we first note that, provided $F<0$ and $h^2<4/(-F\xi^2)$, we have that $\alpha_0<0$ and $\alpha_i>0$ for $i\neq 0$ and thus the matrix A has the sign distribution $a_{ii}>0$ and $a_{ij}\le 0$ for $i\neq j$. In each case, $\Sigma\alpha_i \le 0$, and so, as a result of the sign distribution, $|a_{ii}| = -\alpha_0 \ge 4\alpha_1 + 4\alpha_5 = \Sigma|a_{ij}|$, with strict inequality for equations resulting from stencils with central points adjacent to the boundary, since some of the α_i with $i>0$ are equal to zero there. Thus, since the linear system is obviously irreducible, A is irreducibly diagonally dominant. Part c follows from the fact that an irreducibly diagonally dominant symmetric matrix with positive diagonal entries is positive definite (see [5], pg. 23). Finally, monotonicity follows from b and the sign distribution (see [5], pg. 85). ∎

One consequence of this theorem is that any of the point or block Jacobi, Gauss-Seidel or SOR iterative methods are convergent when applied to any HODIE method of the classes considered here.

3. Error Analysis for Members of the Fourth Order Families

There are many fourth order HODIE discretizations with five auxiliary points. The first non-zero term in an expansion of the truncation error, $T_h \equiv I_h L - L_h$, provides some insight about which of these methods are superior.

Let $X^i Y^j$ denote the differentiation operator of order i in x and j in y, and let I denote the identity operator. We define several intermediate operators and give expansions of the result of applying the operators to some sufficiently smooth function u.

$$A_h u_0 \equiv (-20u_0 + 4(u_1+u_2+u_3+u_4) + 1(u_5+u_6+u_7+u_8))/6h^2$$
$$= [(X^2+Y^2) + (h^2/12)(X^4+Y^4)$$
$$+ (1/360)h^4(X^6+5X^4Y^2+5X^2Y^4+Y^6)]u_0 + O(h^6)$$
$$B_h u_0 \equiv 4(3-\xi^2)u_0 + \xi^2(u_1+u_2+u_3+u_4)$$
$$= [12I + \xi^2h^2(X^2+Y^2) + (1/12)\xi^2h^4(X^4+Y^4)]u_0$$
$$+ O(h^6)$$
$$C_h u_0 \equiv 2(6-\xi^2)u_0 + \tfrac{1}{2}\xi^2(u_5+u_6+u_7+u_8)$$
$$= [12I + \xi^2h^2(X^2+Y^2)$$
$$+ (1/12)\xi^2h^4(X^4+6X^2Y^2+Y^4)]u_0 + O(h^6)$$
$$D^P_{h,\xi} v_0 \equiv 4(3\xi^2-1)v_0 + 1(v_1+v_2+v_3+v_4)$$
$$= [12\xi^2 I + \xi^2h^2(X^2+Y^2)$$
$$+ (1/12)\xi^2h^4(X^4+Y^4)]v_0 + O(h^6)$$
$$D^Q_{h,\xi} v_0 \equiv 2(6\xi^2-1)v_0 + \tfrac{1}{2}(v_1+v_2+v_3+v_4)$$
$$= [12\xi^2 I + \xi^2h^2(X^2+Y^2)$$
$$+ (1/12)\xi^2h^4(X^4+6X^2Y^2+Y^4)]v_0 + O(h^6)$$

In terms of these operators the families of discretization operators can be expressed as

$$L^P_{h,\xi} = A_h + (F/24)B_h - (1/12)(1-\xi^2)F^2h^2 I$$
$$I^P_{h,\xi} = 1/(12\xi^2)D^P_{h,\xi} - (1/12)(1-\xi^2)Fh^2 I$$
$$L^Q_{h,\xi} = A_h + (F/24)C_h - (1/12)(1-\xi^2)F^2h^2 I$$
$$I^Q_{h,\xi} = 1/(12\xi^2)D^Q_{h,\xi} - (1/12)(1-\xi^2)Fh^2 I$$

Note that $L^P_{h,\xi}$ and $L^Q_{h,\xi}$ as well as $I^P_{h,\xi}$ and $I^Q_{h,\xi}$ are identical except that C_h and $D^Q_{h,\xi}$ each have one fourth order cross derivative term.

Next we can compute the truncation operators directly.

Theorem 3-1(a)

The truncation operator of the HODIE discretization with auxiliary point set $Q(h,\xi)$ is

$$T^{Q}_{h,\xi} = I^{Q}_{h,\xi}L - L^{Q}_{h,\xi}$$
$$= (h^4/720)[\ (5\xi^2-2)(X^6+Y^6) + 5(7\xi^2-2)(X^4Y^2+X^2Y^4)\ ] + O(h^6)$$

Theorem 3-1(b)

The truncation operator of the HODIE discretization with auxiliary point set $P(h,\xi)$ is

$$T^{P}_{h,\xi} = I^{P}_{h,\xi}L - L^{P}_{h,\xi}$$
$$= (h^4/720)[\ (5\xi^2-2)(X^6+Y^6) + 5(\xi^2-2)(X^4Y^2+X^2Y^4)\ ] + O(h^6)$$

We can now compare the truncation operators of the HODIE discretizations displayed in the last section. They are

$$T^{Q}_{h,1} = (h^4/720)[\ 3(X^6+Y^6) + 25(X^4Y^2+X^2Y^4)\ ] + O(h^6)$$
$$T^{Q}_{h,1/2} = (h^4/720)[\ -(3/4)(X^6+Y^6) - (5/4)(X^4Y^2+X^2Y^4)\ ] + O(h^6)$$
$$T^{P}_{h,1} = (h^4/720)[\ 3(X^6+Y^6) - 5(X^4Y^2+X^2Y^4)\ ] + O(h^6)$$
$$T^{P}_{h,1/2} = (h^4/720)[\ -(3/4)(X^6+Y^6) - (35/4)(X^4Y^2+X^2Y^4)\ ] + O(h^6)$$

For the Helmholtz operator it is not difficult to relate the truncation error to the discretization error $e = U - u$, where u is the true solution and U is the approximate solution. We first note that in each case the truncation operator is of the form

$$T_{h,\xi} = (h^4/720)\ [\ r(\xi)(X^6+Y^6) + s(\xi)(X^4Y^2+X^2Y^4)\ ] + O(h^6)$$

where $r(\xi)$ and $s(\xi)$ are constants depending on the parameter ξ.

Theorem 3-2

Let R be a rectangle $R = [a,b] \times [c,d]$ for which the ratio $(b-a)/(d-c)$ is rational. For integers $n,m\leq 2$, suppose that $h = (b-a)/n = (d-c)/m$ and let $x_i = a + ih$, $y_i = c + ih$. For given functions G and H, let u denote the solution to

$$\nabla^2 u + Fu = G \quad \text{on interior}(R)$$
$$u = H \quad \text{on boundary}(R)$$

where $F\leq 0$ is a constant. Suppose further that G and H are sufficiently smooth so that $u \in C^6(R)$. Denote $u(x_i,y_j)$ by u_{ij}. If U_{ij} is an approximation to u_{ij} by one of the HODIE discretizations $P(h,\xi)$ or $Q(h,\xi)$ for $0<\xi\leq 1$, then, for h sufficiently small,

$$|e_{ij}| \equiv |u_{ij} - U_{ij}| \leq h^4 K(\xi)$$

for $i=1,2,\ldots,n-1$ and $j=1,2,\ldots,m-1$, where $K(\xi)$ is a constant depending only on u, ξ and whether the auxiliary point set was of type P or Q.

Proof :

Let $w(x,y) = \gamma(t^2 - (x-x_0)^2 - (y-y_0)^2)$, where $(x_0,y_0) = ((a+b)/2,(c+d)/2)$ and γ and t are constants. Choose t large enough that $t^2 - (x-x_0)^2 - (y-y_0)^2 > 0$ on R and choose γ small enough that $w(x,y) \le 1$ on R. Then for all sufficiently small h,

$$|L_{h,\xi} w_{ij}| \ge M > 0$$

for some constant M independant of i, j and h. To see this, note that $L_{h,\xi} = (\alpha_0+4\alpha_1+4\alpha_5)w - 4h^2(\alpha_1+2\alpha_5)$. We have seen before that $\alpha_0+4\alpha_1+4\alpha_5 \le 0$ and that if $F<0$ and $h^2<4/(-F\xi^2)$ then both α_1 and α_5 are positive. Thus $L_{h,\xi}w$ is of one sign on R, which shows the existence of the required $M>0$. Let M_1 and M_2 be constants chosen so that

$$|(X^6+Y^6)u| \le M_1 \quad \text{and} \quad |(X^4Y^2+X^2Y^4)u| \le M_2$$

in R. In addition, let the constant $K(\xi)$ be chosen so large that for $h^2<4/(-F\xi^2)$,

$$(|r(\xi)|M_1 + |s(\xi)|M_2)/720M + |O(h^6)| \le K(\xi)$$

Then,

$$\begin{aligned} |L_{h,\xi} e_{ij}| &= |L_{h,\xi} U_{ij} - L_{h,\xi} u_{ij}| \\ &= |I_{h,\xi} G_{ij} - L_{h,\xi} u_{ij}| = |T_{h,\xi} u_{ij}| \\ &\le (h^4/720)(|r(\xi)|M_1 + |s(\xi)|M_2) + |O(h^6)| \\ &\le h^4 K(\xi) M \le h^4 K(\xi) |L_{h,\xi} w_{ij}| \\ &= |L_{h,\xi}(h^4 K(\xi) w_{ij})| \end{aligned}$$

Since, as we have shown above, for h chosen as above, the operator $L_{h,\xi}$ is of monotone type, this implies that (see [0], pg. 43)

$$|e_{ij}| \le h^4 K(\xi) w_{ij} \le h^4 K(\xi) \quad \blacksquare$$

Inspecting the proof one sees that if the truncation error constants $|r(\xi)|$ and $|s(\xi)|$ are reduced, then the discretization error bound $K(\xi)$ can also be reduced. In particular, using the truncation error constants previously displayed, we see immediately that the smallest error bound arrived at this way is attained with the $Q(h,\frac{1}{2})$ method and the largest with the $Q(h,1)$ method.

We now consider the value of ξ which produces the smallest error bound. Since we do not know anything about derivatives of u in general, it is natural to seek that value of ξ that minimizes $\max(|r(\xi)|,|s(\xi)|)$. (We note that other norms could be used as well and, in general, would lead to different theoretical conclusions.) This determines a set of auxiliary points which are optimal with respect to this choice of norm. Figure 1 shows the region bounded by the curves $\xi=0$, $|r(\xi)|$, $|s(\xi)|$, and $\xi=1$ for $Q(h,\xi)$ and Figure 2 shows the curves for $P(h,\xi)$. From the diagrams it is clear that the minima occur at the vertex where $6-15\xi^2 = 105\xi^2-30$ in Figure 1 and at $\xi = 1$ in Figure 2. Thus the min-max value of the parameter is $\xi^* = \sqrt{0.3}$ for $Q(h,\xi)$ and it is $\xi^+=1$ for $P(h,\xi)$. These yield the truncation operators

$$T^Q_{h,\xi^*} = (h^4/720)[\ -\tfrac{1}{2}(X^6+Y^6) - \tfrac{1}{2}(X^4Y^2+X^2Y^4)\ ] + O(h^6)$$

$$T^P_{h,\xi^+} = (h^4/720)[\ 3(X^6+Y^6) - 5(X^4Y^2+X^2Y^4)\ ] + O(h^6)$$

from which the discretization with $Q(h,\xi^*)$ gives the smaller bound on the discretization error.

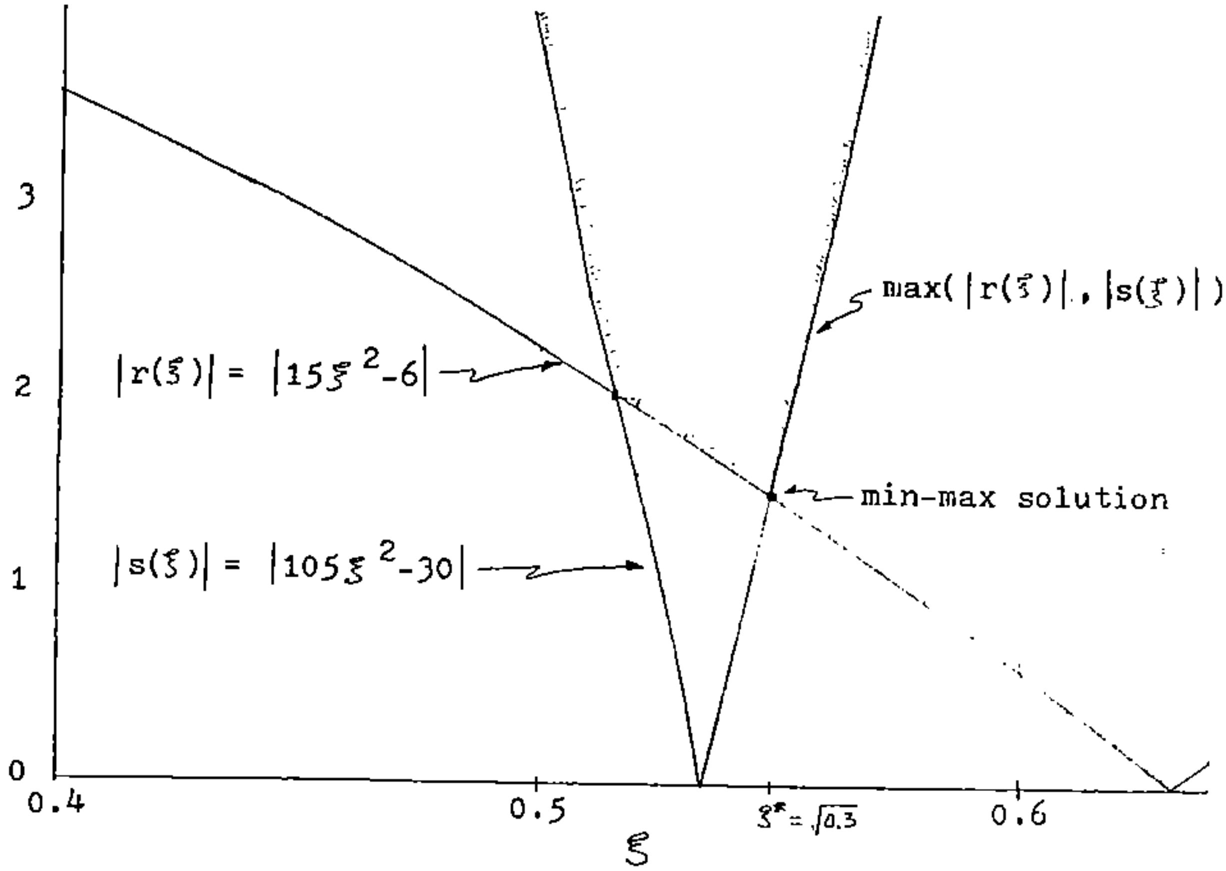


Figure 1 : Min-max Values of $Q(h,\xi)$

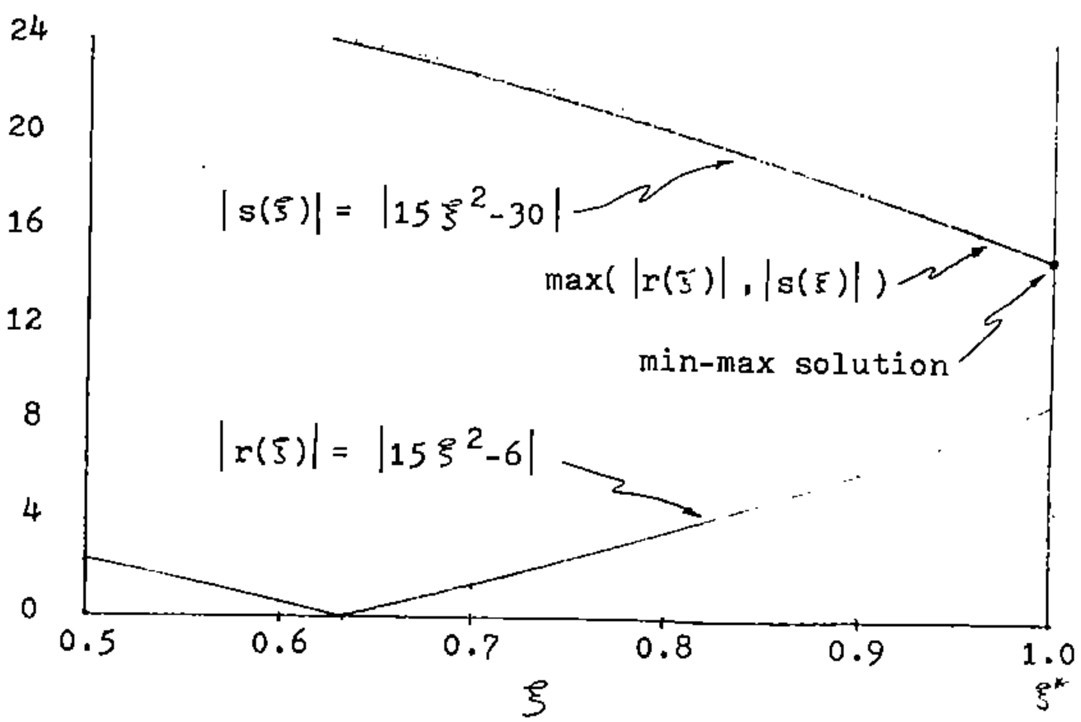


Figure 2 : Min-max Values of $P(h,\xi)$

Finally we note that the discretizations with $\xi=\frac{1}{2}$ or $\xi=1$ appear to be the only HODIE schemes of practical value in these families since in these cases the number of evaluations of the function G per grid point can be reduced from an average of five to one or two. However, the fact that $\xi^*\sim 0.55$ indicates that the discretization using the auxiliary point set $Q(h,\frac{1}{2})$ might be the best of those considered for the constant coefficients case.

4. Extensions to Variable F

We now briefly consider Helmholtz type operators $L = \nabla^2 + F$ in which F is a function of x and y. A simple extension of the constant coefficient case is a result of the following theorem.

Theorem 4-1

Let L_h and I_h be finite difference operators such that $L_h s = I_h \nabla^2 s$ for all $s \in \mathbb{P}_n$. Then the discretization

$$(L_h + I_h F)u = I_h G$$

of the equation $Lu = (\nabla^2 + F)u = G$ with variable $F \leq 0$ is exact on $\mathbb{P}_n$.

Proof :

Choose $s \in \mathbb{P}_n$. Then $Ls = \nabla^2 s + Fs$ and so adding $I_h Fs$ to both sides of $L_h s = I_h \nabla^2 s$ gives

$$(L_h + I_h F)s = I_h(\nabla^2 s + Fs) = I_h Ls = I_h G. \quad \blacksquare$$

This theorem is of practical use only when the auxiliary points are a subset of the nine mesh points of a single grid element, since otherwise extra unknowns are introduced into the system of difference equations.

Corollary

The HODIE discretization of $Lu = \nabla^2 u + Fu = G$ with variable F using the auxiliary point sets $P(h,1)$ and $Q(h,1)$ are exact on $\mathbb{P}_5$ and the truncation errors are the same as for the constant coefficient case.

5. A Sixth-Order HODIE Discretization

We next observe that in the case of constant coefficients there is a HODIE method which has sixth order accuracy. This was first observed in the case F=0 by Lynch and Rice in [3].

Theorem 5-1

The finite difference approximation

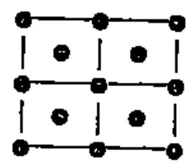
$$(48L^{Q}_{h,\frac{1}{2}} + 4L^{Q}_{h,1} + 8L^{P}_{h,1})u = (48I^{Q}_{h,\frac{1}{2}} + 4I^{Q}_{h,1} + 8I^{P}_{h,1})G$$

of the Helmholtz equation $\nabla^2 u + Fu = G$ with constant F is exact for all $u \in \mathbb{P}_7$.

Proof :

A simple computation shows that $48T^{Q}_{h,\frac{1}{2}} + 4T^{Q}_{h,1} + 8T^{P}_{h,1} = O(h^6)$ and hence the discretization above has truncation error $O(h^6)$. ■

This discretization has the auxiliary point set $Q(h,\frac{1}{2}) \cup Q(h,1) \cup P(h,1)$ which gives a set of auxiliary points of the form



and hence requires an average of two evaluations of G per grid point. This discretization requires only slightly more computation (to evaluate the right side of the difference equations) than the fourth order method with the set $Q(h,\frac{1}{2})$, a fact that makes the sixth order method quite attractive in the case of constant coefficients.

6. Numerical Results

Several FORTRAN programs were written to verify the derived properties of the HODIE schemes. These programs have been interfaced with the ELLPACK 77 system [4] for the solution to elliptic partial differential equations and were run on the Purdue University CDC 6500 computing system. The CDC 6500 uses floating point numbers accurate to about one part in 10^{-14}. The Minnesota FORTRAN compiler was used in the tests.

One of our programs generates a discretization based on the fourth order $P(h,\xi)$ or $Q(h,\xi)$ schemes. Because the parameter ξ is arbitrary here, the function G is evaluated five times per grid point in this program. In order to assess the savings possible in a more efficient implementation of the schemes P(h,1), Q(h,1), P(h,½), and Q(h,½), an efficient version of P(h,1) written by R. E. Lynch was also used. We call this program "Fast P(h,1)". We also tested the sixth order HODIE code available in the March 27, 1978 version of ELLPACK 77. In this experimental code the coefficients of the method are initially derived (though they are known for the cases we consider here) and then 13 evaluations of G are made per grid point. Finally, we also ran the bi-cubic Hermite ($\mathbb{P}_3C^1$) collocation module available in the same version of ELLPACK 77 as a benchmark. In each case the ELLPACK general banded linear system solver was used to solve the system of difference equations.

Here we report on some of the experiments which were carried out. The methods P(h,1), P(h,½), Q(h,1), Q(h,½), $Q(h,\xi^*)$, Fast P(h,1), $\mathbb{P}_3C^1$ collocation, and the sixth order HODIE were each run on a set of seven problems taken from the PDE population study [2], including three Poisson equations, two Helmholtz equations with constant F and two with variable F. In each case the region R is the unit square. The methods with auxiliary points not at grid points were not used in the latter two cases.

The problems are (problem numbers are taken from [2]):

- Problem : 3

  Operator : $\nabla^2 u = G(x,y)$

  Solution : $u = 3\exp(x+y)xy(1-x)(1-y)$

  Features : Analytic solution

- Problem : 7

  Operator : $\nabla^2 u - 100u = G(x,y)$

  Solution : $u = \frac{1}{2}(\cosh(10x)/\cosh(10)$

  $+ \cosh(ay)/\cosh(a))$

  Parameter values : $a=20$

  Features : Boundary layer

- Problem : 17

  Operator : $\nabla^2 u = G(x,y)$

  Solution : $u = \sin(x-y+\frac{1}{2})$

  $+ \exp(-y^2-(ab^3x^3/(1+b^3x^3))^2)$

  Parameter values : $a=5$, $b=3$

  Features : Ridge in solution

- Problem : 38

  Operator : $\nabla^2 u = G(x,y)$

  Solution : $u = (xy)^{**}(\frac{1}{2}a)$

  Parameter values : $a=\frac{1}{2}$

  Features : Solution has derivatives which are singular at the boundary

- Problem : 41

  Operator : $\nabla^2 u - au = G(x,y)$

  Solution : $u = \cos(by) + \sin(b(x-y))$

  Parameter values : $a=10$, $b=10$

  Features : Oscillatory solution

- Problem : 6

  Operator : $\nabla^2 u - F(x,y)u = G(x,y)$

  $F(x,y) = 100 + \cos(2\pi x) + \sin(3\pi y)$

  Solution : $u = \psi(x)\psi(y)\sin(\pi x)y(y-1) \cdot (1/(1+\phi(x,y)^4)-\frac{1}{2})$

  $\psi(z) = 5.4 - \cos(4\pi z)$

  $\phi(x,y) = 4((x-\frac{1}{2})^2 + (y-\frac{1}{2})^2)$

  Features : Oscillatory solution

- Problem : 20

  Operator : $\nabla^2 u - F(x,y)u = G(x,y)$

  Solution : $u = 10\phi(x)\phi(y) + a$

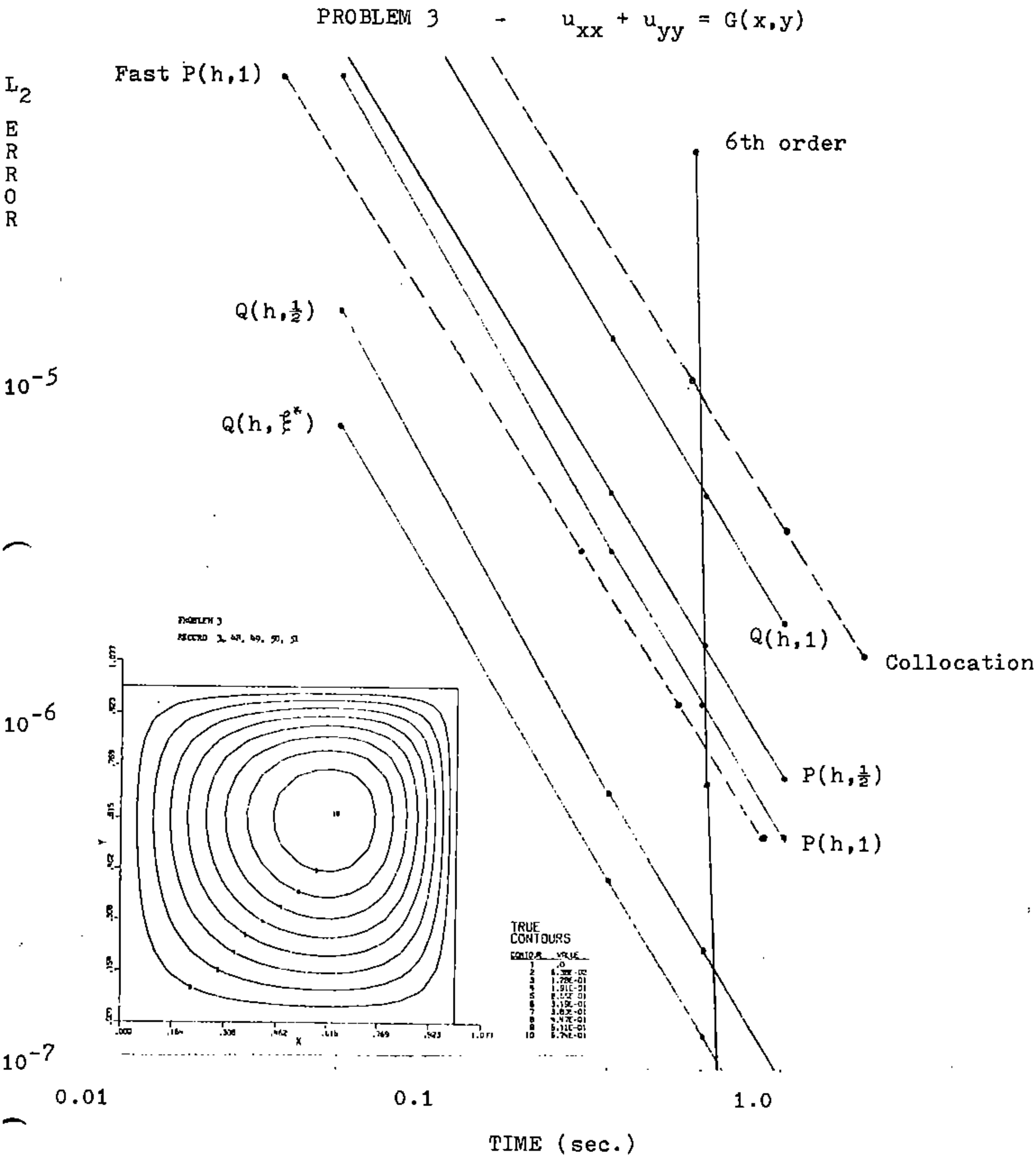
  $\phi(z) = z(z-1)\exp(-100(z-\frac{1}{2})^2)$

  Parameter values : $a=10$

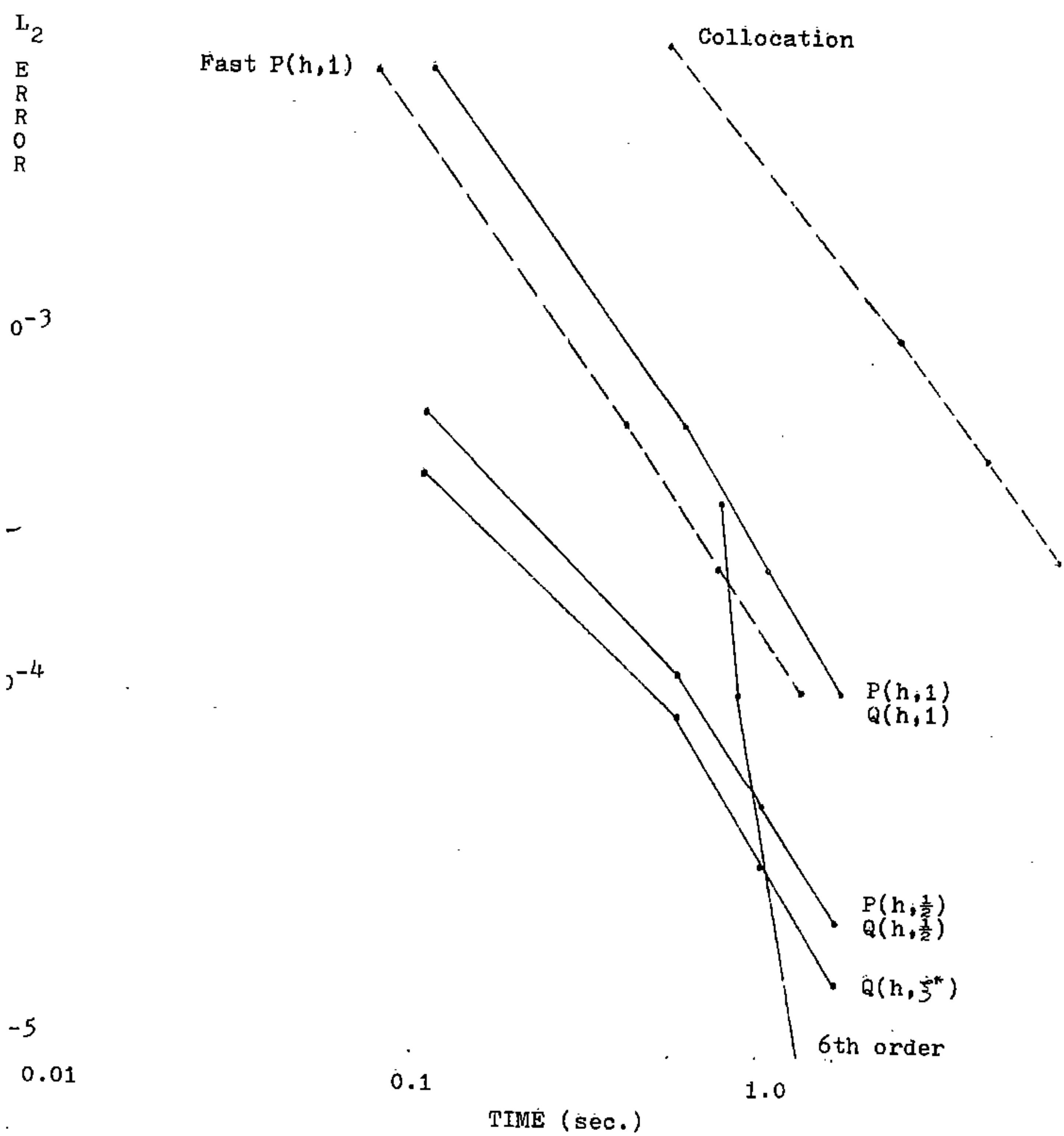
  Features : Sharp peak in solution

The problems were solved on square grids in which the number of grid lines in each direction was 5, 9, 11 and 13 for the fourth order HODIE methods, 3, 5, 6 and 7 for the collocation method and 3, 5, 9 and 11 for the sixth order HODIE method. The results are summarized on the following pages which show graphs of $\log_{10}(|\text{error}|)$ versus $\log_{10}(\text{execution time})$ where error denotes the discrete L_2 error at the grid points.

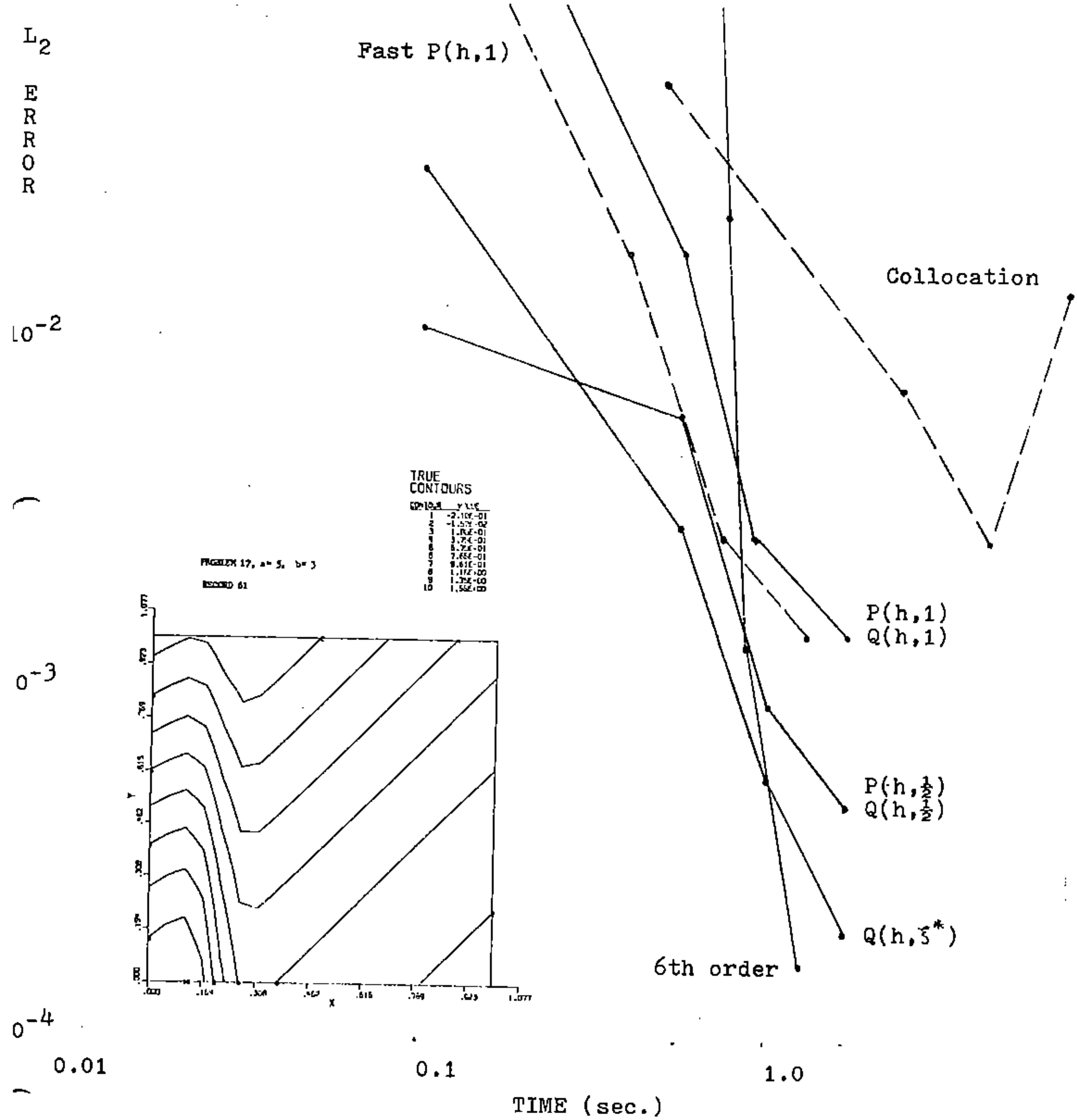
For the case of constant F the optimal scheme $Q(h,\xi^*)$ was uniformly superior, that is, for a given error, it required less execution time than the others. The $Q(h,½)$ method was the next most efficient. The Fast $P(h,1)$ method, although clearly better than its inefficient counterpart $P(h,1)$, was never better than the $Q(h,½)$ scheme, even though the implementation of the latter uses five evaluations of G per grid point. The $Q(h,1)$ choice of auxiliary points was the least effective of the HODIE methods tested, although it still performed better than than collocation in each case. It should be emphasized that most of the test problems do not have homogeneous boundary conditions and no problems were run with non-uniform grids, two cases in which the relative performance of collocation improves. Finally we see that the sixth order HODIE method was usually superior for higher accuracies, although a more efficient implementation of the sixth order method (reducing the number of evaluations of G at each mesh point from 13 to 2) would make this method competitive even for moderate accuracy.


PROBLEM 3 - $u_{xx} + u_{yy} = G(x,y)$
L_2 ERROR
Fast P(h,1)
6th order
Q(h,½)
10^{-5}
$Q(h,\xi^*)$
Q(h,1)
Collocation
10^{-6}
P(h,½)
P(h,1)
TRUE CONTOURS
10^{-7}
0.01
0.1
1.0
TIME (sec.)

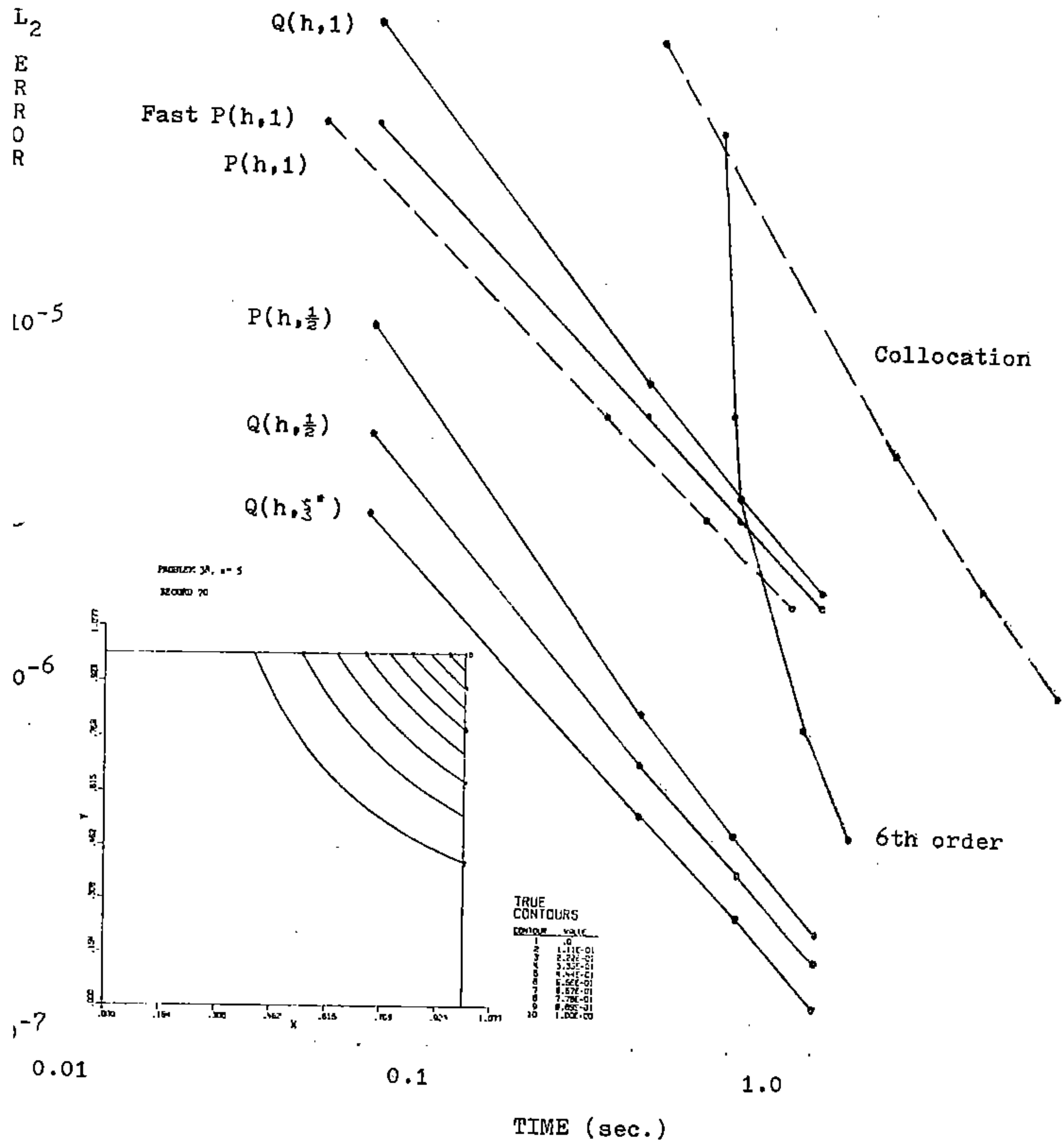
PROBLEM 7 - $u_{xx} + u_{yy} - 100u = G(x,y)$



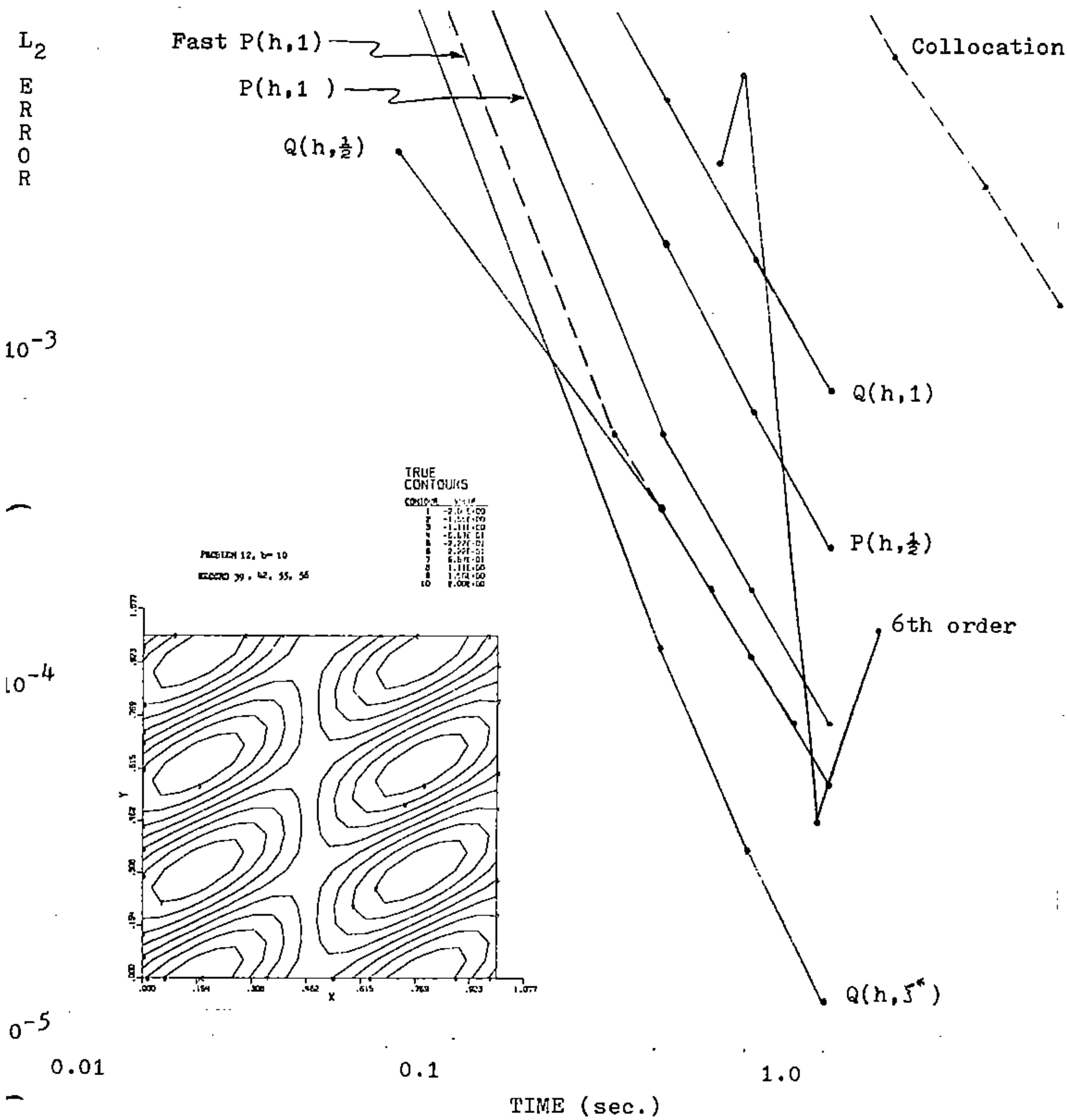
PROBLEM 17 - $u_{xx} + u_{yy} = G(x,y)$



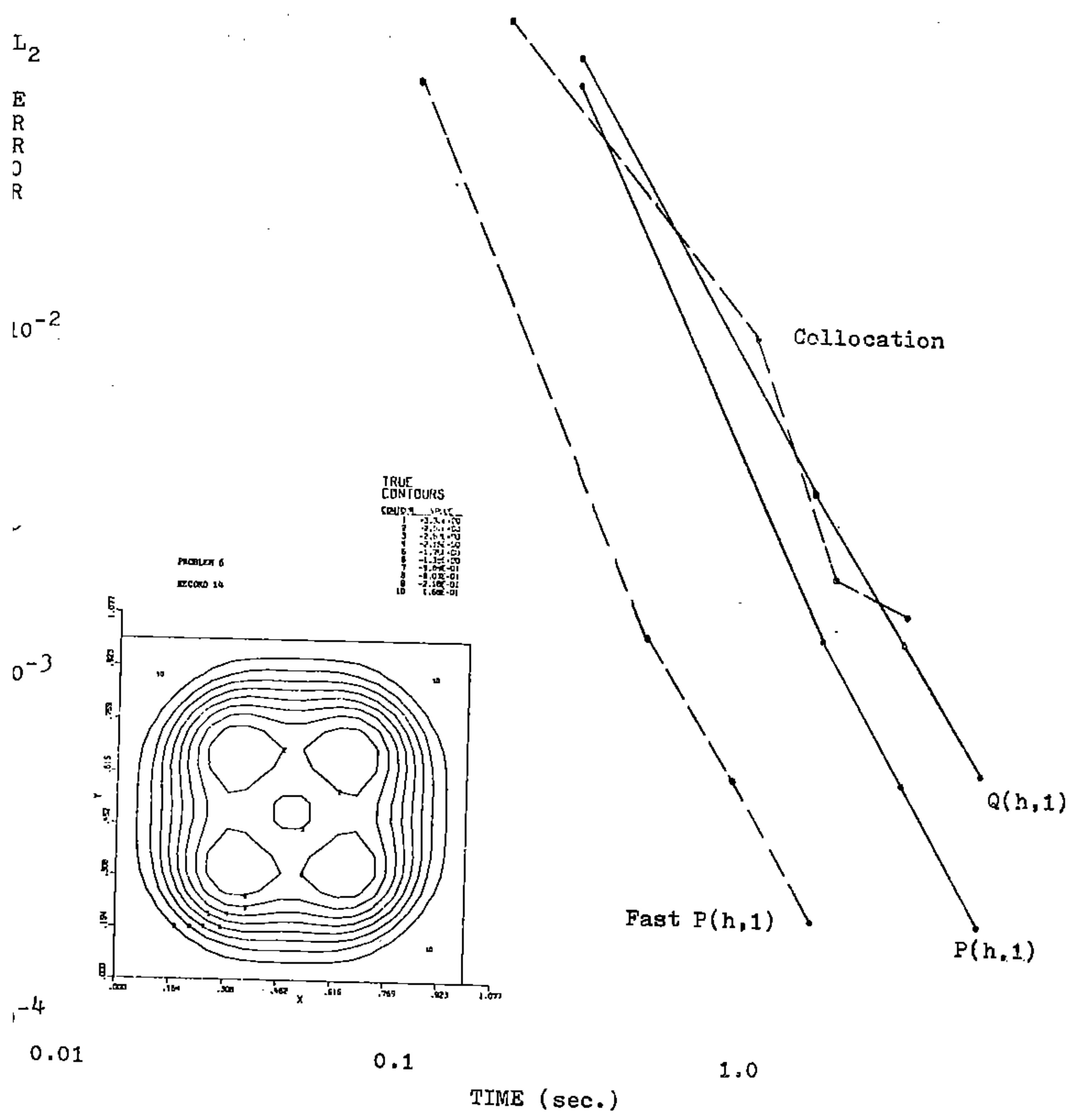
PROBLEM 38 - $u_{xx} + u_{yy} = G(x,y)$



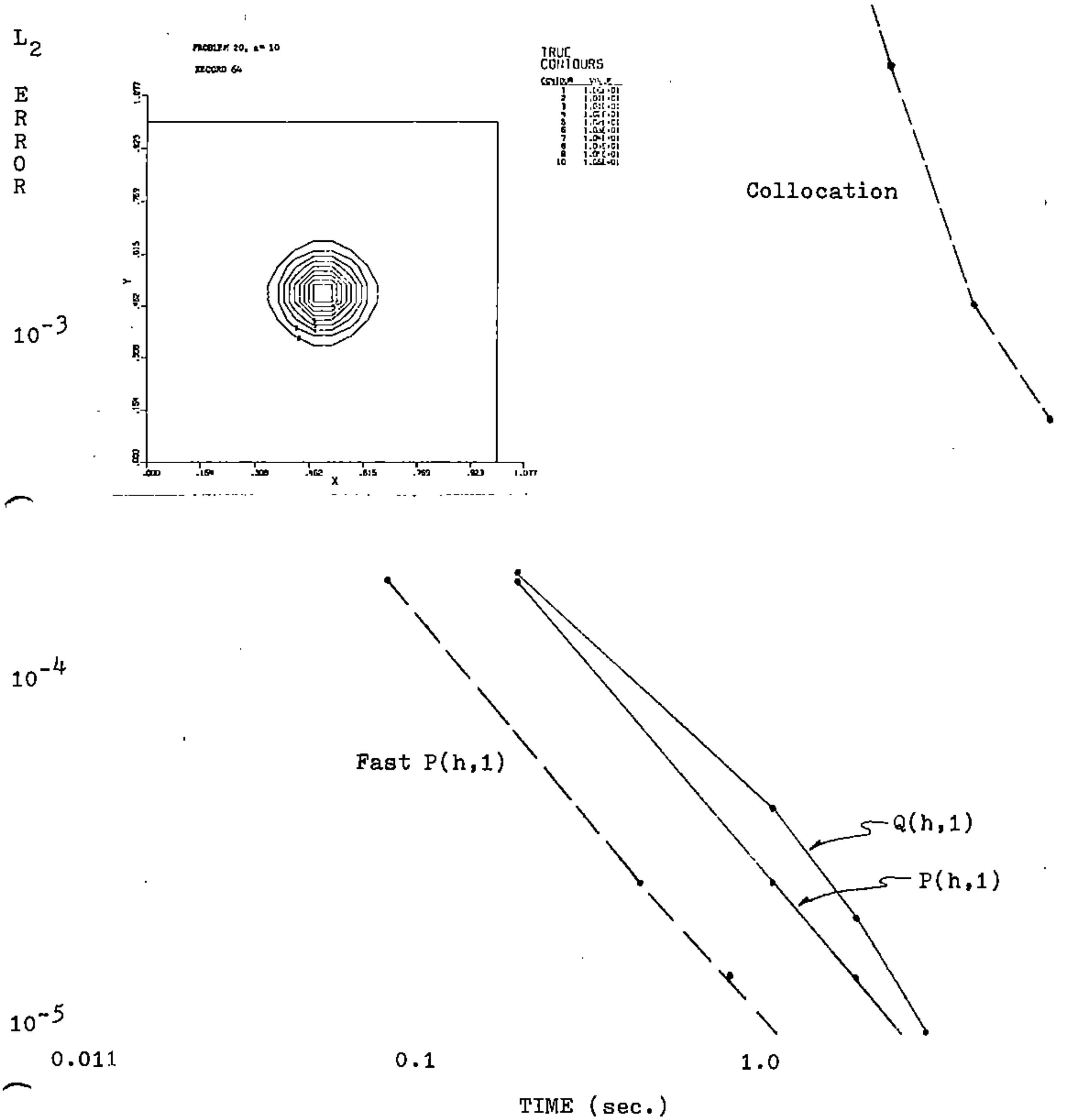
PROBLEM 41 - $u_{xx} + u_{yy} - 10u = G(x,y)$



PROBLEM 6 - $u_{xx} + u_{yy} + F(x,y)u = G(x,y)$



PROBLEM 20 - $u_{xx} + u_{yy} + F(x,y)u = G(x,y)$



7. Conclusions

We have shown that the choice of the auxiliary points in the HODIE method can significantly affect its performance. In particular, we see that, of the schemes tested, the one with the auxiliary point set Q(h,½) is the most efficient practical fourth order method for the equation $\nabla^2 u + Fu = G$ with constant F, its performance being nearly optimal in its class. In addition, we have found that the sixth order method presented here is quite attractive for problems in which moderate to high accuracy is required. We conjecture that, for F constant, an implementation of these methods in conjunction with the Fast Fourier Transform method of solving the resulting system of difference equations as in [1] will yield a method with even more efficiency for solving this class of problems.

For the case of variable F we have found the HODIE method with auxiliary point set P(h,1) superior to the alternative Q(h,1). For the set of problems presented here, the HODIE method performs better than the $\mathbb{P}_3C^1$ collocation method.

It is natural to ask whether the superiority of the HODIE methods with auxiliary points that are not all grid points extends to cases where F is variable or the principal part of the operator is not the Laplacian. In these cases more than five auxiliary points are required to attain fourth order accuracy. We have not yet treated this situation.

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