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# Lateral Moisture Movements In Unsaturated Anisotropic Media

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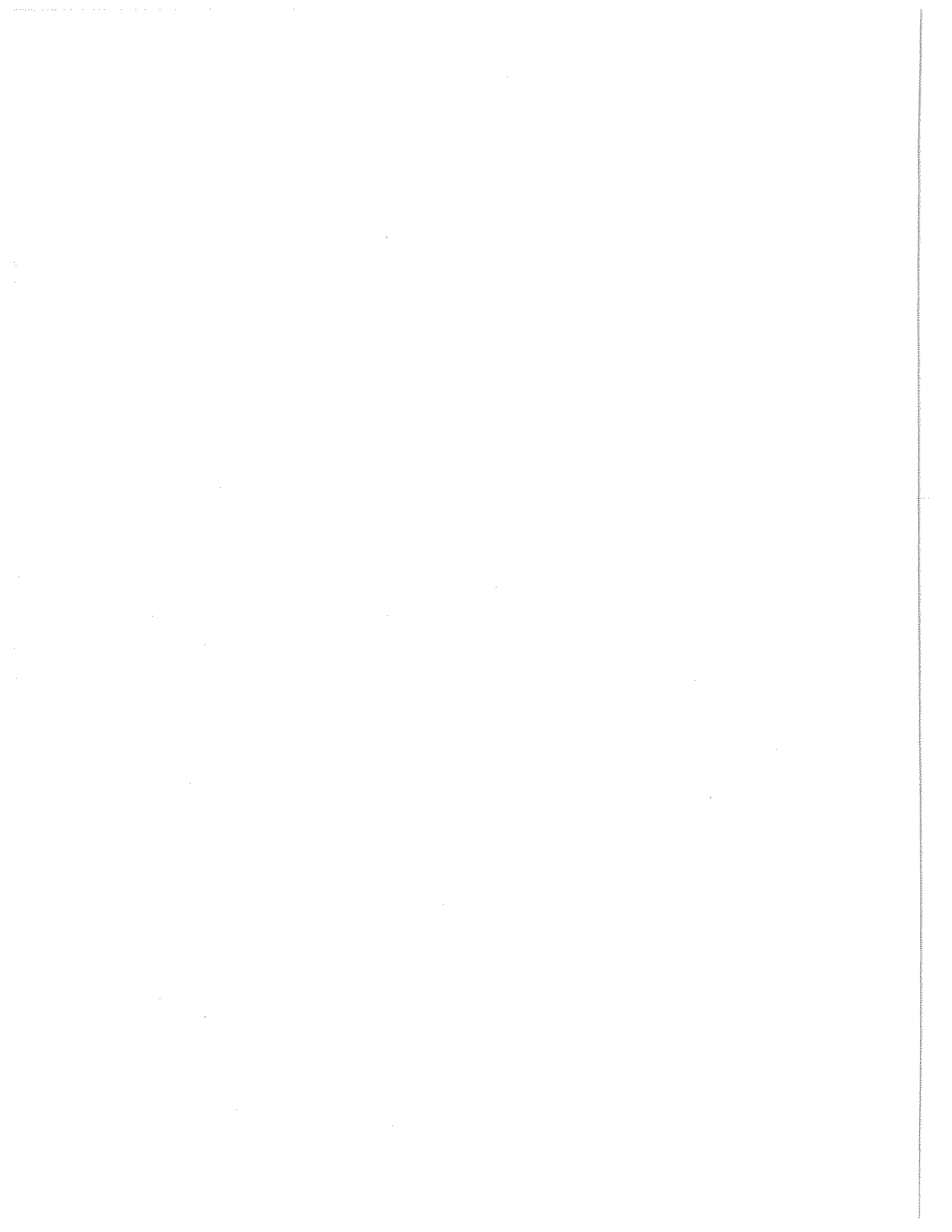
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Technical Report No. 169

LATERAL MOISTURE MOVEMENTS IN UNSATURATED  
ANISOTROPIC MEDIA

by

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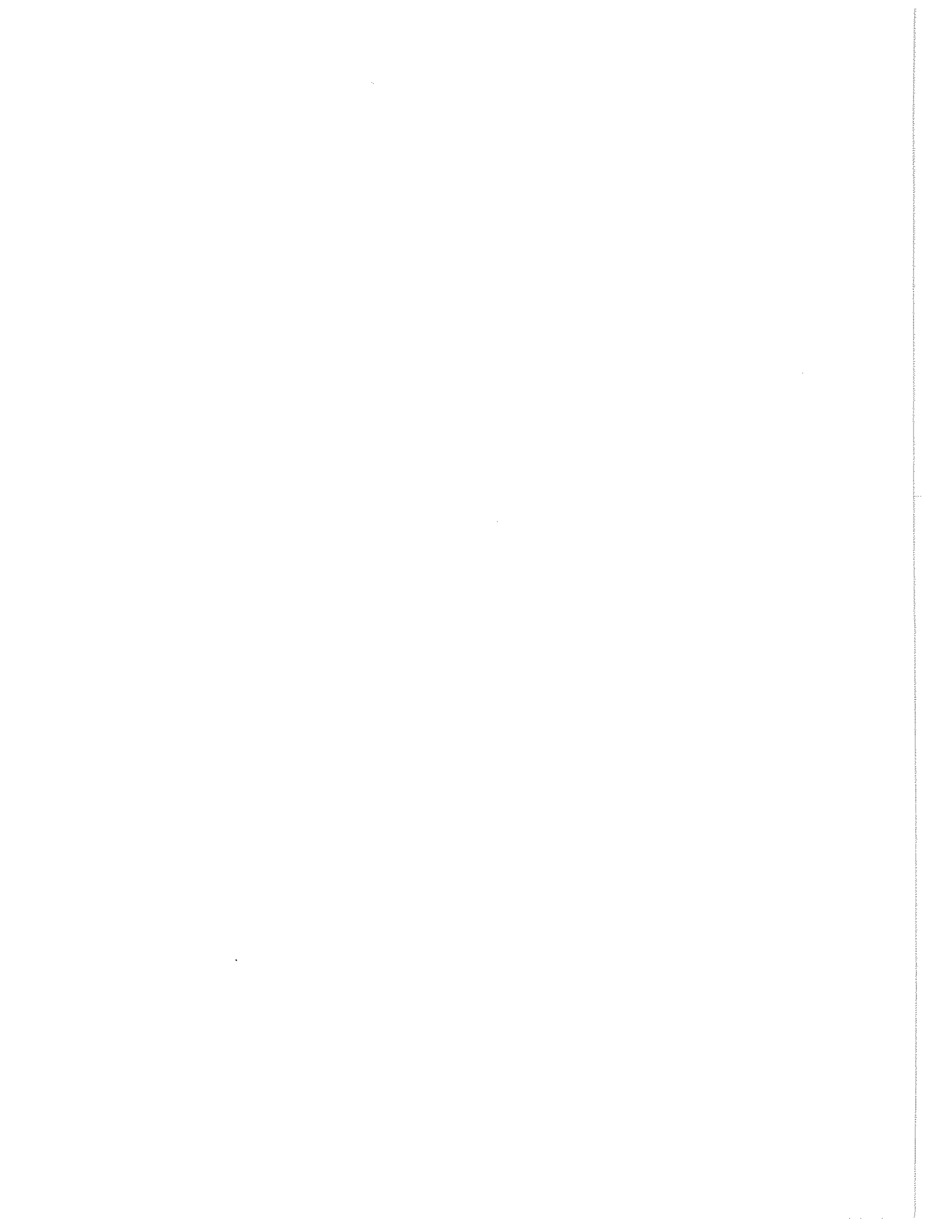
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## ABSTRACT

It is recognized that contamination of surface and subsurface water resources, both in the United States and abroad, is becoming a problem posing a serious threat to the health of our natural and human resources. Viewed in this light, any study which deepens our insight in the mechanisms responsible for the transportation and distribution of those contaminants increases the number of ways by which contamination of our environment can be controlled and prevented.

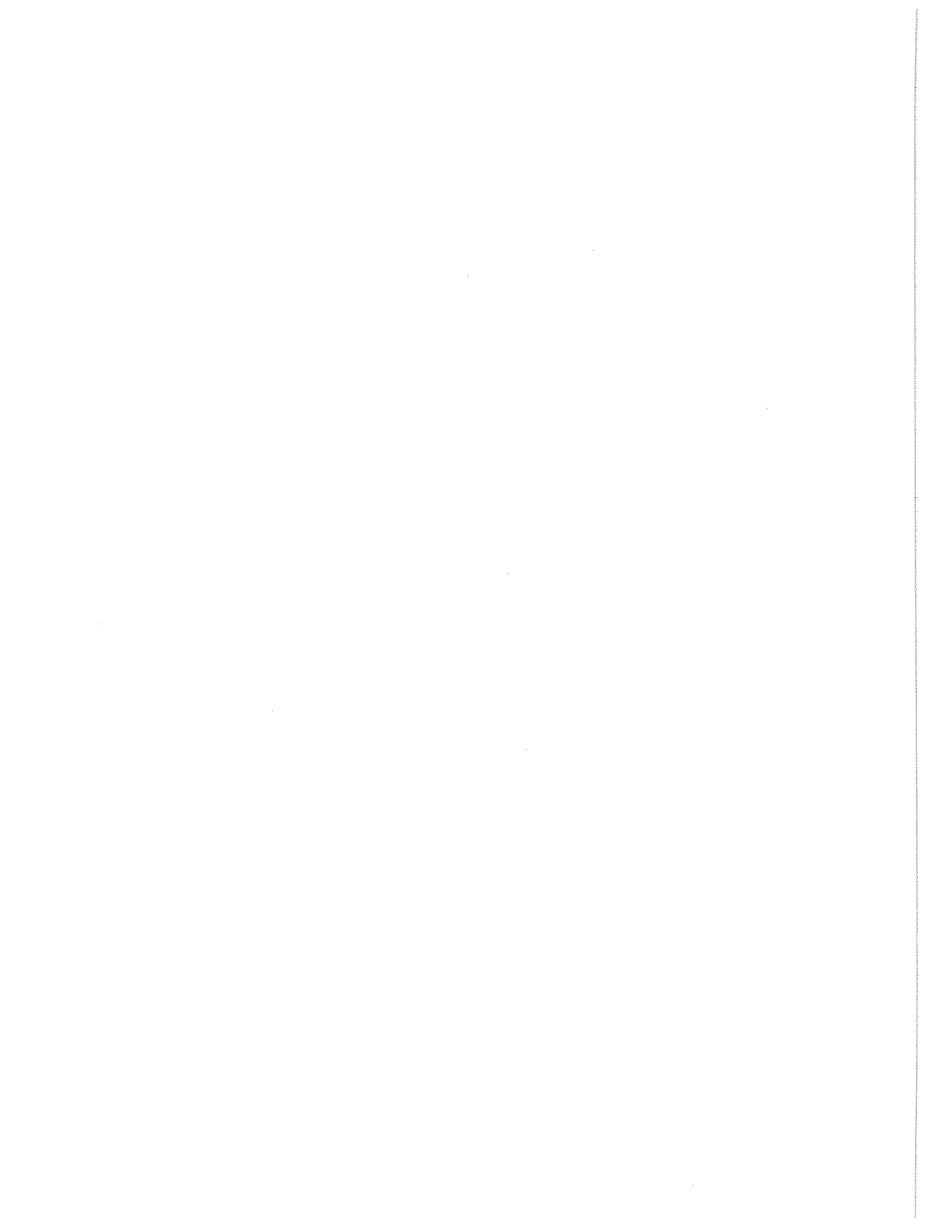
Traditionally, in the area of water resources, interests are focused on contamination of groundwater basins and surface water basins, and the subsequent distribution of those contaminants in our environment.

In the study presented in this report, it is emphasized that the contamination processes involve, additionally, the vadose zone below the soil surface. In this respect, the classical approach, which assumes only vertical infiltration, is abandoned and a more comprehensive theory which includes inclination of the soil surface and anisotropic hydraulic conductivity, is developed. For these more general conditions, an infiltration model is presented, which is in form and structure similar to the equation of solely vertical infiltration. It is shown that, under particular conditions which are chosen so as to emphasize the importance of the problem, lateral infiltration occurs and that it may be more predominant than vertical infiltration.

Because the proposed infiltration model is similar in form and structure to the vertical infiltration model, a numerical solution of the latter is presented by means of an explicit finite difference scheme. The boundary conditions constraining the solution are relaxed with respect to soil moisture content at the surface for the beginning of the infiltration process. This condition supports the philosophy underlying this study, namely the formulation of a general infiltration model as opposed to a simple vertical infiltration model.

## FOREWORD

Since groundwater contamination is becoming recognized as a serious problem both in the United States and abroad, and since the contamination processes involve the vadose zone below the soil surface, the question on whether infiltration may occur in directions other than vertical acquires a place of utmost importance. In this report it is shown that, under particular conditions, which are chosen so as to emphasize the importance of the problem, lateral infiltration occurs and it may be more predominant than vertical infiltration.



## CHAPTER I: INTRODUCTION

### 1.1 Problem Definition and Research Objectives

The phenomena related to surface runoff have been and still are the subject of many hydrologic studies. The primary objective is to obtain a better understanding of the underlying hydrologic and hydraulic mechanisms and then to develop a mathematical theory which is coherent with the phenomenological observations. According to the classical approach (e.g. Rouse, 1950, Horton, 1935, 1949), surface runoff can be explained by one of the following mechanisms:

- 1.) The precipitation rate exceeds the infiltration capacity of the soil, implying that at a lower rate of precipitation no runoff would occur;
- 2.) The groundwater table is built up by water infiltrating into the soil. When a hydraulic gradient is developed and when the soil surface is intersected, then an outflow or seepage is produced.

According to this formulation, lateral subsurface flow takes place only in the saturated zone of the soil and is part of the mechanism described under 2). It is known in the literature as the base flow component of the runoff hydrograph and is characterized by a large time-lag between the rainstorm and its (subsurface) runoff. In this respect it is interesting to know that time-lags with a magnitude in the order of 50-100 years have been observed.

In the hydrologic models developed thus far, it is tacitly assumed that the infiltration mechanism listed under 1) manifests itself only in the vertical direction. This implies the inconsequentiality of hydraulic gradients in directions other than the vertical. Extensive research in this respect has been done by e.g. Philip (1969).

However, when the soil is anisotropic with respect to its hydraulic conductivity the classical approach is highly objectionable. In fact, conditions of anisotropy are responsible for the creation of a lateral flow component in the unsaturated zone of the soil.

It has only recently been recognized that the lateral flow component in the unsaturated zone of the soil is an important element in the overland subsurface flow hydraulics, and therefore, should be incorporated in the mathematical models dealing with overland-subsurface flow phenomena. The complete theory for the overland-subsurface hydraulics, viewed in this light, is at present non-existent.

Zaslavsky and Sinai (1981) observed runoff during rainstorms under conditions in which the infiltration capacity of the soil was not exceeded by the intensity of the rainstorm. They explain this phenomenon by the existence of lateral flow components in the unsaturated zone of the soil. Also the concentration of water in concave parts of the topography are attributed to this particular component. In their attempt to arrive at a mathemati-

cal formulation of the lateral flow component, they simulate anisotropy by means of alternate layers of isotropic soils with different hydraulic conductivities. However, no general model is proposed.

The objectives of the research effort reported in the following chapters are:

- I. To present a general theory of the vadose zone hydraulics, with directionally variable soil hydraulic conductivity, with sloping soil surface.
- II. To generate elementary exact solutions which may be significant on their own or because they will constitute a test-bench for computational models.
- III. To generate numerical solutions for a number of elementary problems which are not amenable to analytical treatment.

The research described in this report has a direct impact on several problems related to the general area of water resource engineering. Obviously, a soil without directional preference in hydraulic conductivity is an idealized concept and more often we are dealing with stratified soils. Under these conditions, they exhibit almost always a directional preference to infiltration. Thus a theory that takes into account anisotropy of a soil to the process of infiltration is bound to have a strong impact on any engineering problem dealing with infiltration. In addition, it should be noted that soil stratification does not necessarily



imply layering parallel to the slope of the soil surface. In fact, in this study, layering parallel to the slope of the soil surface will be considered as a special case of the general condition where the direction of stratification makes an angle with the slope of the soil surface. Zaslavsky and Sinai (1981) considered only situations with soil layering parallel to the slope of the soil surface, and therefore, their model is very restricted. Special areas in water resource engineering which are relevant to the study reported here, are:

I. Groundwater contamination and transport of pollutants.

1. Tracking pollutants through the unsaturated zone of the soil to groundwater and/or surface streams.
2. Assessing the impact of the disposal of municipal and industrial wastes and effluents, including nuclear wastes, on groundwater basins and surface streams.
3. Evaluating sources of recharge of the principal aquifers.

While this study does not address the above mentioned items specifically, it is obvious that an overland-subsurface theory which takes soil anisotropy into account is to be preferred above the conventional hydrologic models. A more complete picture will be obtained with respect to the tracking of pollutants through the soil system and with respect to the transportation of water through the soil system to groundwater basins and overland stream flows.

II. Erosion prediction and control.

1. Predicting implications of new agricultural technology on surface and groundwater resources.
2. Locating areas in the general topography which are especially susceptible to erosion.

The qualification and quantification of the lateral flow component in the vadose zone of the soil, as developed in this study, has the potential of improving soil management and soil conservation techniques.

III. Adverse water resources impacts of energy production and mining.

1. Evaluation of drainage systems of lands which have been subjected to mining activities.

Two aspects are of importance with respect to this problem. The first aspect is that of drainage requirements, which can be formulated as the space-time distribution of rainfall-caused runoff, evaluated within the framework of a hydrologic model which includes the concept of directional preference of infiltration in the soil system. The second aspect is of design and can be formulated as the improvement of the hydraulic characteristics of reclaimed areas by judicious design of soil stratification and surface curvature. Because it has been shown that the soil curvature has a significant effect on the soil moisture distribution, the importance of the knowledge of the relationship between

the two quantities is related to the importance of the improvement of the hydraulic characteristics of the reclaimed areas.

More specifically, since stratified soils under concave surfaces tend to accumulate moisture and stratified soils under convex surfaces tend to behave as if they were impermeable, these elements can be exploited in the design of the land forms in order to achieve either of the two goals: increased moisture retention by layered concave regions or waterproofing by layered convex regions.

### 1.2 Organization of the Report

The organization of this report is as follows. Chapter II is a review of the general theory of infiltration. This includes relationships such as: soil moisture content - capillary pressure head, soil moisture content - soil moisture diffusivity. Concepts such as sorptivity, isotropy and anisotropy, highly significant to the theory of infiltration are also introduced here. Additionally, the basic mechanisms responsible for the transportation of water in the soil system are discussed.

In Chapter III the general formulation of infiltration into anisotropic soil structures is developed under general conditions with respect to soil slope and direction of soil stratification. The tensor formulation for hydraulic conductivity in unsaturated soils is introduced. In Chapter IV the general infiltration equation, developed in Chapter III, is solved numerically. In

Chapter V, the results obtained in Chapter IV are discussed, followed by suggestions and recommendations for future research.



## CHAPTER II: THE INFILTRATION PROBLEM

In this chapter the basic elements, indispensable for any discussion on the theory of infiltration of water into soils, are presented. The flow of water through porous media, e.g., soils, follows the empirical law of Darcy and will be discussed in its most general form, thereby using the tensorial form of the hydraulic conductivity. Because Darcy's law was originally developed for saturated soil moisture conditions, a modified form applicable to unsaturated soil moisture conditions will be included. This formulation forms the basis for all subsequent mathematical treatments of the infiltration phenomenon and depends strongly on the relationship between soil moisture content and the hydraulic conductivity of the soil. Important concepts such as soil moisture diffusivity and sorptivity are subsequently introduced.

The general formulation of Darcy's law is closely related to the concept of soil anisotropy. Because this study focuses on the infiltration phenomenon under anisotropic soil conditions as opposed to isotropic soil conditions, the difference between the two concepts is briefly explained for the sake of completeness, although many textbooks provide excellent definitions (e.g. Muskat 1937, Maasland 1957, Bear 1972).

Isotropy refers to the situation where the soil does not exhibit any directional preference in hydraulic conductivity, as opposed to anisotropy where there is a definite directional

preference in the hydraulic conductivity of the soil. This should not be confused with non-homogeneity, which refers to the spatial variation of the hydraulic conductivity. If no spatial variation of hydraulic conductivity is present, the soil is called homogeneous. Thus a perfectly homogeneous soil may have a tendency to conduct more in one direction than in another, and is therefore anisotropic. On the other side, a non-homogeneous soil may be locally isotropic everywhere. Anisotropy can be expressed only by considering hydraulic conductivity as a tensor and not as a scalar, as it is the case for isotropic soils. Thus anisotropy of a (homogeneous or non-homogeneous) soil is accounted for by (spatially invariant or variable) tensor conductivity. In the subsequent sections of this chapter, these concepts will be explored further, where unsaturated soil moisture conditions are emphasized.

### 2.1 Darcy's Law

In 1856, the Frenchman Darcy conducted a series of experiments for the public water supply system of the city of Dijon, concerning the flow of water through a column of sand. Darcy concluded from his experiments (Figure 2.1) that the flow rate  $Q$  (volume per unit of time)

- is proportional to the constant cross-sectional area  $A$
- is proportional to  $\phi_1 - \phi_2$
- is inversely proportional to the length  $L$  of the sand column. This proportionality can be expressed mathematically in the form

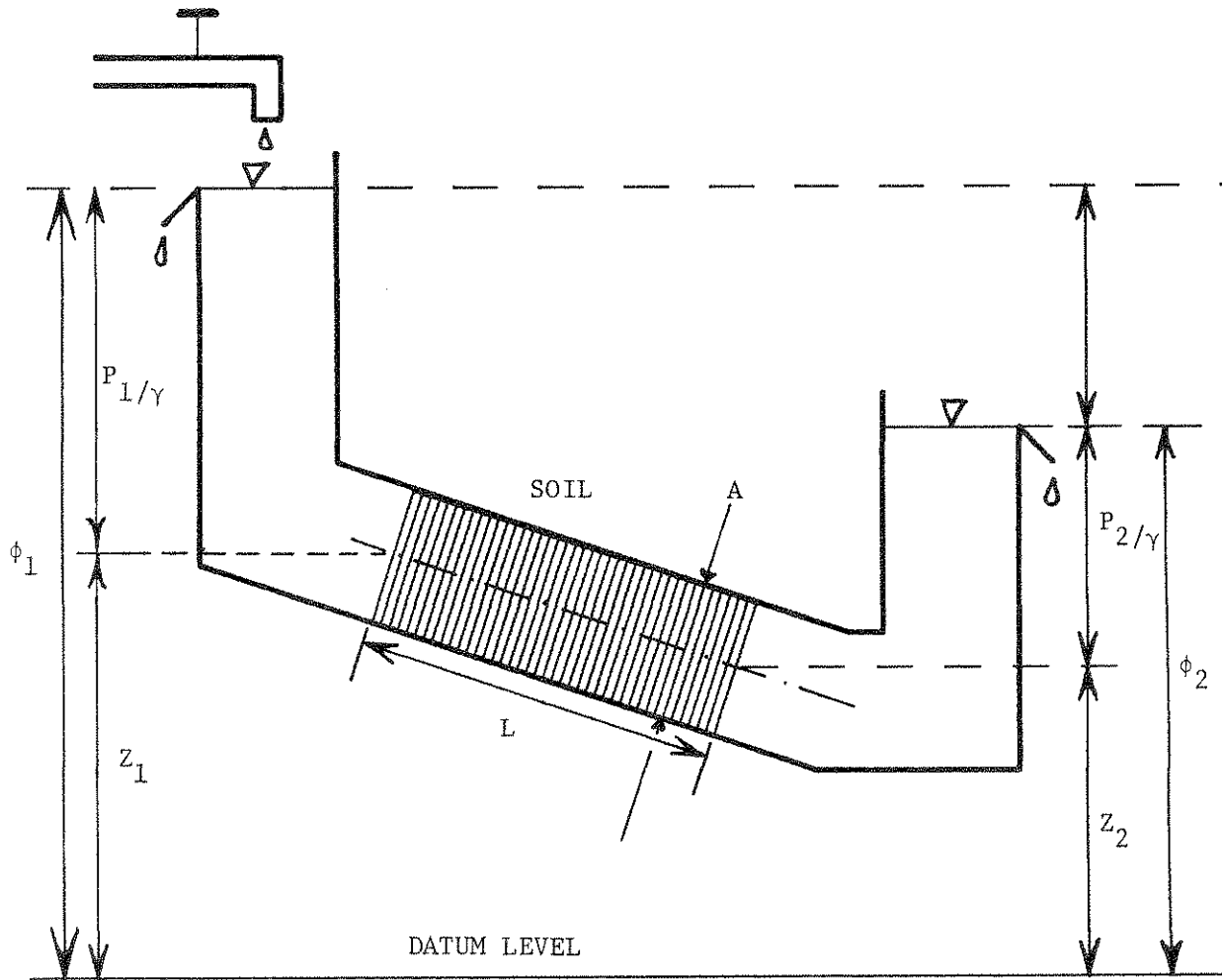


Figure 2.1. Sketch of experimental apparatus for demonstration of Darcy's Law.



$$Q = -KA(\phi_2 - \phi_1)/L \quad (2.1)$$

where

$\phi_1, \phi_2$  = piezometric head at sections 1 and 2, [L],

L = the length of the sand column, [L],

A = the cross-sectional area of the sand column, [L<sup>2</sup>],

K = a proportionality constant, called the coefficient of permeability or hydraulic conductivity, [L/T],

Q = discharge, [L<sup>3</sup>/T].

Although Darcy performed his experiment on a vertical column of sand, his formula may be extended to flow through an inclined porous medium as shown in Figure 2.1. The term  $(\phi_2 - \phi_1)/L$ , representing the change of the piezometric head per unit of distance along the column of sand, is known as the hydraulic gradient and will be denoted by the symbol J. Therefore, Darcy's law states

$$q = -KJ \quad (2.2)$$

where

q = discharge per unit cross-sectional area or specific discharge vector, [L/T]

In its original form, Darcy's law applies only to one-dimensional flow in a homogeneous and isotropic medium under saturated soil conditions.

For three-dimensional flow, we describe the specific

discharge vector  $\bar{q}$  by its three components  $q_x$ ,  $q_y$ , and  $q_z$  in the x, y, and z directions, respectively, of the Cartesian coordinate system. Generalization of Darcy's law gives us

$$\bar{q} = -K\bar{J} = -K\bar{\nabla}\phi \quad (2.3)$$

where

$\bar{\nabla}\phi$  is the hydraulic gradient vector with components  $J_x$ ,  $J_y$ , and  $J_z$ . For a homogeneous and isotropic medium K is a scalar constant and, accordingly,  $\bar{q} \equiv (q_x, q_y, q_z)$  may be written as

$$\begin{aligned} q_x &= -KJ_x & J_x &= \partial\phi/\partial x \\ q_y &= -KJ_y & \text{where } J_y &= \partial\phi/\partial y \\ q_z &= -KJ_z & J_z &= \partial\phi/\partial z \end{aligned} \quad (2.4)$$

and  $\phi = p/\gamma + z =$  piezometric head, or the sum of the pressure head ( $p/\gamma$ ) and elevation head ( $z$ ),

where

$\gamma =$  specific weight of water,  $[F/L^3]$ ,  
 $p =$  pressure,  $[F/L^2]$ .

In this form, we can apply Darcy's law also to non-homogeneous and isotropic media by expressing the spatial variability of the hydraulic conductivity as  $K = K(x, y, z)$ . Not many media are isotropic with respect to their hydraulic conductivity. In stratified soils, for instance, the water movement parallel to the soil layers encounters much less resistance to flow than the water movement perpendicular to the layers.

In 1957, E.C. Childs prefaced his paper on anisotropic hydraulic conductivity of soil with the following statement. "It has long been recognized that soils may have the property of anisotropic hydraulic conductivity, that is to say they may conduct water more readily in certain directions than in others. The absence until recently of a ready means of measuring anisotropic conductivity in the field has resulted in a situation where on the one hand speculation tended to lay great emphasis on the frequency and magnitude of anisotropy and yet, on the other hand, quantitative practical matters such as the design of drainage systems have inevitably proceeded without any reference to such a property."

This statement could be repeated now, mutatis mutandis, for the situation of the hydraulic conductivity of unsaturated anisotropic soils, with a curious twist; on the one hand there exists a strong practical urgency for the use of anisotropic hydraulic conductivity in unsaturated flows to better explain lateral subsurface flows on hillslopes (Atkinson, 1978; Zaslavsky and Sinai, 1981), and yet, on the other hand, no recent literature can be found that presents actual solution, however elementary of the differential equations provided by theory.

As for E.C. Childs' statement, while true at the time of its writing, it was rendered moot by the appearance of a lucid study by Maasland (1957) on the effects of soil anisotropy on land drainage systems design. This study has been more or less directly the model for all subsequent treatises of textbooks

which deal with flow through saturated anisotropic soils, and it contains the most extensive bibliography of the contemporary state-of-the art on the flow of water in saturated anisotropic soils. In the following presentation, we will draw substantially from Maasland's (1957) study as far as references presentation is concerned. Maasland's (1957) approach to the solution of applied problems relies on the fact that the equation of motion of a liquid in a homogeneous anisotropic porous medium, when written in Cartesian coordinates coincident with the principal directions of anisotropy, can be reduced to the Laplace equation by suitable deformation of each coordinate. Maasland presents his theory of fluid flow through anisotropic media in the form of five general theorems which are used throughout his study and in the applications. We will present the theorems here in their original formulation, since doing so will simplify their development's attributions in the annotated references.

Theorem I. A porous medium, consisting of any number of arbitrarily directed sets of parallel, elementary flow tubes, can always be replaced by an equivalent, fictitious, porous medium of equal size with three, mutually perpendicular, uniquely directed systems of pore tubes. In this fictitious medium, the net flow per unit area is the same in every direction as in the actual medium, provided that the hydraulic head is the same everywhere in the fictitious medium as in the actual medium.

Theorem II. The effect of an anisotropy in the hydraulic

conductivity is equivalent to the effect of shrinkage or expansion of the coordinates of a point in the flow system. That is, one can, by suitably shrinking or expanding the coordinates of each point in an anisotropic medium, obtain an equivalent, homogeneous, isotropic system.

Theorem III. The hydraulic conductivity,  $k$ , for the equivalent homogeneous isotropic medium into which the anisotropic medium may be expanded or shrunk is related to the hydraulic conductivities of the actual anisotropic system by the relation

$$k = (k_x k_y k_z / k_0)^{1/2},$$

where  $k_0$  is an arbitrary constant, and  $k_x$ ,  $k_y$ , and  $k_z$  are the hydraulic conductivities for the principal directions of the actual anisotropic medium.

Theorem IV. If the square root of the directional hydraulic conductivity (that is, the hydraulic conductivity in the flow direction) is plotted in all the corresponding directions at a point of an anisotropic medium, then one obtains an ellipsoid; that ellipsoid is called the ellipsoid of direction.

Theorem V. The equipotentials in an anisotropic medium are conjugate to the flow lines with regard to the

As it is clear from the above statements, the theory of fluid flow through anisotropic media that leads to them does not utilize a general tensor formulation, and it is therefore of little use for generalization to moisture flows in unsaturated

anisotropic soils. Toward this end the theory developed by Ferrandon (1948, 1954) is far more useful and it has historically fathered that.

The most significant contribution to the theory of fluid flow through anisotropic saturated media are those of Versluys (1915), Samsioe (19), Schaffernak (1933), Dachler (1933, 1936), Vreedenburgh (1935, 1936, 1937), Vreedenburgh and Stevens (1936), Muskat (1937), Aravin (1937), Yang (1948, 1949, 1953), Ferrandon (1948, 1954), Ghizetti (1949), Litwiniszyn (1950), Irmay (1951), Schneebeili (1953), Scheidegger (1954, 1955, 1960), Maasland and Kirkham (1955), Hall (1956), Edwards (1956), Childs (1957a, 1957b), Maasland (1957), Liakopoulos (1962, 1965a, 1965b), Bear (1972), Bouman (1979), and Falade (1981).

The relationship between micro-stratification and anisotropy has been explored experimentally and theoretically by Dachler (1933), Schaffernak (1933), Vreedenburgh (1937), and Maasland (1957) who has furthermore presented the law of refraction in two anisotropic media. Stevens (1936, 1938) has given a general example of the transformation of an anisotropic two-layer system and he has developed an electrical analog for seepage problems.

The factors affecting soil micro-stratification, be it natural or artificial, have been analyzed by Graton and Frazer (1935), Frazer (1935), Russell and Taylor (1937), Dapples and Rominger (1945), Johnson and Hughes (1948).

Other early research on anisotropic porous media has been performed by researchers of the textile industry like Fowler and Hertel (1940), Sullivan and Hertel (1940), and Sullivan (1941).

As for measurement techniques and actual measurements in the field or in the laboratory, evidence of directional preference (usually horizontal) have been found by Thiem (1907), Fraser (1935), Muskat (1937), Russell and Taylor (1937), Reeve and Kirkham (1951), Johnson and Breston (1951), Childs (1952), Childs, Cole, and Edwards (1953), Yang (1953), DeBoodt and Kirkham (1953), Maasland and Kirkham (1955, 1959), Childs, Collis-George, and Holmes (1957), Marcus (1962), Wilkinson and Shipley (1972), and Irmay (1980).

For the case of fluid flow in unsaturated anisotropic porous media, the literature is very scarce and is limited to the contribution of Liakopoulos (1964), Childs (1969), Burejev and Burejeva (1969), Whistler and Klute (1969), Cisler (1972), Shul'gin (1973), Sawhney, Parlange, and Turner (1976), Dirksen (1978), Akan and Yen (1981), Yeh and Gelhar (1982), and Yen and Akan (1983). On the other side, in the closely related field of dispersion in porous media, work done with the tensor formulation is rather substantial. It suffice to mention here the contributions of Bear (1961), Scheidegger (1961), De Josselin De Jong (1961), and Bachmat and Bear (1964).

In order to account for the possibilities of the soil's anisotropic properties, Darcy's law should be written as

$$q_i = -K_{ij} J_j \quad (2.5)$$

where

$$K_{ij} = \begin{vmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{vmatrix}, \quad (2.6)$$

and

$$J_j = \begin{vmatrix} J_x \\ J_y \\ J_z \end{vmatrix}. \quad (2.7)$$

The equation (2.5) represents the following system:

$$\begin{aligned} q_x &= -K_{xx} J_x - K_{xy} J_y - K_{xz} J_z \\ q_y &= -K_{yx} J_x - K_{yy} J_y - K_{yz} J_z \\ q_z &= -K_{zx} J_x - K_{zy} J_y - K_{zz} J_z. \end{aligned} \quad (2.8)$$

It was shown by e.g. Childs (1957) and Liakopoulos (1965), that the hydraulic conductivity tensor given by (2.6) is symmetric, implying  $K_{xy} = K_{yx}$ ,  $K_{xz} = K_{zx}$ , and  $K_{yz} = K_{zy}$ . Thus, of the nine components of  $K$  in three-dimensional space, only six are different and, of the four components of  $K$  in two-dimensional space, only three are different. Darcy's law for homogeneous and anisotropic soils may easily be extended to the non-homogeneous case by using  $K_{ij} = K_{ij}(x, y, z)$ . It is emphasized again that formulation (2.5) applies only to saturated media.



## 2.2 Validity of Darcy's Law

Darcy's law indicates a linear relationship between the specific discharge vector  $\bar{q}$  and the hydraulic gradient vector  $\bar{J}$ . Many investigators have shown that, with increasing specific discharge, this linear relationship becomes invalid. Because the phenomenon of flow through a porous medium is that of a fluid moving through the small pores between solid particles, this suggests a comparison with the flow of a fluid through a tube. A well-known law, describing the steady movement of viscous fluid through a simple, straight capillary tube of diameter  $d$ , is the Hagen-Poiseuille law. According to this law, the average velocity in the tube  $q$  is given by

$$q = - \frac{d^2}{32} \frac{\rho g}{\mu} \frac{dh}{ds}$$

where

$\frac{dh}{ds}$  = hydraulic gradient

$d$  = diameter of the capillary tube,

$\rho$  = fluid's density

$\mu$  = fluid's dynamic viscosity

$g$  = acceleration due to gravity.

If we consider the quantities  $d$ ,  $\rho$ ,  $g$ , and  $\mu$  as constants, and we write  $\frac{dh}{ds}$  as  $J$ , we obtain an expression  $q = -cJ$ , which is similar to Darcy's law. The Hagen-Poiseuille formula is based on the assumption that the flow is laminar. This is the case if a certain dimensionless parameter, the Reynolds number, does not exceed a critical value. Beyond this critical value, turbulent

effects will dominate and considerable deviations from the laminar flow equation will result. Because the same phenomenon can be expected in flow through porous media, it seems appropriate to define the validity of Darcy's law by means of the Reynolds number. For porous media, the Reynolds number is defined as

$$Re = qd/v, \text{ where}$$

$q$  = specific discharge

$v$  = kinematic viscosity of the fluid

$d$  = some characteristic length of the porous medium; for soils, this may be the average grain diameter.

Experiments have shown that Darcy's law is valid as long as the Reynolds number, as defined above, does not exceed some value between 1 and 10.

### 2.3 The Permeability of Porous Media

The coefficient of proportionality  $K$ , as it appears in the various forms of Darcy's law, is called the coefficient of permeability or hydraulic conductivity. Its dimension is length/time and it expresses the ease with which a particular fluid flows through a certain porous medium. Therefore, it depends on both the fluid and solid matrix properties. The fluid properties influencing the value of  $K$  are its density  $\rho$  and its kinematic viscosity  $v$ . The properties of the solid matrix involved are pore-size distribution, shape of the grains or pores, tortuosity, specific surface and porosity. From dimensional analysis, it can be shown that the hydraulic conductivity

can be expressed as  $K = k\rho g/\mu$  where the term  $\rho g/\mu$  represents the effect of the fluid properties and  $k$  represents the combined effect of the porous medium matrix. This factor is called the intrinsic permeability and is a property of the medium only. Its dimension is  $L^2$ . It is this factor which is responsible for anisotropic behavior of the medium. Many attempts have been made to find correlations between the intrinsic permeability and the properties of the porous medium matrix. The simplest way to attempt to establish correlations, theoretically, is to represent the medium by a model, which can be treated mathematically. Without going into detail, a number of these models are listed below:

---Capillary tube models

- \* Straight capillary tube models
- \* Parallel capillary tube models
- \* Serial capillary tube models
- \* Hydraulic radius models
- \* Network models

---Fissure models

---Resistance-to-flow models

---Averaging the Navier-Stokes Equation

---Ferrandon's model.

#### 2.4 The Principal Directions

Given the system (2.8), it is possible (by means of a rotation in space of the  $x,y,z$  coordinate axes) to obtain a new sys-

tem  $x'$ ,  $y'$ ,  $z'$  such that

$$\begin{aligned}q_{x'} &= -K'_{xx} J_{x'} \\q_{y'} &= -K'_{yy} J_{y'} \\q_{z'} &= -K'_{zz} J_{z'}\end{aligned}\tag{2.9}$$

where  $K'_{xx}$ ,  $K'_{yy}$ , and  $K'_{zz}$  are hydraulic conductivities in the directions  $x'$ ,  $y'$ ,  $z'$ .

Obviously, in the system given by (2.9),  $K'_{xy} = K'_{xz} = K'_{yx} = K'_{yz} = K'_{zx} = K'_{zy} = 0$ . It is said that  $x'$ ,  $y'$ , and  $z'$  form the principal directions. The physical consequence is that a gradient in any of the principal directions produces a specific discharge vector only in that direction. Thus, values of the hydraulic conductivity may be specified for those directions and, if they are different in each direction, the medium is anisotropic. It is obvious that for such a medium a gradient in a direction other than the principal directions produces a specific discharge vector in a direction other than the gradient vector. Restricting ourselves in what follows to two-dimensional flow analysis ( $x, z$ ), which is the focus of our attention in this study, this fact is demonstrated in Figure 2.2.

Generalizing now our discussion, we let the  $(x', z')$ -system be any orthogonal coordinate system (not necessarily principal system), which is rotated over an angle  $\phi$  with respect to the  $(x, z)$ -system. Then the following relationships hold for the components of the hydraulic conductivity tensor in both systems:

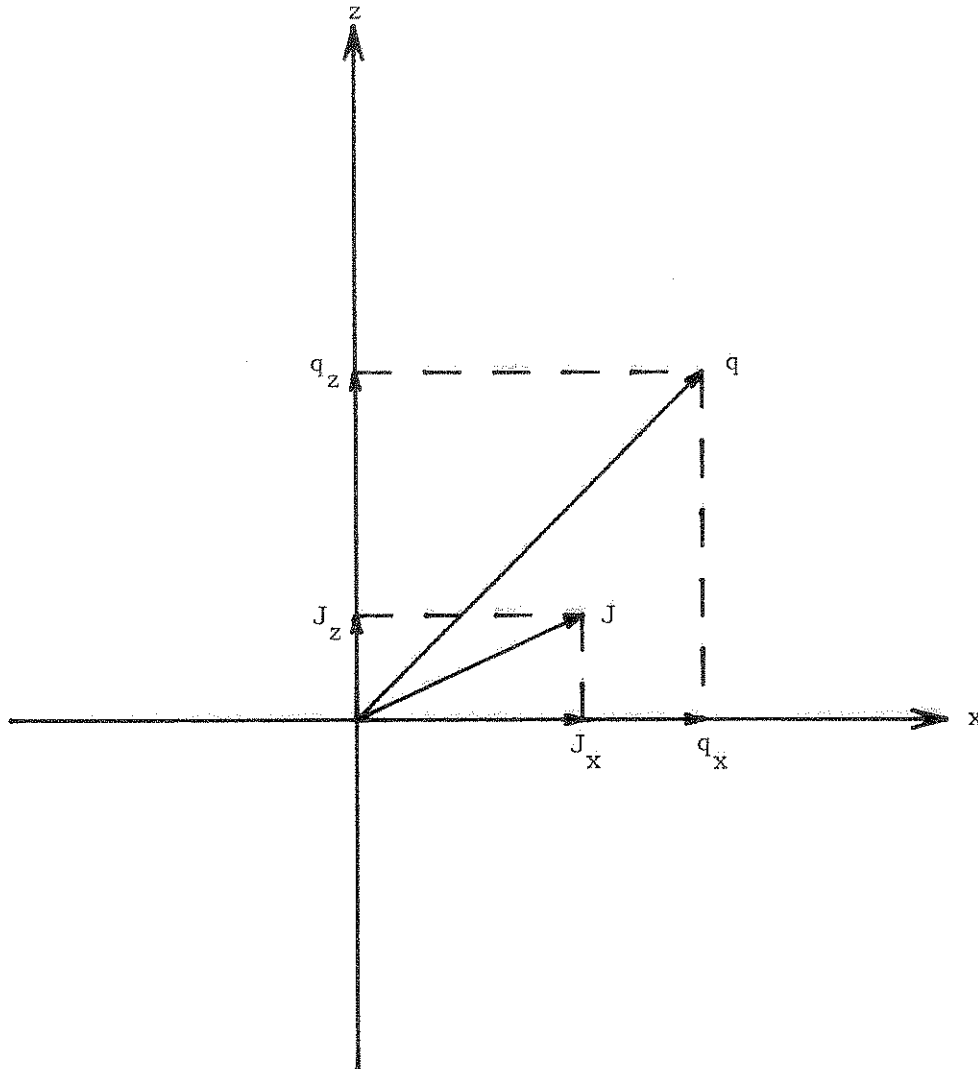


Figure 2.2. Direction of piezometric head gradient and specific flow rate vector in anisotropic soils.

$$K'_{xx} = \frac{K_{xx} + K_{zz}}{2} + \frac{K_{xx} - K_{zz}}{2} \cos 2\phi + K_{xz} \sin 2\phi$$

$$K'_{zz} = \frac{K_{xx} + K_{zz}}{2} - \frac{K_{xx} - K_{zz}}{2} \cos 2\phi - K_{xz} \sin 2\phi \quad (2.10)$$

$$K'_{xz} = K'_{zx} = -\frac{K_{xx} - K_{zz}}{2} \sin 2\phi + K_{xz} \cos 2\phi$$

If  $(x, z)$  is the principal system, (2.10) changes by letting  $K_{xz} = 0$ . If, however,  $(x', z')$  is the principal system, but  $(x, z)$  not, then the angle  $w$  between the  $(x', z')$ -system and the  $(x, z)$ -system is given by

$$\tan 2\phi = 2K_{xz} / (K_{xx} - K_{zz}) \quad (2.11)$$

Consequently, (2.10) changes into:

$$K'_{xx} = \frac{K_{xx} + K_{zz}}{2} + \left| \left| \frac{K_{xx} - K_{zz}}{2} \right|^2 + K_{xz}^2 \right|^{1/2}$$

and

$$K'_{zz} = \frac{K_{xx} + K_{zz}}{2} - \left| \left| \frac{K_{xx} - K_{zz}}{2} \right|^2 + K_{xz}^2 \right|^{1/2} \quad (2.12)$$

For an isotropic medium,  $K_{xx} = K_{yy} = K_{zz} = K$ , and every direction is principal direction.

The transformation of the hydraulic conductivity tensor from one system to another can conveniently be done by making use of Mohr's circle. The principle of this technique, for the case given by (2.12), is illustrated in Figure 3.3.

As stated before, in an anisotropic medium, the specific discharge vector  $\bar{q}$  and the gradient vector  $\bar{J}$  are not parallel,

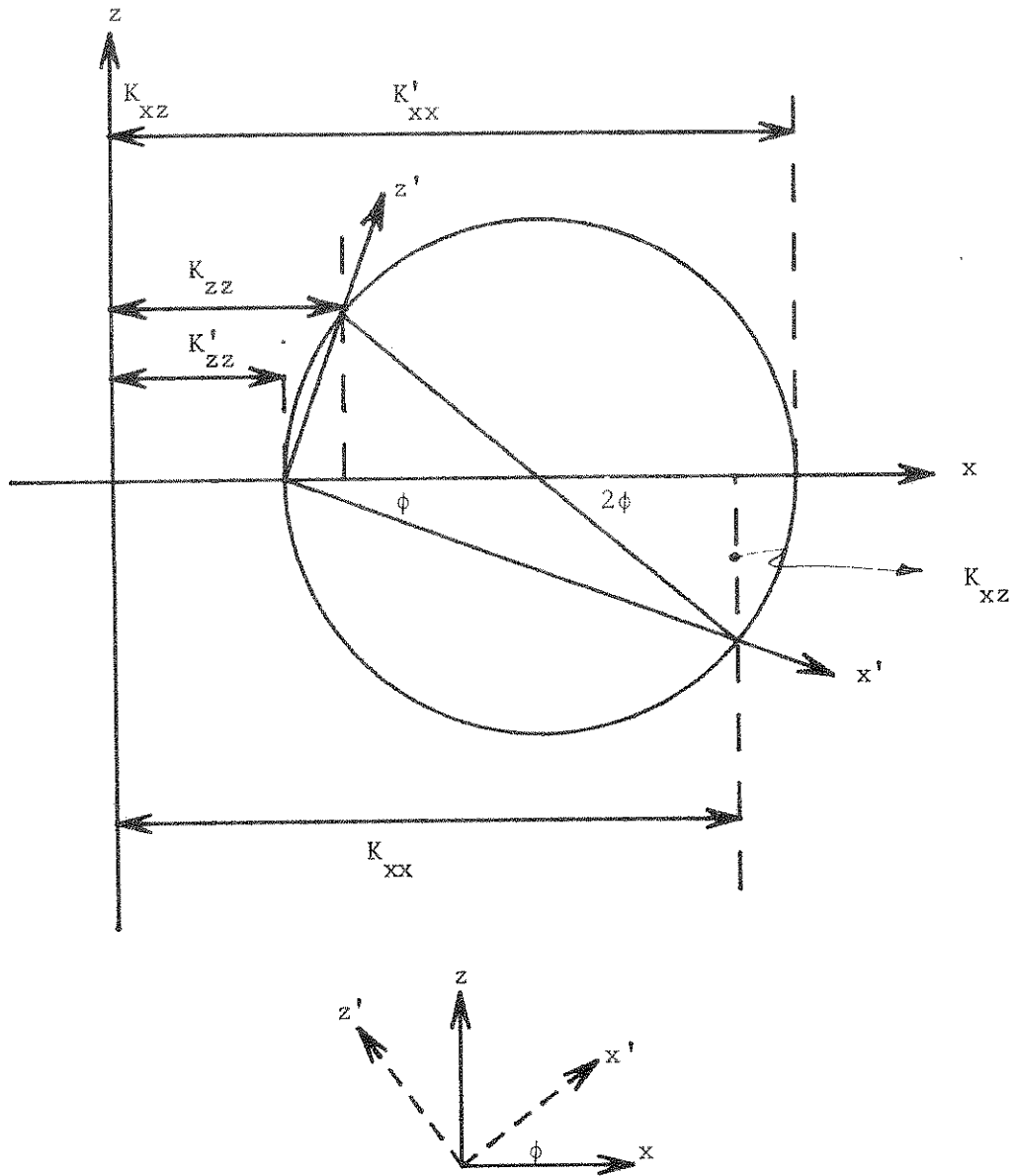


Figure 2.3. Mohr's circle for determination of hydraulic conductivity.

except for the case when they are in one of the principal directions. This gives rise to two important cases, which are briefly discussed below. For a more detailed discussion on this subject as well as the application of Mohr's circle to transform the components of the hydraulic conductivity tensor, the reader is referred to Bear (1972).

### Case 1

Suppose that the direction of the hydraulic gradient vector  $\bar{J}$  is known, one may wish to determine the hydraulic conductivity in this direction. According to Darcy's law one can define the hydraulic conductivity  $K_J$  in the direction of  $\bar{J}$  as

$$K_J = |q| \cos \alpha / |J| \quad (2.13)$$

where  $\alpha$  is the angle between the directions of the vectors  $\bar{q}$  and  $\bar{J}$ . According to this stipulation, one can derive the relationship

$$\frac{x^2}{1/K_{xx}} + \frac{z^2}{1/K_{zz}} = 1, \quad (2.14)$$

where  $\sqrt{1/K_{xx}}$  and  $\sqrt{1/K_{zz}}$  are the semi-axes of an ellipse. Using a graphical technique, one can derive the magnitude of  $K_J$ .

### Case 2

Suppose that the direction of the specific discharge vector  $\bar{q}$  is known. In a way similar to Case 1, one can obtain the magnitude of the hydraulic conductivity  $K_a$  in the direction of  $q$ .



Defining

$$K_q = \frac{|q|}{|J| \cos \alpha} \quad (2.15)$$

where  $\alpha$  is the angle between the vectors  $\bar{q}$  and  $\bar{J}$ , one can obtain the relationship

$$\frac{x^2}{K_{xx}} + \frac{z^2}{K_{zz}} = 1. \quad (2.16)$$

Equation (2.15) represents an ellipse with semi-axes  $\sqrt{K_{xx}}$  and  $\sqrt{K_{zz}}$ . Again, using a graphical technique, the magnitude of  $K_q$  can be obtained.

## 2.5 Unsaturated Soil Moisture Conditions

### 2.5.1 The $\phi$ - $\theta$ Relationship

In section 2.1, the piezometric head  $\phi$ , was defined as  $\phi = p/\gamma + z$ , the sum of the pressure head and elevation head. This applied to saturated porous media. We extend this definition here to account for conditions of non-saturation:

$$\phi = \psi + z \quad (2.17)$$

where

$\psi = p/\gamma =$  pressure head, [L]

$z =$  elevation head, [L]

$\phi =$  matrix flux potential, [L].

The following cases can be distinguished:

$p > 0 \quad \rightarrow \quad \theta = \theta_s, \text{ below groundwater table}$

$p = 0 \quad \rightarrow \quad \theta = \theta_s, \text{ at groundwater table}$

$p < 0 \quad \rightarrow \quad \theta < \theta_s, \text{ above groundwater table}$

where

$\theta = \text{soil moisture content, usually on a volumetric basis, } [L^3/L^3]$

$\theta_s = \text{soil moisture content at saturation, } [L^3/L^3].$

Thus the relationship (2.17) covers the whole range of possible conditions encountered in a soil system. It is obvious that, when  $p < 0$ , the physical process involved is one of suction. In other words, the soil exerts a force on the moisture in it and the quantity  $\psi$  is then referred to as capillary head, although capillary pressure is not the only force responsible for it. In an unsaturated soil, water tends to flow from the moist regions to the dryer regions. The capillary head  $\psi$  is usually expressed positively in centimeters of water column and depends on the moisture content of the soil and whether the soil is being wetted or dried. When the moisture content  $\theta$  increases, the value of  $\psi$  decreases, and vice versa. The phenomenon that the value of  $\psi$  is higher for a soil in the drying phase than for a soil in the wetting phase, at a given moisture content, is called hysteresis. Due to the large range of  $\psi$ -values, the relationship between  $\theta$  and  $\psi$ , which is usually referred to as the soil-water retention curve, makes use of a logarithmic scale according to

$$pF = \log(\psi) \quad (2.18)$$

where

$\psi$  = capillary head in units length of  
watercolumn, [L]

Theoretically, when  $\psi \rightarrow 0$ , the pF-value  $\rightarrow -\infty$ , and  $\theta = \theta_s$ , where  $\theta_s$  is the saturated soil moisture content. At pF =  $-\infty$ , the pF- $\theta$  curve becomes vertical. This relationship is shown in Figure 2.4. In practice, however, at  $\psi = 1$  cm the value of  $\theta$  is about equivalent to  $\theta_s$  if no air is included in the soil, and, therefore, at this point, the pF-curve intersects the  $\theta$ -axis almost perpendicularly. For agricultural purposes, the range of pF=2 to pF=4.2 is of particular interest. An increasing value of  $\psi$  makes the intake of soil moisture by the plant more difficult. At  $\psi \approx 15\text{atm}$  ( $\approx 15,000$  cm water column), the intake becomes impossible. This point is called the permanent wilting point and is reached for a pF value of 4.2. A lower bound on  $\psi$  is determined by considering the availability of air for the plant. At  $\psi = 1$  cm,  $\theta \approx \theta_s$ , and no air is available. When  $\psi > 100\text{cm}$  (pF=2), the inter-aggregate pores are in most cases empty and supply the necessary amount of air. The difference between pF- $\theta$  relationship of different soil is clearly illustrated in Figure 2.5. The clay soil displays a more or less continuous pore size distribution; whereas, in the sand soil, a particular pore size is dominant.

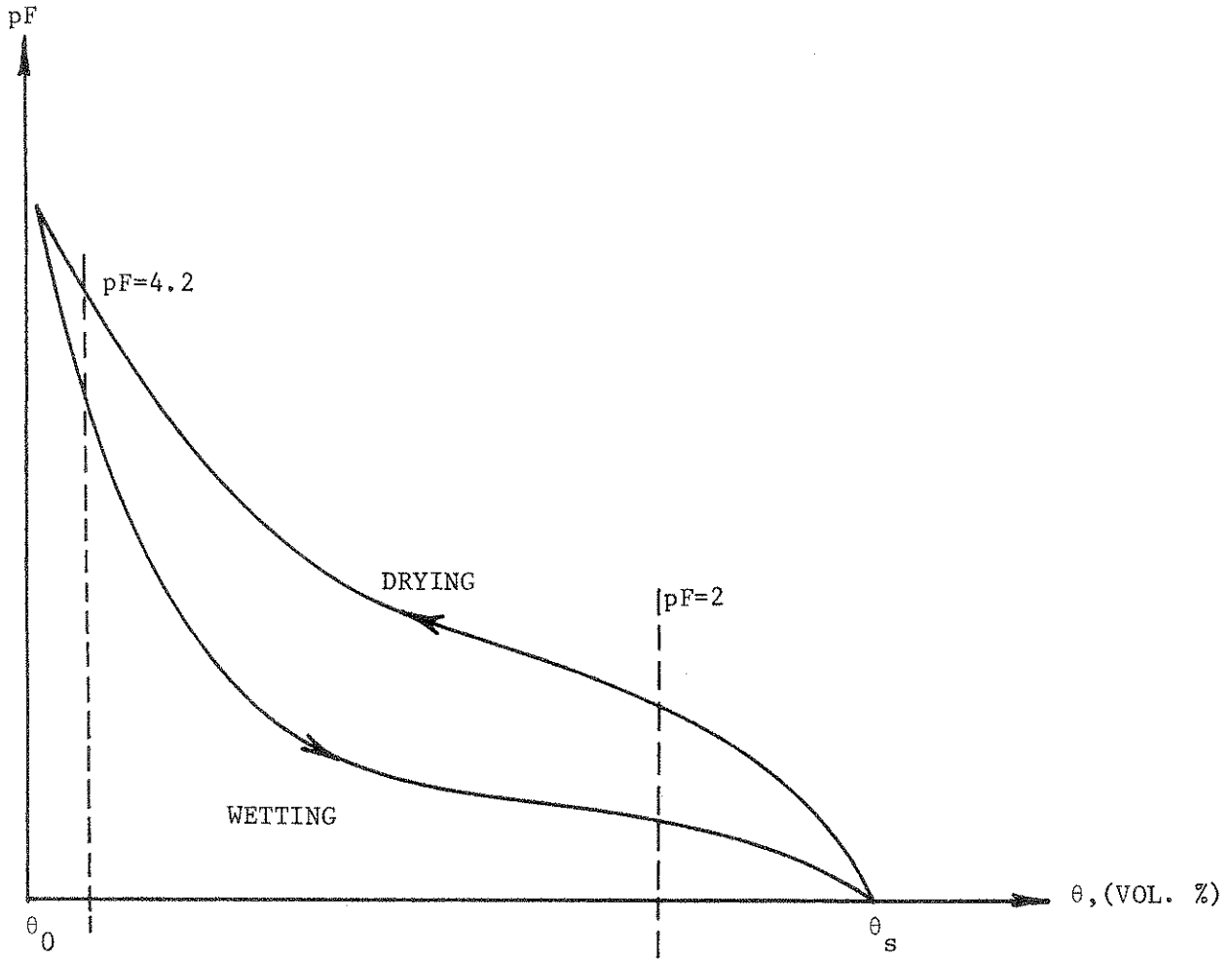


Figure 2.4. Hysteresis in soil moisture retention curve.

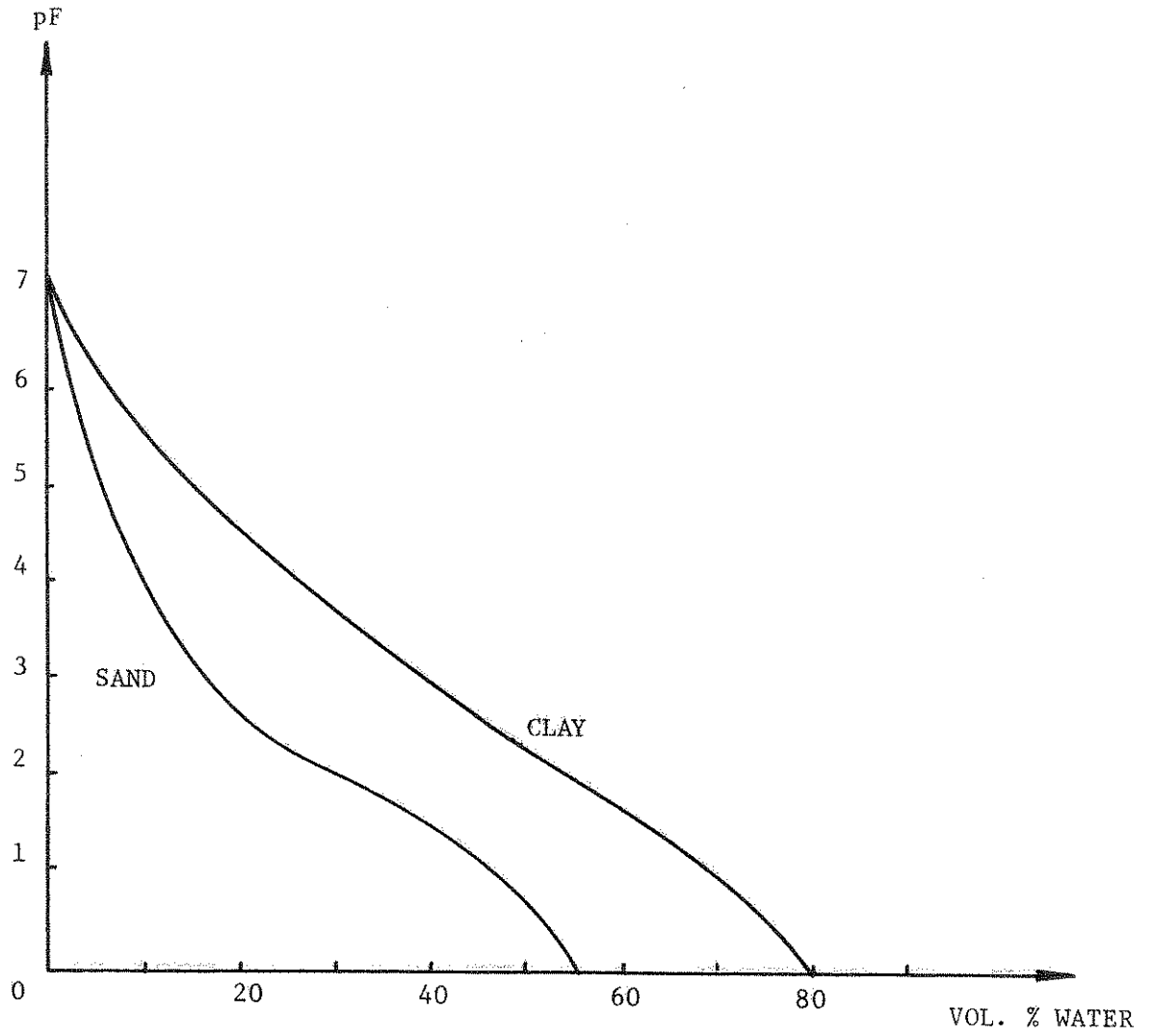


Figure 2.5. Soil moisture retention curves for different types of soils.

Various mathematical models exist in order to simulate the  $\psi - \theta$  relationship. Campbell (1974) proposed the relationship

$$\psi = \psi_e (\theta/\theta_s)^{-b} \quad (2.19)$$

where

$\psi_e$  = air-entry matrix suction, [L]

b = slope of the plot of  $\log \theta$  vs.  $\log \psi$

$\theta$  = volumetric moisture content, [ $L^3/L^3$ ]

$\theta_s$  = soil moisture content at saturation, [ $L^3/L^3$ ].

For practical purposes, a lower bound on the soil-moisture content is defined as the residual moisture content,  $\theta_r$ . At this value for  $\theta$ , the hydraulic conductivity is extremely small, and, from a practical point of view, equals zero. Values for  $\theta_r$  may be determined on a very dry soil, e.g.,  $\psi = 15000$  cm. At this point, the gradient  $\frac{d\theta}{d\psi}$  becomes practically zero. Taking this into account, the soil moisture content is frequently defined as

$$\theta = \frac{\theta - \theta_r}{\theta_s - \theta_r} \quad (2.20)$$

where

$\theta$  = effective saturation

$\theta$  = actual soil moisture content, [ $L^3/L^3$ ].

Based on (2.14), Brooks and Corey (1964) proposed

$$\psi = \psi_e \left| \frac{\theta - \theta_r}{\theta_s - \theta_r} \right|^{1/\lambda} \quad (2.21)$$

where  $\psi$ ,  $\psi_e$ ,  $\theta$ ,  $\theta_r$ , and  $\theta_s$  are defined as previously and  $\lambda$  is a soil characteristic. Another frequently used class of models (Van Genuchten, 1980) is given by

$$\theta = \left| \frac{1}{1+(\alpha\psi)^n} \right|^m \quad (2.22)$$

where  $\alpha$ ,  $n$  and  $m$  are parameters to be estimated.

### 2.5.2 The K- $\theta$ Relationship

In an unsaturated soil, part of the pore space is filled with air, thereby reducing the volume of the soil which contributes to the transport of moisture. Therefore, the hydraulic conductivity of an unsaturated soil depends to a large extent on the soil moisture content, which may be expressed as

$$K=K(\theta). \quad (2.23)$$

A typical K- $\theta$  relationship is shown in Figure 2.6. Because  $\theta$  is dependent on the capillary head  $\psi$ , K may also be given as a function of  $\psi$ ,

$$K=K(\psi). \quad (2.24)$$

Numerous mathematical models have been proposed to estimate the hydraulic conductivity (e.g., Campbell 1974, Mualem 1976, Van Genuchten 1980). Without going into great detail, the ones which are most widely used are presented here.

Campbell (1974) presented a model for estimating the hydraulic conductivity of unsaturated soils given by

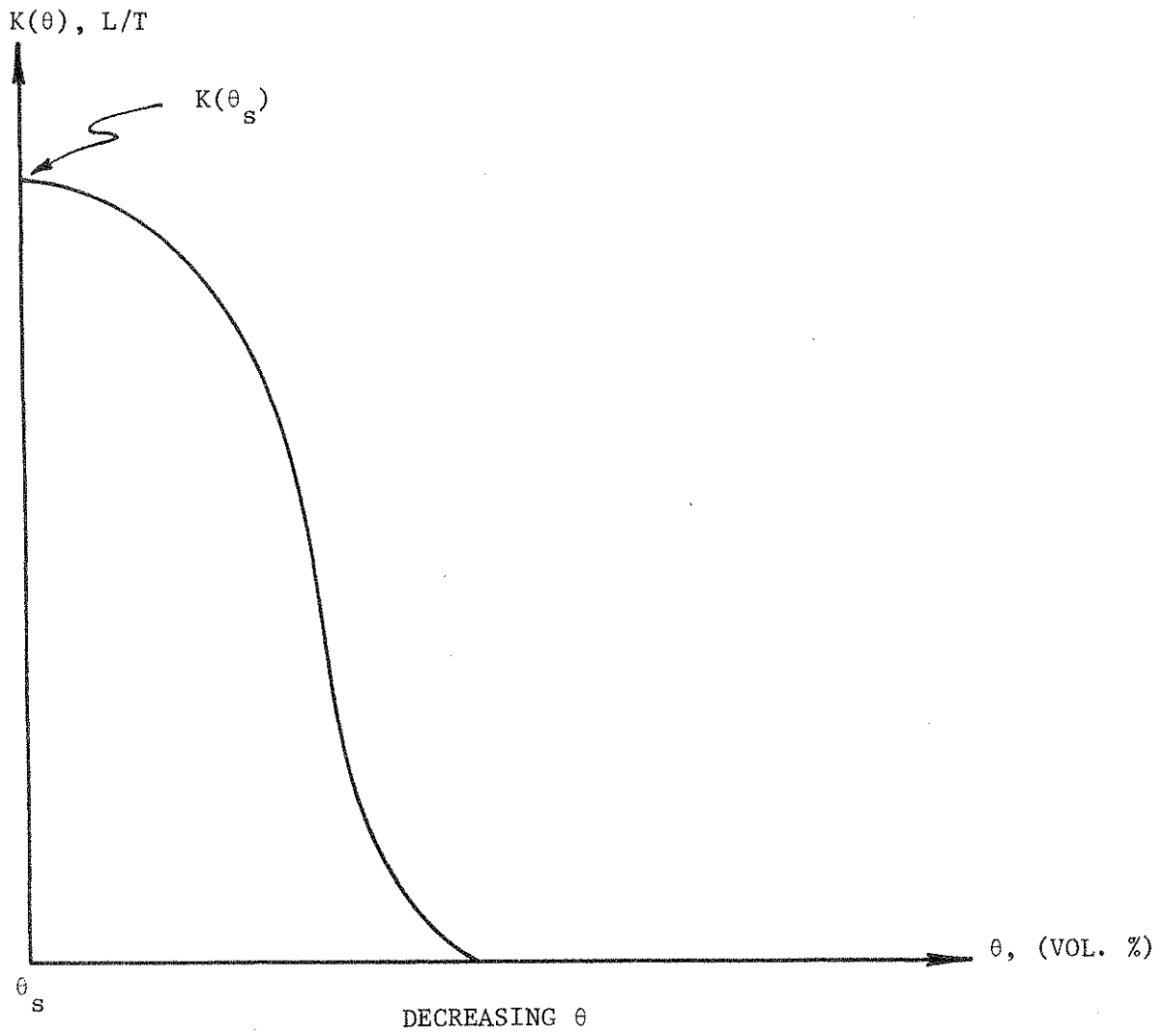


Figure 2.6. A typical  $K(\theta)$  relationship.



$$K(\theta) = K_s (\theta/\theta_s)^{2b+3} \quad (2.25)$$

where

$K$  = hydraulic conductivity at moisture content  $\theta$ , [L/T]

$K_s$  = hydraulic conductivity at saturation  $\theta_s$ , [L/T]

$\theta$  = soil moisture content, [ $L^3/L^3$ ]

$b$  = slope of the plot  $\log \theta$  vs.  $\log \psi$

$\psi$  = capillary head, [L].

A very common model is given by the expression

$$K(\theta) = K_s \left| \frac{\theta - \theta_r}{\theta_s - \theta_r} \right|^N \quad (2.26)$$

where  $K$ ,  $K_s$ ,  $\theta$  and  $\theta_s$  are defined as in (2.25) and  $\theta_r$  is the residual soil moisture content. The exponent  $N$  depends on the specific soil properties and has a value 3 according to Irnay (1954), although other investigators propose different values (Wang et. al., 1964). Mualem (1976) uses  $N = 2.5 + 2/\lambda$ , where  $\lambda$  is a soil characteristic parameter.

The hydraulic conductivity may also be expressed as a function of the capillary head,  $\psi$ . Van Genuchten (1980) presents the following formulation:

$$K_r(\psi) = \frac{\left| 1 - (\alpha\psi)^{n-1} [1 + (\alpha\psi)^n]^{-m} \right|^2}{\left| 1 + (\alpha\psi)^n \right|^{m/2}} \quad (2.27)$$

where

$K_r$  = relative hydraulic conductivity,  $\frac{K(\theta)}{K(\theta_s)}$

$\psi$  = capillary head, [L]

$\alpha, n$  = soil characteristic properties, [ $\alpha = L^{-1}$ ]

$m = 1-1/n$ .

The parameters  $\alpha$  and  $n$  have to be estimated from the observed soil-moisture retention data. When  $\psi=0$  in (2.27),  $K_r = 1$ , and when  $\psi \rightarrow \infty$ ,  $K_r \rightarrow 0$ . Brooks and Corey (1966) proposed

$$K_r(\psi) = (\alpha\psi)^{-2-3\lambda}, \quad (2.28)$$

while based on Mualem's theory, Van Genuchten (1980) presents

$$K_r(\psi) = (\alpha\psi)^{-2-5\lambda/2}. \quad (2.29)$$

Both formulations (2.28) and (2.29) are inaccurate when  $\theta$  is relatively high (low  $\psi$ -value).

It is obvious that the discussion in this section is restricted to isotropic soils, because, when dealing with anisotropic media, the hydraulic conductivity tensor has to be considered. In that case, it is not immediately clear how this tensor is affected by the different soil-moisture conditions. More on this subject will be said in section 2.7.

## 2.6 One Dimensional Infiltration

Darcy's law for the flow of water in unsaturated isotropic media can be expressed as

$$\bar{q} = -K(\theta)\bar{\nabla}\phi \quad (2.30)$$

where

$\bar{q}$  = specific discharge vector,  $[L/T^{-1}]$

$K(\theta)$  = hydraulic conductivity at moisture content  $\theta$   $[L/T^{-1}]$

$\bar{\nabla}\phi = \left( \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right)$  = hydraulic gradient vector

$\phi = \psi(\theta) + z$  = matrix flux potential,  $[L]$

$\psi(\theta)$  = capillary head at moisture content  $\theta$ ,  $[L]$

$z$  = elevation head,  $[L]$ .

Combining (2.30) with the continuity equation

$$\frac{\partial\theta}{\partial t} = - \bar{\nabla} \cdot \bar{q} \quad (2.31)$$

gives

$$\frac{\partial\theta}{\partial t} = \bar{\nabla} \cdot (K(\theta)\bar{\nabla}\phi). \quad (2.32)$$

Substituting  $\phi = \psi(\theta) + z$  into (2.32), we then have

$$\frac{\partial\theta}{\partial t} = \bar{\nabla} \cdot [K(\theta)\bar{\nabla} (\psi(\theta)+z)] \quad (2.33)$$

which can be expanded as

$$\frac{\partial\theta}{\partial t} = \bar{\nabla} \cdot (K\bar{\nabla}\psi) + \frac{\partial K}{\partial z} \quad (2.34)$$

Equation (2.34) holds for homogeneous as well as for heterogeneous soils. For a homogeneous soil, the equation (2.34) can be simplified in the following manner. If  $K$  and  $\psi$  are single-valued functions of  $\theta$ , the following quantity can be defined

$$D(\theta) = K(\theta) \frac{\partial \psi}{\partial \theta} \quad (2.35)$$

where

$D$  = moisture diffusivity,  $[L^2/T]$ .

Within the range of  $pF=2$  to  $pF=4.2$ ,  $D$  may vary with as much as a factor  $10^4$ . Substituting (2.35) into (2.34) gives

$$\frac{\partial \theta}{\partial t} = \bar{\nabla} \cdot [D(\theta) \bar{\nabla} \theta] + \frac{\partial K(\theta)}{\partial \theta} \frac{\partial \theta}{\partial z}. \quad (2.36)$$

Equation (2.36) is a non-linear Fokker-Planck equation. The one dimensional form of (2.36) is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ D(\theta) \frac{\partial \theta}{\partial z} \right] - \frac{\partial K(\theta)}{\partial \theta} \frac{\partial \theta}{\partial z} \quad (2.37)$$

where  $z$  is now taken positive downward. The second term on the right hand side of (2.37) accounts for the gravity effects on infiltration, whereas the first term on the right hand side accounts for the effects of moisture gradients. In cases where the influence of moisture gradients is much more important than gravity, e.g., infiltration into fine textured soils, or where gravity can be neglected altogether, such as infiltration into horizontal systems, equation (2.37) becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left( D(\theta) \frac{\partial \theta}{\partial x} \right) \quad (2.38)$$

and describes absorption.

Philip (1969) solves (2.38) numerically by using the Boltzmann substitution  $\sigma = xt^{-1/2}$ , thereby replacing the two independent variables  $x$  and  $t$  by one independent variable  $\sigma$ . He

derives, then, the following relationship

$$i = St^{1/2} \tag{2.39}$$

where

$i$  = cumulative infiltration, [L]

$S$  = sorptivity, [L/T<sup>1/2</sup>]

$t$  = time, [T].

The sorptivity is defined as

$$S = \int_{\theta_0}^{\theta_1} \sigma(\theta) d\theta \tag{2.40}$$

where

$\theta_1$  = soil moisture content at saturation, ( $\sigma = 0$ )

$\theta_0$  = initial soil moisture content, ( $\sigma = \infty$ )

The sorptivity is a soil characteristic and equals the total infiltration at  $t=1$ .

From (2.39) it appears that a proportionality exists between the cumulative infiltration and the square root of time. When gravity plays a role, such as in vertical infiltration, the distance traveled by the wetting front will be greater than according to (2.39), whereas for capillary rise, this distance will be smaller. Equation (2.37) was also solved by Philip in a numerical fashion by using the following expansion

$$z(\theta, t) = \sigma_1 t^{1/2} + \sigma_2 t + \sigma_3 t^{3/2} + \dots \quad (2.41)$$

For details, the reader is referred to Philip (1969).

For the case that  $\frac{\partial \theta}{\partial t} = 0$ , equation (2.37) gives the solution  $\partial \psi = -\partial z$ . Thus the soil moisture gradient is compensated for by the gradient in the gravity field. This situation may occur in practice when the ground water table is not too deep.

### 2.7 The Hydraulic Conductivity Tensor for Unsaturated Soils

The solution of (2.37) and (2.38) applies only to isotropic soil conditions. As was explained in section 2.1, for anisotropic soils, the hydraulic conductivity  $K$  in (2.3) has to be replaced by the hydraulic conductivity tensor  $K$ , to obtain (2.5). The question arises if such a change is also valid for unsaturated soils. Restricting ourselves to the two-dimensional case, the hydraulic conductivity tensor for an unsaturated soil would be expressed as

$$K_{ij}(\theta) = \begin{vmatrix} K_{xx}(\theta) & K_{xz}(\theta) \\ K_{zx}(\theta) & K_{zz}(\theta) \end{vmatrix} \cdot \quad (2.42)$$

There is no evidence that the tensor given by (2.42) is symmetrical, as was the case for saturated soils. Also, at different moisture contents, the ratio of the coefficients may change. It is therefore not obvious that the transformations described by (2.10) are appropriate. Very little is known in the literature about the properties of (2.42). Liakopoulos (1964), Childs (1969) and Cisler (1972) are a few of the investigators who

discussed the hydraulic conductivity of unsaturated and anisotropic soils. However, their reports are by no means conclusive. In this study, therefore, the following approach is taken. Starting with the principal directions at a certain moisture content  $\theta$ , (2.42) may be written as

$$K_{ij}(\theta) = \begin{vmatrix} K_{xx}(\theta) & 0 \\ 0 & K_{zz}(\theta) \end{vmatrix} \quad (2.43)$$

where it is assumed that  $x$  and  $z$  are now principal directions. Letting  $K_{zz}(\theta) = K(\theta)$ , and  $K_{xx}(\theta) = \lambda(\theta)K(\theta)$ , we obtain

$$K_{ij}(\theta) = \begin{vmatrix} \lambda(\theta)K(\theta) & 0 \\ 0 & K(\theta) \end{vmatrix}. \quad (2.44)$$

A simplification can be obtained by letting  $\lambda(\theta)$  be independent of  $\theta$ , a property which has not physically been demonstrated. This gives

$$K_{ij}(\theta) = \begin{vmatrix} \lambda K(\theta) & 0 \\ 0 & K(\theta) \end{vmatrix}. \quad (2.45)$$

### 2.8 Hydraulic Conductivity of Stratified Soils

Suppose we have a soil consisting of  $n$  layers of various thickness  $d_i$  ( $i=1, \dots, N$ ), each of which is isotropic in itself with a hydraulic conductivity  $K_i$  ( $i=1, \dots, N$ ). In a direction parallel to the strata, the hydraulic conductivity of the soil system is given by (Bear, 1972):

$$K_p = \frac{\sum_{i=1}^N d_i K_i}{\sum_{i=1}^N d_i} \quad (2.46)$$

and, in a direction perpendicular the soil strata, the hydraulic conductivity is given by

$$K_n = \frac{\sum_{i=1}^N d_i}{\sum_{i=1}^N \frac{d_i}{K_i}} \quad (2.47)$$

Thus, as a system, the stratified soil behaves itself as an anisotropic medium.

### 2.9 Summary

In this chapter, the basic elements of the infiltration theory were presented. Darcy's law was discussed in great extent, covering saturated, unsaturated, isotropic and anisotropic soil conditions. In this respect, the various forms of the hydraulic conductivity are essential, such as scalar versus tensor form. Various mathematical relationships between  $K$  and  $\theta$ , and  $\psi$  and  $\theta$  were introduced, which are essential for any numerical solution of the infiltration problem. The non-linear Fokker Planck equation was presented as the general mathematical formulation of infiltration in isotropic soils. Finally, it was indicated how, in the next chapters of this report, the anisotropic case will be approached.





## CHAPTER III: INFILTRATION IN ANISOTROPIC SOILS

### 3.1 Introduction

The only references which address the occurrence of lateral flow in unsaturated soils, that is, the occurrence of nonvertical infiltration are those of Atkinson (1978) and Zaslavsky and Sinai (1981). We purposely omit mentioning the few contributions to the study of the lateral movement of vadose water in the capillary fringe just above the phreatic surface.

The fact that traditionally infiltration is assumed to take place vertically stems from the fact that it is usually assumed that the porous medium is isotropic. Even when the soil is recognized to be anisotropic, if one of the principal axes of anisotropy coincides with the direction of gravity, then infiltration takes place vertically. It is when the layers of stratification have been disturbed from their original horizontal direction that lateral flows are possible.

Since groundwater contamination, particularly from hazardous wastes, has recently been recognized as a very serious national problem [RCRA (1976), USEPA (1978, 1980), Winograd (1981), Wood, Ferrara, Gray and Pinder (1984)], it is of utmost importance to know whether special configurations of soil layers through which contaminants percolate allow lateral flows in the vadose zone and to estimate the amount of such lateral flows.

This contribution wants to address that problem by present-

ing the general tensor formulation of the flow within an unsaturated anisotropic soil under a sloping surface, and by focusing on a special case with the purpose of attracting attention to the occurrence of lateral flows.

### 3.2 Problem Formulation

Our interest will be confined to two-dimensional problems definable in a plane containing the gravitational field vector. The (vertical) ordinate, directed against the gravitational field, is called  $z$  and the (horizontal) abscissa is called  $s$ .

We define further an axis  $x$  which forms an angle  $\alpha$  with the axis  $s$  and an axis  $y$  orthogonal to  $x$ . The axis  $x$  ( $y = 0$ ) defines the boundary between porous medium (free surface) and the ambient. The coordinate systems  $z, s$  and  $y, x$  are congruent.

Another coordinate system is furthermore defined as  $y', x'$ , coherent with the previous systems, such that the angle between  $x$  and  $x'$  is  $\phi$ . The axis  $x'$  ( $y = 0$ ) defines the directions of the soil layers.

The three coordinate systems are illustrated in Figure 3.1.

The coordinate system  $x, y$  defines the symmetry of the problems we wish to consider. In fact we will agree that the only type of problems we will examine in this report are problems where all physical quantities are independent of  $x$ . This means that boundaries which are not at infinity must be parallel to the

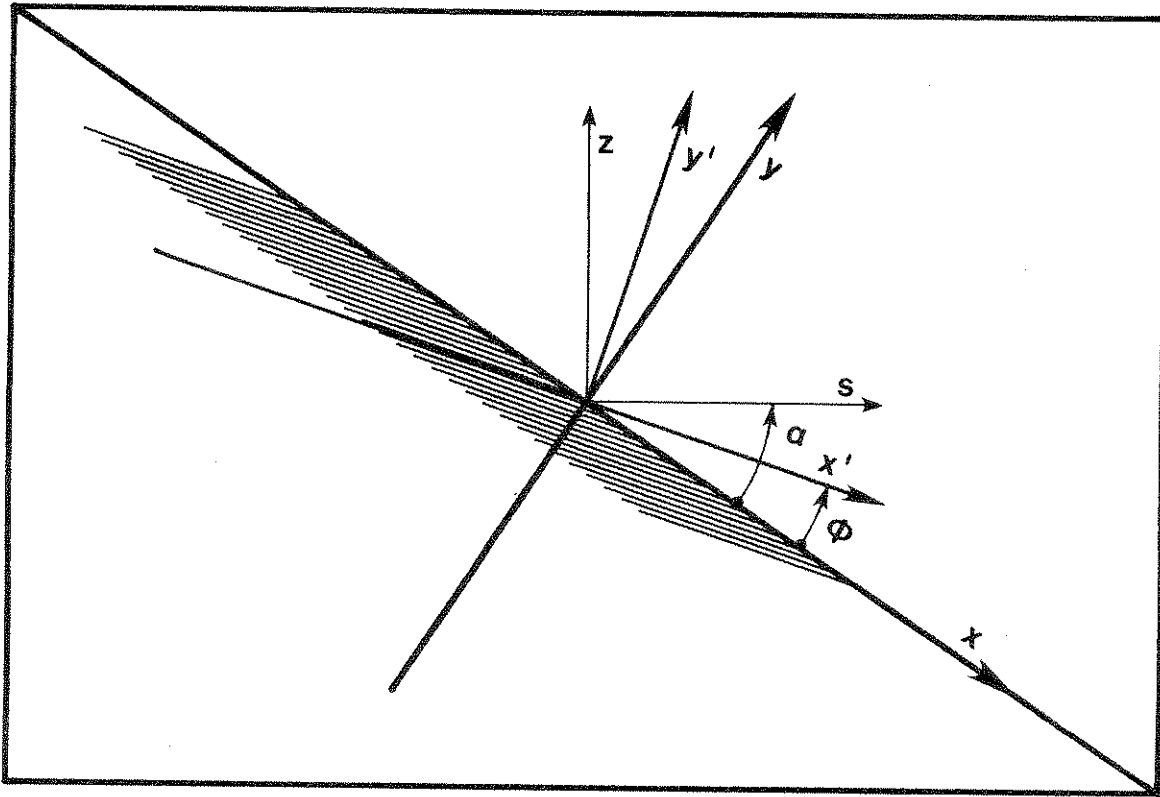


Figure 3.1. The three coordinate systems defined in the problem.

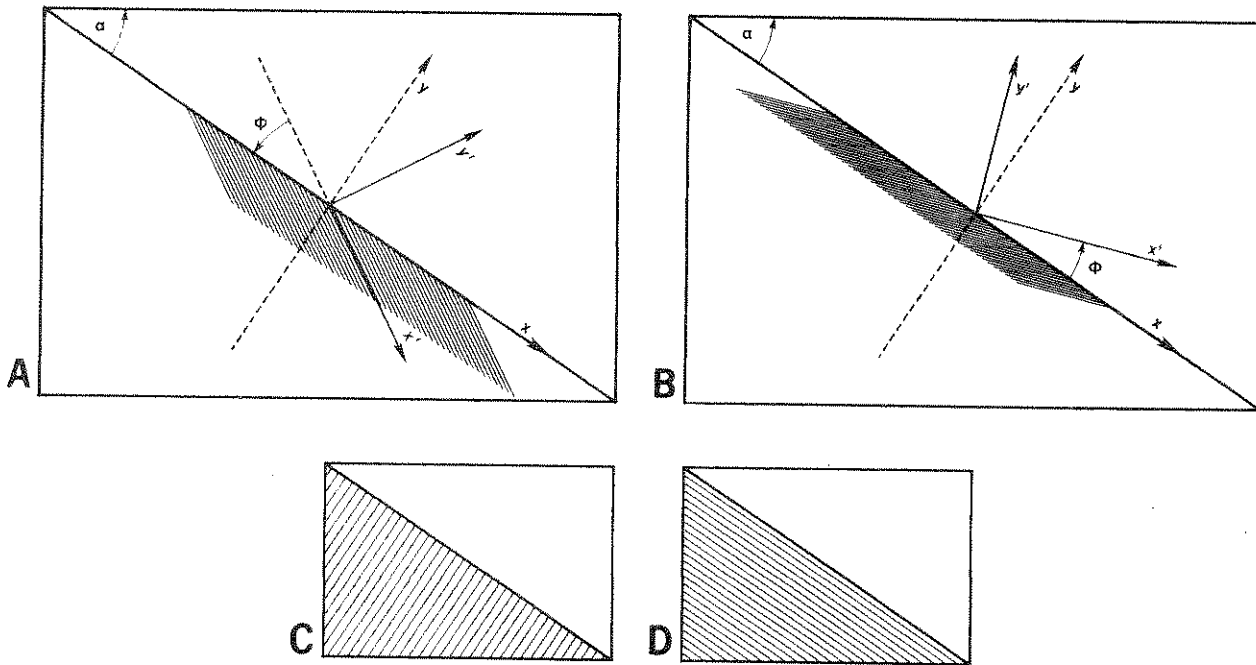


Figure 3.2. A.) Case of positive lateral flow due to absorption;  
B.) Case of negative lateral flow;  
C.) and D.) Cases of null lateral flow.

x axis.

Notice that the x axis is not horizontal and that there is therefore a "driving force" due to gravity in both x and y directions.

The coordinate system  $x', y'$  coincides with the principal axes of the hydraulic conductivity tensor  $K'_{ij}$ . With this notation the hydraulic conductivity along the  $x'$  axis is,  $K'_{11}$ , the hydraulic conductivity along the  $y'$  axis is  $K'_{22}$ .

We will assume, without loss of generality, that the  $K'_{11}$  is larger than the  $K'_{22}$  component. In fact we will assume that  $K'_{11} = \lambda K'_{22}$  where  $\lambda$  is at most a function of the moisture content  $\theta$ .

The tensor  $K'_{ij}$  is therefore expressible as

$$\begin{aligned} K'_{11} &= \lambda K \\ K'_{22} &= K \\ K'_{12} &= K'_{21} = 0. \end{aligned} \tag{3.1}$$

We will recall, for reference ease, the definitions and the equations which we shall use in the remainder of this article. They are:

- 1) The generalization of Darcy's law

$$q_i = - K_{ij} \frac{\partial \Psi}{\partial x_j}, \tag{3.2}$$

where  $q_i$  is the specific discharge vector,  $x_i$  is any orthog-

onal Cartesian coordinate system, and  $\Psi$  is the piezometric head  $z + p/\gamma$  for saturated soils and  $z + \psi$  for unsaturated soils. The physical quantity  $\psi$  is the negative pressure head due to the capillary effects of the soil porosity, and it is called interchangeably as capillary pressure head, moisture potential, moisture suction, or negative pressure head.

2) The continuity equation

$$\frac{\partial \theta}{\partial t} + \frac{\partial q_i}{\partial x_i} = 0. \quad (3.3)$$

Upon insertion of (3.2) into (3.3) we obtain

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x_i} \left| K_{ij} \frac{\partial \phi}{\partial x_j} \right|, \quad (3.4)$$

or

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x_i} \left| K_{ij} \frac{\partial \psi}{\partial x_j} \right| + \frac{\partial}{\partial x_i} \left| K_{ij} \frac{\partial z}{\partial x_j} \right|. \quad (3.5)$$

The first term on the right-hand side of (3.5) represents the capillary effects on the fluid motion, the second term represents the effects of gravity.

If we now assume that the  $x_1, x_2$  coordinate system coincides with  $x, y$ , we obtain

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= \frac{\partial}{\partial x} \left| K_{11} \frac{\partial \psi}{\partial x} + K_{12} \frac{\partial \psi}{\partial y} \right| + \frac{\partial}{\partial y} \left| K_{21} \frac{\partial \psi}{\partial x} + K_{22} \frac{\partial \psi}{\partial y} \right| \\ &= \frac{\partial}{\partial x} \left| K_{11} \frac{\partial z}{\partial x} + K_{12} \frac{\partial z}{\partial y} \right| + \frac{\partial}{\partial y} \left| K_{21} \frac{\partial z}{\partial x} + K_{22} \frac{\partial z}{\partial y} \right|, \end{aligned} \quad (3.6)$$

and the expressions for the specific discharge along  $x$  and  $y$  are

$$q_x = - K_{11} \frac{\partial \psi}{\partial x} - K_{12} \frac{\partial \psi}{\partial y} - K_{11} \frac{\partial z}{\partial x} - K_{12} \frac{\partial z}{\partial y} \quad (3.7)$$

$$q_y = - K_{21} \frac{\partial \psi}{\partial x} - K_{22} \frac{\partial \psi}{\partial y} - K_{21} \frac{\partial z}{\partial x} - K_{22} \frac{\partial z}{\partial y}, \quad (3.8)$$

where  $q_x$  is the lateral specific flowrate and  $q_y$  is the normal specific flowrate.

Since the elevation  $z$  can be expressed in terms of  $x$  and  $y$  as

$$z = y \cos \alpha - x \sin \alpha, \quad (3.9)$$

we can rewrite (3.6), (3.7) and (3.8) as follows

$$\begin{aligned} \frac{\partial \theta}{\partial t} = & \frac{\partial}{\partial x} \left| K_{11} \frac{\partial \psi}{\partial x} + K_{12} \frac{\partial \psi}{\partial y} \right| + \frac{\partial}{\partial y} \left| K_{21} \frac{\partial \psi}{\partial x} + K_{22} \frac{\partial \psi}{\partial y} \right| \\ & + \cos \alpha \left| \frac{\partial K_{12}}{\partial x} + \frac{\partial K_{22}}{\partial y} \right| - \sin \alpha \left| \frac{\partial K_{11}}{\partial x} + \frac{\partial K_{21}}{\partial y} \right| \end{aligned} \quad (3.10)$$

and

$$q_x = - K_{11} \frac{\partial \psi}{\partial x} - K_{12} \frac{\partial \psi}{\partial y} + K_{11} \sin \alpha - K_{12} \cos \alpha \quad (3.11)$$

$$q_y = - K_{21} \frac{\partial \psi}{\partial x} - K_{22} \frac{\partial \psi}{\partial y} + K_{21} \sin \alpha - K_{22} \cos \alpha. \quad (3.12)$$

The above equations are drastically simplified by our condition that all the fields be  $x$  independent: equations (3.10), (3.11), and (3.12) become, respectively

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial y} \left| K_{22} \frac{\partial \psi}{\partial y} \right| + \cos \alpha \frac{\partial K_{22}}{\partial y} - \sin \alpha \frac{\partial K_{21}}{\partial y} \quad (3.13)$$

$$q_x = - K_{12} \frac{\partial \psi}{\partial y} + K_{11} \sin \alpha - K_{12} \cos \alpha \quad (3.14)$$

$$q_y = - K_{22} \frac{\partial \psi}{\partial y} + K_{21} \sin \alpha - K_{22} \cos \alpha. \quad (3.15)$$

The components  $K_{ij}$  of the hydraulic conductivity tensor can be written in terms of the principal components  $K'_{11} = \lambda K$ ,  $K'_{22} = K$  as follows

$$K_{11} = \frac{K'_{11} + K'_{22}}{2} + \frac{K'_{11} - K'_{22}}{2} \cos 2\phi \quad (3.16)$$

$$K_{22} = \frac{K'_{11} + K'_{22}}{2} - \frac{K'_{11} - K'_{22}}{2} \cos 2\phi \quad (3.17)$$

$$K_{21} = K_{12} = - \frac{K'_{11} - K'_{22}}{2} \sin 2\phi \quad (3.18)$$

or

$$K_{11} = K \left| \frac{\lambda+1}{2} + \frac{\lambda-1}{2} \cos 2\phi \right| = \xi K \quad (3.19)$$

$$K_{22} = K \left| \frac{\lambda+1}{2} - \frac{\lambda-1}{2} \cos 2\phi \right| = \eta K \quad (3.20)$$

$$K_{21} = K_{12} = K \left| \frac{\lambda-1}{2} \sin 2\phi \right| = \chi K \quad (3.21)$$

The above equations give implicitly the definitions of the functions  $\xi$ ,  $\eta$ , and  $\chi$  in terms of  $\lambda$  and  $\phi$ .

Substituting (3.19), (3.20) and (3.21) into (3.13), (3.14) and (3.15) we obtain, assuming  $\lambda$  independent of  $\theta$ ,

$$\frac{\partial \theta}{\partial t} = \eta \frac{\partial}{\partial y} \left| K \frac{\partial \psi}{\partial y} \right| + \eta \cos \alpha \frac{\partial K}{\partial y} - \chi \sin \alpha \frac{\partial K}{\partial y} \quad (3.22)$$

and

$$\begin{aligned} q_x &= - \chi K \frac{\partial \psi}{\partial y} + (\xi \sin \alpha - \chi \cos \alpha) K \\ q_y &= - \eta K \frac{\partial \psi}{\partial y} - (-\chi \sin \alpha + \eta \cos \alpha) K \end{aligned} \quad (3.23)$$

or



$$q_x = \frac{\chi}{\eta} q_y + \frac{\lambda}{\eta} \sin \alpha K. \quad (3.24)$$

If we define

$$\begin{aligned} \mu &= -\chi \sin \alpha + \eta \cos \alpha \\ \nu &= \xi \sin \alpha - \chi \cos \alpha, \end{aligned} \quad (3.25)$$

equations (3.22) and (3.23) can be rewritten as

$$\frac{\partial \theta}{\partial t} = \eta \frac{\partial}{\partial y} \left| K \frac{\partial \psi}{\partial y} \right| + \mu \frac{\partial K}{\partial y} \quad (3.26)$$

$$q_x = -\chi K \frac{\partial \psi}{\partial y} + \nu K = q_{ax} + q_{gx}$$

$$q_y = -\eta K \frac{\partial \psi}{\partial y} - \mu K = q_{ay} + q_{gy}, \quad (3.27)$$

where we have implicitly defined  $q_{ax}$ ,  $q_{ay}$  as the absorption contribution to the specific flowrate and  $q_{gx}$ ,  $q_{gy}$  as the gravitation contributions to the specific flowrate.

If we introduce the definitions

$$\begin{aligned} \bar{K} &= \frac{\mu}{\eta} K \\ Y &= \frac{\mu}{\eta} y, \end{aligned} \quad (3.28)$$

(3.26) can be rewritten as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial y} \left| \bar{K} \frac{\partial \psi}{\partial y} \right| + \frac{\partial \bar{K}}{\partial y}, \quad (3.29)$$

which has the same form as the equation of vertical infiltration (through a horizontal surface) for isotropic media.

Once (3.28) is solved, then  $q_x$  and  $q_y$  can be found via (3.23) and (3.24).

The absorption contribution  $q_{ax} = -\chi K \partial\psi/\partial y$  to the lateral flow is null when  $\chi = 0$ , that is either when the porous medium is isotropic or when  $\sin 2\phi = 0$ , that is when the principal axes of anisotropy coincide with the  $y, x$  axes. For other values of  $\phi$ , it may be positive or negative. For the particular case where  $\partial\psi/\partial y > 0$  (which corresponds to situations where the moisture is gradually decreasing from the surface of the soil downward)  $q_{ax}$  is positive for  $-\pi/2 < \phi < 0$  and it is negative for  $0 < \phi < \pi/2$ . These two cases are illustrated in Figure 3.2, where the direction of the hatching coincides with the principal direction with maximum hydraulic conductivity.

The absorption contribution  $q_{ay} = -\eta K(\partial\psi/\partial y)$  to the normal flow is always negative when  $\partial\psi/\partial y > 0$  and has extreme values (corresponding to  $\eta = 1$  and to  $\eta = \lambda$ ) for  $\phi = 0$  and for  $\phi = \pi/2$  respectively.

The two components of the absorption contribution to the flow can be found by using the geometrical construction shown in Figure 3.3 (Mohr circle). From the construction it can be seen that there are instances in which  $|q_{ax}|$  can be larger than  $|q_{ay}|$ . This occurs only for

$$\lambda > 3 + 2\sqrt{2} = 5.83. \quad (3.30)$$

For values of  $\lambda$  satisfying (3.30), the range of the angle  $\phi$  for which  $|q_{ax}| > |q_{ay}|$  is given by the two values

$$\phi_{1,2} = \frac{1}{2} \sin^{-1} \left| \frac{\lambda + 1 \mp \sqrt{\lambda^2 - 6\lambda + 1}}{2(\lambda - 1)} \right| \quad (3.31)$$



and the angle  $\phi^*$  for which the ratio  $|q_{ax}|/|q_{ay}|$  is a maximum is

$$\phi^* = \frac{1}{2} \cos^{-1} \left| \frac{\lambda - 1}{\lambda + 1} \right|. \quad (3.32)$$

As for the gravitation contribution to the specific discharge, its geometrical construction can be more conveniently carried out in terms of the horizontal component  $q_{gs}$  and of the vertical component  $q_{gz}$ . The general expressions in terms of lateral and normal components are given by

$$\begin{aligned} q_s &= q_x \cos \alpha + q_y \sin \alpha \\ q_z &= -q_x \sin \alpha + q_y \cos \alpha, \end{aligned} \quad (3.33)$$

which lead to the particular expressions

$$\begin{aligned} q_{as} &= -(\chi \cos \alpha + \eta \sin \alpha) K \frac{\partial \psi}{\partial y} = -\tau K \frac{\partial \psi}{\partial y} \\ q_{az} &= -(-\chi \sin \alpha + \eta \cos \alpha) K \frac{\partial \psi}{\partial y} = -\mu K \frac{\partial \psi}{\partial y}, \end{aligned} \quad (3.34)$$

where the first equation implicitly defines  $\tau$ , and

$$\begin{aligned} q_{gs} &= \frac{\lambda - 1}{2} \sin (2\alpha - 2\phi) K \\ q_{gz} &= \left| -\frac{\lambda + 1}{2} + \frac{\lambda - 1}{2} \cos (2\alpha - 2\phi) \right| K. \end{aligned} \quad (3.35)$$

Since  $(\alpha - \phi)$  is the angle that the layers form with the horizontal, it follows that the geometrical construction (Mohr's circle) illustrated in Figure 3.4 gives the gravitation contribution to horizontal and vertical flow.

The gravitation contribution  $q_{gs}$  to the horizontal flow is null either when the porous medium is isotropic or when  $\alpha = \phi$  or  $\alpha = \psi + \frac{\pi}{2}$ , that is when one of the principal axes of anisotropy coincides with the slope. When  $0 < \alpha - \phi < \pi/2$  then  $q_{gs}$  is

positive and when  $\pi/2 < \alpha - \phi < \pi$  then  $q_{gs}$  is negative. These four cases are illustrated in Figure 3.5.

The gravitation contribution  $q_{gz}$  to the vertical flow is always negative and has extreme values for  $\alpha - \phi = 0$  and  $\alpha - \phi = \pi/2$  respectively. Analogously to what has been shown for the absorption contribution to the flow, there are instances when  $|q_{gs}|$  is larger than  $|q_{gz}|$ . This occurs only for  $\lambda > 3 + 2\sqrt{2} = 5.83$  and the range  $(\alpha - \phi)_1, (\alpha - \phi)_2$  for which this occurs is given by the same expression (3.31) found for  $\phi_1$  and  $\phi_2$ . Furthermore, the value  $(\alpha - \phi)^*$  for which the ratio  $|q_{gs}/q_{gy}|$  is maximal is the same as the expression (3.32) for  $\phi^*$ .

### 3.3 The Absorption Simulation Case

A case of particular interest is the one for which  $\mu = 0$ . In fact in this case the infiltration equation (3.26) reduces to the much simpler pure absorption equation

$$\frac{\partial \theta}{\partial t} = \eta \frac{\partial}{\partial y} \left| K \frac{\partial \psi}{\partial y} \right|. \quad (3.36)$$

We will call this case the absorption simulation case, and we will study it in detail because it is one of the extreme cases of lateral flow.

The condition  $\mu = 0$  implies first that a relationship between  $\alpha$ ,  $\phi$ , and  $\lambda$  must exist. This is

$$\tan \alpha = \frac{\eta}{\chi} = \frac{(\lambda + 1) - (\lambda - 1) \cos 2\phi}{(\lambda - 1) \sin 2\phi}. \quad (3.37)$$

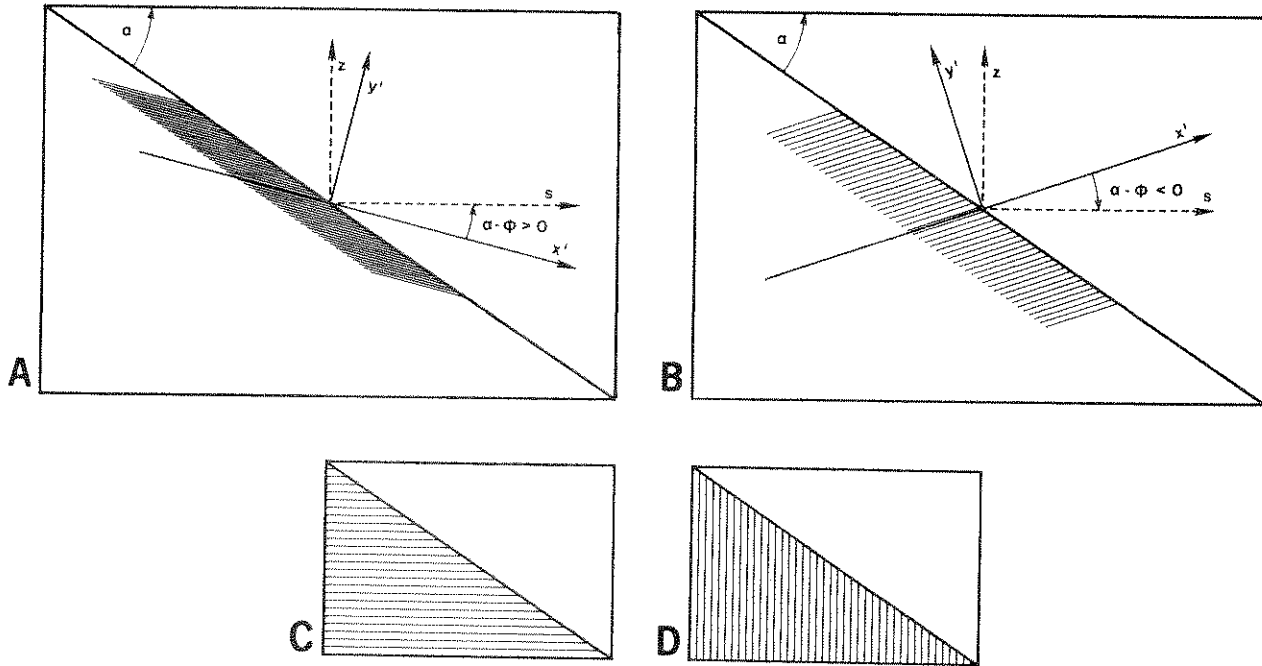


Figure 3.5. A.) Case of positive horizontal flow due to gravity;  
B.) Case of negative horizontal flow;  
C.) and D.) Cases of null horizontal flow.

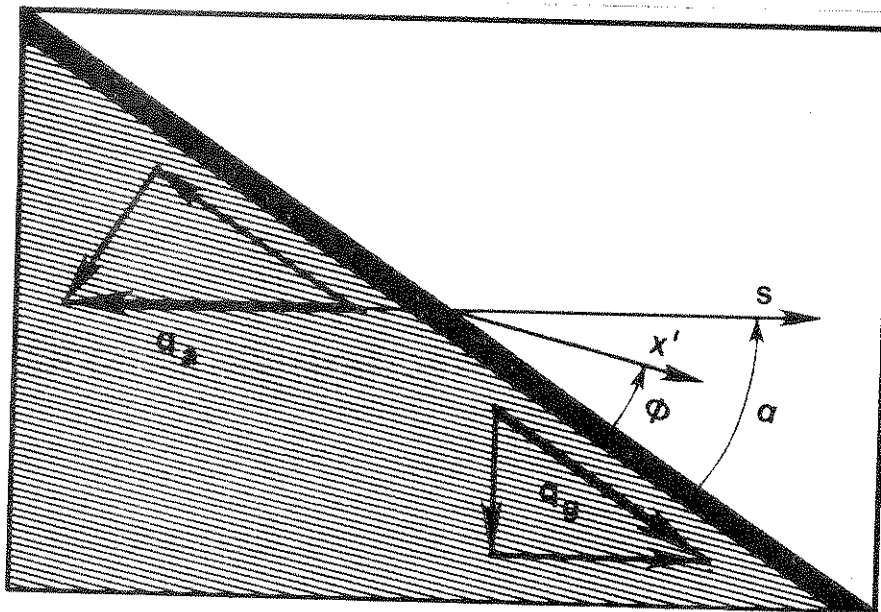


Figure 3.6. In the absorption simulation case, the absorption contribution to the flow is purely horizontal and the gravitational contribution to the flow is purely lateral.

Since the numerator is always positive and we consider only positive values of  $\alpha$ , then  $\sin 2\phi$  must be positive and so must be  $\phi$ .

In order to have a full perception of the geometrical meaning of (3.37), we will write the expressions for the specific flowrates in the particular case  $\mu = 0$ .

$$\begin{aligned} q_{ax} &= -\chi K \frac{\partial \psi}{\partial y} & q_{ay} &= -\eta K \frac{\partial \psi}{\partial y} \\ q_{as} &= -\bar{\tau} K \frac{\partial \psi}{\partial y} & q_{az} &= 0 \end{aligned} \quad (3.38)$$

$$\begin{aligned} q_{gx} &= \bar{v}K & q_{gy} &= 0 \\ q_{gs} &= \bar{v}K \cos \bar{\alpha} & q_{gz} &= -\bar{v}K \sin \bar{\alpha}, \end{aligned} \quad (3.39)$$

where  $\bar{\alpha}$  is the value of  $\alpha$  given by (3.37) and  $\bar{\tau}$  and  $\bar{v}$  are the values of  $\tau$  and  $v$  for  $\alpha = \bar{\alpha}$ .

One can demonstrate that it is

$$\bar{\tau} = \sqrt{\chi^2 + \eta^2} = \sqrt{\frac{\lambda^2 + 1}{2} - \frac{\lambda^2 - 1}{2} \cos 2\phi}, \quad (3.40)$$

$$\bar{v} = \frac{\xi \eta - \chi^2}{\sqrt{\chi^2 + \eta^2}} = \lambda / \bar{\tau}. \quad (3.41)$$

From equations (3.38) and (3.39) it results clearly that in the absorption simulation case, the infiltration flow is made up of a purely horizontal (negative) absorption flow and of a purely lateral (positive) gravitational flow. In a language echoing the Green Ampt infiltration model, we can say that in the case of impulsively started runoff<sup>1</sup> a wet front propagates from the free

1. Impulsively started runoff problem is the slope analog of the instantaneously started ponding for horizontal surfaces.

surface in the negative horizontal direction. In the region within the free surface and the front, a purely lateral gravitational flow takes place. The above statements contain in embryo the results of this contribution, which will be presented later in detail.

Figure 3.6 and Figure 3.7 illustrate the absorption simulation case and presents the pertinent geometrical constructions by means of the Mohr's circles.

The parametric analysis of the absorption simulation case is rather easy. If we plot  $\alpha$  versus  $\phi$  for several values of the parameter  $\lambda$ , we obtain the curves of Figure 3.8. For each given value of  $\lambda$  there is a minimum value  $\alpha_{\min}$  of  $\alpha$  below which absorption simulation cannot occur. This value is given by

$$\alpha_{\min} = \tan^{-1} \frac{2\sqrt{\lambda}}{\lambda - 1}, \quad (3.42)$$

and the locus of the minima is the straight line  $\alpha = 2\phi$ . On the other hand, given any value of  $\alpha$ , there is a value of  $\lambda^*$  of  $\lambda$  below which no simulation of absorption can occur. This value is

$$\lambda^* = \frac{1 + \cos \alpha}{1 - \cos \alpha}. \quad (3.43)$$

For  $\lambda > \lambda^*$  there are two possible value  $\phi_1^*$  and  $\phi_2^*$  of  $\phi$  for which absorption simulation occurs. These two values satisfy the following inequality

$$0 < \phi_1^* < \frac{\alpha}{2} < \phi_2^* < \alpha$$

and are given by



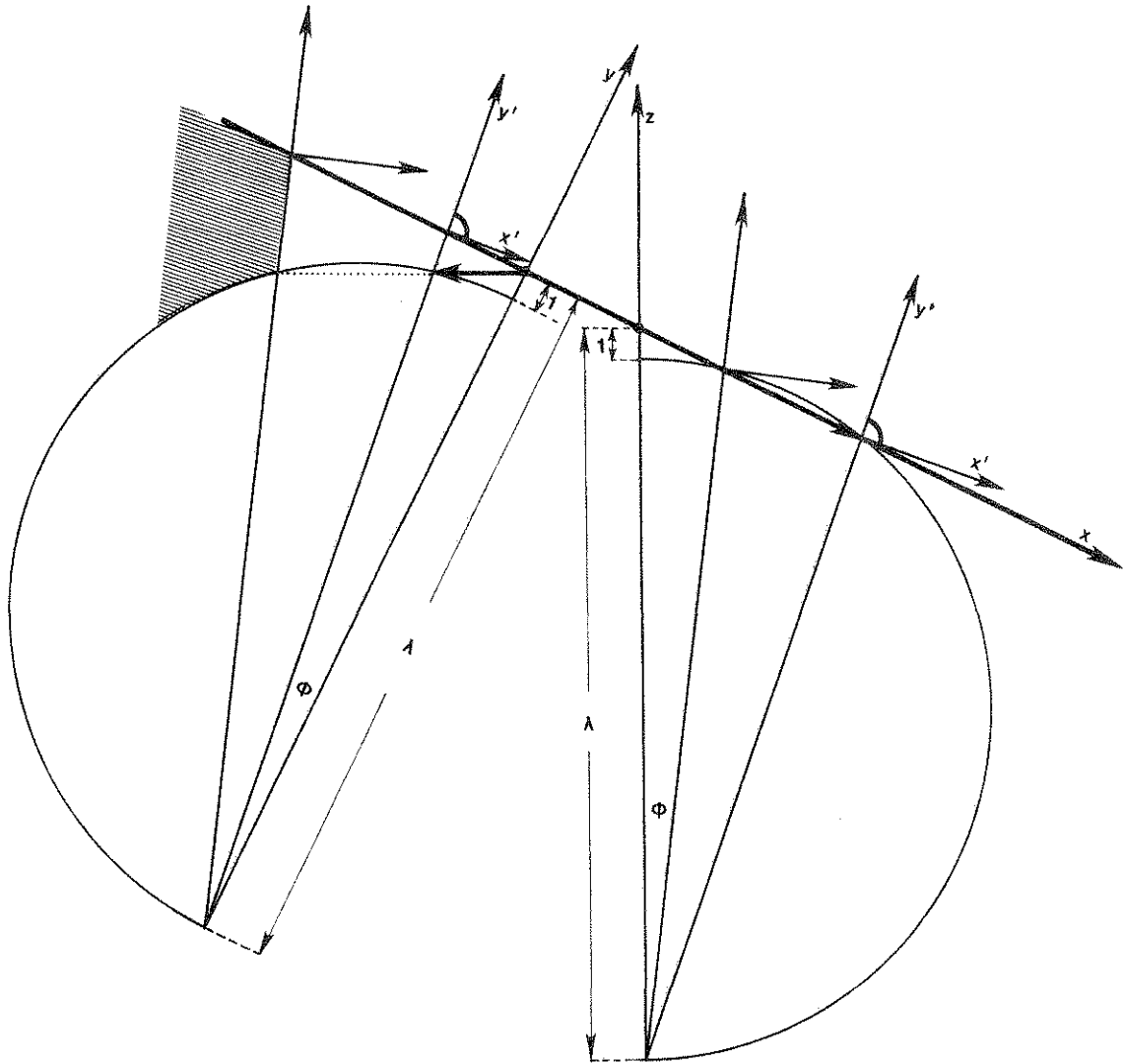


Figure 3.7. Geometrical constructions for the absorption simulation case.

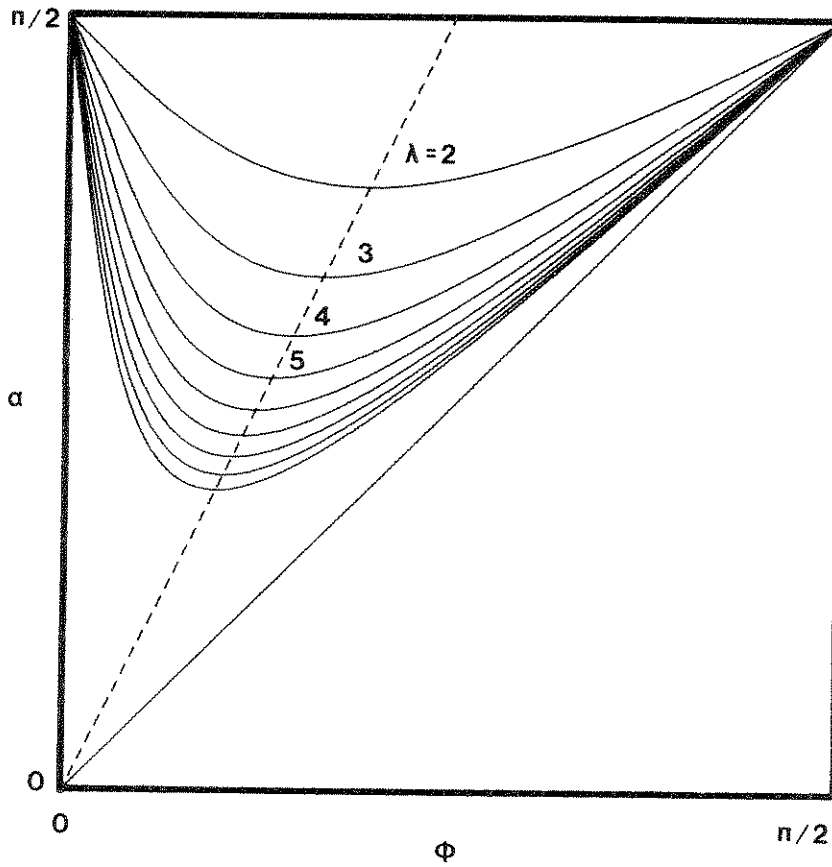


Figure 3.8. Relationship between  $\alpha$ ,  $\phi$ , and  $\lambda$  in the absorption simulation case.

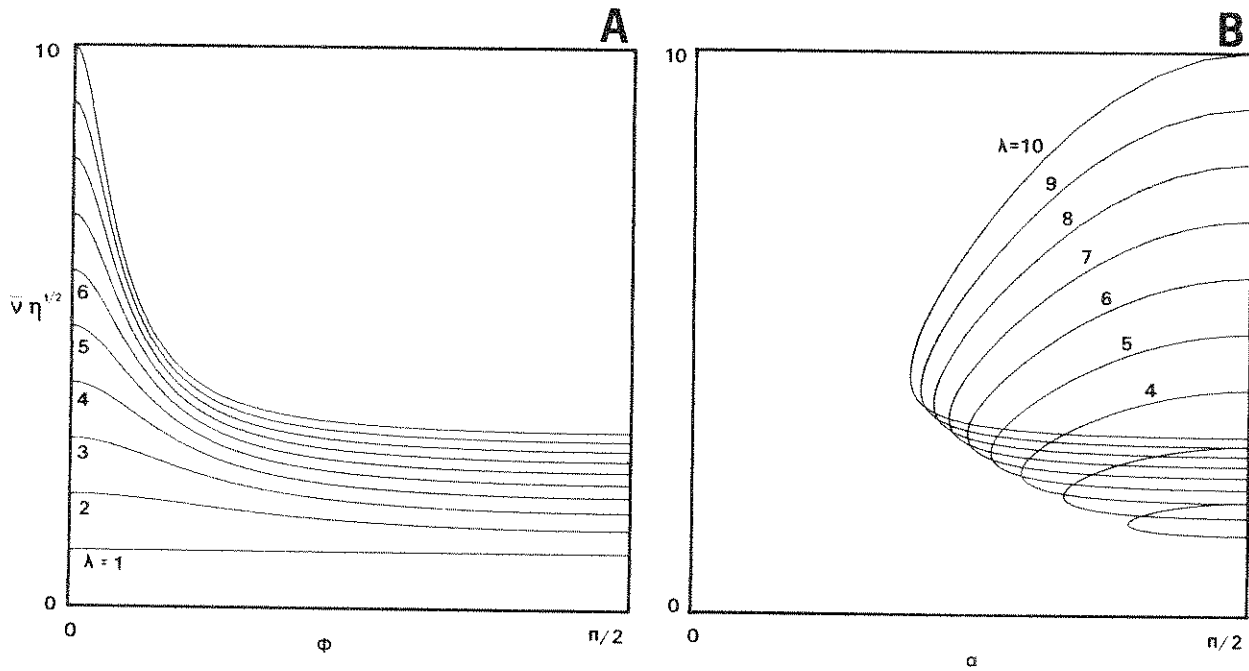


Figure 3.9. Magnification factor for lateral flow due to gravity.

$$\phi_1^*, \phi_2^* = \tan^{-1} \left| \frac{\tan \alpha}{2} \frac{\lambda - 1}{\lambda} \mp \sqrt{\left| \frac{\tan \alpha}{2} \frac{\lambda - 1}{\lambda} \right|^2 - \frac{1}{\lambda}} \right|. \quad (3.44)$$

Of the two cases, the one corresponding to  $\phi = \phi_1^*$  is the most interesting for the purpose of lateral flow. In fact, that case has a rather large lateral flow with respect to the absorption flow.

### 3.4 Subsurface Lateral Flow in the Case of Absorption Simulation

We formulate our problem in the following fashion. We choose an arbitrary value of  $\lambda > 1$ , and then a value of  $\phi$  that satisfies the following inequality

$$0 < \phi < \frac{1}{2} \tan^{-1} \frac{2\sqrt{\lambda}}{\lambda - 1}, \quad (3.45)$$

so that we are sure that we are in the most propitious conditions for lateral flow, then we calculate the value of  $\alpha$  by means of (3.37). The absorption simulation case is therefore assured.

We set as initial condition

$$t = 0, y > 0, \theta = \theta_0 \quad (3.46)$$

and as boundary condition

$$t > 0, y = 0, \theta = \theta_1 \quad (3.47)$$

The initial condition implies an initial uniform flow, due to gravitation,  $q_{x0} = \bar{v} K(\theta_0)$ .

Following the suggestion of Philip (1969), we define the cumulative absorption into the porous medium as

$$i = \int_{-\infty}^0 (\theta - \theta_0) dy = - \int_{-\infty}^0 y d\theta$$

$$i = \eta^{1/2} t^{1/2} \int_{\theta_0}^{\theta_1} \phi(\theta) d\theta, \quad (3.48)$$

where  $y(\theta) = -\eta^{1/2} t^{1/2} \phi(\theta)$  and where the integral appearing in (3.48) is the sorptivity  $S(\theta_0, \theta_1)$  of an isotropic porous medium of hydraulic conductivity  $K$ , from  $\theta_0$  to  $\theta_1$ . The absorption rate, in the  $y$  direction, is then

$$v = - \frac{1}{2} \eta^{1/2} t^{-1/2} S(\theta_0, \theta_1). \quad (3.49)$$

The novel feature of the hillslope case on an anisotropic medium is constituted by the lateral flow. If we call  $Q_x$  the excess of lateral flow with respect to the initial condition, we obtain

$$Q_x = \int_{-\infty}^0 (q_x - q_{x0})$$

or

$$Q_x = \int_{-\infty}^0 \left[ -\chi D \frac{\partial \theta}{\partial y} + \bar{v}(K - K(\theta_0)) \right] dy$$

or

$$Q_x = -\chi \int_{\theta_0}^{\theta_1} D(\theta) d\theta + \bar{v} \eta^{1/2} t^{1/2} \int_{\theta_0}^{\theta_1} \phi(\theta) \frac{\partial K}{\partial \theta} d\theta. \quad (3.50)$$

It is to be noted that  $Q_x$  is made up of two contributions: a negative absorption contribution which is time independent and which is conceptually justified by the fact that a negative  $q_{ax}$  proportional to  $t^{-1/2}$  takes place in a region of thickness proportional to  $t^{1/2}$ , and a positive gravity induced contribution proportional to  $t^{1/2}$ .

We will call the integral

$$C(\theta_0, \theta_1) = \int_{\theta_0}^{\theta_1} D(\theta) d\theta \quad (3.51)$$

the (non-directional) absorption conveyance of the medium and the integral

$$G(\theta_0, \theta_1) = \int_{\theta_0}^{\theta_1} \phi(\theta) \frac{\partial K}{\partial \theta} d\theta \quad (3.52)$$

the (non-directional) gravitation conveyability of the medium.

With these definitions, (3.50) can be expressed in terms of  $\lambda$  and  $\phi$ , as

$$Q_x = - \frac{\lambda - 1}{2} \sin 2\phi C(\theta_0, \theta_1) + \lambda \sqrt{\frac{\lambda+1 - (\lambda-1)\cos 2\phi}{\lambda^2+1 - (\lambda^2-1)\cos 2\phi}} G(\theta_0, \theta_1) t^{1/2}$$

or

$$Q_x = - \chi C(\theta_0, \theta_1) + \bar{v} \eta^{1/2} G(\theta_0, \theta_1) t^{1/2}. \quad (3.53)$$

Figure 3.9.A presents the coefficient  $\bar{v} \eta^{1/2}$  in terms of  $\lambda$  and Figure 3.9.B presents  $\bar{v} \eta^{1/2}$  in terms of  $\lambda$  and  $\alpha$ .

If we consider  $i$  as the thickness of the region where flow occurs, we can define the average velocity  $v_x$  at any time  $t$  as

$$v_x = \frac{Q_x}{i} = - \chi \eta^{-1/2} t^{-1/2} \frac{C}{S} + \bar{v} \frac{G}{S}, \quad (3.54)$$

and if we respect the directions of the  $q_a$  (horizontal) and  $q_g$  (lateral) found before, we obtain

$$v_s = - \frac{\chi}{\cos \alpha} \eta^{-1/2} t^{-1/2} \frac{C}{S} + \bar{v} \frac{G}{S} \cos \alpha \quad (3.55)$$

$$v_z = - \bar{v} \frac{G}{S} \sin \alpha$$

which, once integrated in  $t$ , yield

$$\begin{aligned} s &= s_0 - 2 \frac{\lambda}{\cos \alpha} \eta^{-1/2} t^{-1/2} \frac{C}{S} + \frac{v}{S} \frac{G}{S} \cos \alpha t \\ z &= z_0 - \frac{v}{S} \frac{G}{S} \sin \alpha t \end{aligned} \quad (3.56)$$

as the approximate trajectory of a particle that at time  $t = 0$  leaves the point  $s_0, z_0$  on the surface of the incline.

A measure of the closeness of the trajectories to the incline is given by the elongation  $E$ , defined as the distance between the point of departure  $s_0, z_0$  and the next point in the trajectory where  $s = s_0$ . The elongation is given by

$$E = \frac{C^2}{SG} \frac{1}{\lambda} \left| \frac{\lambda^2 + 1 - (\alpha^2 - 1) \cos 2 \phi}{(\lambda - 1) \sin 2 \phi} \right|^2. \quad (3.57)$$

Figure 3.10 illustrates the trajectories described above for three values of the parameter  $E$ .

### 3.5 Summary

It has been shown that in the case of nonisotropic soils, infiltration through an incline may lead to lateral flow. For those geometrical combinations of parameters which lead to absorption simulation the lateral flow can be calculated in terms of those geometrical parameters, of Philip's sorptivity, and of two new properties of porous media which have been called absorption conveyance and gravitation conveyability.

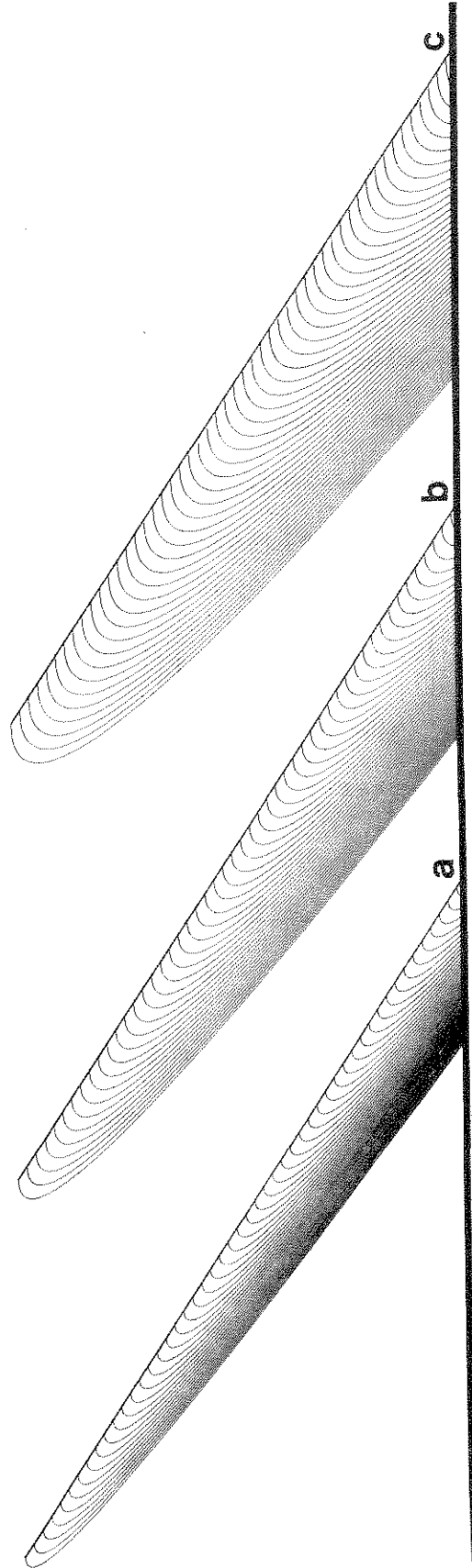


Figure 3.10. Approximate trajectories of fluid particles released at time  $t=0$  from the incline's surface. The three cases illustrated correspond to elongations in the ratio 1:2:4.

## CHAPTER IV: NUMERICAL ANALYSIS

In this chapter the one-dimensional Fokker-Planck equation is solved numerically, using an explicit finite difference scheme. The Fokker-Planck equation is the mathematical model for the infiltration process in a porous medium. It is a second order partial differential equation of the parabolic type.

A second order explicit finite difference scheme is used for the spatial dependency and a second order Runge-Kutta algorithm is used for the time dependency.

Due to the fact that at the beginning of the rainfall the surface of the soil is saturated or unsaturated, both cases have been considered. It is of great importance to study the behavior of the infiltration pattern in the initial stages before saturation has been reached. The present model can handle these conditions, and the method and assumptions used are discussed in detail later.

In the following paragraphs a brief description of the integration scheme used for the numerical solution, is given.

### 4.1 Basic Equations

Starting point of the numerical analysis is equation (2.34), which for the sake of reference ease is repeated here

$$\frac{\partial \theta}{\partial t} = \nabla(K \nabla \psi) + \frac{\partial K}{\partial z} . \quad (4.1)$$



Given that  $\psi = \psi(\theta)$  and that  $\theta = \theta(x,y,z)$ , it can easily be shown that  $\nabla\psi = \frac{\partial\psi}{\partial\theta}\nabla\theta$ ; multiplying both sides by K, and defining the term  $K\frac{\partial\psi}{\partial\theta}$  as the moisture diffusivity D, then the Fokker-Planck equation can be written as

$$\frac{\partial\theta}{\partial t} = \nabla(D\nabla\theta) + \frac{\partial K}{\partial z} . \quad (4.2)$$

The problem of concern in the present study is the one-dimensional infiltration. Taking now z as positive downward the one-dimensional Fokker-Planck equation becomes

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}(D\frac{\partial\theta}{\partial z}) - \frac{\partial K}{\partial z} . \quad (4.3)$$

In order to solve equation (4.3) it is required to know the variation of D and K with respect to  $\theta$ . Ward, Wells and Phillips (1983) reported that these function are given different expressions by different authors. In this study Burdine's approximation was chosen for the function K, namely

$$K = K_s \left| \frac{\theta - \theta_r}{\theta_s - \theta_r} \right|^{2.5 + 2/\lambda} . \quad (4.4)$$

where  $K_s$  and  $\theta_s$  are the hydraulic conductivity and soil moisture content at saturation respectively,  $\theta_r$  is the residual moisture content and  $\lambda$  is a soil characteristic.

The Brooks-Corey (1966) model was chosen for the function  $\psi$ , namely

$$\psi = \psi_e \left| \frac{\theta - \theta_r}{\theta_s - \theta_r} \right|^{-1/\lambda} , \quad (4.5)$$

where  $\psi_e$  is the air entry matrix suction occurring when  $\theta = \theta_e$ .

Both models are simple to implement and although  $\frac{\partial \theta}{\partial \psi}$  becomes zero at  $\theta = \theta_r$ , it can be shown that D does not. Substituting equations (4.4) and (4.5) into the definition of D we obtain

$$D(\theta) = - \frac{K_s \psi_e}{\lambda} \frac{(\theta - \theta_r)^{1.5 + 1/\lambda}}{(\theta_s - \theta_r)^{2.5 + 1/\lambda}} \quad (4.6)$$

The derivative  $\frac{\partial D}{\partial \theta}$  can be evaluated analytically as

$$\frac{\partial D}{\partial \theta} = - \frac{(1.5 + 1/\lambda)K_s \psi_e}{\lambda} \frac{(\theta - \theta_r)^{.5 + 1/\lambda}}{(\theta_s - \theta_r)^{2.5 + 1/\lambda}} \quad (4.7)$$

#### 4.2 Boundary and Initial Conditions

In order to obtain a numerical solution of equation (4.3), boundary conditions and initial conditions have to be specified.

Initial conditions are of the following form:

$$t = 0, \theta = \theta_o \text{ for all } z, \text{ where } \theta_i < \theta_o < \theta_s.$$

Normally,  $\theta_o$  is very small (dry soil), and the initial infiltration of water in the soil is mainly governed by moisture gradients (negligible gravity effect).

The boundary conditions are formulated as follows:

$$t > 0, \theta = \theta_i \text{ for } z=0.$$

A common assumption is that  $\theta_i = \theta_s$ , but in this study, this assumption is relaxed:

$$\theta_r < \theta_i < \theta_s$$

The initial condition must be supplied numerically and can be of any shape, satisfying always the conditions that the soil moisture content must be between  $\theta_r$  and  $\theta_s$ . Fig. 4.1 shows an initial conditions set up.

### 4.3 Numerical Scheme

The following two situations are considered:

- 1) The surface of the soil is saturated,
- 2) The surface of the soil is unsaturated.

#### 4.3.1 Saturated Soil Surface Conditions

Since gravity and capillary suction are equally important, equation (4.3) is written as

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial z^2} + \frac{\partial D}{\partial \theta} \left( \frac{\partial \theta}{\partial z} \right)^2 - \frac{\partial K}{\partial z} . \quad (4.8)$$

Spatial discretization. Using a finite difference central scheme, the derivatives can be written in the following manner

$$\frac{\partial \theta}{\partial z} = \frac{\theta_{i+1} - \theta_{i-1}}{2 \Delta z} \quad (4.9a)$$

$$\frac{\partial^2 \theta}{\partial z^2} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta z)^2} \quad (4.9b)$$

$$\frac{\partial K}{\partial z} = \frac{K_{i+1} - K_{i-1}}{2 \Delta z} \quad (4.9c)$$

$$\frac{\partial D}{\partial z} = \frac{K_{i+1} - K_{i-1}}{2 \Delta z} \quad (4.9d)$$

At the surface the soil moisture content is at saturation level and is not changing with time, which implies that for  $i$

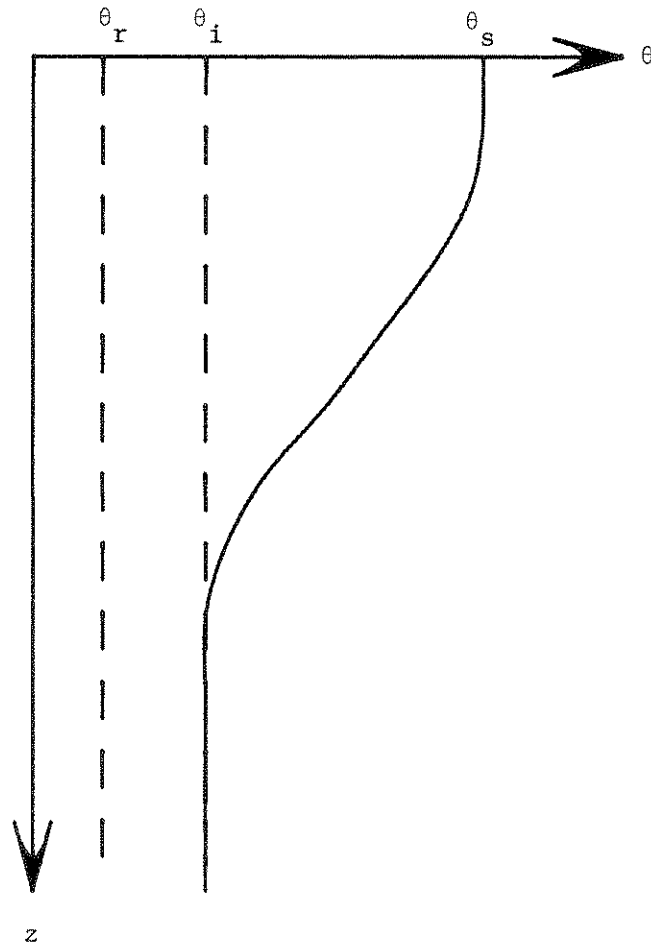


Figure 4.1. Typical initial condition for vertical infiltration problems.

equal to 2 up to n-1 the preceding scheme can be used without any problem.

Time discretization. A second order Runge-Kutta algorithm is used for the time advancement in order to calculate the new value of  $\theta_i$ , where  $i$  is the discrete space position:

$$\theta_i^{j+1} = \theta_i^j + \Delta t \left. \frac{\partial \theta}{\partial t} \right|_i^{j+\frac{1}{2}} \quad (4.10)$$

and

$$\theta_i^{j+\frac{1}{2}} = \theta_i^j + \frac{\Delta t}{2} \left. \frac{\partial \theta}{\partial t} \right|_i^j \quad (4.11)$$

#### 4.3.2 Unsaturated Soil Surface Condition

In this case a different scheme is used, due to the fact that the moisture content at the surface of the soil is changing with time. Therefore, the discretized Fokker-Planck equation cannot be used for the first two points of the spacing grid.

For  $i=1$  one cannot compute the second derivative; application of a skewed scheme would prove unsuccessful. For  $i=2$  one must take into account that the first derivative of the soil moisture content function at  $i=1$  ( $z=0$ ) is dictated by the rainfall intensity  $I$ . When saturation has not been reached all the rainfall is absorbed by the soil, then the infiltration flow is

$$I = - \left( D(\theta) \frac{\partial \theta}{\partial z} - K(\theta) \right) \quad (4.12)$$

from this equation the derivative  $\frac{\partial \theta}{\partial z}$  at the surface is calculated given the rainfall intensity  $I$ . To evaluate the soil moisture

content at the new time step, the Fokker-Planck equation can be expressed as

$$\frac{\partial \theta}{\partial z} = \frac{\partial}{\partial z} \left( D \frac{\partial \theta}{\partial z} + K \right) \quad (4.13)$$

or explicitly

$$\frac{\partial \theta}{\partial t} \Big|_2 = [D(3) \frac{\partial \theta}{\partial z} \Big|_3 - D(1) \frac{\partial \theta}{\partial z} \Big|_1] \frac{1}{2 \Delta z} - \frac{K(3) - K(1)}{2 \Delta z} \quad (4.14)$$

where

$$\frac{\partial \theta}{\partial z} \Big|_1 = - \frac{(Q - K(1))}{D(1)}$$

and

$$\frac{\partial \theta}{\partial z} \Big|_2 = \frac{\theta(4) - \theta(2)}{2 \Delta z} .$$

At the surface, where  $i=1$ , the right hand side of equation (4.12) cannot be developed, because the second derivative cannot be evaluated by numerical discretization. In this case the soil moisture content  $\theta$  is updated for each time step using the bulk continuity equation, namely

$$I \Delta t = \int_0^{\infty} \theta(z) dz \quad (4.15)$$

where  $\theta(z)$  is given for all points but for the first ( $i=1$ ). Then using the trapezoidal rule,  $\theta(1)$  is evaluated in the following manner

$$\theta(1) = 2 \left[ \frac{I \Delta t}{\Delta z} - \sum_{i=2}^{n-1} \theta_i - \frac{\theta_n}{2} \right]. \quad (4.16)$$

For all the points from  $i=3$  to  $i=n-1$ , the same scheme is used as before. The advance in time is similar to the saturation scheme, namely using the second order Runge-Kutta scheme.

#### 4.4 Computer Program

In APPENDIX I a listing is provided of the computer code used in the present study. A brief discussion follows.

The main program is designed to control the input and output facilities. Two sort of outputs are generated. The first one writes the numerical results in the input/output file DATA and the second one plots the results. The boundary conditions are set directly through the main program and the initial conditions are computed or read from the DATA file. Variables NP and NDP control respectively the number of curves to be plotted and the time increments between plotted curves. The product NP\*NDP is the total integration time.

Subroutine "FUNK" calculates the values of the hydraulic conductivity K and of the soil moisture difusivity D, given the moisture content  $\theta$ , by using equations (4.4) and (4.5).

Subroutine "MARCH" evaluates the right-hand side of equations (4.3) or (4.14) as required. Variable KEY3 indicates which equation to use, as far as saturation of the soil surface is concerned.

Subroutines "PLT" and "CURVE" are plotting subroutines. They are directed to the PURDUE PLOTTING LIBRARY.

Subroutine "UPDATE" performs the integration scheme which evaluates the soil moisture content at the surface when dealing with unsaturated conditions.

Fig. 4.2 shows a simple flowchart of the program algorithm.

#### 4.5 Results and Conclusions

The program was used to simulate infiltration patterns of four types of soils, the characteristics of which were taken from Ward, Welles and Phillips (1983). It was not possible to simulate sample bin No. 6 because the hydraulic conductivity was so high, that the moisture diffusivity  $D$  was out of range of the model used in this study.

The characteristics of the samples used in the simulation are summarized in Table I.

Graphical representations of the results are shown in Figures 4.3 to 4.9. Two types of curves are plotted, the first one represents the soil moisture content as a function of the depth and the second one represents the hydraulic conductivity as a function of the depth. Odd numbered figures concern the unsaturated condition, and even numbered figures concern the saturated condition.

In the unsaturated condition, one notices the discontinuity in the relationship between the soil moisture content and the depth at the very beginning of the infiltration process. This is due to the fact that the soil reaches saturation at the upper boundary faster than it lets moisture penetrate to sublayers.



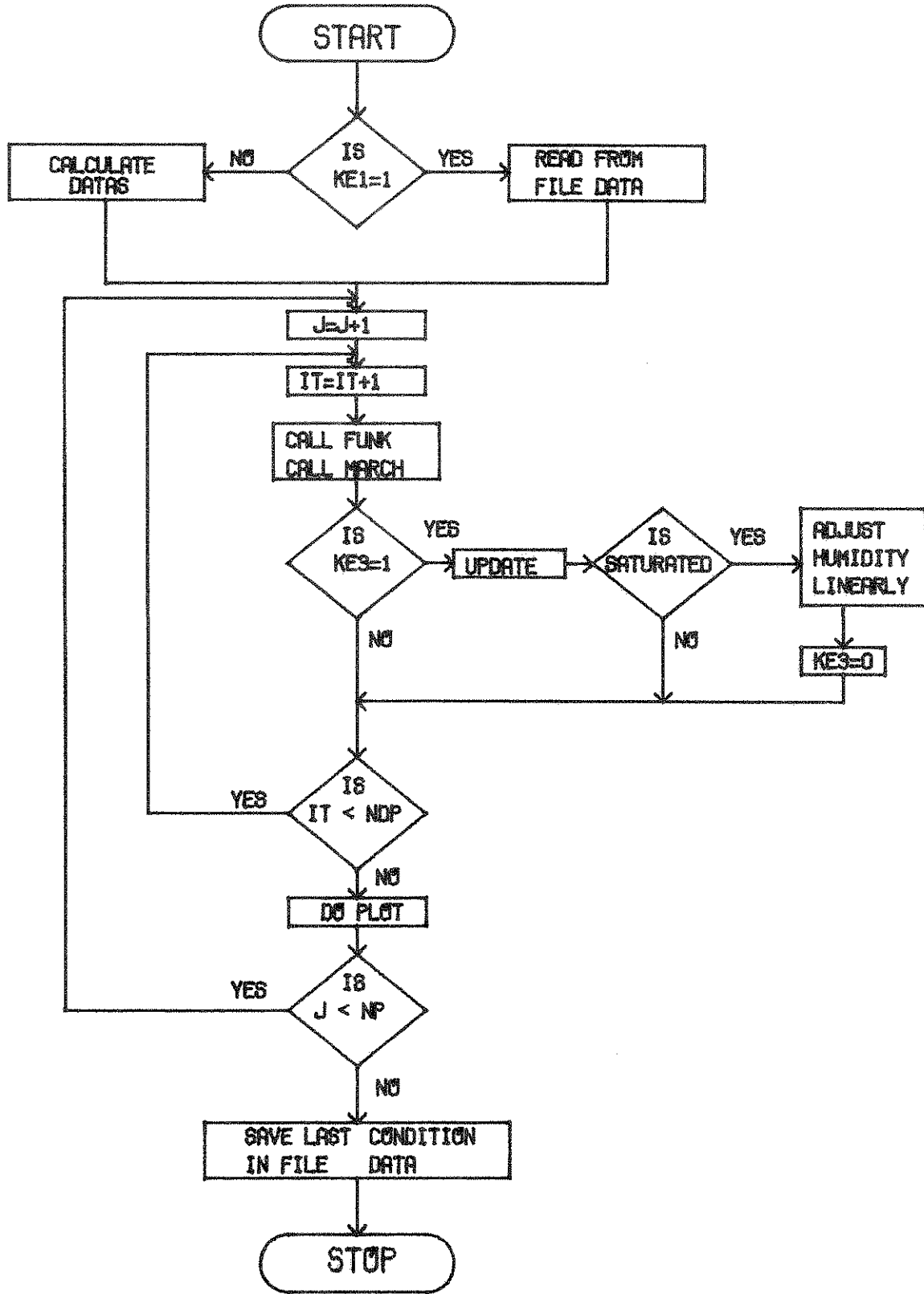


Figure 4.2. Flow chart for the program algorithm.

TI= 0.0000

T = .0039

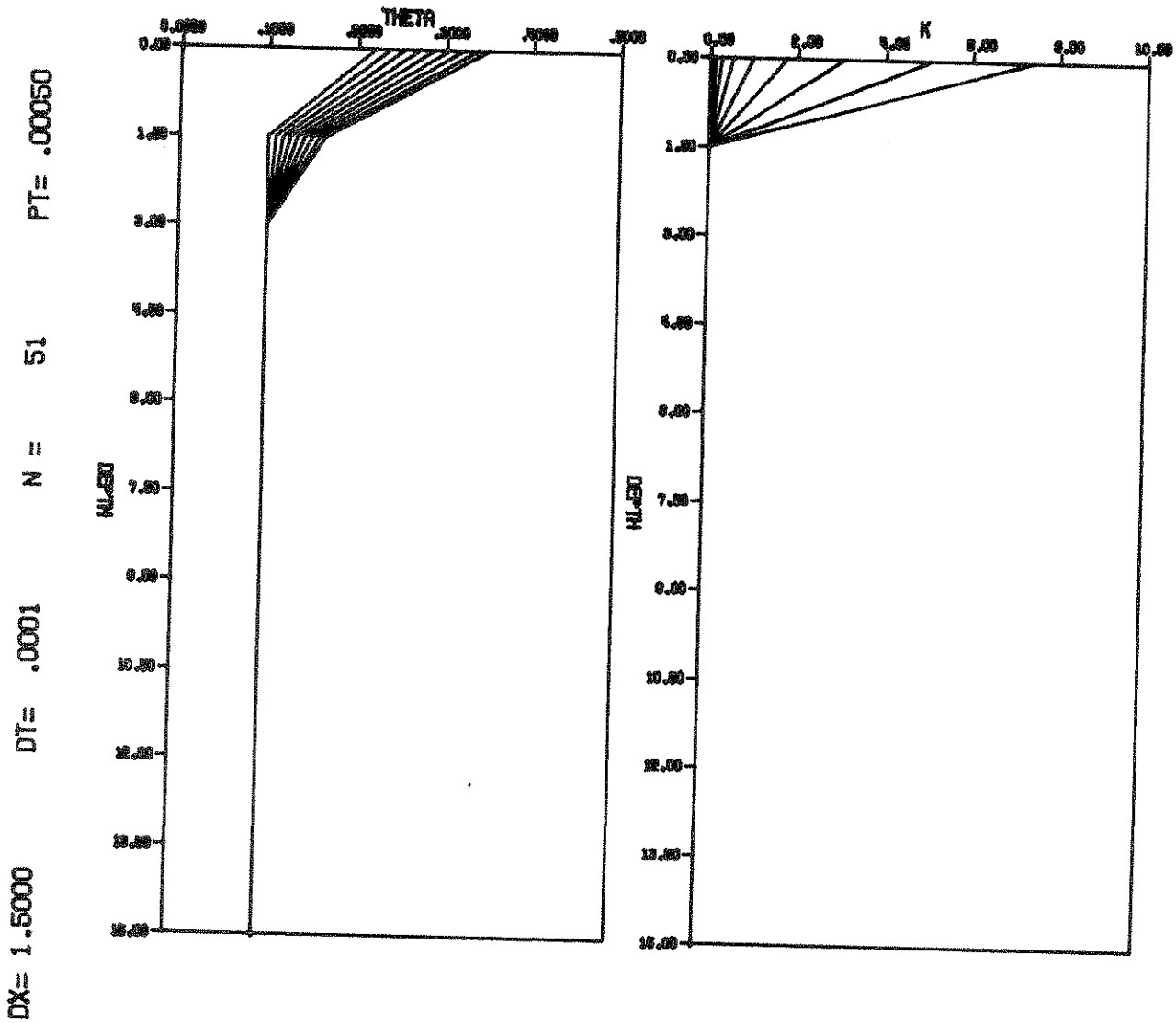


Figure 4.3. Time variation of soil moisture content and of hydraulic conductivity from t=0 until ponding for sample bin 1.

T = .0039

TF = .8000

DX = 1.5000  
DT = .00010  
N = 101  
PT = .04000

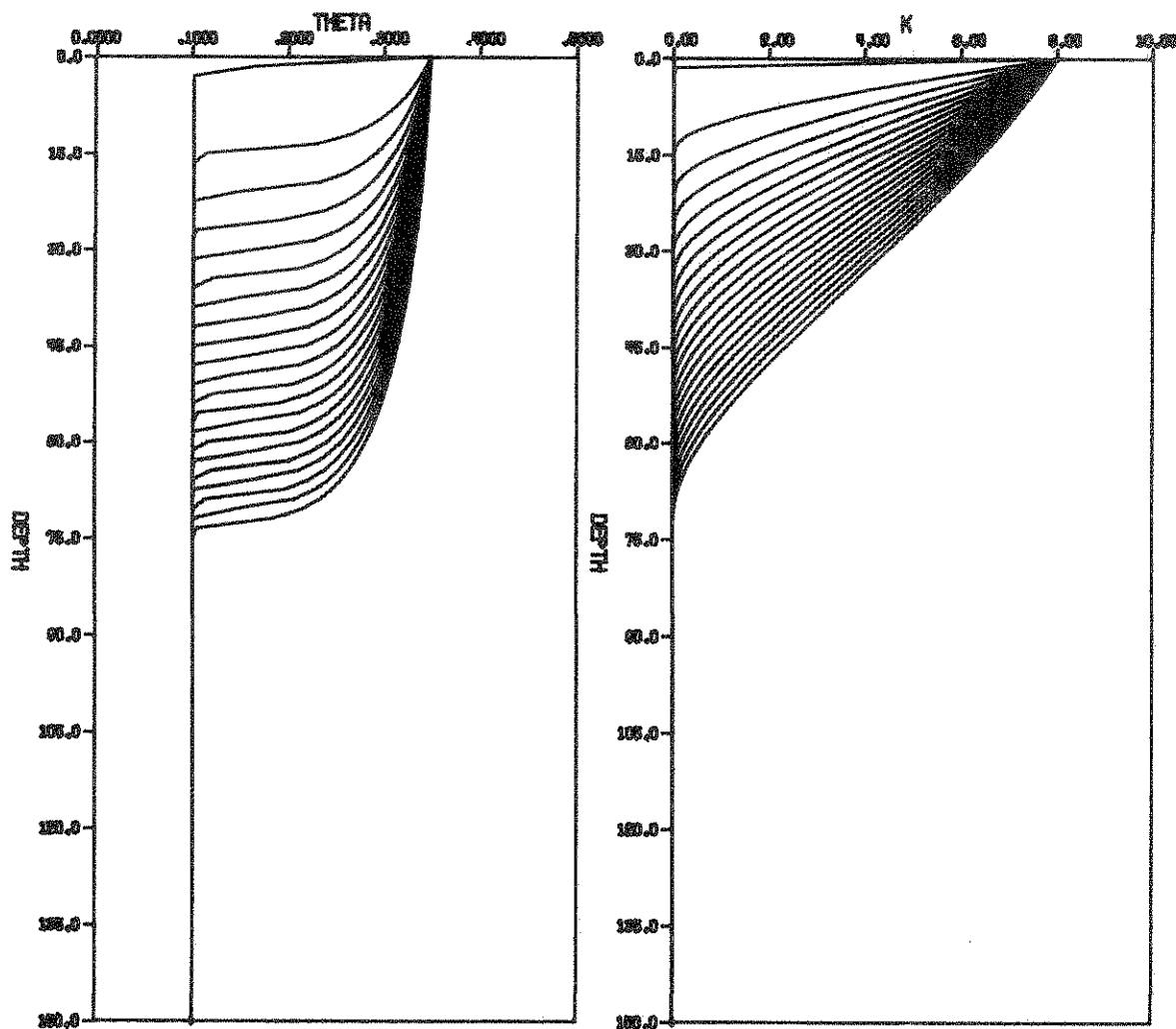


Figure 4.4. Time variation of soil moisture content and of hydraulic conductivity from initial ponding to  $t=0.8$  for sample bin 1.

TI= 0.0000

T = .0085

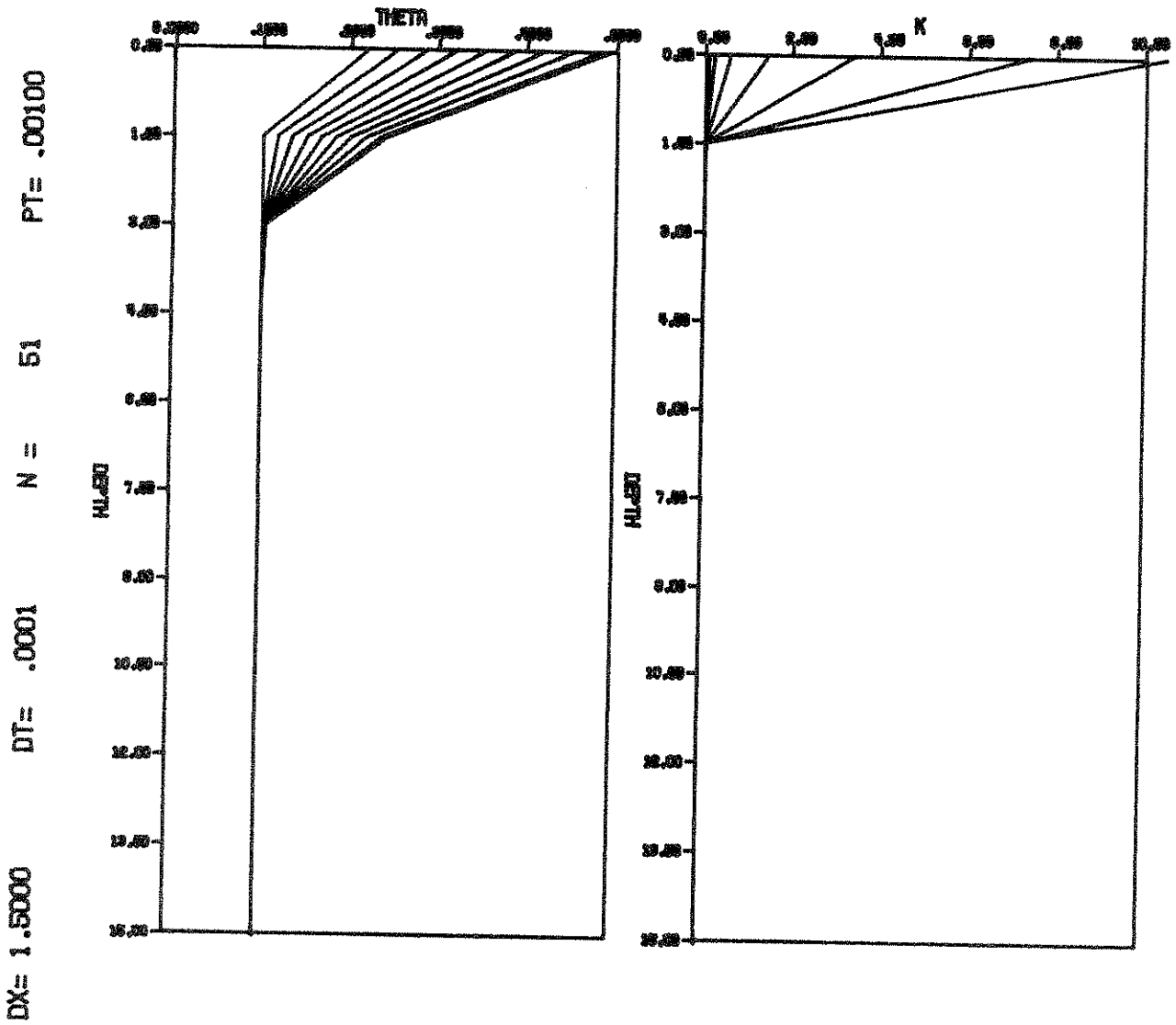


Figure 4.5. Time variation of soil moisture content and of hydraulic conductivity from t=0 until ponding for sample bin 3.

T = .0085

TF = .8000

DX = 1.5000  
DT = .00010  
N = 101  
PT = .04000

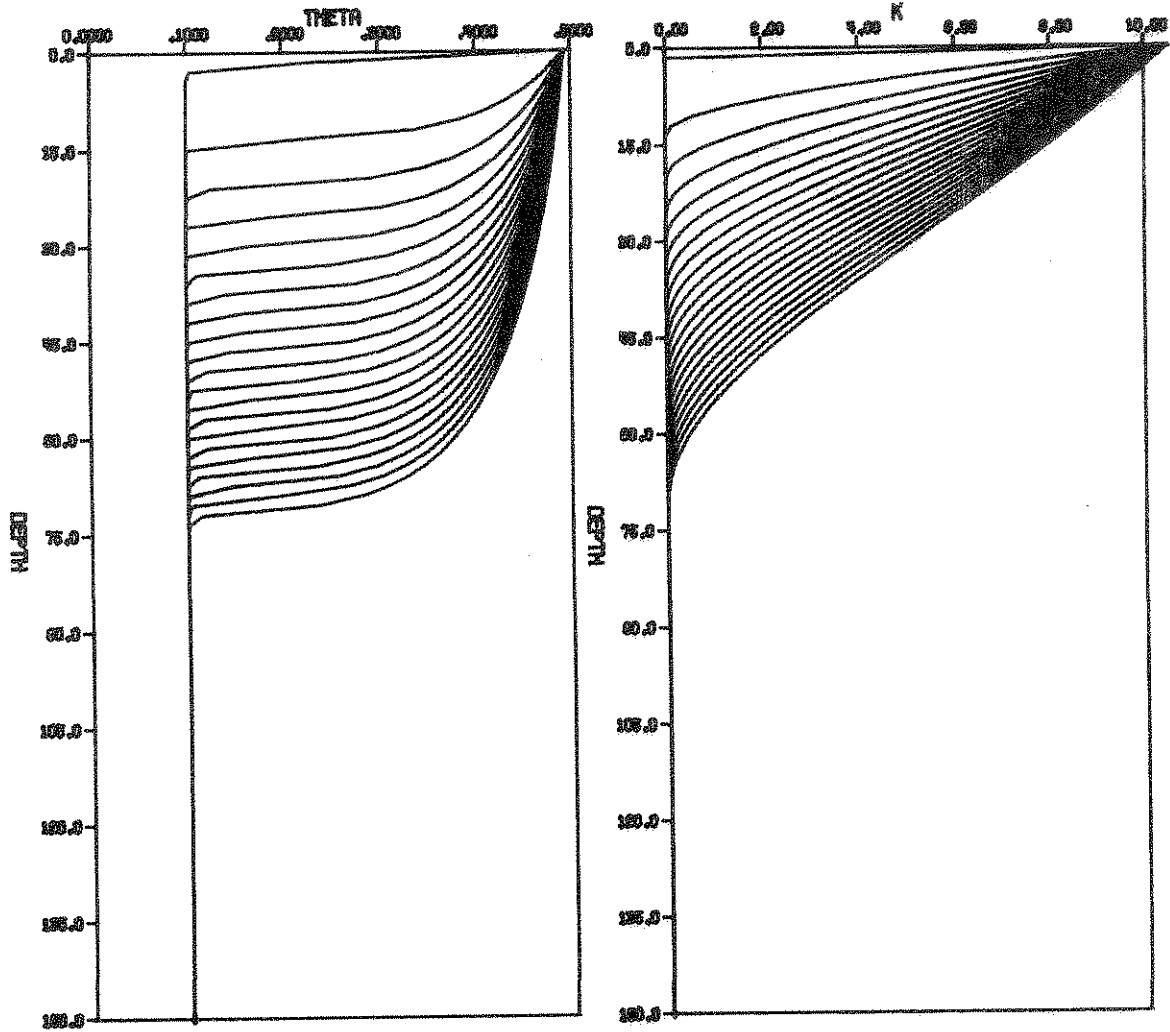


Figure 4.6. Time variation of soil moisture content and of hydraulic conductivity from initial ponding to t=.8 for sample bin 3.

TI= 0.0000

T = .0026

DX= 1.5000 DT= .0001 N = 51 PT= .00050

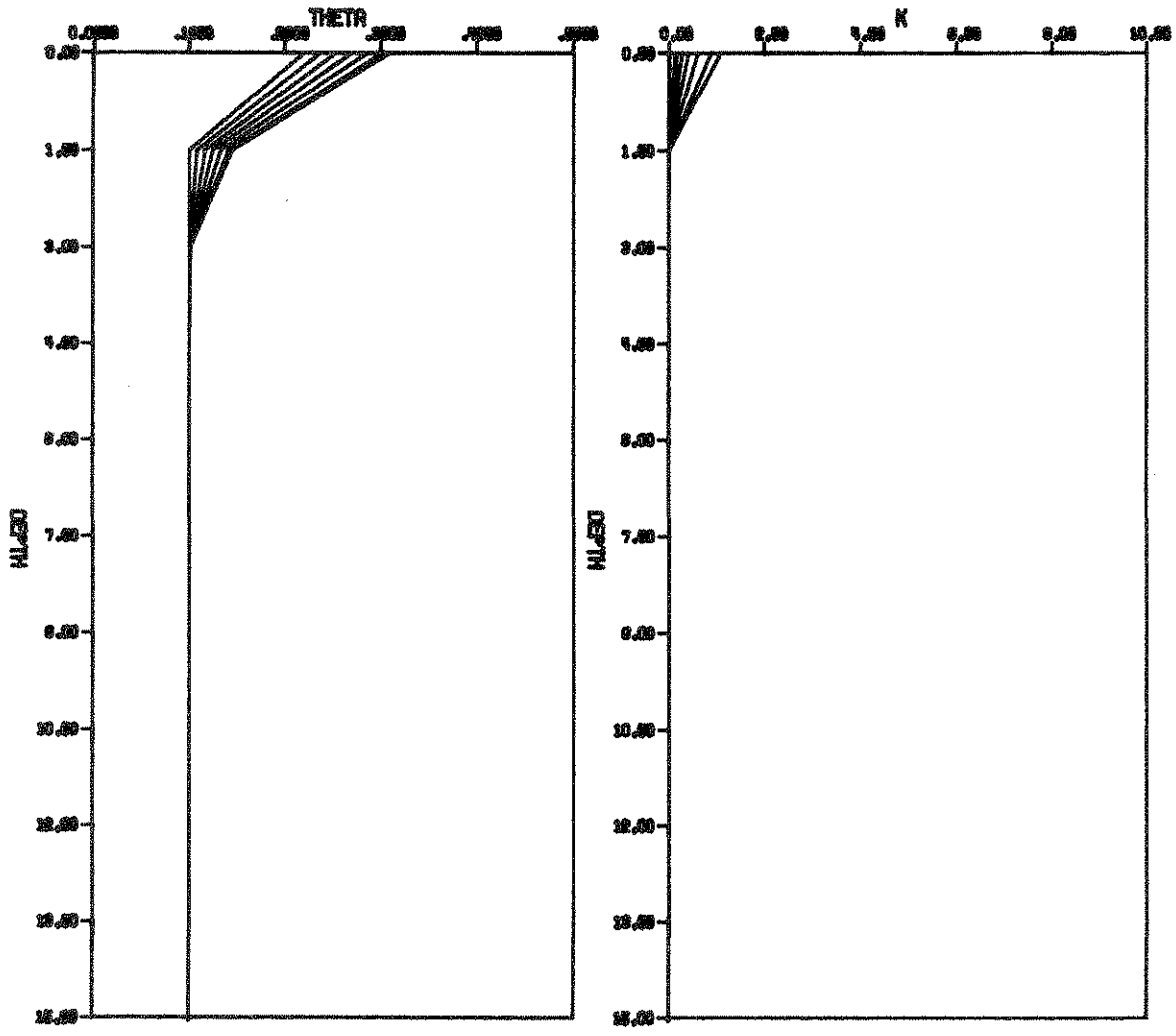


Figure 4.7. Time variation of soil moisture content and of hydraulic conductivity from t=0 until ponding for sample bin 4.

T = .0028

TF = .8000

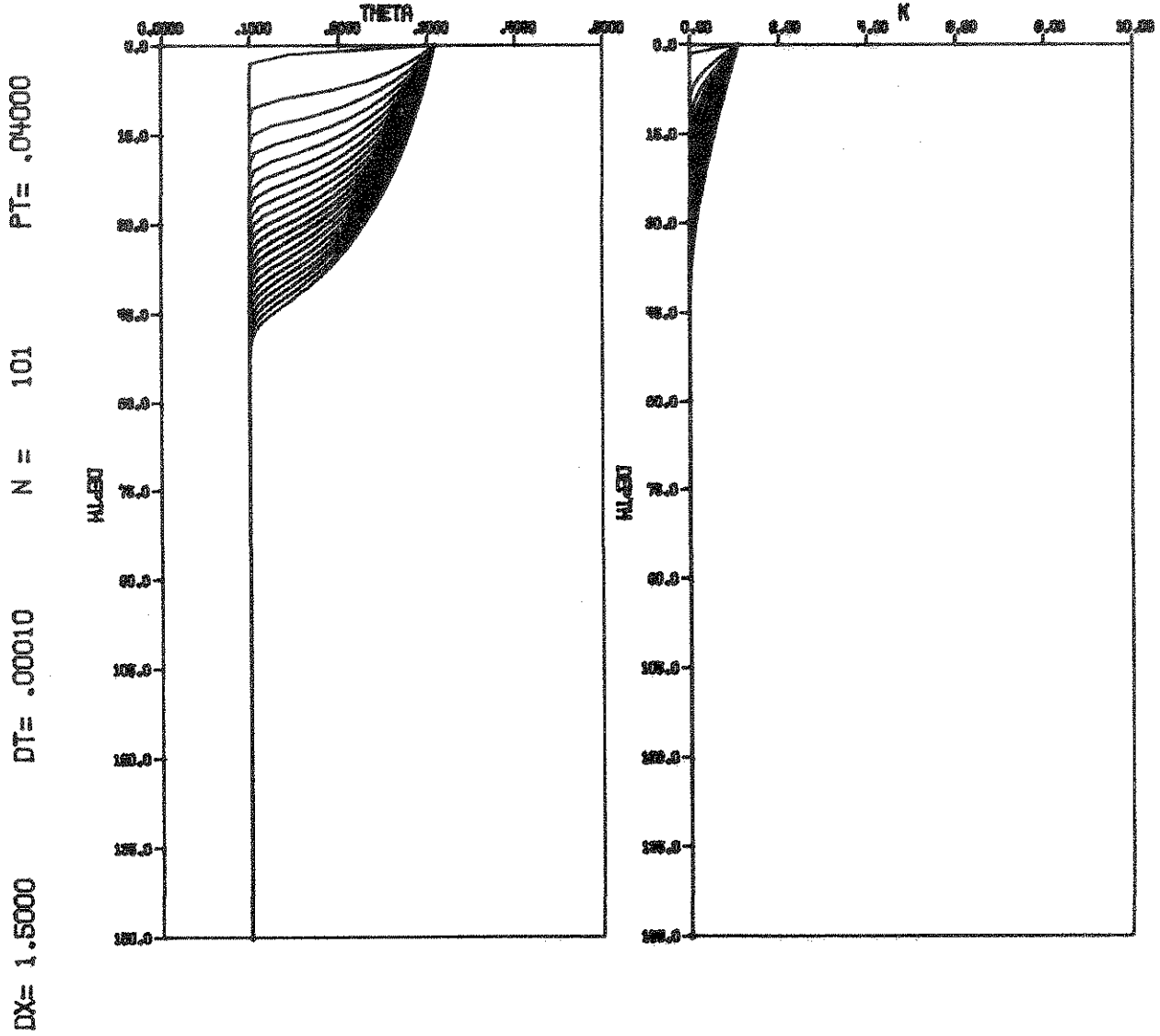


Figure 4.8. Time variation of soil moisture content and of hydraulic conductivity from initial ponding to  $t=.8$  for sample bin 4.

TI= 0.0000

T = .0072

DX= 1.5000 DT= .0001 N = 51 PT= .00100

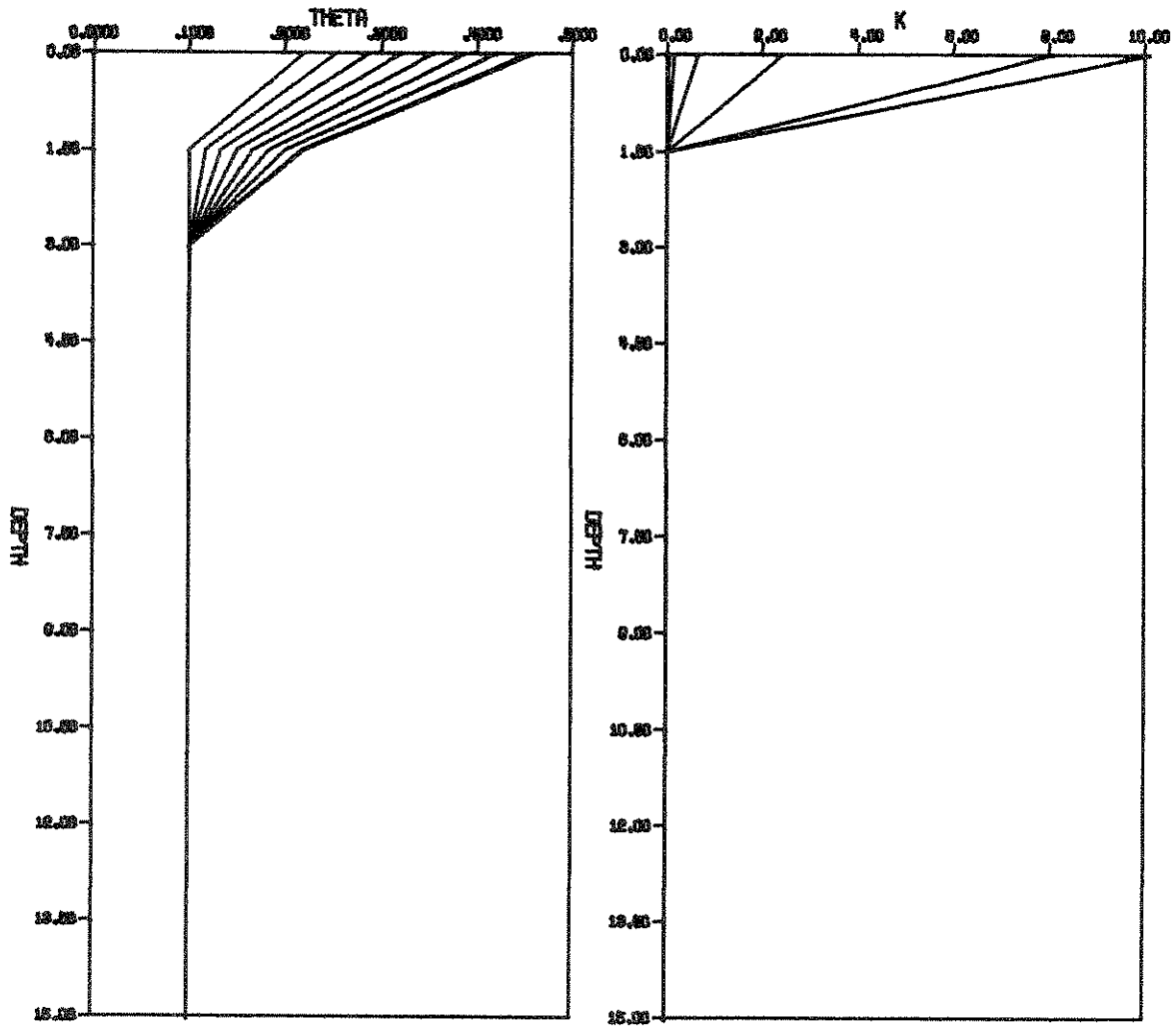


Figure 4.9. Time variation of soil moisture content and of hydraulic conductivity from t=0 until ponding for sample bin 5.



Sample bin	Type	$\theta_s$	$\psi_e$	$\lambda$	$\theta_r$	$K_s$
Units	--	vol/vol	mm	--	--	mm/hr
1	Topsoil	.348	113	.33	.09	8.0
3	Spoil	.458	101	.15	.00	11.0
4	Spoil	.309	289	.49	.00	1.1
5	Topsoil	.493	119	.22	.00	10.5

For the saturated condition, the behavior of the soil moisture content with respect to depth is parabolic, although it is exponential at the tail, in accordance with the curves obtained by Cheng (1975).

The examples studied here were restricted to the cases where the soil obeys the Brook-Corey and Burdine models.

The computer code presented here can easily handle other model to simulate the hydraulic conductivity  $K$ , the suction  $\psi$  and the moisture diffusivity  $D$ . The only change would be in the subroutine "FUNK".

## CHAPTER V: CONCLUSION AND RECOMMENDATIONS

The principal result of this study has been the demonstration that, under particular configurations of soil surface slope and soil layers orientation, infiltration can occur parallel to the soil surface.

Albeit this special configurations may occur in nature with small probability, the fact that they are possible was the main thrust of the study. Since most macroscopic hydrologic, or ecologic models usually consider infiltration as taking place vertically, and since this is the tacit assumption that is always done when infiltration is considered, the fact that infiltration may occur in the direction of the steepest descent of the soil surface has the obvious purpose of attracting attention to likely frequent deviations from the simple-minded vertical infiltration approximation.

Having done so, it is the ambition of this report to suggest avenues for further analytic and numerical study.

Since the constitutive equation  $\Psi = \Psi(\theta)$  and  $K = K(\theta)$  have usually rather complicated analytic expressions (and they are at most best fits of experimental data), it seems that comprehensive analytical studies can be ruled out in favor of computer simulations. Along these lines the authors of this report wish to suggest the following research directions:

A. The real trajectories of fluid particles should be studied

in the particular case shown in this report instead of the "averaged" trajectories as presented here. While the averaged trajectories have a definite descriptive value, they should be considered as shorofcut versions at the true trajectories, which can be obtained only by computational means. The computer program presented in Chapter IV can be used for such undertaking and that was the reason for the program's development.

- B. The asymptotic behavior of the fluid particle trajectories should be studied for the possible configurations of soil surface slope, soil layers orientation and degree of anisotropy. This requires only some patient work of cataloguing what is already contained in nuce in the reports.
  
- C. The effect of soil surface curvature should be studied with utmost care for isotropic and nonisotropic soils. The fact that surface curvature and soil anisotropy combined may cause the vadose water to migrate in nonhomogeneous fashion, creating regions of higher humidity and dry regions, should be considered of extreme importance in all those situations in which the water resources are linked to (or limited to) the vadose zone.

The first most important problems to be analyzed in such context are the following:

- a. Infiltration in nonisotropic soils where the soil surface is a cylindrical sinusoidal form;

- b. Infiltration in nonisotropic soils when the soil surface is the sum of two cylindrical sinusoidal forms with orthogonal generatrices.

The result of these studies should be in the form of moisture maps showing the evolution of vadose water concentrations.

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APPENDIX I: COMPUTER CODE

```

program main (output,data,plot,tape6=output,tape10=data)
parameter (nn=101)
real theta(nn),k(nn),d(nn),aux(nn),dy(nn),thf(nn)
real lamda,ks
common /aa/ lamda,dx,dt,thetas,thetar,psie,ks,q
common /ab/ ke2,ke3,x1,fae
common /bb/ t,ar

```

```

c
c*****
c
c      np      number of plotting
c
c      ndp     number of iterations between printings
c              of t printings or plottings are wanted
c
c      n       number of positions in the z direction
c              (n-1) must be multiple of 5
c
c      lamda   coefficient of uniformity
c
c      psie    suction pressure at saturation
c
c      ks      hydraulic conductivity at saturation
c
c      dx      length increments
c
c      dt      time increments
c
c      q       rain intensity
c
c      k       hydraulic conductivity
c
c      d       difusivity
c
c      theta   humidity
c
c      thetar  residual humidity
c
c      thetas  humidity at saturation
c
c      t       actual time
c
c      kel     key to indicate if
c
c      0       initial condition is calculated into the program
c      1       initial condition is read from the file "data"
c
c      ke2     key to indicate
c

```



```
c      0 no axis to be plotted in subroutine plt
c      1 axis will be plotted in subroutine plt
c
c      ke3      key to indicate
c
c      0 humidity at the upper boundary is at saturation
c      1 humidity at the upper boundary in not at saturation
c
c*****
c
      kel = 0
      np = 20
      ndp = 100
      n = 101
      lamda = .33
      thetar = .05
      thetas = .348
      thetai = .1
      thetaw = .22
      psie = -128.
      ks = 8.
      dx = 1.5
      dt = 1.e-4
      q = 50.
c
      fac = ndp*dt
      ke3 = 0
      xl = dx*(n-1)
c
c*****
c
      set initial conditions
c
c*****
c
      if (kel.eq.0) then
          t = 0.
          do 10 i = 1,n
              theta(i) = thetai
              if (i.eq.1) theta(i) = thetaw
10          continue
      else
          read (10,120) n,dx,dt,t
          do 20 i = 1,n
              read (10,130) x,theta(i)
20          continue
      endif
c
      sum = 0.
      inx = (n-1)/2
      do 30 i = 1,inx
          il = 2*i
          i2 = il+1
```

```

    sum = sum+4.*theta(i1)+2.*theta(i2)
30 continue
    ar = (theta(1)-theta(n)+sum)*dx/3.
    if (theta(1).lt.thetas) then
        do 40 i = 1,n
            aux(i) = theta(i)
40     continue
        ke3 = 1
    endif
    call funk (theta,k,d,n)
    call plots
    call factor (.50)
    call number (1.5,13.,.2,t,0.,'3hti=,f7.4')
    ke2 = 1
    call plt (theta,k,n)
    ke2 = 0
    it = 0
c
c*****
c
c   initiate calculations
c
c*****
c
    thf(n-1) = thetai
    thf(n) = thetai
    do 100 j = 1,np
50     call funk (theta,k,d,n)
        it = it+1
        t = t+dt
        call march (theta,k,d,dy,n)
        thf(1) = theta(1)
        do 60 i = 2,n-2
            thf(i) = theta(i)+dy(i)/2
60     continue
        call march (thf,k,d,dy,n)
        do 70 i = 2,n-2
            theta(i) = theta(i)+dy(i)
70     continue
        if (ke3.eq.1) then
            call update (theta,k,n)
            if (theta(1).lt.thetas) then
                do 80 i = 1,n-2
                    aux(i) = theta(i)
80             continue
            else
                fact = (thetas-aux(1))/(theta(1)-aux(1))
                theta(1) = thetas
                do 90 i = 2,n-2
                    theta(i) = aux(i)+fact*(theta(i)-aux(i))
90             continue
            call plt (theta,k,n)
            call number (9.,12.,.2,t,0.,'3ht =,f7.4')
```

```

        call plot (15.,-1.,-3)
        call number (1.5,13.,.2,t,0.,'3ht =,f7.4')
        ke3 = 0
        ke2 = 1
        call plt (theta,k,n)
        ke2 = 0
    endif
endif
if (it.lt.ndp) go to 50
call plt (theta,k,n)
it = 0
100 continue
c
c*****
c
c    save results in file data
c
c*****
c
    rewind 10
    write (10,120) n,dx,dt,t
    do 110 i = 1,n
        x = (i-1)*dx
        write (10,130) x,theta(i),k(i),d(i)
110 continue
120 format (i10,4e15.5)
130 format (4e15.5)
    call number (9.,12.,.2,t,0.,'3htf=,f7.4')
    call plot (0.,0.,999)
    stop
    end
c
c*****
c
c    subroutine "funk" calculates the values of the
c    hydraulic contivity k and the difusivity d.
c
c*****
c
    subroutine funk (theta,k,d,n)
    real theta(n),k(n),d(n)
    real lamda,ks
    common /aa/ lamda,dx,dt,thetas,thetar,psie,ks,q
    do 20 i = 1,n
        tt = theta(i)-thetar
        if (tt.gt.0.) go to 10
        k(i) = 0.
        d(i) = 0.
        go to 20
10    k(i) = ks*(tt/(thetas-thetar))**(2.5+2./lamda)
        d(i) = -(ks*psie/(lamda*(thetas-thetar)))*(tt/
        * (thetas-thetar))**(1.5+1./lamda)
20 continue
```

```
return
end
```

```
c
c*****
c
```

```
c subroutine "march" advance in time of the value
c of the humidity theta
```

```
c
c*****
c
```

```
subroutine march (theta,k,d,dy,n)
real theta(n),k(n),d(n),dy(n)
real lamda,ks
common /aa/ lamda,dx,dt,thetas,thetar,psie,ks,q
common /ab/ ke2,ke3,xl,fac
if (ke3.eq.1) then
  dlth1 = -(q-k(1))/d(1)
  dlth2 = (theta(4)-theta(2))* .5/dx
  d2th = (d(3)*dlth2-d(1)*dlth1)* .5/dx
  dlk = (k(3)-k(1))* .5/dx
  dy(2) = (d2th-dlk)*dt
else
  dld = (d(3)-d(1))* .5/dx
  dlth = (theta(3)-theta(1))* .5/dx
  d2th = (theta(3)-2.*theta(2)+theta(1))/(dx)**2
  dlk = (k(3)-k(1))* .5/dx
  dy(2) = (dld*dlth+d(2)*d2th-dlk)*dt
endif
do 10 i = 3,n-2
  dlth = (theta(i+1)-theta(i-1))* .5/dx
  d2th = (theta(i+1)-2.*theta(i)+theta(i-1))/(dx*dx)
  dlk = (k(i+1)-k(i-1))* .5/dx
  dld = (d(i+1)-d(i-1))* .5/dx
  dy(i) = (d(i)*d2th+dld*dlth-dlk)*dt
10 continue
return
end
```

```
c
c*****
c
```

```
c subroutine "plt" plot the partial results of the
c calculations. theta and k are plotted.
```

```
c
c*****
c
```

```
subroutine plt (theta,k,n)
real theta(n),k(n)
real xx1(200),xx2(200),yy(200)
common /aa/ lamda,dx,dt,thetas,thetar,psie,ks,q
common /ab/ ke2,ke3,xl,fac
if (ke2.eq.0) go to 10
if (ke3.eq.1) then
  xx1 = xl/50.
```

```
else
  xx1 = x1/10.
endif
call number (0.,0.,.2,dx,90.,'3hdx=,f7.4')
call number (0.,3.,.2,dt,90.,'3hdt=,f7.4')
call number (0.,6.,.2,n,90.,'3hn =,i7')
call number (0.,9.,.2,fac,90.,'3hpt=,f7.5')
call plot (1.5,1.,-3)
call axis (0.,10.,' depth',-6,10.,-90.,0.,xx1,1)
call axis (6.,10.,' depth',-6,10.,-90.,0.,xx1,1)
call axis (0.,10.,' theta',6,5.,0.,0.,.1,0)
call axis (6.,10.,' k      ',6,5.,0.,0.,2.,0)
call plot (0.,0.,3)
call plot (5.,0.,2)
call plot (5.,10.,2)
call plot (6.,0.,3)
call plot (11.,0.,2)
call plot (11.,10.,2)
10 if (ke3.eq.1) then
  n2 = n/5+1
  ff = 50.
else
  n2 = n
  ff = 10.
endif
do 20 i = 1,n2
  yy(i) = 10.-ff*(i-1)*dx/x1
  xx1(i) = theta(i)*10.
  xx2(i) = k(i)/2.+6.
20 continue
call curve (xx1,yy,n2)
call curve (xx2,yy,n2)
return
end

c
c*****
c
c  subroutine "curve" draw a curve given an ordered
c  pair x(i),y(i)
c
c*****
c
c  subroutine curve (x,y,n)
c  real x(n),y(n)
c  do 10 i = 1,n
c    if (i.eq.1) call plot (x(1),y(1),3)
c    if (i.gt.1) call plot (x(i),y(i),2)
10 continue
return
end

c
c*****
c
```

```
c      subroutine "update" evaluate the value of the humidity
c      at the surface using the continuity equation
c      in integral form
c
c*****
c
      subroutine update (theta,k,n)
      real theta(n),k(n)
      common /aa/ lamda,dx,dt,thetas,thetar,psie,ks,q
      real lamda,ks
      common /bb/ t,ar
c
      sum = 0.
      inx = (n-1)/2
      do 10 i = 1,inx
         il = 2*i
         i2 = il+1
         sum = sum+4.*theta(il)+2.*theta(i2)
10 continue
      sum = (sum-theta(n))*dx/3.
      theta(1) = ((q-k(n))*t+ar-sum)*3./dx
      return
      end
```

