Free inverse monoids up to rewriting

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Abstract. In this paper, generalizing the study of free partially commutative inverse monoids [5], for any rewriting system T over an alphabet A, we define the notion of T-compatible inverse A-generated monoids, we show there is a free T-compatible monoid $FIM(A, T)$ generated by A and we provide an explicit construction of this monoid. Then, as examples, free partially commutative inverse monoid and free partially semi-commutative inverse monoids are studied and shown to have effective representations.

1 Introduction

Halfway between groups and semigroups [14], inverse semigroups can be seen as an algebraic theory of overlapping structures.

This is especially true in view of the presentations available for certain inverse semigroups, called E-unitary, which elements can simply be seen as connected birooted subgraphs of the Cayley graph of a group [13]. In this case, the product of two birooted graphs just amounts to glue the output root of the first graph with the input root of the second, as in sort of a synchronization operation, and then, to propagate this gluing along automata transition edges, performing a fusion operation in order to recover bideterministic graphs. This notion of two steps product has been generalized [9] in link with graph transformation theory [3, 2].

In the case of E-unitary inverse monoid, the a product of two birooted (subgraphs of Cayley) graphs is depicted in Figure 1. In this figure, we consider the freiest E-unitary inverse semigroup with group image the free abelian group generated by $A = \{a, b\}$ that satisfies the commutation equation $ab = ba$. The birooted graph representation of x and $x \cdot x$ are depicted in the case x is generated by the product of generators $bab^{-1}baa^{-1}$.

More generally, Stephen's representation theorem [19] states that every element of any given A-generated inverse monoid can be represented by a bideterministic automata on the alphabet $A + \bar{A}$. Then, these bideterministic automata can be combined or compared providing geometrical views and understanding of the underlying abstract algebra based concepts.

These representation results generalize Scheiblich-Munn theorem [17, 15] that provides a canonical representation of the elements of the free inverse monoid:

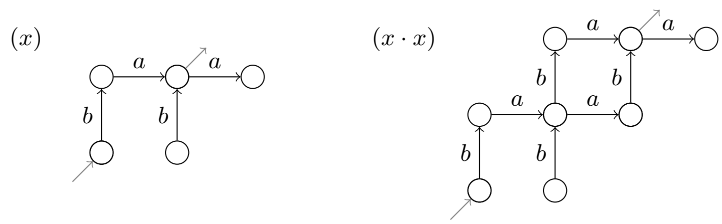

Fig. 1. A birooted graphs product in a E-unitary inverse semigroup.

birooted trees. They also lead to various development for studying presentations of more complex inverse monoids [12] or even languages of birooted trees themselves [18].

More recently, Stephen's techniques have been developed towards a decidable presentation of free partially commutative inverse monoids [4, 5] achieving thus a connection between inverse semigroup theory in algebra and the quite developed theory of partially commutative traces in concurrency theory [6].

In these new developments, traces are still modeled by means of bideterministic automata on the alphabet $A + \bar{A}$, but the geometrical interpretation of their product is a bit more involved than in the above case. With partially commutative traces, an additional completion operation is required that take into account partial commutation rules. These rules induce some completion of the birooted graphs resulting from a product.

Such a product of birooted graph representations of partially commutative traces is depicted in Figure 2. In this figure, we consider the free partially commutative inverse monoid (as defined in [6]) generated by $A = \{a, b\}$ and the commutation rules $ab = ba$ and $ab^{-1} = b^{-1}a$. The birooted graph representa-

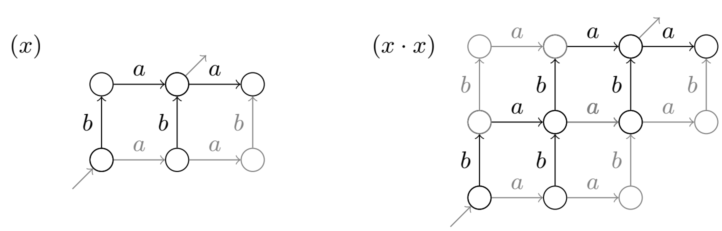

Fig. 2. A birooted graphs product in the partially commutative inverse monoid.

tions of x and $x \cdot x$ are still depicted in the case x is generated by the product of generators $bab^{-1}baa^{-1}$. We observe that the completion process has been applied to the two graphs. In this figure, the vertices and edges resulting from such a process are depicted in gray.

In other words, in free partially commutative inverse monoids [5], birooted graphs and birooted graphs product are defined *up to* the equational system that is induced by the commutation rules.

In this paper, we aim at generalizing such a construction to arbitrary equational systems or, to be more precise, arbitrary rewriting systems, and to examine some cases where such a construction is effective as in [5].

More precisely, given a finite alphabet A and its dual copy $\bar{A}$, given a rewriting relation $\Rightarrow_T$ defined by a finite set of rewriting rules $T \subseteq (A + \bar{A})^* \times (A + \bar{A})^*$, we define the notion of T-compatible inverse monoid. Then, generalizing Wagner equational characterization of the free inverse monoid generated by A, we characterize the free inverse monoid up to the rewriting system T. In this approach, the partial order induced by the rewriting system T is handled via the natural order in inverse monoids.

Then, together with the notion of birooted graphs, that is, automata on the alphabet $A + \bar{A}$, called inverse automata, that are bideterministic reversible automata, we review Stephen's presentation theorem [19] and show how it allows us to give a geometrical interpretation of the free inverse semigroup up to the rewriting system T.

As application instances, we recover effective presentations of free inverse partially commutatives, as in [5], and free inverse semi-commutative monoids, which, to the best of our knowledge, was not known before.

2 Inverse monoids and free inverse monoids

We review in this section the notion of inverse monoids and free inverse monoids. In this presentation, a special emphasis is put on Wagner preorder that plays a prominent role in the quotient by rewriting systems presented in the next section.

A *semigroup* S is a set of elements equipped with an associative product $x \cdot y$, also simply written xy, for every $x, y \in S$. A semigroup S is a *monoid* when there is an element $1 \in S$ such that $1x = x1 = x$ for every $x \in S$. The monoid completion S^1 of a semigroup S is the monoid S in the case S has a neutral element or the monoid $S^1 = S \cup \{1\}$ with obvious extension of the product otherwise.

A semigroup *congruence* over S is an equivalence relation $\simeq$ stable under product, that is, for every $u, v \in S$ if $u \simeq v$ then $xuy \simeq xvy$ for every $x, y \in S^1$. The *quotient* $S/\simeq$ of the semigroup S by $\simeq$ is defined as the set of equivalence classes $[x]_\simeq = \{y \in S : x \simeq y\}$ with $x \in S$ equipped with the product defined for every $x, y \in S$ by $[x]_\simeq \cdot [y]_\simeq = [xy]_\simeq$.

An *ordered* (resp. *preordered*) semigroup is a semigroup S equipped with a order (resp. preorder) relation $\leq$ that is stable under product, that is, if $x \leq y$ then $z_1 x z_2 \leq z_1 y z_2$ for every $z_1, z_2 \in S^1$. The equivalence relation $\sim \, = \, \leq \cap \geq$ induced by the preorder in a preordered semigroup is a congruence. Such a preorder is thus also called a *precongruence*.

Let S be a semigroup and let $u \in S$. A *semigroup inverse* of u is an element $v \in S$ such that $uvu = u$ and $vuv = v$. When every element $u \in S$ has a unique inverse, the semigroup S is an *inverse semigroup* and the inverse of u is denoted by u^{-1}. Equivalently (see [11]), a semigroup S is an inverse semigroup when every element has an inverse and its idempotents commute.

The *natural order* over an inverse semigroup S is defined by $x \leq y$ when $x = xx^{-1}y$, or, equivalently, $x = yx^{-1}x$. The *upward closure* in the natural order of a set $X \subseteq S$ is denoted by $X^{\uparrow}$.

The mapping $x \mapsto x^R = xx^{-1}$ (resp. $x \mapsto x^L = x^{-1}x$) is called the *right projection* (resp. the *left projection*) mapping. For all $x, y \in S$, we have $x \leq y$ if and only if $x = x^R y x^L$.

These projections are related with Green relations as follows. For all $x, y \in S$, we have $x^R = y^R$ if and only if $x\mathcal{R}y$, and $x^L = y^L$ if and only if $x\mathcal{L}y$, where the $\mathcal{R}$ and $\mathcal{L}$ relation in an semigroup S are defined, for every $x, y \in S$, by $x\mathcal{R}y$ when $xS^1 = yS^1$, and by $x\mathcal{L}y$ when $S^1x = S^1y$.

Let $A = \{a, b, c, \cdots\}$ be a finite alphabet, the free monoid generated by A is denoted by A^*, with the empty word still denoted by 1. Let $\bar{A} = \{\bar{a}, \bar{b}, \bar{c}, \cdots\}$ be a disjoint copy of A and let $(A + \bar{A})^*$ be the free monoid generated by $A + \bar{A}$.

The *syntactic inverse* $\overline{w}$ of a word $w \in (A + \bar{A})^*$ is inductively defined by $\overline{1} = 1$, $\overline{a \cdot v} = \overline{v} \cdot \bar{a}$ and $\overline{\bar{a} \cdot v} = \overline{v} \cdot a$ for every $a \in A$ and $v \in (A + \bar{A})^*$. The syntactic inverse mapping $w \mapsto \overline{w}$ is an involutive monoid anti-isomorphism, that is, we have $\overline{u \cdot v} = \overline{v} \cdot \overline{u}$ and $\overline{\overline{u}} = u$ for every $u, v \in (A + \bar{A})^*$.

The *Wagner congruence* $\simeq_\rho$ is defined over $(A+\bar{A})^*$ to be the least congruence such that:

$$u \simeq_\rho u\overline{u}u \qquad \text{and} \qquad u\overline{u}v\overline{v} \simeq_\rho v\overline{v}u\overline{u}$$

for every $u, v \in (A + \bar{A})^*$. Let then $FIM(A) = (A + \bar{A})^* / \simeq_\rho$ and let $\theta : (A + \bar{A})^* \to FIM(A)$ be the induced monoid morphism.

Theorem 1 (Wagner). *The monoid $FIM(A)$ is the free monoid generated by A. The monoid morphism θ is inverse preserving in the sense that, for every $u \in (A + \bar{A})^*$, we have $\theta(\overline{u}) = \theta(u)^{-1}$, i.e. the image of the syntactic inverse of a word u is the semigroup inverse of the image of u.*

Proof. For every $u \in (A + \bar{A})^*$, we simply write $[u] = \{v \in (A + \bar{A})^* : u \simeq_\rho v\}$ the equivalence class of u under Wagner equivalence.

Let $[u] \in FIM(A)$. By the congruence property, we have $[u] \cdot [\overline{u}] \cdot [u] = [u\overline{u}u]$. Hence, by the first equation of Wagner, we have $[u] \cdot [\overline{u}] \cdot [u] = [u]$. Since this holds both for u and $\overline{u}$, it follows that every element $[u] \in FIM(A)$ admits a semigroup inverse $[\overline{u}]$.

The unicity of inverse is then proved by proving that idempotent commutes. We first prove that idempotents are self inverse. Let $[u] \in FIM(A)$ such that $[u] = [u][u]$. Since $[\overline{u}] = [\overline{u}u\overline{u}]$ this implies that $[\overline{u}] = [\overline{u}uu\overline{u}]$. But we also have $[\overline{u}uu\overline{u}] = [u\overline{u}\overline{u}u]$ hence $[\overline{u}] = [u\overline{u}\overline{u}u]$. It follows that $[u\overline{u}] = [uu\overline{u}\overline{u}u]$ hence, since $[uu] = [u]$, we have $[u\overline{u}] = [\overline{u}]$. And, similarly, we have $[\overline{u}u] = [u\overline{u}\overline{u}uu]$ henceforth $[\overline{u}u] = [\overline{u}]$. Altogether, since $[\overline{u}] = [\overline{u}uu\overline{u}]$, it follows that $[\overline{u}] = [\overline{u}\overline{u}]$. By replacing

u by $\overline{u}$ from the beginning, since $\overline{\overline{u}} = u$, we immediately deduce that $[u\overline{u}] = [u]$ and $[\overline{u}u] = [u]$ hence $[u] = [\overline{u}]$. In other words, we have prove that for every $u \in (A + barA)^*$, if $[u]$ is idempotent, then $[u] = [u\overline{u}]$.

Let then two idempotents $[u]$ and $[v]$ in $FIM(A)$. By the result above, we have $[uv] = [u\overline{u}v\overline{v}]$. By the second equation of Wagner, this implies that $[uv] = [v\overline{v}u\overline{u}]$ hence $[uv] = [vu]$, that is $[u][v] = [v][u]$. This concludes the proof that idempotents in $FIM(A)$ commute and thus $FIM(A)$ is an inverse monoid.

The fact $[u]^{-1} = [\overline{u}]$ for every $u \in (A + \bar{A})^*$ immediately follows from the unicity of inverses since we have shown above that $[\overline{u}]$ is an inverse of $[u]$.

Let M be an inverse monoid finitely generated (as an inverse monoid) by some subset $A \subseteq M$. Let $\varphi : (A + \bar{A})^* \to M$ be the induced onto morphism defined by mapping very letter $a \in A$ to the element $a \in M$ and every inverse letter $\bar{a} \in \bar{A}$ to the inverse element $a^{-1} \in M$ of $a \in M$.

It is immediate that for every $u, v \in (A + \bar{A})^*$, we have $\varphi(u\bar{u}u) = \varphi(u)$ and that both $\varphi(u\bar{u})$ and $\varphi(v\bar{v})$ are idempotent hence $\varphi(u\overline{u}v\overline{v}) = \varphi(v\overline{v}u\overline{u})$. It follows that the congruence $\simeq_\varphi$ defined, for every $u, v \in (A + \bar{A})^*$, by $u \simeq_\varphi v$ when $\varphi(u) = \varphi(v)$ satisfies both Wagner equation, hence, by minimality, $\simeq_\rho \subseteq \simeq_\varphi$ and thus, the mapping $\psi : FIM(A) \to M$ defined, for every $[u] \in FIM(A)$ by $\psi([u]) = \varphi(u)$ is well defined and we have $\varphi = \psi \circ \theta$.

□

The *Wagner precongruence* $\preceq_\rho$ is defined over $(A + \bar{A})^*$ to be the least preorder relation, stable under product, such that:

$$u \preceq_\rho u\overline{u}u\,, \quad u\overline{u}v\overline{v} \preceq_\rho v\overline{v}u\overline{u} \quad \text{and} \quad u\overline{u} \preceq_\rho 1$$

for every $u, v \in (A + \bar{A})^*$. The relationship between Wagner precongruence and Wagner congruence is detailled in the following lemma.

Lemma 2. *The natural order on $FIM(A)$ equals the order induced by the precongruence, that is, for every $u \in (A + \bar{A})^*$, we have $u \preceq_\rho v$ if and only if $\theta(u) \leq \theta(v)$. In particular, we have $\simeq_\rho = \preceq_\rho \cap \succeq_\rho$.*

Proof. Let $\preceq_\theta$ be the preorder relation defined by $u \preceq_\theta v$ when $\theta(u) \leq \theta(v)$ for all $u, v \in (A + \bar{A})^*$. Clearly, the relation $\preceq_\theta$ satisfies all properties defining the Wagner preorder hence, by minimality of the Wagner preorder, we have $\preceq_\rho \subseteq \preceq_\theta$. This means that, for every $u, v \in (A + \bar{A})^*$, if $u \preceq_\rho v$ then $\theta(u) \leq \theta(v)$.

Conversely, assume that $\theta(u) \leq \theta(v)$. By definition of the natural order, this means that $\theta(u) = \theta(u)\theta(u)^{-1}\theta(v)$. Since $\theta(u)^{-1} = \theta(\overline{u})$ and θ is a morphism, it follows that $\theta(u) = \theta(u\overline{u}v)$, or, in other words, $u \simeq_\rho u\bar{u}v$. Since $u\bar{u}v \preceq_\rho v$, it thus suffices to prove that $\simeq_\rho \subseteq \preceq_\rho$ in order to conclude. But this immediately follows from the fact that, for every $u \in (A + \bar{A})^*$, we have $u\bar{u} \preceq_\rho 1$. Indeed, by stability under product, we also have $u\bar{u}u \preceq_\rho u$. This proves that the congruence $\sim_\rho = \preceq_\rho \cap \succeq_\rho$ induced by $\preceq_\rho$ satisfies all Wagner equations hence, by minimality, we have $\simeq_\rho \subseteq \sim_\rho \subseteq \preceq_\rho$ hence we have $u \preceq_\rho v$ and thus $\preceq_\varphi \subseteq \preceq_\rho$.

□

Corollary 3. *Let M be an inverse monoid and let $\varphi : (A+\bar{A})^* \to M$ be an inverse preserving onto morphism. T hen, for every $u, v \in (A+\bar{A})^*$, if $u \simeq_\rho v$ then we have $\varphi(u) = \varphi(v)$ and if $u \preceq_\rho v$ then we have $\varphi(u) \leq \varphi(v)$.*

3 Inverse monoids and rewriting systems

We define in this section the notion of T-compatible inverse monoid generated by A where T is a finite rewriting system and, we prove that there is a free T-compatible monoid generated by A.

Let $T \subseteq (A+\bar{A})^* \times (A+\bar{A})^*$ be a set of *rewriting rules* assumed to be closed under inverse, that is, for every $(u, v) \in T$ we have $(\overline{u}, \overline{v}) \in T$. Let $\Rightarrow_T$ be the *induced rewriting relation* defined as the least reflexive and transition relation over $(A+\bar{A})^*$ that is stable under product and inverse and such that $T \subseteq \Rightarrow_T$. The *closure* $T(X)$ of a set $X \subseteq (A+\bar{A})^*$ is defined by $T(X) = \{v \in (A+\bar{A})^* : \exists u \in X, u \Rightarrow_T v\}$. The set X is *T-closed* when $X = T(X)$.

Let M be an inverse monoid finitely generated by some $A \subseteq M$. Let $\theta_M : (A+\bar{A})^* \to M$ be the induced canonical inverse preserving monoid morphism. The monoid M is said *T-compatible* when, for every $u, v \in (A+\bar{A})^*$, if $u \Rightarrow_T v$ then $\theta_M(u) \leq \theta_M(v)$.

Lemma 4. *Let M be an A-generated inverse monoid with its canonical inverse preserving onto monoid morphism $\theta_M : (A+\bar{A})^* \to M$. Then the following properties are equivalent:*

(1) *M is T-compatible,*
(2) *the language $\theta_M^{-1}(x^\uparrow)$ is T-closed for every $x \in M$,*
(3) *$\theta_M(u) \leq \theta_M(v)$ for every $(u, v) \in T$.*

Proof.

(1) $\Rightarrow$ (2)**.** Assume that M is T-compatible. Let $x \in M$. Let $u \in \theta_M^{-1}(x)$. We have $x \leq \theta_M(u)$. Let $v \in (A+\bar{A})^*$ such that $u \Rightarrow_T v$. Since M is T-compatible, we have $\theta_M(u) \leq \theta_M(v)$ hence $x \leq \theta_M(v)$ and thus $v \in \theta_M^{-1}(x)$. It follows that $\theta_M^{-1}(x^\uparrow)$ is T-closed.

(2) $\Rightarrow$ (3)**.** Assume that (2) holds. Let $(u, v) \in T$. We have $u \Rightarrow_M v$. Let $x = \theta_M(u)$. Since $\theta_M^{-1}(x^\uparrow)$ is T-closed, this means that $v \in \theta_M^{-1}(x^\uparrow)$ hence $x \leq \theta_M(v)$ and thus $\theta_M(u) \leq \theta_M(v)$.

(3) $\Rightarrow$ (1)**.** Assume that (3) holds. Let $\preceq_M$ be the relation defined by $u \preceq_M v$ when $\theta_M(u) \leq \theta_M(v)$ for every $u, v \in (A+\bar{A})^*$. Clearly, relation $\preceq_M$ is reflexive and transitive. Moreover, since θ_M is a monoid morphism and the natural order is stable under product, then relation $\preceq_M$ is also stable under product. By (3), we know that $u \preceq_M v$ for every $(u, v) \in T$. It follows, by minimality of the relation $\Rightarrow_T$ that we have $\Rightarrow_T \subseteq \preceq_M$. This implies that for every $u, v \in (A+\bar{A})^*$, if $u \Rightarrow_T v$ then we have $u \preceq_M v$ hence $\theta_M(u) \leq \theta_M(v)$.

$\square$

Let $\preceq_T$ be the least preorder relation over $(A+\bar{A})^*$ that is stable under product and that contains both the Wagner preorder, that is, $\preceq_\rho \subseteq \preceq_T$, and the rewriting relation induced by T, that is, $\Rightarrow_T \subseteq \preceq_T$. Let $\simeq_T = \preceq_T \cap \succeq_T$ the induced congruence. Let $FIM(A,T) = (A+\bar{A})^* / \simeq_T$ and let $\theta_T : (A+\bar{A})^* \to FIM(A,T)$ the induced canonical monoid morphism.

Theorem 5. *The monoid $FIM(A,T)$ is an T-compatible inverse monoid. The monoid morphism θ_T is inverse preserving and, for every $u, v \in (A+\bar{A})$, we have $u \preceq_T v$ if and only if $\theta_T(u) \leq \theta_T(v)$ in the natural order. Moreover, $FIM(A,T)$ is the free T-compatible inverse monoid generated by A.*

Proof. Since $\preceq_T$ contains the Wagner preorder, this implies, by Lemma 2, that $\simeq_T$ contains the Wagner equivalence, hence, by applying Wagner's theorem (Theorem 1), we deduce that $FIM(A,T) = (A+\bar{A})^* / \simeq_T$ is an inverse monoid and the induced canonical monoid morphism $\theta_T : (A+\bar{A})^* \to FIM(A,T)$ is an onto inverse preserving morphism.

Let then $u, v \in (A+\bar{A})^*$. Assume that $u \preceq_T v$. By stability under product, this first implies that $u\overline{u}u \preceq_T u\overline{u}v$. Since $\preceq_\rho \subseteq \preceq_T$ and $u \preceq_\rho u\overline{u}u$, we thus have $u \preceq_T u\overline{u}v$. Conversely, by stability under product, this also implies that $\overline{v}u \preceq_T \overline{v}v$ hence, since $\preceq_\rho \subseteq \preceq_T$ and $\overline{v}v \preceq_\rho 1$ we have $\overline{v}u \preceq_T 1$ and thus, by stability under inverse, $\overline{u}v \preceq_T 1$. Then, again by multiplying on the left by u, we have $u\overline{u}v \preceq_T u$. This proves that $u\overline{u}v \simeq_T u$ hence, by definition of the quotient, we have $\theta_T(u) = \theta_T(u\overline{u}v)$ and thus, since θ_T is inverse preserving, we have $\theta_T(u) = \theta_T(u)^R\theta_T(v)$ hence, by definition of the natural order, we have $\theta_T(u) \leq \theta_T(v)$.

Conversely, assume that $\theta_T(u) \leq \theta_T(v)$. By definition of the natural order, this means that $u \simeq_T u\overline{u}v$ but since $\preceq_\rho \subseteq \preceq_T$ with $u\overline{u} \preceq_\rho 1$ this implies, by stability under product, that $u\overline{u}v \preceq_T v$ hence, by transitivity, $u \preceq_T v$.

As a particular case, this implies that for every $u \in (A+\bar{A})^*$, the set $\theta_T^{-1}(\theta_T(u)^\uparrow)$ is closed under $\preceq_T$ henceforth it is T-closed and thus, by applying Lemma 4, we conclude that $FIM(A,T)$ is T-compatible.

Let then M be an A-generated T-compatible monoid with $\theta_M : (A+\bar{A})^* \to M$ the canonical onto morphism. Let $\preceq_M$ be the relation over $(A+\bar{A})^*$ defined by $u \preceq_M v$ when $\theta_M(u) \leq \theta_M(v)$ for every $u, v \in (A+\bar{A})^*$.

Clearly, since the natural order over M is reflexive and transitive then so is relation $\preceq_M$. Since φ is a morphism and the natural order is stable under product, the relation $\preceq_M$ is stable under product. Since φ is inverse preserving, $\preceq_\rho \subseteq \preceq_M$. Last, by definition of T-compatibility, for every $u, v \in (A+\bar{A})^*$, if $u \Rightarrow v$ then $\theta_M(u) \leq \theta_M(v)$ hence $u \preceq_M v$.

It follows, by minimality $\preceq_T$, that we have $\preceq_T \subseteq \preceq_M$ hence $\simeq_T \subseteq \simeq_M$ and there is thus an inverse preserving morphism $\psi : FIM(A,T) \to M$ such that $\theta_M = \psi \circ \theta_T$.

□

4 Automata for inverse monoids

We review in this section some basic notion concerning automata and we present some of the main concepts of Stephen's work that relate automata and finitely generated inverse monoids [19].

As a matter of notations, an *automaton* on the alphabet $A+\bar{A}$ is a quadruple $\mathcal{A} = \langle Q, \delta, I, T\rangle$ with set of states Q, transition function $\delta : A+\bar{A} \to \mathcal{P}(Q \times Q)$, initial states $I \subseteq Q$ and terminal states $T \subseteq Q$. For every set of states $X \subseteq Q$, every word $w \in A+\bar{A}^*$, the set of states $X \cdot w$ *reachable* by $\mathcal{A}$ from X reading w is inductively defined by

$$X \cdot 1 = X \quad \text{and} \qquad X \cdot av = \{q' \in Q : \exists q \in X, (q, q,') \in \delta(a)\} \cdot v$$

for every $a \in A + \bar{A}$ and $v \in A + \bar{A}^*$. Since $X \cdot (uv) = (X \cdot u) \cdot v$ for every $u, v \in A + \bar{A}^*$, reachability induces an action from the monoid $A + \bar{A}^*$ on the powerset $\mathcal{P}(Q)$ and parenthesis can be omitted. In order to simplify notation, we may also write $p \cdot u$ in place for $\{p\} \cdot u$. Then, the *language* of words on the alphabet $A + \bar{A}$ recognized by the automaton $\mathcal{A}$ is defined by

$$L(\mathcal{A}) = \{w \in A + \bar{A}^* : I \cdot w \cap T \neq \emptyset\}$$

Clearly, some states of an automaton $\mathcal{A}$ may be useless in defining $L(\mathcal{A})$. The automaton $\mathcal{A}$ is *trim* when for every $q \in Q$ there exists $u, v \in A + \bar{A}^*$ such that $q \in I \cdot u$ and $q \cdot v \cap F \neq \emptyset$.

We say that the automaton $\mathcal{A}$ is *deterministic* (resp. *co-deterministic*) when I is a singleton (resp. when T is a singleton) and, for every $x \in A+\bar{A}$, the relation $\delta(x)$ (resp. the relation $\delta(x)^{-1}$) is functional, that is, for every $(p, q), (p', q') \in \delta(x)$, if $p = p'$ then $q = q'$ (resp. if $q = q'$ then $p = p'$). Additionally, we say that the automaton $\mathcal{A}$ is *symmetric* when for every $x \in A+\bar{A}$, we have $\delta(\overline{x}) = \delta^{-1}(x)$, that is, for every $p, q \in Q$, $(p, q) \in \delta(\overline{x})$ if and only if $(q, p) \in \delta(x)$. We say that the automaton $\mathcal{A}$ is *reversible* when it is symmetric and, for every $x \in A+\bar{A}$, the relation $\delta(x)$ induces a partial bijection, that is, for every $(p, q), (p', q') \in \delta(x)$, we have $p = p'$ if and only if $q = q'$. Last, we say that the automaton $\mathcal{A}$ is *inverse* when it is symmetric and both deterministic and co-deterministic. We refer the reader to [16] for further detail on reversible automata. We refer the reader to [18] for a study of deterministic reversible automata.

We say that a language $X \subseteq (A + \bar{A})^*$ is a *inverse language* (resp. a *closed inverse language*) when it is closed with respect to $\simeq_\rho$ (resp. when it is upward closed with respect to preorder $\preceq_\rho$) or, equivalently, when we have $X = \theta^{-1}(\theta(X))$ (resp., or, equivalently, thanks to Lemma 2, when we have $X = \theta^{-1}(\theta(X)^{\uparrow})$).

Clearly, a language recognizable by an inverse automaton is a closed inverse language. However, the converse is false as shown by the closed inverse language $a(\overline{a}a)^* + b(\overline{b}b)^*$. Languages recognizable by inverse automata are called *prime closed inverse*. The following properties, proved by many authors, shows the tight connection between inverse automata and prime closed inverse languages.

Lemma 6. *Let $\mathcal{A}$ be a trim inverse automaton. Then $\mathcal{A}$ is minimal.*

Proof. For every state $q \in Q$, let $L(q) = \{w \in (A + \bar{A})^* : t \in p \cdot w\}$. Let then $p, q \in Q$ such that $L(p) = L(q)$. Since $\mathcal{A}$ is trim, these languages are non empty. Let then $w \in L(p) \cap L(q)$. We have $t \in p \cdot w$ and $t \in q \cdot w$. Since $\mathcal{A}$ is symmetric, we have $p \in t \cdot \overline{w}$ and $q \in \overline{w}$. Since $\mathcal{A}$ is deterministic, this implies that $t \cdot w$ is a singleton, henceforth $p = q$. This concludes the proof that $\mathcal{A}$ is minimal.

□

As an immediate consequence:

Corollary 7. *Let $\mathcal{A}_1 = \langle Q_1, \delta_1, i_1, t_1\rangle$ and $\mathcal{A}_2 = \langle Q_2, \delta_2, i_2, t_2\rangle$ be two trim inverse automata. Then $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$ if and only if there is an automata morphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$, that is, a mapping $\varphi : Q_1 \to Q_2$ such that $\varphi(i_1) = i_2$, $\varphi(t_1) = t_2$ and $\varphi(\delta_1(x)) \subseteq \delta_2(x)$ for every $x \in A + \bar{A}$.*

The following definition and lemma allows us to build a canonical inverse automaton from any symmetric automaton. Let $\mathcal{A} = \langle Q, \delta, I, T\rangle$ be a symmetric automaton. Let $\sim$ be the least equivalence over Q such that:

(1) both $I \times I \subseteq \sim$ and $T \times T \subseteq \sim$,
(2) for every $p, q, p', q' \in Q$ and every $x \in A + \bar{A}$, if $p \sim q$, $(p, p') \in \delta(x)$ and $(q, q') \in \delta(x)$ then $p' \sim q'$.

The automaton $\mathcal{A}^i = \langle Q', \delta', I', , T'\rangle$, called the *inverse normalization* of the automaton $\mathcal{A}$, is defined by the set of states $Q' = Q/{\sim}$, the set of transition $\delta'(x) = \{(X, Y) \in Q' \times Q' : \exists p \in X, q \in Y, (p, q) \in \delta(x)\}$ for every $x \in A + \bar{A}$, the set of initial states $I' = \{X \in Q' : X \cap I \neq \emptyset\}$ and the set of terminal states $T' = \{Y \in Q' : X \cap T \neq \emptyset\}$.

Lemma 8 (Inverse normalization). *Let $\mathcal{A} = \langle Q, \delta, I, T\rangle$ be a symmetric automaton. Then, the automaton $\mathcal{A}^i$ is an inverse automaton and $L(\mathcal{A}^i)$ is the least prime closed inverse language that contains $L(\mathcal{A})$.*

Proof. Let $\mathcal{A} = \langle Q, \delta, I, T\rangle$. First, let $\mathcal{A}'$ be the automaton obtained from $\mathcal{A}$ just by gluing all states in I and all states in T. It is an easy observation that $L(\mathcal{A}') = L(\mathcal{A}) \cdot (\overline{L(\mathcal{A})} \cdot L(\mathcal{A}))^*$ hence, by Lemma 6, $L(\mathcal{A}')$ is included into any prime closed language that contains $L(\mathcal{A})$.

Having said so, we may assume that both I and T are singletons with $I = \{i\}$ and $T = \{t\}$. Let then $\mathcal{A}^i = \langle Q', \delta', I', T'\rangle$ as defined above. Since $\sim$ is an equivalence relation, this means that both I' and T' are singletons, with $I' = \{[i]_\sim\}$ and $T = \{[t]_\sim\}$. Moreover, since $\mathcal{A}$ is symmetric, then so is $\mathcal{A}^i$. In order to prove that $\mathcal{A}$ is inverse, it thus remains to prove that $\mathcal{A}$ is reversible.

Let us prove that automaton $\mathcal{A}^i$ is reversible. Let $x \in A + \bar{A}$. By symmetry, it suffices to prove the relation $\delta(x)$ is functional. Let $X, Y_1, Y_2 \in Q' = Q/{\sim}$ such that $(X, Y_1) \in \delta'(x)$ and $(X, Y_2) \in \delta'(x)$. Then, by definition of δ', there exists $x_1, x_2 \in X$, $y_1 \in Y_1$ and $Y_2 \in Y_2$ such that $(x_1, y_1) \in \delta(x)$ and $(x_2, y_2) \in \delta(x)$. But since $x_1 \sim x_2$, this implies that $y_1 \sim y_2$ hence $Y_1 = Y_2$.

In other words, the automaton $\mathcal{A}^i$ is an inverse automaton. Now, it suffices to show that $L(\mathcal{A}^i)$ is the least closed inverse language that contains $L(\mathcal{A})$. Since $\mathcal{A}^i$ is inverse, this will ensures that $L(\mathcal{A}^i)$ is the least prime closed invers language containing $L(\mathcal{A})$.

For such a purpose, we use a more effective definition of relation $\sim$ on the automaton $\mathcal{A}$ with a single initial and terminal state.

Let then $\sim'$ be the relation over Q defined by $p \sim' q$ when there exists $u \in (A+\bar{A})^*$ such that $u \preceq_\rho 1$, or, equivalently, such that u reduces to 1 via the rewriting systems defined by $a\bar{a} \to 1$ and $\bar{a}a \to 1$ for every $a \in A$, and such that $p \in q \cdot u$ and thus, by symmetry of the automaton $\mathcal{A}$, such that $q \in p \cdot \overline{u}$.

Then we prove that $\sim = \sim'$. Let us first show that $\sim' \subseteq \sim$ by induction on the number n of rules applied to reduce u into 1. In the case $n = 0$, we have $p = q$ hence $p \sim q$. Assume this is true for all word v that reduces to 1 in strictly less than n steps and assume that u reduces to 1 in n steps. This means that $u = xu_1\bar{x}u_2$ for some $x \in A + \bar{A}$ with both u_1, u_2 that reduces to 1 in strictly less than n steps. Since $p \in q \cdot u$ this implies there is $q_1, q_2, q_3 \in Q$ such that $(q, q_1)\delta(x)$, $q_2 \in q_1 \cdot u_1$, $(q_2, q_3) \in \delta(\overline{x})$ and $p \in q_3 \cdot u_2$. It follows that, by applying the induction hypothesis, we have $q_1 \sim q_2$ and $q_3 \sim p$. But, by definition of $\sim$, since $(q_1, q) \in \delta(\overline{x})$ and $(q_2, q_3) \in \delta(\overline{x})$ this implies that $q \sim q_3$ and thus, by transitivity, $q \sim p$.

Conversely, the relation $\sim'$ is clearly an equivalence relation. Since both I' and T' are singletons, we have $I' \times I' \subseteq \sim'$ and $T' \times T' \subseteq \sim'$ hence $\sim'$ satisfies the condition **(1)** defining $\sim$. Let then $p, q, p', q' \in Q$ and $x \in A + \bar{A}$. Assume that $p \sim' q$, $(p, p') \in \delta(x)$ and $(q, q') \in \delta(x)$. Since $p \sim' q$, there is $u \in (A + \bar{A})^*$ such that $p \in q \cdot u$ with $\theta(u) \leq 1$. Since $(p, p') \in \delta(x)$ and $(q, q') \in \delta(x)$, hence, $(q', q) \in \delta(\overline{x})$, it follows that $p' \in q' \cdot \overline{x}ux$. We conclude by observing that, since $u \preceq_\rho 1$, we have $\overline{x}ux \preceq_\rho \overline{x}x$ and, since $\overline{x}x \preceq_\rho 1$, we thus have $\overline{x}ux \preceq_\rho 1$. Since we have proved that $p' \in q' \cdot \overline{x}ux$, this implies that $p' \sim' q'$ hence $\sim'$ satisfies the condition **(2)** defining $\sim$. It follows, by minimality of $\sim$, that we have $\sim \subseteq \sim'$.

Altogether, we thus have proved that $\sim = \sim'$. Let then X be a closed inverse language such that $L(\mathcal{A}) \subseteq X$. Let $u \in L(\mathcal{A}^i)$. Assume that $u = x_1x_2 \cdots x_n$ with $x_i \in A + \bar{A}$ for every $1 \leq i \leq n$. By definition of the automaton $\mathcal{A}^i$, this means that there is $(p_i, q_i) \in \delta(x_i)$ with $1 \leq i \leq n$ such that $i \sim p_1$, $q_n \sim t$ and $q_i \sim p_{i+1}$ for every $1 \leq i < n$.

But, since we have proved that $\sim'$ equals $\sim$, this means that there is $u_0, u_1, \cdot u_n \in (A + \bar{A})^*$ with $u_i \preceq_\rho 1$ for $0 \leq i \leq n$ such that, on the automaton $\mathcal{A}$, we have $p_1 \in p \cdot u_0$, $q \in q_n \cdot u_n$ and $p_{i+1} \in q_i \cdot u_i$ for every $1 \leq i < n$.

This means that $q \in p \cdot v$ on the automaton $\mathcal{A}$ with $v = u_0x_1u_1 \cdots x_nu_n$ hence $v \in L(\mathcal{A})$. Since $v \preceq_\rho u$ and X is a closed inverse language that contains $L(\mathcal{A})$, this proves that $u \in X$. Since this holds for every $u \in L(\mathcal{A}^i)$ we thus have proved that $L(\mathcal{A}^i)$ is indeed the least closed inverse language that contains $L(\mathcal{A})$.

□

Remark 1. As already observed in [9], in the case $\mathcal{A} = \langle Q, \delta, I, T \rangle$ is finite, computing the relation $\sim$ can be done in time quasi-linear in the size of $\mathcal{A}$

since it just amounts to compute the least fixed-point of the mapping $f(R) = R \cup \{(p,q) \in Q : \exists x \in A+\bar{A}, \exists (p',q') \in R, (p,p'),(q,q') \in \delta(x)\}$ that contains the equality, $I \times I$ and $T \times T$. It follows that classical minimization like techniques can be applied.

The following construction and associated result that appeared in [19], induces an automata theoretic presentation of all finitely generated inverse semigroup.

Let M be an inverse monoid and let $\varphi : (A + \bar{A})^* \to M$ be an inverse preserving monoid morphism. For every $x \in M$, we define the *Schützenberger automaton* $\mathcal{A}_\varphi(x)$ induced by x to be the automaton $\mathcal{A}_\varphi(x) = \langle Q, \delta, I, T\rangle$ defined by the set of states $Q = \{y \in M : y^R = x^R\}$, the transition function $\delta(z) = \{(y_1, y_2) \in Q \times Q : y_1 \cdot \varphi(z) = y_2\}$ for every $z \in A + \bar{A}$, the set of initial states $I = \{x^R\}$ and the set of terminal states $T = \{x\}$.

Lemma 9 (Stephen [19]). *Let M be an inverse monoid and let $\varphi : (A+\bar{A})^* \to M$ be an inverse preserving monoid morphism. Let $x \in M$. Then $\mathcal{A}_\varphi(x)$ is a trim inverse automaton with $L(\mathcal{A}_\varphi(x)) = \varphi^{-1}(x^\uparrow)$. As a consequence, if $\mathcal{A}_\varphi(x)$ and $\mathcal{A}_\varphi(y)$ are isomorphic automata, then $x = y$.*

Proof. Let $\mathcal{A}_\varphi(x) = \langle Q, \delta, I, T\rangle$ be the automaton defined above. Let $z \in (A + \bar{A})^*$. The relation $\delta(z)$ is functional. Let us show it is symmetric. Assume that $(y_1, y_2) \in \delta(z)$. By definition, we have $y_1 \cdot \varphi(z) = y_2$. This implies that $y_1 \cdot \varphi(z) \cdot \varphi(z)^{-1} = y_2 \cdot \varphi(z)^{-1}$. Since φ is an inverse preserving morphism this implies that $y_1 \cdot \varphi(z\overline{z}) = y_2 \cdot \varphi(\overline{z})$. Now, by definition again, we know that $y_1^R = y_2^R$. This means that $y_1 \cdot \varphi(z) \cdot \varphi(\overline{z}) \cdot y_1^{-1} = y_2 y_2^{-1}$ hence, by multiplying both side by y_1 and by commutation of idempotent, we have $y_1 y_1^{-1} y_1 \cdot \varphi(z\overline{z}) = y_2 y_2^{-1} y_1$ hence, because $y_1 = y_1 y_1^{-1} y_1$ and $y_2 y_2^{-1} = y_2^R = y_1^R$, we have $y_1 \cdot \varphi(z\overline{z}) = y_1$ and thus $y_1 = y_1 \cdot \varphi(\overline{z})$, or, in other words, $(y_2, y_1) \in \delta(\overline{z})$. In other words, we have proved that $\mathcal{A}_\varphi(x)$ is an inverse automaton.

Let us prove automaton $\mathcal{A}_\varphi(x)$ is trim. Let $y \in Q$ that is $y^R = x^R$. Since φ is onto, there is $v \in (A + \bar{A})^*$ such that $y = \varphi(v)$ and we have $\varphi^R(v) = x^R$. Since $\varphi(v)^R \varphi(v) = \varphi(v)$ this means that $x \cdot \varphi(v) = \varphi(v)$ hence, we can easily prove, by induction on the length of v, that $y = \varphi(v) \in x^R \cdot v$ in the automaton $\mathcal{A}_\varphi(x)$. In particular, given $u \in \varphi^{-1}(x)$, we have $x \in x^R \cdot u$ and, since the automaton $\mathcal{A}_\varphi(x)$ is reversible, we also have $x^R \in y \cdot \overline{v}$, henceforth $x \in y \cdot \overline{v}u$, which concludes the proof that $\mathcal{A}_\varphi(x)$ is trim.

Let then $u \in (A + \bar{A})^*$. We easily prove, by induction on the length of u that $x \in x^R \cdot u$ in the automata $\mathcal{A}_\varphi(x)$, that is, there is a run in the automaton $\mathcal{A}_\varphi$ from x^R to x reading v if and only if $x^R \cdot \varphi(u) = x$, that is, if and only if $x \leq \varphi(u)$ in the natural order . This proves that we have $L(\mathcal{A}_\varphi(x)) = \varphi^{-1}(x^\uparrow)$. □

Though not necessarily effective, the mapping $x \mapsto \mathcal{A}_\varphi(x)$ is one to one. This justifies the following presentation [19].

Let M be an inverse monoid. Let $\varphi : (A + \bar{A})^* \to M$ be an onto inverse preserving monoid morphism. Let $Aut(M)$ be the set of (isomorphic classes of) inversible automata of the form $\mathcal{A}_\varphi(x)$ for $x \in M$ and let $\cdot$ be the product defined by $\mathcal{A}_\varphi(x) \cdot \mathcal{A}_\varphi(y) = \mathcal{A}_\varphi(xy)$. Then the set $Aut(M)$ equipped with the above automaton product is a well defined monoid, it is an inverse monoid and the mapping $\mathcal{A}_\varphi : M \to Aut(M)$ is an inverse monoid isomorphism.

In view of the fact that, thanks to Lemma 6 and Lemma 9, the automaton $\mathcal{A}_\varphi(\varphi(u))$ is the (unique) minimal automaton that recognizes $\varphi^{-1}(\varphi(u)^\uparrow)$. This suggest that Stephen's presentation theorem's may indeed offers effective language theoretical way to describe inverse monoids.

Various cases of free monoids $FIM(A,T)$ up to some rewriting systems T are discussed in the next section.

5 Partial commutation and partial semi-commutation

Following [6], a irreflexive relation $I \subseteq A \times A$ is called a partial semi-commutation system. A irreflexive and symmetric relation $I \subseteq A \times A$ is called a commutation system. Let then $T_I = \{(ab, ba) \in A^* \times A^* : (a, b) \in I\}$ be the corresponding commutation rewriting rules. Given $u \in A^*$, the I-trace $tr(u)$ induced by u with the commutation rules I is defined as the T_I-closure of the language $\{u\}$.

Example 1. As a running example, assume that $A = \{a, b, c\}$. Let $I_1 = \{(b, c)\}$ with corresponding rules $T_1 = \{(bc, cb)\}$ be the partial semi-commutation system stating that when b precedes c then they commute. Let $I_2 = \{(b, c), (c, b)\}$ with corresponding rules $T_2 = \{(bc, cb), (bc, cb)\}$ the partial commutation system stating that b and c commute in any context. Then, given $u = abcbab$, the trace $tr_1(u)$ induced by u and I_1 and the trace $tr_2(u)$ induced u and I_2 are given by

$$tr_1(u) = \{abcbab, acbbab\} \quad \text{and} \quad tr_2(u) = \{abcbab, acbbab, abbcab\}$$

As a matter of fact, both these traces are recognized by inverse automata on the alphabet A that are depicted in Figure 3.

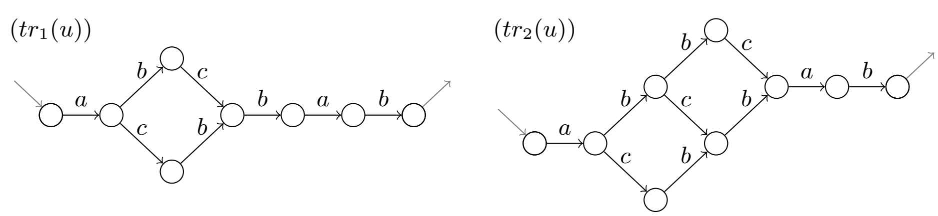

Fig. 3. Inverse automata for partially semi-commutative and commutative traces.

This examples leads to the more general fact that partially commutative and semi-commutative trace free monoids can be defined as submonoids of the corresponding free inverse monoid up to the corresponding rewriting rules.

More precisely, let $I \subseteq A \times A$ be an irreflexive relation. Let $T^i(I)$ be the least inverse closed set of rewriting rules that contains the rewriting rules induced by I over A^*, that is,

$$T^i(I) = \bigcup_{(a,b)\in I} \{(ab, ba), (\bar{b}\bar{a}, \bar{a}\bar{b})\}$$

Then we have:

Theorem 10. *The free partially semi-commutative monoid $FM(A, I)$ generated by A and the semi-commutation I equals the submonoid of $FIM(A, T^i(I))$ generated by $\theta_{T^i(I)}(A)$.*

Proof. Immediate from the definitions and the fact that the inverse automata generated by traces are acyclic w.r.t. to positive paths, that is, paths labeled by words of A^*.

□

However, as soon as one allows backward letters in traces, then, as argued in [5] in the partially commutative case, the inverse closure $T^i(I)$ of the rewriting rules induced by I used above may be non satisfactory. Indeed, what could be the trace of a path of the form $cb\bar{c}$ according to the commutations defined by I_1 or by I_2 ? The obvious interpretation of a trace is that it models all possible linearized executions of a concurrent system and such a backward c would certainly lead to an action with a dangling source state.

A remedy to this fact can be done as follows consists in allowing mixed positive and negative letter to commute both ways, as soon as they are involved in a partial commutation. More precisely, let $I \subseteq A \times A$ be an irreflexive relation. We define the set $T(I)$ be the set of mixed rewriting rules by

$$T(I) = \bigcup_{(a,b)\in I} \{(ab, ba), (\bar{b}\bar{a}, \bar{a}\bar{b}), (b\bar{a}, \bar{a}b), (\bar{a}b, b\bar{a})\}$$

Then it make sense to define $FIM(T(I))$ to be the free partially semi-commutative inverse monoid generated by A and induced by I. When I is symmetric, that is when semi-commutation are commutation, this definition coincides with the one proposed in [5]. Now, due to the very peculiar form of these rewriting rules, we have:

Theorem 11. *Let $I \subseteq A \times A$ be an irreflexive relation. Then, the free partially semi-commutative inverse monoid $FIM(A, T(I))$ generated by A and induced by I admits an effective representation by finite inverse automata.*

Proof. (sketch of) We observe that all rewriting rules of $T(I)$ are lengths preserving and apply only to reduced words of $u \in (A + \bar{A})$, that is, words that contains no factors of the form $a\bar{a}$ nor $\bar{a}a$ for $a \in A$. This implies that, for every $u \in (A + \bar{A})$, the Schützenberger automaton $\mathcal{A}(\theta_{T(I)}(u))$ is finite and acyclic in A^*.

It follows that, for every $u \in (A + \bar{A})$, the Schützenberger automaton $\mathcal{A}(\theta_{T(I)}(u))$ can effectively be constructed by successive application of inverse normalizations (see Lemma 8) while incrementally computing the upward closure of u under $\preceq_{T(I)}$ (Theorem 5) that equals the language recognized by $\mathcal{A}(\theta_{T(I)}(u))$ (Lemma 9).

□

Example 2. Continuing the examples on the alphabet $A = \{a, b, c\}$ with irreflexive relations $I_1 = \{(b, c)\}$ and $I_2 = \{(b, c), (c, b)\}$, we give in Figure 4 below some examples of inverse automata, denoting, for every $u \in (A + \bar{A})^*$, by $Aa_1(u)$ (resp. by $Aa_2(u)$) the Schützenberger automaton induced by u in $FIM(A, T(I_1))$ (resp. in $FIM(A, T(I_2))$). We may also use the notation $\mathcal{A}_{1,2}(u)$ to denote the situation when $\mathcal{A}_1(u)$ and $\mathcal{A}_2(u)$ are isomorphic.

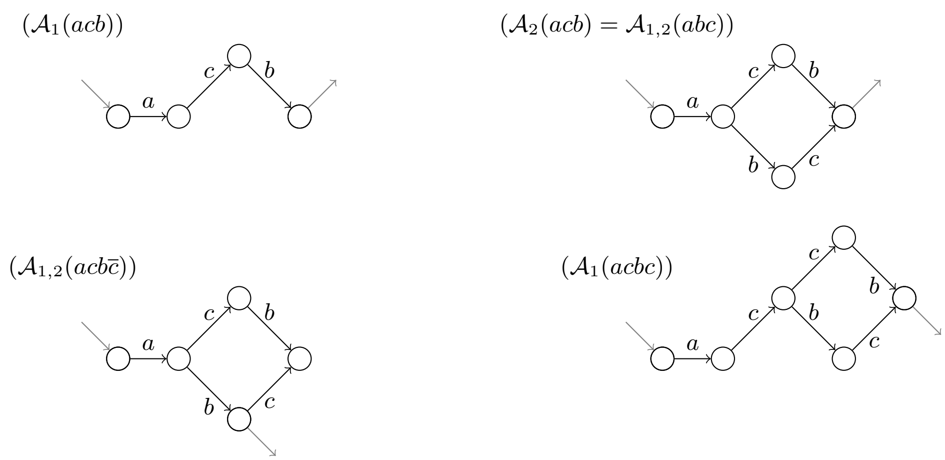

Fig. 4. Some inverse automata in $FIM(A, T(I_1))$ and $FIM(A, T(I_2))$.

In all these exemples, we can observe that $\mathcal{A}_1(u)$ is a subautomaton of $\mathcal{A}_2(u)$ which follows from the fact that $T(I_1) \subseteq T(I_2)$.

6 Conclusion

We thus have shown that for every set of rewriting rules $T \subseteq (A + \bar{A})^* \times (A + \bar{A})^*$ there is a notion of free T-compatible inverse monoid $FIM(A, T)$ generated by A. Applied to rewriting systems induced by partial semi-commutation rules, this leads to the effective construction of free inverse partially semi-commutative inverse monoids, henceforth generalizing the framework proposed in [5].

However, the study of the free partially semi-commutative monoids proposed here is far from being as deep as the one proposed in [5]. Complexity issues are not addressed, neither are considered further quotients of these monoids by equations of the form $e = f$ for idempotent birooted trees e and f. We believe it can be pursued with techniques quite similar to the ones developed in [5], but this remains to be checked.

With a view towards application, the study of free inverse monoids up to rewriting systems initiates a complement of the categorical study of birooted graphs, provided in [9], by enriching the induced modeling tools by the possibility of quotienting these birooted graphs by rewriting systems. Doing so, we pursue our goal to provide solid mathematical basis to the notion of tiled modeling and tiled programming experiments that are conducted in parallel in the field of interactive temporal media systems [1, 10, 7, 8].

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