When Conditional Logic and Belief Revision Meet Substructural Logics

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Abstract

Two threads of research have been pursued in parallel in logic and artificial intelligence. On the one hand, in artificial intelligence, logic-based theories have been developed to study and formalize belief change and the so-called "common sense reasoning", *i.e.* the actual reasoning of humans. On the other hand, in logic, substructural logics, *i.e.* logics lacking some of the structural rules of classical logic, have been studied in depth from a theoretical point of view. However, the powerful (proof-theoretical) techniques and methods developed in logic have not yet been applied to artificial intelligence. Conditional logic and belief revision theory are prominent theories in artificial intelligence dealing with common sense reasoning. We show in this article that they can both be embedded within the framework of substructural logics and can both be seen as extensions of the Lambek calculus. This allows us to compare and relate them to each other systematically, via a natural formalization of the Ramsey test.

1 Introduction

In everyday life, the way we update and revise our beliefs plays an important role in our representation of the surrounding world and therefore also in our decision making process. This has lead researchers in artificial intelligence and computer science to develop logic-based theories that study and formalize belief change and the so-called "common sense reasoning". The rationale underlying the development of such theories is that it would ultimately help us understand our everyday life reasoning and the way we update our beliefs, and that the resulting work could subsequently lead to the development of tools that could be used for example by artificial agents in order to act autonomously in an uncertain and changing world.

A number of theories have been proposed to capture different kinds of updates and the reasoning styles that they induce, using different formalisms and under various assumptions: dynamic epistemic logic [van Benthem, 2011], default and non-monotonic logics [Makinson, 2005], belief revision theory [Gärdenfors, 1988], conditional logic [Nute and Cross, 2001], *etc.*. . . However, a generic and general framework encompassing all these theories is still lacking. Instead, the current state of the art is such that we are left with various formalisms which are difficult to relate formally to each other despite numerous attempts [Makinson and Gärdenfors, 1989; Aucher, 2004; Baltag and Smets, 2008], partly because they rely on different kinds of formalisms. This is problematic if logic is to be viewed ultimately as a unified and unifying field and if we want to avoid that logic goes on "riding off madly in all directions" (a metaphor used by van Benthem [2011]).

Our objective in this article is to show that conditional logic and belief revision can be reformulated meaningfully and naturally within the very general framework of substructural logics [Restall, 2000]. More specifically, we will show that conditional logic and belief revision theory are extensions of the well-known Lambek calculus with appropriate structural inference rules. This will allow us to compare and relate them to each other systematically. In particular, our approach will shed new lights on Gärdenfors' impossibility theorem that draws attention to certain formal difficulties in defining a conditional connective from a revision operation, via the Ramsey test. We will also pinpoint the key to non-monotonicity and we will show that it depends crucially on a constrained application of the (left) weakening rule.

Other proof theoretical approaches to non-monotonic reasoning have already been proposed, notably by Bonatti and Olivetti [2002][1992]. However, they deal with non-monotonicity at the meta-logical level by introducing specific inference relations like $\mid\!\sim$ or $\rhd$. Instead of it, we will deal with non-monotonicity at the object-language level by means of the substructural connective $\supset$ and the introduction of appropriate structural rules.

The article is organized as follows. In Section 2, we briefly recall elementary notions of substructural logics and we observe that the ternary relation can be interpreted intuitively as a kind of update. In Section 3, we recall the basics of conditional logic and belief revision theory and recall how they are formally connected. In Section 4, we show how each of them can be embedded within the framework of substructural logic that was introduced in Section 2 by adding specific structural inference rules. In Section 5, we discuss Gärdenfors' impossibility theorem. Finally, we conclude in Section 6.

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2 Substructural Logics

Substructural logics are a family of logics lacking some of the structural rules of classical logic. A *structural rule* is a rule of inference which is closed under substitution of formulas. The structural rules for classical logic are given in Fig. 1: they are called *Weakening*, *Contraction*, *Permutation* and *Associativity* (see Definition 2 for explanations about the notations used). The comma in these sequents has to be interpreted as a conjunction in an antecedent and as a disjunction in a consequent. While *Weakening* and *Contraction* are often dropped like in relevance logic and linear logic, the rule of *Associativity* is often preserved. When some of these rules are dropped, the comma ceases to behave as a conjunction (in the antecedent) or a disjunction (in the succedent). In that case, the comma corresponds to other substructural connectives and we often introduce new punctuation marks which do not fulfill all these structural rules to deal with these new substructural connectives.

Our exposition of substructural logics is based on [Restall, 2000, 2006] (see also Ono [1998] for a general introduction).

2.1 Syntax and Semantics

In the sequel, $\mathbb{P}$ is a non-empty and *finite* set of propositional letters.

Definition 1 (Language $\mathcal{L}_{\circ,\supset,\subset}$)**.** The *language* $\mathcal{L}_{\circ,\supset,\subset}$ is defined inductively by the following grammar in BNF:

$$\varphi \ ::= \ \bot \ \mid \ p \ \mid \ (\varphi \wedge \varphi) \ \mid \ (\varphi \to \varphi) \ \mid \ (\varphi \circ \varphi) \ \mid \ (\varphi \supset \varphi) \ \mid \ (\varphi \subset \varphi)$$

where p ranges over $\mathbb{P}$. Also, $\to$ is material implication whereas $\supset$ and $\subset$ are Lambek implications. If $\mathsf{Con} \subseteq \{\wedge, \to, \circ, \supset, \subset\}$, then the language $\mathcal{L}_{\mathsf{Con}}$ is the language $\mathcal{L}_{\circ,\supset,\subset}$ restricted to the connectives of Con. The propositional language $\mathcal{L}_{\mathsf{PL}}$ is the language $\mathcal{L}_{\mathsf{Con}}$ with $\mathsf{Con} := \{\wedge, \to\}$.

We will use the following abbreviations: $\neg\varphi := \varphi \to \bot$, $\top := \neg\bot$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, and $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$. We use the following ranking of binding strength for parenthesis: $\neg, \circ, \supset, \subset, \wedge, \vee, \to, \leftrightarrow$. □

Definition 2 ($\mathcal{L}_{\mathsf{Con}}$–structure, $\mathcal{L}_{\mathsf{Con}}$–sequent and $\mathcal{L}_{\mathsf{Con}}$–hypersequent)**.** Let $\mathsf{Con} \subseteq \{\wedge, \to, \circ, \supset, \subset\}$. *$\mathcal{L}_{Con}$–structures* are defined by the following grammar in BNF:

$$\mathcal{S}_L^{\mathcal{L}_{\mathsf{Con}}} : \ X \ ::= \ \varphi \ \mid \ (X, X) \ \mid \ (X \,;\, X)$$
$$\mathcal{S}_R^{\mathcal{L}_{\mathsf{Con}}} : \ Y \ ::= \ \varphi \ \mid \ (Y, Y)$$

where φ ranges over $\mathcal{L}_{\mathsf{Con}}$. $\Gamma[X]$ denotes a $\mathcal{L}_{\mathsf{Con}}$–structure containing as substructure the $\mathcal{L}_{\mathsf{Con}}$–structure X, and $\Gamma[Z]$ denotes the $\mathcal{L}_{\mathsf{Con}}$–structure $\Gamma[X]$ where X is uniformly substituted by the structure Z. $\mathcal{L}_{\mathsf{Con}}$–structures are denoted U, X, Y or Z and we write $\varphi \in X$ when φ is a substructure of X.

A *$\mathcal{L}_{Con}$–sequent* is an expression of the form $X \vdash Y$, $\vdash Y$ or $X \vdash$ where $X \in \mathcal{S}_L^{\mathcal{L}_{\mathsf{Con}}}$, $Y \in \mathcal{S}_R^{\mathcal{L}_{\mathsf{Con}}}$. A *$\mathcal{L}_{\mathsf{Con}}$–hypersequent* has the form $X_1 \vdash Y_1 \mid \ldots \mid X_n \vdash Y_n$ where $X_1 \vdash Y_1, \ldots X_n \vdash Y_n$ are $\mathcal{L}_{\mathsf{Con}}$–sequents.

The *depth of a $\mathcal{L}_{Con}$–structure*, denoted $d(X)$, is defined inductively as follows: $d(\varphi) := 0$, $d((X, Y)) = max\{d(X), d(Y)\}$ and $d((X \,;\, Y)) := max\{d(X), d(Y)\} + 1$. The *depth of a $\mathcal{L}_{Con}$–sequent* $X \vdash Y$ is defined by $d(X \vdash Y) := max\{d(X), d(Y)\}$. □

The semantics of substructural logics is based on the ternary relation of the frame semantics for relevant logic originally introduced by Routley and Meyer [1972a,b, 1973]; Routley *et al.* [1982].

Definition 3 (Point set)**.** A *point set* $\mathcal{P} = (P, \sqsubseteq)$ is a set P together with a partial order $\sqsubseteq$ on P. We abusively write $x \in \mathcal{P}$ for $x \in P$. □

The partial order $\sqsubseteq$ (introduced for dealing with intuitionistic reasoning) will not be used in this article.

Definition 4 (Model)**.** A *model* is a tuple $\mathcal{M} = (\mathcal{P}, \mathcal{R}, \mathcal{I})$ where:

- $\mathcal{P} = (P, \sqsubseteq)$ is a point set;
- $\mathcal{I} : P \to 2^{\mathbb{P}}$ is an interpretation function;
- $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P} \times \mathcal{P}$ is a ternary relation on $\mathcal{P}$.

We abusively write $x \in \mathcal{M}$ for $x \in \mathcal{P}$, and $(\mathcal{M}, x)$ is called a *pointed model*. □

A model stripped out from its interpretation corresponds to a *frame* as defined in [Restall, 2000, Def. 11.8] without *truth sets* (defined in [Restall, 2000, Def. 11.7]). Truth sets are not needed for what concerns us here.

Definition 5 (Truth conditions)**.** Let $\mathcal{M}$ be a model, $x \in \mathcal{M}$ and $\varphi \in \mathcal{L}_{\circ,\supset,\subset}$. The relation $\mathcal{M}, x \Vdash \varphi$ is defined inductively as follows:

$\mathcal{M}, x \Vdash \bot$		never
$\mathcal{M}, x \Vdash p$	iff	$p \in \mathcal{I}(x)$
$\mathcal{M}, x \Vdash \varphi \wedge \psi$	iff	$\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$
$\mathcal{M}, x \Vdash \varphi \to \psi$	iff	if $\mathcal{M}, x \Vdash \varphi$ then $\mathcal{M}, x \Vdash \psi$
$\mathcal{M}, x \Vdash \varphi \circ \psi$	iff	there are $y, z \in \mathcal{P}$ such that $\mathcal{R}yzx$, $\mathcal{M}, y \Vdash \varphi$ and $\mathcal{M}, z \Vdash \psi$
$\mathcal{M}, x \Vdash \varphi \supset \psi$	iff	for all $y, z \in \mathcal{P}$ where $\mathcal{R}xyz$, if $\mathcal{M}, y \Vdash \varphi$ then $\mathcal{M}, z \Vdash \psi$
$\mathcal{M}, x \Vdash \psi \subset \varphi$	iff	for all $y, z \in \mathcal{P}$ where $\mathcal{R}yxz$ if $\mathcal{M}, y \Vdash \varphi$ then $\mathcal{M}, z \Vdash \psi$

Let $\mathsf{Con} \subseteq \{\wedge, \to, \circ, \supset, \subset\}$. We extend the scope of the relation $\Vdash$ to also relate points to $\mathcal{L}_{\mathsf{Con}}$–structures:

$\mathcal{M}, x \Vdash X, Y$	iff	$\mathcal{M}, x \Vdash X$ and $\mathcal{M}, x \Vdash Y$
$\mathcal{M}, x \Vdash X \,;\, Y$	iff	there are $y, z \in \mathcal{M}$ such that $\mathcal{R}yzx$, $\mathcal{M}, y \Vdash X$ and $\mathcal{M}, z \Vdash Y$

Let $X \vdash Y$ be a $\mathcal{L}_{\mathsf{Con}}$–sequent and let $(\mathcal{M}, x)$ be a pointed model. We say that $X \vdash Y$ *is true at* $(\mathcal{M}, x)$, written $\mathcal{M}, x \Vdash X \vdash Y$, when the following holds:

$$\mathcal{M}, x \Vdash X \vdash Y \quad \text{iff} \quad \text{if } \mathcal{M}, x \Vdash X, \text{ then there is } \varphi \in Y \text{ such that } \mathcal{M}, x \Vdash \varphi.$$

We say that *the $\mathcal{L}_{Con}$–sequent* $X \vdash Y$ *is valid*, written $X \Vdash Y$, when for all pointed models $(\mathcal{M}, x)$, $\mathcal{M}, x \Vdash X \vdash Y$. We say that *the $\mathcal{L}_{Con}$–hypersequent* $X_1 \vdash Y_1 \mid \ldots \mid X_n \vdash Y_n$ *is valid*, written $X_1 \Vdash Y_1 \mid \ldots \mid X_n \Vdash Y_n$, when $X_1 \Vdash Y_1$ or ... or $X_n \Vdash Y_n$. □

Here is a key inference of substructural logics, more precisely of the Lambek Calculus:

$$\chi \,;\, \varphi \Vdash \psi \quad \text{iff} \quad \chi \Vdash \varphi \supset \psi \qquad \text{(RT1)}$$

2.2 Updates as Ternary Relations

The ternary relation of the Routley and Meyer semantics was introduced originally for technical reasons: any 2-ary (n-ary) connective of a logical language can be given a semantics by resorting to a 3-ary (resp. $n + 1$-ary) relation on worlds. Subsequently, a number of philosophical interpretations of this ternary relation have been proposed (see [Beall *et al.*, 2012; Restall, 2006; Mares and Meyer, 2001] for more details). However, one has to admit that providing a non-circular and conceptually grounded interpretation of this relation remains problematic.

I proposed in [Aucher, 2014] a new *dynamic* interpretation. The proposal is based on the observation that an update can be represented abstractly as a ternary relation: the first argument of the ternary relation represents the initial situation/state, the second the event that occurs in this initial situation (the informative input) and the third the resulting situation/state after the occurrence of the event. With this interpretation in mind, $\mathcal{R}xyz$ reads as 'the occurrence of event y in world x results in the world z' and the corresponding conditional $\varphi \supset \psi$ reads as 'the occurrence in the current world of an event satisfying property φ results in a world satisfying ψ'.

This interpretation is coherent with a number of interpretations of the ternary relation proposed in substructural logic. Keeping in mind the truth conditions for the connective $\supset$ of Definition 5, the following quote makes perfect sense:

> "To be committed to $A \supset B$ is to be committed to B whenever we gain the information that A. To put it another way, a body of information warrants $A \supset B$ if and only if whenever you *update* that information with new information which warrants A, the resulting (perhaps new) body of information warrants B." (my emphasis) [Restall, 2006, p. 362]

2.3 Proof Systems

Our sequent calculus extends the Lambek calculus with propositional connectives.

Definition 6 (Sequent calculi $\mathsf{L}_{\mathsf{Con}}$)**.** Let $\mathsf{Con} \subseteq \{\wedge, \rightarrow, \circ, \supset, \subset\}$. The *sequent calculus for* $\mathcal{L}_{\mathsf{Con}}$, denoted $\mathsf{L}_{\mathsf{Con}}$, is the sequent calculus of Fig. 1 whose logical rules are restricted to the rules for the connectives of Con. The sequent calculus for propositional logic (where $\mathsf{Con} := \{\wedge, \rightarrow\}$) is denoted L_{PL}.

A $\mathcal{L}_{\mathsf{Con}}$–sequent $X \vdash Y$ is *provable in* $\mathsf{L}_{\mathsf{Con}}$, written $X \vdash^{\mathsf{L}_{\mathsf{Con}}} Y$, when it can be derived from the axioms and inference rules of $\mathsf{L}_{\mathsf{Con}}$ in a finite number of steps. A formula $\varphi \in \mathcal{L}_{\mathsf{Con}}$ is $\mathsf{L}_{\mathsf{Con}}$–consistent when it is not the case that $\varphi \vdash^{\mathsf{L}_{\mathsf{Con}}}$. We also write $\vdash^{\mathsf{L}_{\mathsf{Con}}} X$ for $\top \vdash^{\mathsf{L}_{\mathsf{Con}}} X$. □

Note that the following rules are derivable in L_{PL}:

$$\frac{X, \varphi \vdash Y}{X \vdash Y, \neg\varphi}\ \neg_R \qquad \frac{\Gamma[\varphi] \vdash U \quad \Gamma[\psi] \vdash U}{\Gamma[\varphi \vee \psi] \vdash U}\ \vee_L.$$

Theorem 1 (Cut elimination [Restall, 2000])**.** *Let* $\mathsf{Con} \subseteq \{\wedge, \rightarrow, \circ, \supset, \subset\}$*. The cut rule can be eliminated from any proof of* $\mathsf{L}_{\mathsf{Con}}$*.*

Theorem 2 (Soundness and completeness [Restall, 2000])**.** *Let* $\mathsf{Con} \subseteq \{\wedge, \rightarrow, \circ, \supset, \subset\}$*. Then, for all* $\mathcal{L}_{\mathsf{Con}}$*-sequents* $X \vdash Y$*, it holds that* $X \vdash^{\mathsf{L}_{\mathsf{Con}}} Y$ *iff* $X \Vdash Y$*.*

Axioms:

$$p \vdash p \qquad\qquad \bot \vdash$$

Structural Rules:

Weakening: *Contraction:*

$$\frac{\Gamma[X] \vdash U}{\Gamma[(Y, X)] \vdash U}\ \mathsf{W}_L \qquad \frac{\Gamma[(X, X)] \vdash U}{\Gamma[X] \vdash U}\ \mathsf{C}_L$$

$$\frac{U \vdash \Gamma[X]}{U \vdash \Gamma[(X, Y)]}\ \mathsf{W}_R \qquad \frac{U \vdash \Gamma[(Y, Y)]}{U \vdash \Gamma[Y]}\ \mathsf{C}_R$$

Permutation: *Associativity:*

$$\frac{\Gamma[(Y, X)] \vdash Z}{\Gamma[(X, Y)] \vdash Z}\ \mathsf{P}_L \qquad \frac{\Gamma[X, (Y, Z)] \vdash U}{\Gamma[(X, Y), Z] \vdash U}\ \mathsf{B}_L$$

$$\frac{U \vdash \Gamma[(Y, X)]}{U \vdash \Gamma[(X, Y)]}\ \mathsf{P}_R \qquad \frac{U \vdash \Gamma[(X, (Y, Z))]}{U \vdash \Gamma[((X, Y), Z)]}\ \mathsf{B}_R$$

Cut Rule:

$$\frac{X \vdash \varphi \quad \Gamma[\varphi] \vdash Y}{\Gamma[X] \vdash Y}\ \mathsf{Cut}$$

Logical Rules:

Propositional Connectives:

$$\frac{X, \psi \vdash \varphi}{X \vdash \psi \rightarrow \varphi}\ \rightarrow_R \qquad \frac{Y \vdash \psi \quad \Gamma[\varphi] \vdash X}{\Gamma[(\psi \rightarrow \varphi, Y)] \vdash X}\ \rightarrow_L$$

$$\frac{X \vdash \varphi \quad Y \vdash \psi}{X, Y \vdash \varphi \wedge \psi}\ \wedge_R \qquad \frac{\Gamma[(\varphi, \psi)] \vdash U}{\Gamma[(\varphi \wedge \psi)] \vdash U}\ \wedge_L$$

Substructural Connectives:

$$\frac{X \,;\, \psi \vdash \varphi}{X \vdash \psi \supset \varphi}\ \supset_R \qquad \frac{Y \vdash \psi \quad \Gamma[\varphi] \vdash X}{\Gamma[(\psi \supset \varphi \,;\, Y)] \vdash X}\ \supset_L$$

$$\frac{X \vdash \varphi \quad Y \vdash \psi}{X \,;\, Y \vdash \varphi \circ \psi}\ \circ_R \qquad \frac{\Gamma[(\varphi \,;\, \psi)] \vdash U}{\Gamma[(\varphi \circ \psi)] \vdash U}\ \circ_L$$

$$\frac{\varphi \,;\, Y \vdash \psi}{Y \vdash \psi \subset \varphi}\ \subset_R \qquad \frac{X \vdash \varphi \quad \Gamma[\psi] \vdash Y}{\Gamma[(X \,;\, \psi \subset \varphi)] \vdash Y}\ \subset_L$$

Figure 1: Our Sequent Calculus

3 Conditional Logic and Belief Revision

Default reasoning, sometimes identified with *non-monotonic reasoning* and formalized by conditional logics, involves making default assumptions and reasoning with the most typical or "normal" situations. Belief revision, on the other hand, deals with the representation of mechanisms for revising our beliefs. Even if the phenomena that are studied seem to be different, we will see in Section 3.3 that default reasoning and belief revision are in fact "two sides of the same coin".

3.1 Conditional Logic

Default reasoning arises frequently in everyday life. It involves leaping to conclusions. For example, if an agent sees a bird, she may conclude that it flies. However, not all birds fly: penguins and ostriches do not fly, nor do newborn birds, dead birds, or birds made of clay. Nevertheless, birds *typically* fly, and by default, in everyday life, we often reason with such abusive simplifications that are revised only after we receive more information. This explains informally why default reasoning is *non-monotonic*: adding new information may withdraw and invalidate some of our previous inferences.

Definition 7 (Language for defaults $\mathcal{L}_{\mathsf{DEF}}$)**.** The language for defaults is defined by $\mathcal{L}_{\mathsf{DEF}} := \{\varphi, \varphi \supset \psi \mid \varphi, \psi \in \mathcal{L}_{\mathsf{PL}}\}$. □

The formula $\varphi \supset \psi$ can be read in various ways, depending on the application. For example, it can be read as "if φ (is the case) then typically ψ (is the case)", "if φ, then normally ψ", "if φ, then by default ψ", and "if φ, then ψ is very likely".

Numerous semantics have been proposed for default statements, such as preferential structures [Kraus *et al.*, 1990], ϵ-semantics [Adams, 1975], the possibilistic structures [Dubois and Prade, 1991] and κ-ranking [Spohn, 1988]. They all have in common that they define the same set of validities axiomatized by the same proof system P (originally introduced by Kraus *et al.* [1990]). This remarkable fact is explained by Friedman and Halpern [2001][2003].

Definition 8 (System P)**.** The proof system P for $\mathcal{L}_{\mathsf{DEF}}$ is defined by the following axiom and inference rules, where all formulas are propositional.

$$\text{If } \vdash^{\mathsf{L_{PL}}} \varphi \leftrightarrow \varphi', \text{ then from } \varphi \supset \psi \text{ infer } \varphi' \supset \psi \quad \text{(LLE)}$$

$$\text{If } \vdash^{\mathsf{L_{PL}}} \psi \rightarrow \psi', \text{ then from } \varphi \supset \psi \text{ infer } \varphi \supset \psi' \quad \text{(RW)}$$

$$\varphi \supset \varphi \quad \text{(REF)}$$

$$\text{From } \varphi \supset \psi_1 \text{ and } \varphi \supset \psi_2 \text{ infer } \varphi \supset \psi_1 \wedge \psi_2 \quad \text{(AND)}$$

$$\text{From } \varphi_1 \supset \psi \text{ and } \varphi_2 \supset \psi \text{ infer } \varphi_1 \vee \varphi_2 \supset \psi \quad \text{(OR)}$$

$$\text{From } \varphi \supset \psi_1 \text{ and } \varphi \supset \psi_2 \text{ infer } \varphi \wedge \psi_2 \supset \psi_1. \quad \text{(CM)}$$

□

3.2 Belief Revision

In the so-called AGM belief revision theory of Alchourrón *et al.* [1985], the beliefs of the agent are represented by a *belief set*, denoted $\mathcal{K}$. These propositional formulas represent the beliefs of the agent. The revision of $\mathcal{K}$ with φ, written $\mathcal{K} * \varphi$, consists of adding φ to $\mathcal{K}$, but in order that the resulting set be consistent, some formulas are removed from $\mathcal{K}$. Because this can be done in various ways, 8 AGM rationality postulates have been elicited as reasonable principles for revision.

Formally, A *belief set* $\mathcal{K}$ is a set of propositional formulas of $\mathcal{L}_{\mathsf{PL}}$ such that $Cn(\mathcal{K}) = \mathcal{K}$ (where $Cn(\mathcal{K}) := \{\varphi \in \mathcal{L}_{\mathsf{PL}} \mid \varphi_1, \ldots, \varphi_n \vdash^{\mathsf{L_{PL}}} \varphi \text{ for some } \varphi_1, \ldots, \varphi_n \in \mathcal{K}\}$). Let $\mathcal{K}$ be a belief set and let $\varphi \in \mathcal{L}_{\mathsf{PL}}$. As argued by Katsuno and Mendelzon, because $\mathbb{P}$ is finite, a belief set $\mathcal{K}$ can be equivalently represented by a mere propositional formula χ. This formula is also called a *belief base*. Then, $\varphi \in \mathcal{K}$ if and only if $\varphi \in Cn(\chi)$.

We define the *expansion of* $\mathcal{K}$ *by* φ, written $\mathcal{K} + \varphi$, as follows: $\mathcal{K} + \varphi = Cn(\mathcal{K} \cup \{\varphi\})$. Then, one can easily show that $\psi \in \mathcal{K} + \varphi$ if and only if $\psi \in Cn(\chi \wedge \varphi)$. So, in this approach, the expansion of the belief base χ by φ is the belief base $\chi \wedge \varphi$, which is possibly an inconsistent formula. Now, given a belief base χ and a formula φ, $\chi \circ \varphi$ denotes the *revision* of χ by φ. But in this case, $\chi \circ \varphi$ is supposed to be consistent if φ is. Given a revision operation $*$ on belief sets, one can define a corresponding revision operation $\circ$ on belief bases as follows: $\chi \circ \varphi \rightarrow \psi$ if, and only if, $\psi \in Cn(\chi) * \varphi$. Then, we have that:

Lemma 3 (Katsuno and Mendelzon 1992)**.** *Let * be a revision operation on belief sets and $\circ$ its corresponding operation on belief bases. Then * satisfies the 8 AGM postulates if, and only if, $\circ$ satisfies the postulates* (R1)–(R6) *below:*

$$\chi \circ \varphi \rightarrow \varphi \quad \text{(R1)}$$

$$\textit{if } \chi \wedge \varphi \textit{ is } \mathsf{L_{PL}}\textit{–consistent, then } \chi \circ \varphi \leftrightarrow \chi \wedge \varphi \quad \text{(R2)}$$

$$\textit{If } \varphi \textit{ is } \mathsf{L_{PL}}\textit{–consistent, then } \chi \circ \varphi \textit{ is also } \mathsf{L_{PL}}\textit{–consistent} \quad \text{(R3)}$$

$$\textit{If } \chi \leftrightarrow \chi' \textit{ and } \varphi \leftrightarrow \varphi', \textit{ then } \chi \circ \varphi \leftrightarrow \chi' \circ \varphi' \quad \text{(R4)}$$

$$(\chi \circ \varphi) \wedge \varphi' \rightarrow \chi \circ (\varphi \wedge \varphi') \quad \text{(R5)}$$

$$\textit{If } (\chi \circ \varphi) \wedge \varphi' \textit{ is } \mathsf{L_{PL}}\textit{–consistent, then } \chi \circ (\varphi \wedge \varphi') \rightarrow (\chi \circ \varphi) \wedge \varphi' \quad \text{(R6)}$$

3.3 "Two Sides of the Same Coin": Ramsey Test

A well-known result, originally suggested by Ramsey [1929], connects closely non-monotonic reasoning with belief revision. Informally, from φ I can non-monotonically infer ψ if, and only if, I believe ψ after revising my belief base with φ. This lead Makinson and Gärdenfors [1989][1991] to show formally that non-monotonic reasoning and belief revision are "two sides of the same coin".

Theorem 4 (Halpern 2003)**.**

- *Suppose that a revision operation $\circ$ satisfies* (R1)–(R6)*. Fix a belief base χ, and define a relation $\supset$ on propositional formulas by taking $\varphi \supset \psi$ to hold iff $\chi \circ \varphi \rightarrow \psi$. Then, $\supset$ satisfies all the properties of P as well as Rational Monotonicity:*

$$\textit{if } \varphi \supset \psi_1 \textit{ and not } \varphi \supset \neg\psi_2, \textit{ then } \varphi \wedge \psi_2 \supset \psi_1 \quad (\mathsf{Rat})$$

  Moreover, $\varphi \supset \bot$ if, and only if, φ is not satisfiable.
- *Conversely, suppose that $\supset$ is a relation on formulas that satisfies the properties of P and Rational Monotonicity* (Rat)*, and $\varphi \supset \bot$ if, and only if, φ is not satisfiable. Let $\mathcal{K} = \{\psi \in \mathcal{L}_{\mathsf{PL}} \mid \top \supset \psi\}$. Then, $\mathcal{K}$ is a belief set. Let χ be its corresponding belief base. Then, if $\circ$ is defined by taking $\chi \circ \varphi \rightarrow \psi$ if, and only if, $\chi \rightarrow (\varphi \supset \psi)$, then the postulates* (R1)–(R6) *hold for χ and $\circ$.*

$$\dfrac{X \vdash Y}{(Z\,;\,X) \vdash Y}\ \mathsf{R}_1 \qquad \dfrac{(X\,;\,Y) \vdash U}{((X,Z)\,;\,Y) \vdash U}\ \mathsf{W}'_L$$

$$\dfrac{(W\,;\,X) \vdash Y}{(W\,;\,(X,Z)) \vdash Y \;\big|\; ((W\,;\,X),Z) \vdash}\ \ \mathrm{RM}$$

Figure 2: Structural Rules for $\mathsf{L_{P^+}}$ and $\mathsf{L_{NMR}}$

4 Sequent Calculi for Belief Revision and Conditional Logic

We show that the main systems of common sense reasoning can be reformulated in the proof-theoretical setting of substructural logics.

4.1 Conditional Logic in Substructural Logic

Because we resort to the structural connective ; we need to introduce and add the logical connective $\circ$ to the system P:

Definition 9 (Proof systems P^+ and $\mathsf{L_{P^+}}$)**.**

- The *calculus* P^+ *for* $\mathcal{L}_{\circ,\supset}$ is the calculus P to which we add the following (bidirectional) inference rule:

$$\chi \circ \varphi \to \psi \quad \text{iff} \quad \chi \to \varphi \supset \psi \qquad \text{(RT2)}$$

- The *sequent calculus* $\mathsf{L_{P^+}}$ *for* $\mathcal{L}_{\supset}$ is the sequent calculus $\mathsf{L}_{\supset}$ where the structural rules are restricted to the $\mathcal{L}_{\supset}$–sequents of depth 0, with rules R_1 and W'_L of Fig. 2. □

Proposition 5. *The cut rule can be eliminated from any proof of* $\mathsf{L_{P^+}}$*. Moreover, all the rules of* $\mathsf{L_{P^+}}$ *are invertible.*

Proof sketch. It can be adapted from [Restall, 2000, Th. 6.11] and [Troelstra and Schwichtenberg, 2000, Prop. 3.5.4]. □

Theorem 6. *Let* $\chi, \varphi, \psi \in \mathcal{L}_{\mathsf{PL}}$*. Then, the following holds:*

$$\chi \vdash_{\mathsf{L_{P^+}}} \varphi \supset \psi \quad \textit{iff} \quad \chi \vdash_{\mathsf{P^+}} \varphi \supset \psi$$

Proof. In this proof and the following, we will use the mappings $t_1 : \mathcal{S}_L^{\mathcal{L}_{\mathsf{Con}}} \to \mathcal{L}_{\mathsf{Con}}$ and $t_2 : \mathcal{S}_R^{\mathcal{L}_{\mathsf{Con}}} \to \mathcal{L}_{\mathsf{Con}}$ defined inductively as follows:

$$\begin{array}{lll lll} t_1(\varphi) & := & \varphi & t_2(\varphi) & := & \varphi \\ t_1(X,Y) & := & t_1(X) \wedge t_1(Y) & t_2(X,Y) & := & t_2(X) \vee t_2(Y) \\ t_1(X\,;\,Y) & := & t_1(X) \circ t_1(Y) & & & \end{array}$$

We can prove by induction on the number of steps used that

$$X \vdash_{\mathsf{L_{Con}}} Y \quad \text{iff} \quad t_1(X) \vdash_{\mathsf{L_{Con}}} t_2(Y) \qquad (1)$$

The proof of Theorem 6 is by induction on the number of inference steps used in a proof. This boils down to show that each rule of inference and each axiom of $\mathsf{L_{P^+}}$ is derivable in P^+, and, vice versa, each rule and each axiom of P^+ is derivable in $\mathsf{L_{P^+}}$. First, we prove that rules (LLE), (RW), (REF), (CM), (AND) and (OR) are derivable in $\mathsf{L_{P^+}}$ (in this order):

$$\dfrac{\dfrac{\varphi' \vdash \varphi \quad \psi \vdash \psi}{\varphi \supset \psi\,;\,\varphi' \vdash \psi}\,\supset_L}{\varphi \supset \psi \vdash \varphi' \supset \psi}\,\supset_R \qquad \dfrac{\dfrac{\varphi \vdash \varphi \quad \psi \vdash \psi'}{\varphi \supset \psi\,;\,\varphi \vdash \psi'}\,\supset_L}{\varphi \supset \psi \vdash \varphi \supset \psi'}\,\supset_R \qquad \dfrac{\dfrac{\varphi \vdash \varphi}{\top\,;\,\varphi \vdash \varphi}\,\mathsf{R}_1}{\top \vdash \varphi \supset \varphi}\,\supset_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi}{\varphi, \psi_2 \vdash \varphi}\,\mathsf{W}_L}{\varphi \wedge \psi_2 \vdash \varphi}\,\wedge_L \quad \psi_1 \vdash \psi_1}{\varphi \supset \psi_1\,;\,\varphi \wedge \psi_2 \vdash \psi_1}\,\supset_L}{\varphi \supset \psi_1 \vdash \varphi \wedge \psi_2 \supset \psi_1}\,\supset_R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi \quad \psi_1 \vdash \psi_1}{\varphi \supset \psi_1\,;\,\varphi \vdash \psi_1}\,\supset_L}{(\varphi \supset \psi_1, \varphi \supset \psi_2)\,;\,\varphi \vdash \psi_1}\,\mathsf{W}'_L \quad \dfrac{\dfrac{\varphi \vdash \varphi \quad \psi_2 \vdash \psi_2}{\varphi \supset \psi_2\,;\,\varphi \vdash \psi_2}\,\supset_L}{(\varphi \supset \psi_1, \varphi \supset \psi_2)\,;\,\varphi \vdash \psi_2}\,\mathsf{W}'_L}{(\varphi \supset \psi_1, \varphi \supset \psi_2)\,;\,\varphi \vdash \psi_1 \wedge \psi_2}\,\wedge_R}{\varphi \supset \psi_1, \varphi \supset \psi_2 \vdash \varphi \supset \psi_1 \wedge \psi_2}\,\supset_R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi_1 \vdash \varphi_1 \quad \psi \vdash \psi}{\varphi_1 \supset \psi\,;\,\varphi_1 \vdash \psi}\,\supset_L}{(\varphi_1 \supset \psi, \varphi_2 \supset \psi)\,;\,\varphi_1 \vdash \psi}\,\mathsf{W}'_L \quad \dfrac{\dfrac{\varphi_2 \vdash \varphi_2 \quad \psi \vdash \psi}{\varphi_2 \supset \psi\,;\,\varphi_2 \vdash \psi}\,\supset_L}{(\varphi_1 \supset \psi, \varphi_2 \supset \psi)\,;\,\varphi_2 \vdash \psi}\,\mathsf{W}'_L}{(\varphi_1 \supset \psi, \varphi_2 \supset \psi)\,;\,\varphi \vee \varphi_1 \vdash \varphi}\,\vee_L}{\varphi_1 \supset \psi, \varphi_2 \supset \psi \vdash \varphi_1 \vee \varphi_2 \supset \varphi}\,\supset_R$$

As for rule (RT2), because the rules of $\mathsf{L_{P^+}}$ are invertible (Proposition 5), we have that $\chi\,;\,\varphi \vdash_{\mathsf{L_{P^+}}} \psi$ iff $\chi \vdash_{\mathsf{L_{P^+}}} \varphi \supset \psi$, by rule $\supset_R$. We derive (RT2) by applying Expression 1. The derivability of modus ponens follows from the cut rule.

Now, consider the right to left direction. Cut elimination holds for $\mathsf{L_{P^+}}$ (Proposition 5) and $\mathsf{L_{P^+}}$ satisfies the subformula property. Because what we prove is of the form $\chi \vdash_{\mathsf{L_{P^+}}} \varphi \supset \psi$ with $\chi, \varphi, \psi \in \mathcal{L}_{\mathsf{PL}}$, this entails that the $\mathcal{L}_{\supset}$–sequent will all be of depth at most 1 (see Definition 2). So, in what follows, we only consider $\mathcal{L}_{\supset}$–structures of depth at most 1.

The rules of $\mathsf{L_{PL}}$ are all derivable in P^+, because $\mathsf{L_{PL}} \subseteq \mathsf{P}^+$. We consider the rules $\supset_R, \supset_L, \mathsf{R}_1$ and W'_L (we do not consider Cut because of Proposition 5). First, we prove that $\supset_R$ is derivable in P^+, the proof for $\supset_L$ is similar. Assume that $X\,;\,\psi \vdash_{\mathsf{L_{P^+}}} \varphi$; we must prove that $X \vdash_{\mathsf{L_{P^+}}} \psi \supset \varphi$. That is, by Expression 1, we must prove that from $t_1(X) \circ \psi \to \varphi$, we can infer $t_1(X) \to (\psi \supset \varphi)$ in P^+. This last inference follows from (RT2) of P^+. Second, we consider R_1. Assume that $X \vdash_{\mathsf{L_{P^+}}} Y$; we must prove that $(Z\,;\,X) \vdash_{\mathsf{L_{P^+}}} Y$. That is, by Expression 1 and (RT2), we must prove that from $t_1(X) \to t_2(Y)$ $(*)$, we can infer $t_1(Z) \to (t_1(X) \supset t_2(Y))$ $(**)$. By (RW) and (REF), we can prove $t_1(X) \supset t_2(Y)$ from $(*)$, and therefore also $(**)$. Third, we consider W'_L. Assume that $(X\,;\,Y) \vdash_{\mathsf{L_{P^+}}} U$, we must prove that $(X,Z)\,;\,Y \vdash_{\mathsf{L_{P^+}}} U$. That is, by Expression 1 and (RT2), we must prove that from $t_1(X) \vdash_{\mathsf{L_{P^+}}} t_1(Y) \supset t_2(U)$ $(*)$ we can prove $t_1(X) \wedge t_1(Z) \vdash_{\mathsf{L_{P^+}}} t_1(Y) \supset t_2(U)$. This is true in $\mathsf{L_{PL}}$. □

4.2 Belief Revision in Substructural Logic

Based on the rationality postulates (R1)–(R6), we can define a Hilbert-*like* proof system. It is not a genuine Hilbert system because inference rules are not of the standard form: some premises refer to the satisfiability of formulas. This drawback is avoided in our sequent calculus reformulation by resorting to *hypersequents* [Pottinger, 1983; Avron, 1996].[1]

Definition 10 (Proof systems AGM and $\mathsf{L_{AGM}}$)**.**

[1]In all the hypersequent calculi that we define in the sequel (based on already defined sequent calculi) we always take the *internal* version of the structural rules. See [Avron, 1996] for details.

$$\frac{X \vdash Y}{(Z\,;X) \vdash Y}\ \mathsf{R}_1 \qquad \frac{(X\,;Y) \vdash Z}{(X,Y) \vdash Z}\ \mathsf{R}_2^a$$

$$\frac{(X\,;Y) \vdash}{Y \vdash}\ \mathsf{R}_3 \qquad \frac{(X,Y) \vdash Z}{(X\,;Y) \vdash Z \mid (X,Y) \vdash}\ \mathsf{R}_2^b$$

$$\frac{(X\,;(Y,Z)) \vdash U}{((X\,;Y),Z) \vdash U}\ \mathsf{R}_5$$

$$\frac{((X\,;Y),Z) \vdash U}{(X\,;(Y,Z)) \vdash U \mid ((X\,;Y),Z) \vdash}\ \mathsf{R}_6$$

Figure 3: Structural Rules for $\mathsf{L}_{\mathsf{AGM}}$

- The *Hilbert-like calculus* AGM *for* $\mathcal{L}_\circ$ is the Hilbert calculus of propositional logic to which we add the axioms and inference rules (R1)–(R6) of Lemma 3 (where L_{PL}–consistency is replaced by AGM–consistency).
- The *hypersequent calculus* L_{AGM} *for* $\mathcal{L}_\circ$ is the sequent calculus $\mathsf{L}_\circ$ to which we add the rules of Figure 3. □

Proposition 7. *The cut rule can be eliminated from any proof of* L_{AGM}*. Moreover, all the rules of* L_{AGM} *are invertible.*

Proof sketch. Similar to the proof of Proposition 5. □

Theorem 8. *Let* $\chi, \varphi, \psi \in \mathcal{L}_{\mathsf{PL}}$*. Then, the following holds:*

$$\chi \circ \varphi \mathrel{\vert\!\!\!\frac{\ \mathsf{L}_{AGM}\ }{}} \psi \quad \textit{iff} \quad \chi \circ \varphi \mathrel{\vert\!\!\!\frac{\ \mathsf{AGM}\ }{}} \psi$$

Proof sketch. The proof follows the same methodology as Theorem 6. For the left to right direction, we prove (R1), (R4), (R2) (*a* and *b*) and (R5) (rule (R6) is proved similarly).

$$\dfrac{\dfrac{\dfrac{\overline{\varphi \vdash \varphi}}{\psi\,;\varphi \vdash \varphi}\ \mathsf{R}_1}{\psi \circ \varphi \vdash \varphi}\ \circ_L}{\vdash \psi \circ \varphi \to \varphi}\ \to_R \qquad \dfrac{\dfrac{\dfrac{\psi_1 \vdash \psi_2 \quad \varphi_1 \vdash \varphi_2}{\psi_1\,;\varphi_1 \vdash \psi_2 \circ \varphi_2}\ \circ_R}{\psi_1 \circ \varphi_1 \vdash \psi_2 \circ \varphi_2}\ \circ_L}{\vdash \psi_1 \circ \varphi_1 \to \psi_2 \circ \varphi_2}\ \to_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\psi \vdash \psi} \quad \overline{\varphi \vdash \varphi}}{\psi\,;\varphi \vdash \psi \circ \varphi}\ \circ_R}{\psi,\varphi \vdash \psi \circ \varphi}\ \mathsf{R}_2^a}{\psi \wedge \varphi \vdash \psi \circ \varphi}\ \wedge_L}{\vdash \psi \wedge \varphi \to \psi \circ \varphi}\ \to_R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\psi \vdash \psi} \quad \overline{\varphi \vdash \varphi}}{\psi,\varphi \vdash \psi \wedge \varphi}\ \wedge_R}{\psi\,;\varphi \vdash \psi \wedge \varphi \mid \psi,\varphi \vdash}\ \mathsf{R}_2^b}{\psi \circ \varphi \vdash \psi \wedge \varphi \mid \psi,\varphi \vdash}\ \circ_L}{\psi \circ \varphi \vdash \psi \wedge \varphi \mid \psi \wedge \varphi \vdash}\ \wedge_L}{\vdash \psi \circ \varphi \to \psi \wedge \varphi \mid \psi \wedge \varphi \vdash}\ \to_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\psi \vdash \psi} \quad \dfrac{\overline{\varphi \vdash \varphi} \quad \overline{\varphi' \vdash \varphi'}}{\varphi,\varphi' \vdash \varphi \wedge \varphi'}\ \wedge_R}{\psi\,;(\varphi,\varphi') \vdash \psi \circ (\varphi \wedge \varphi')}\ \circ_R}{(\psi\,;\varphi),\varphi' \vdash \psi \circ (\varphi \wedge \varphi')}\ \mathsf{R}_5}{(\psi \circ \varphi, \varphi' \vdash \psi \circ (\varphi \wedge \varphi')}\ \circ_L}{(\psi \circ \varphi) \wedge \varphi' \vdash \psi \circ (\varphi \wedge \varphi')}\ \wedge_L}{\vdash (\psi \circ \varphi) \wedge \varphi' \to \psi \circ (\varphi \wedge \varphi')}\ \to_R$$

As for (R3), assume that $\psi \circ \varphi \mathrel{\vert\!\!\!\frac{\ \mathsf{L}_{\mathsf{AGM}}\ }{}}$. Then, because the logical rules are invertible (Proposition 7), we have that $\psi\,;\varphi \mathrel{\vert\!\!\!\frac{\ \mathsf{L}_{\mathsf{AGM}}\ }{}}$. So, by Rule R_3, we have that $\varphi \mathrel{\vert\!\!\!\frac{\ \mathsf{L}_{\mathsf{AGM}}\ }{}}$.

The right to left direction is proved similarly and relies also on the fact that (hyper)sequents are of depth 1 because of our cut elimination result (Proposition 7). □

4.3 The Ramsey Test in Substructural Logic

We are going to reformulate Theorem 4 in our sequent calculi, and obtain a new formalization of the Ramsey test. Note the similarity between Expressions (RT1), (RT2) and (RT3).

Definition 11 (Hypersequent calculus $\mathsf{L}_{\mathsf{NMR}}$)**.** The *hypersequent calculus* L_{NMR} *for* $\mathcal{L}_\supset$ is the sequent calculus $\mathsf{L}_{\mathsf{P}^+}$ to which we add the structural rule (RM) of Fig. 2. □

Theorem 9 (Ramsey test)**.** *Let* $\chi, \varphi, \psi \in \mathcal{L}_{\mathsf{PL}}$*. Then, the following holds:*

$$\chi \mathrel{\vert\!\!\!\frac{\ \mathsf{L}_{NMR}\ }{}} \varphi \supset \psi \quad \textit{iff} \quad \chi \circ \varphi \mathrel{\vert\!\!\!\frac{\ \mathsf{L}_{AGM}\ }{}} \psi \qquad \text{(RT3)}$$

Proof sketch. We define the Hilbert calculus $\mathsf{NMR} := \mathsf{L}_{\mathsf{P}^+} +$ (RM). Theorem 4 can be reformulated in our setting as follows: if $\chi, \varphi, \psi \in \mathcal{L}_{\mathsf{PL}}$, then $\chi \mathrel{\vert\!\!\!\frac{\ \mathsf{NMR}\ }{}} \varphi \supset \psi$ iff $\chi \circ \varphi \mathrel{\vert\!\!\!\frac{\ \mathsf{AGM}\ }{}} \psi$. So, if we prove that NMR and $\mathsf{L}_{\mathsf{NMR}}$ are provably equivalent, then we will have proved the theorem, because we already know by Theorem 6 that AGM and $\mathsf{L}_{\mathsf{AGM}}$ are provably equivalent. To prove that, it suffices to show that the inference rule (RM) is derivable in $\mathsf{L}_{\mathsf{NMR}}$ and, vice versa, that the inference rule (Rat) is derivable in NMR. It is proved without difficulty. □

5 Gärdenfors' Impossibility Result

As to the Ramsey test, a famous result [Gärdenfors, 1988] states a difficulty in introducing a connective $\supset$ such that $\varphi \supset \psi \in \mathcal{K}$ iff $\psi \in \mathcal{K} * \varphi$. Indeed, an immediate consequence means that if $\mathcal{K} \subseteq \mathcal{K}'$ then $\mathcal{K} * \varphi \subseteq \mathcal{K}' * \varphi$, which is a property essentially incompatible with the AGM postulates for $*$. Accordingly, we retrieve Gärdenfors' result as follows:

$$\dfrac{\dfrac{\overline{\top \vdash \top} \quad \overline{\bot \vdash}}{\top \supset \bot\,;\top \vdash}\ \supset_L}{\top \vdash}\ \mathsf{R}_3$$

Inconsistency of $\mathsf{L}_{\circ,\supset}$ extended with R_3 reflects, in a way reminiscent of Gärdenfors' proof, that conditionals capturing defaults do not easily lend themselves to the role of premises.

6 Conclusion

Drawing intuitions from a dynamic interpretation of substructural concepts in terms of updating, we have reformulated conditional logic and belief revision in the substructural framework as extensions of the Lambek calculus. We thus retrieve some well-known results and provide new axiomatizations of belief revision and default reasoning.

In particular, our results show that the key to non-monotonicity is a constrained application of the left weakening rule: our inferences stay the same if our knowledge of the initial situation is made more precise, but we may cancel some of them if we are forced to update our knowledge in face of new information (see rule W'_L and definition of $\mathsf{L}_{\mathsf{P}^+}$).

The range of belief revision and default reasoning clearly calls for further work. For example, in our setting, given our reading of ternary relations as updates and given our truth conditions, the connective $\subset$ represents some sort of abduction (see Definition 5). This notion of abduction can now be studied within our substructural framework in interaction with the notions of revision, update and non-monotonicity.

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