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ABSTRACT

This paper develops a hybrid/gain scheduled control to move a diesel engine through a driving profile represented as a set of 12 operating points in the 7-dimensional state space of a 7^{th} order nonlinear state model. The calculations for the control design are based on a 3^{rd} order(reduced) model of the Diesel engine on which state space is projected the 12 operating points. About each operating point, we generate a 3^{rd} order nonlinear error models of the Diesel engine. Using the error model for each operating point, a control design is set forth as a system of LMI's. The solution of each system of LMI's produces a norm bounded controller guaranteeing that $x_i^d{}_1 \quad x_i^d$ where x_i^d is the i-th desired operating point in the 3-dimensional state space. The control performance is then evaluated on the 7^{th} order model.

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INTRODUCTION.

This paper sets forth the first control design phase in the development of a control law for a diesel engine having a VGT/EGR(Variable Geometry Turbocharger/Exhaust Gas Recirculation) system. [7] For this phase, we assume a linear hybrid/gain-scheduled state feedback control law evaluated on a 7^{th} order nonlinear diesel engine model. The objective of the controller is to drive the engine operating point from an initial value to a desired value along a drive cycle and stabilize the engine around the desired equilibrium. This equilibrium is computed so that it satisfies driver needs while achieving a reasonable trade-off between undesirable emissions of nitrogen oxides (NO_x) and smoke emissions on the one hand, and fuel consumption on the other hand.

The control design will be achieved using polytopic system methods. Here a chain of overlapping compact regions of the state space is formed so that each region contains an equilibrium point common to the next polytopic region in the chain. Given appropriate continuity, the induced image of each region in the model vector fields is bounded by a polytope. Using Lyapunov methods applied to each region, a feedback control and an ellipsoidal domain of attraction is obtained by solving a set of LMIs. Each controller will move the state through the associated region to an operating point common to the domain of attraction of current region and the next region along the chain. The controller for the next equilibrium state is invoked when the system is sufficiently close to the preceding equilibrium state. Thus the control law is hybrid in nature in that invocation of a new control requires knowledge of the current region as well as the next desired region. This of course is the lowest type of hybrid control.

To preserve geometric understanding of the control construction and to simplify the calculations associated with the LMIs, the design will build on a third order nonlinear diesel engine model. However, the control and its performance will be evaluated using a 7^{th} order nonlinear diesel engine model.

2. MODEL OF THE DIESEL ENGINE WITH VGT/EGR.

Over the last few decades, legislated levels of motor vehicle exhaust emissions have been cut dramatically , forcing automotive engineers to focus on emission control more than ever. Diesel engines offer superior fuel economy but their nitrogen oxide (NO_x) emission control remains challenging. This is because the conventional Three-Way Catalyst utilized in gasoline-powered vehicles is not efficient for NO_x conversion at lean air-to-fuel ratios where diesels typically operate. To be competitive with gasoline engines, new generation diesel engines are equipped with exhaust gas recirculation (EGR) systems to reduce NO_x emission and variable geometry turbochargers (VGT) to reduce transient smoke. Combination of EGR and VGT provides an important avenue for NO_x emission reduction. Traditionally, turbocharging has been used to increase the power density of diesel engines. Variable geometry turbocharging is accomplished by a turbine that has a system of movable guide vanes located on the turbine. By adjusting the guide vanes, the exhaust gas energy to the turbocharger can be regulated, thus controlling the compressor mass airflow and exhaust manifold pressure. (See figure 1.)

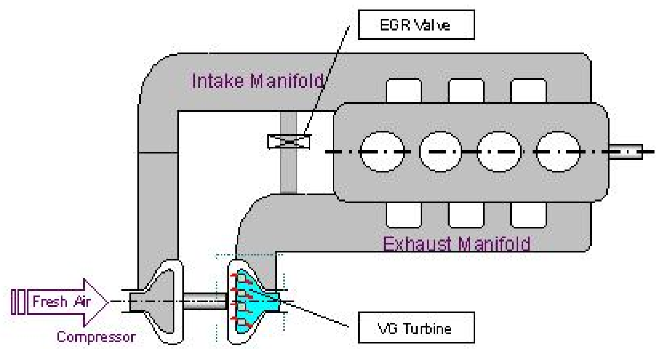

Figure 1: Schematic diagram of VGT/EGR diesel engines

Initially, the rationale for using the variable geometry turbocharging was to increase engine torque output at tip-ins and reduce turbo-lag. Now, variable geometry turbocharging has emerged as an important way to reduce NO_x emissions because it can be used to increase the exhaust gas recirculation rates. The exhaust gas recirculation is accomplished by an EGR valve that directs some of the exhaust gas from the exhaust manifold into the intake manifold. This dilutes the cylinder charge and lowers the combustion temperatures thereby impeding the process of NO_x formation. Because the flow through the EGR valve depends on the pressure drop across the valve, and because the VGT can affect this pressure drop, the turbocharger can also be utilized to increase the EGR flow. Thus, these two devices are strongly coupled and the system exhibits internal instability, requiring advanced control algorithms. The controller has to keep EGR flow rate and air-fuel ratio at the desired levels such that NO_x emission, as well as transient smoke can be lowered to meet future regulations. Additional factors that burden the application of conventional control design methods are:

∑ lack of information about the system states; only a limited number of physical coordinates (air flow rate and intake manifold pressure) can be measured directly, while knowledge of the variable "EGR flow rate" to be controlled is not available;
∑ parameters of the process (intake burnt gas fraction, intake charge flow rate and exhaust pressure) are unknown, due to aging and deposits on the flow ways/valves, and may vary over a wide range for different operation modes;
∑ system behavior is governed by a high order system of nonlinear equations and as a result new control strategies should be developed (traditional methods based on PI or PID controllers are generally no longer effective).

In light of this background, the model of the VGT/EGR Diesel engine ([1],[2],[3]) is obtained through the application of the mass and energy balances for the intake and exhaust manifolds. For control design we will use a simplified 3rd order model whereas the 7th order model will serve to evaluate and fine tune the control design.

2.1 The 7th order model for the diesel engine. [1, 2, 3, 8]

The definitions of the state variables used in the nonlinear state model below are: (i) ρ_1, ρ_2 (gas density in intake (subscript 1) and exhaust (subscript 2) manifold), (ii) F_1, F_2 (burnt gas fractions in intake and exhaust manifolds), (iii) p_1, p_2 (pressures in intake and exhaust manifolds), and (iv) P_C (compressor power). For convenience we define the state vector

$$x(t) = [\rho_1(t), F_1(t), p_1(t), \rho_2(t), F_2(t), p_2(t), P_C(t)]^T .$$

In general, the indices 1, 2, e, and C stand for intake manifold, exhaust manifold, engine and compressor respectively. The symbols W_{ij} and u_{ij} have the meaning of flows from index i to index j where i, j $\in$ {1, 2, e, C}. T_i has the meaning of temperature in "compartment" i, whereas T_{ij} means the temperature of the mixture flowing from i to j.

The intake manifold equations are:

$$\begin{aligned} \dot\rho_1 &= \left[W_C + u_{21}(\chi) - W_{1e} - u_{12}(\chi)\right]/V_1 \\ \dot F_1 &= \left[(F_2 - F_1)u_{21}(\chi) - F_1 W_C\right]/\rho_1 V_1 \\ \dot p_1 &= \gamma R\left[W_C T_C + u_{21}(\chi)T_2 - W_{1e}T_1 - u_{12}(\chi)T_1\right]/V_1 \end{aligned} \tag{2.1}$$

where (i) γ is the specific heat ratio, (ii) R is the universal gas law constant (the difference of specific heats), (iii) V_1, V_2 are the volumes of the intake and exhaust manifolds respectively, and (iv) $u_{12}(\chi)$ and $u_{21}(\chi)$ are EGR flows that depend on the EGR value position χ, $0 \le \chi \le 1$, 0 denoting fully closed and 1 fully open. Note that W_C is the flow from the compressor to the intake manifold, and the mass flow rate of the intake charge to the engine is $W_{1e} = \frac{\eta_{vol}\rho_1 N V_d}{120}$ where (i) η_{vol} is the volumetric efficiency of the engine, (ii) ρ_1 is the density of the gas mixture in the intake manifold, (iii) N is the engine RPM, (iv) V_d is the displacement volume of the engine; alternately we can write $W_{1e} = k_e p_1$ where $k_e = \frac{\eta_{vol} N V_d}{120 \cdot R \cdot T_1}$ is a pumping rate coefficient associated with air flow through the engine.

The exhaust manifold equations are:

$$\begin{aligned} \dot\rho_2 &= \left(W_{e2} - u_{2t}(\xi) - u_{21}(\chi) + u_{12}(\chi)\right)/V_2 \\ \dot F_2 &= \left[(F_{e2} - F_2)W_{e2}(W_f) + (F_1 - F_2)u_{12}(\chi)\right]/\rho_2 V_2 \\ \dot p_2 &= \gamma R\left[W_{e2}(W_f)T_{e2} - u_{2t}(\xi)T_2 - u_{21}(\chi)T_2 + u_{12}(\chi)T_1\right]/V_2 \end{aligned} \tag{2.2}$$

where $u_{2t}(\xi)$ is the controlled flow of the exhaust gases across the turbine veins which depends on the VGT fin position ξ, $0 \le \xi \le 1$. Note that W_f is fuel flow and $W_{e2}(W_f)$ means that the flow from the engine to the exhaust manifold depends on fueling rate W_f.

The power transfer from the turbine to the compressor is described by the first order differential equation:

$$\dot{P}_C = (\quad P_C + \quad {}_m P_t)/ \qquad (2.3)$$

where is turbine to compressor power transfer time constant and ${}_m$ is the turbocharger mechanical efficiency. In actuality, $P_C(t + \quad) = \quad {}_m P_t(t)$. Equation 2.3 arises from a truncated Taylor series for $P_C(t + \quad)$ about t, i.e.,

$$P_C(t + \quad) = P_C(t) + \dot{P}_C(t) \quad + \text{H.O.T.}$$

(H.O.T. means higher order terms) It follows that

$$\dot{P}_C(t) = \frac{1}{}\left[P_C(t + \quad) \quad P_C(t)\right] = \left[\quad P_C(t) + \quad {}_m P_t(t)\right]/$$

Thus as per [8], power transfer between the turbine and the compressor is represented by a first order lag with a time constant . The time constant is assumed constant which in reality is untrue as depends on the turbocharger operating point.

Physically speaking the control inputs are the valve position of the EGR and the fin position of the VGT. Since the valve/fin positions control flow, for convenience we use the actual flows as control variables in our work, i.e., we use $u_{21}(\ ) = W_{egr}(\ )$ and $u_{2t}(\ )$, and desire control values for these quantities as opposed to and . This assumes that the appropriate system states are measurable and it also avoids having to deal with any dynamics and disturbances associated with moving the EGR baffle or adjustment of the veins on the compressor.

Finally, for practical reasons, we set $u_{12}(\ ) = 0$, i.e., the EGR flow from the intake manifold to the exhaust manifold is set to zero, although there are circumstances when this is not the case.

2.2 Reduced order model

A reduced (3rd order) model of the diesel engine follows by differentiating the ideal gas law for the intake and exhaust manifolds under the assumptions that the temperatures in the intake and exhaust manifolds are constant for local operation and that the there is no dependence of thermodynamic properties on composition (i.e. all thermodynamic properties are considered with respect to air $\ = \ _a$ and $R = R_a$). As mentioned, this reduced order model will be used for control design and its evaluation will take place on the 7th order model. Set points for the 3rd order model are the projection of those of the 7th order model. From [1] the 3rd order reduced model has the form:

$$
\begin{aligned}
\dot{p}_1 &= k_1\left(W_C + W_{egr} \quad W_{1e}\right) \\
\dot{p}_2 &= k_2\left(W_{e2}(W_f) \quad W_{egr} \quad u_{2t}(\quad)\right) \\
\dot{P}_C &= \frac{1}{\ }(P_C \quad {}_m P_t)
\end{aligned}
\qquad (2.4)
$$

where $k_1 = \frac{RT_1}{V_1}$, $k_2 = \frac{RT_2}{V_2}$, the flow from the compressor to the intake manifold is represented by $W_C = \frac{{}_C}{T_a C_p} \frac{P_C}{\left(\frac{p_1}{p_a}\right) \quad 1}$, the turbine power is represented by $P_t = {}_t T_2 C_p \left(1 \quad \frac{p_a}{p_1}\right) u_{2t}$, and $W_{1e} = k_e p_1$ where $k_e = \frac{{}_{vol} N V_d}{120 \, R \, T_1}$ is a pumping rate constant which governs the flow rate of air through the engine. Substituting into equation 2.4, leads to the following 3rd order dynamical model

$$
\begin{aligned}
\dot{p}_1 &= k_1 \frac{{}_C}{T_a C_p} \frac{P_C}{\left(\frac{p_1}{p_a}\right) \quad 1} \quad k_1 k_e p_1 + \ k_1 u_{21}(\quad) \\
\dot{p}_2 &= k_2\left(k_e p_1 + W_f^d \quad u_{21}(\quad) \quad u_{2t}(\quad)\right) \\
\dot{P}_C &= \frac{1}{\ }\left(P_C \quad {}_m \, {}_t T_2 C_p \left(1 \quad \frac{p_a}{p_2}\right) u_{2t}(\quad)\right)
\end{aligned}
\qquad (2.5)
$$

where (i) W_f^d is the desired fueling rate, (ii) $u_{21}(\quad) = W_{egr}$ is EGR flow from the exhaust manifold to the intake manifold, and (iii) $u_{2t}(\quad)$ is the flow of the exhaust across the turbocharger veins.

2.3 Operating Points

Generation of the set of operating points for the 7th order system is done by specifying a set of triplets (AFR, EGR, W_f) for a corresponding set of specified engine speeds and loads. Presumably, the triples correspond to emissions that meet appropriate constraints and torque and speed requirements of a driving cycle. The set of triples are then mapped into a new set of triples corresponding to desired flow rates: $\left(W_f^d,\ W_C^d,\ W_{egr}^d\right)$. The appropriate mappings can be found in [1,8] Given these flow rates, it is then necessary to determine consistent equilibrium states in the 7th order model computed in the usual fashion by setting the derivative of the state vector to zero. The set of these 7th order equilibrium states are then projected onto the state space of the 3rd order model with the additional assumption of locally constant T_1 and T_2 for each operating point.

Details of the computation of the operating points for the seventh and thus third order models can be found in Appendix 1.

2.4 Remarks

The 7th and 3rd order models have independent derivations in contrast to the customary reduced order model which is computed as a some type of projection of the higher order model onto a lower dimensional state space or is possibly a singularly perturbed model reduction. Further, there does not appear to be any work which guarantees that a control developed on the 3rd order model will work for the 7th order model. Rather, experience via simulators has shown that controllers so developed in fact work well on the 7th order model. The control of course uses full state feedback for the 3rd order model, but only partial state feedback (the same variables, p_1, p_2, and P_C, for the 7th order model.

Although it might be possible to rigorously demonstrate a projective relationship between the 7th and 3rd order models, an investigation into an alternate approach to control design (found to be useful for linear controller design [10]) might prove more beneficial. The essential idea here is to develop a first control on the 3rd order model. Apply the control to the 7th order model and evaluate its performance. Store the essential input-output data associated with the performance evaluation. Return to the 3rd order model with this input-output data. With the control in place, execute a parameter ID on the controlled third order model so that its behavior "better" conforms to that of the 7th order model as per the input-output data. With these new parameter values, redo the control design and repeat the process. The process can be shown to converge in certain linear cases. Investigation of convergence in the general nonlinear case remains an open frontier. Nevertheless, one intuitively expects such a process to provide incremental improvements in the controller design.

3. A LYAPUNOV/LMI METHOD FOR STABILIZING POLYTOPIC NONLINEAR SYSTEMS

Several notions and theorems from Lyapunov stability theory [9] underlie the theoretical results of this section. To set forth these results in a straightforward manner, we consider the usual nonlinear state model

$$\dot{x}(t) = f(t, x(t)), x(t_0) = x_0 \tag{3.1}$$

where $x(t) \quad R^n$ and $t \quad R$ and assume that the origin is an equilibrium point, i.e. $f(t,0) = 0$ for all t. Our interest is in the stability of the system about the origin. The origin is *uniformly exponentially stable* with rate of convergence > 0 if there exists R > 0 and > 0 such that whenever $\|x(t_0)\| < R$ then $\|x(t)\| < \quad \|x(t_0)\| e^{\quad (t \quad t_0)}$ for all $t \geq t_0$. Assuming the origin is an exponentially stable equilibrium point for 3.1, the set is a *region of attraction* for the origin if $x(t_0) \qquad \lim_{t \bullet} x(t) \quad 0$. Finally, a subset of the state space R^n is called an *invariant set* for 3.1 if $x(t_0) \qquad x(t) \qquad$ for all $t \geq t_0$. These notions underlie the following theorem of Lyapunov stability theory:

Theorem 1. [9] With respect to the system 3.1, suppose that there is a continuously differentiable function $V(x)$ and positive scalars $\ ,\ _1,\ _2$, and c such that for all x in $\ = \{x \quad R^n : V(x) \quad c\},\ _1\|x\|^2 \quad V(x) \quad _2\|x\|^2$ and for all t

$$\frac{\partial V(x)}{\partial t} = \frac{\partial V(x)}{\partial x}\dot{x} = \frac{\partial V(x)}{\partial x} f(t,x) \quad 2 \quad V(x). \tag{3.2}$$

Then the origin is a uniformly exponentially stable equilibrium point with invariant region of attraction and rate of convergence .

Because $\dot{V}(x(t)) \quad 2 \quad V(x(t))$. It follows that $V(x(t)) = V(x(0))e^{\ 2\ t}$ meaning that the energy of the system as measured by the Lyapunov function $V(x)$ diminishes to zero with exponential rate 2 . However, the state trajectory has may have a different rate of convergence. Specifically, we assume that $_1\|x\|^2 \quad V(x) \quad _2\|x\|^2$. Hence, $_1\|x\|^2 \quad V(x) \quad V(x(0))e^{\ 2\ t}$ implies that $\|x\| \quad \frac{V(x(0))}{_1} e^{\ \ t}$, i.e., the state trajectory converges to zero with minimum rate of convergence .

A second result important to our formulation is the Shur complement [6, pp. 7, 28] for converting nonlinear (convex) inequalities to LMI form: let $Q(x) = Q^T(x)$, $P(x) = P^T(x)$, and $S(x)$ depend affinely on x; the inequalities

$$P(x) > 0, \quad Q(x) \quad S(x)P^{\ 1}(x)S^T(x) > 0$$

is equivalent to the LMI

$$\begin{matrix} Q(x) & S(x) \\ S^T(x) & P(x) \end{matrix} > 0$$

This equivalence proves useful when converting constraint equations to LMIs.

The main theoretical result of this paper states sufficient LMI conditions for the origin to be uniformly exponentially stable for a system having the following polytopic form:

$$\dot{x}(t) = A(t,x)x(t) + B(t,x)u(t), x(t_0) = x_0 \tag{3.3}$$

where the time/state dependent matrices have the following structure

$$A(t,x) = A_0 + \ (t,x) \ A,\ B(t,x) = B_0 + \ (t,x) \ B \tag{3.4}$$

for constant matrices A_0, B_0, $\ A$, and $\ B$ and scalar valued function $\ (t,x)$. Here $\ (t,x)$ has the property that whenever

$$\|Cx\| \tag{3.5a}$$

for an appropriate matrix C and a positive scalar , then the following holds:

$$a \qquad (t,x) \quad b \tag{3.5b}$$

for two constants a and b.

Equation 3.4 represents a non-unique decomposition. The following theorem presumes such a decomposition. Different decompositions may lead to existence or nonexistence of the LMI set forth in the theorem.

Theorem 2 [5]. With respect to the system 3.3 having properties 3.4 and 3.5, suppose there exists a matrix L, a symmetric positive definite matrix S, an appropriate matrix C, and a scalar > 0 (as per equation 3.5) such that

$$A_1 S + S A_1^T + B_1 L + L^T B_1^T < 0 \tag{3.6}$$
$$A_2 S + S A_2^T + B_2 L + L^T B_2^T < 0 \tag{3.7}$$
$$CSC^T \quad 1 \tag{3.8}$$

where $A_1 = A_0 + a \quad A$, $A_2 = A_0 + b \quad A$, $B_1 = B_0 + a \quad B$, and $B_2 = B_0 + b \quad B$. Then for the closed loop system,

$$\dot{x}(t) = \left(A(t,x) + B(t,x)K\right) x(t) \tag{3.9}$$

obtained from system 3.3 with the linear state feedback $u = Kx = LS^{\ 1}x$, the origin is a uniformly exponentially stable equilibrium point with $\ = \{x \quad R^n : x^T S^{\ 1} x \quad {}^2\}$ an invariant region of attraction.

Proof. For convenience we will use the notation $\frac{\partial V}{\partial x} \equiv DV$. From the inequalities 3.6 and 3.7, it follows that there exists a positive definite matrix Q > 0 such that

$$A_1 S + B_1 L + S A_1^T + L^T B_1^T \qquad Q \quad 0 \tag{3.10}$$
$$A_2 S + B_2 L + S A_2^T + L^T B_2^T \qquad Q \quad 0 \tag{3.11}$$

With $V(x) = x^T S^{\ 1} x$ as a candidate Lyapunov function, the goal is to find a continuous function $W(\circ)$ > 0 such that, whenever $x \qquad \backslash\{0\}$, $DV(x) f(t,x) \qquad W(x) \quad 0$ for all t.

Let, $x \qquad = \{x \quad R^n : x^T S^{\ 1} x \quad {}^2\}$. From Cauchy-Schwartz, the existence of a symmetric S > 0, and inequality 3.8, we have

$$\|Cx\|^2 = \left\|CS^{0.5} S^{\ 0.5} x\right\|^2 \quad \left\|CS^{0.5}\right\|^2 \quad \left\|S^{\ 0.5} x\right\|^2 = \quad \left\|CSC^T\right\| \quad \left\|x^T S^{\ 1} x\right\| \qquad {}^2 \tag{3.12}$$

i.e., $\|Cx\|$ for all x . This relation is required for showing that for $x \qquad \backslash\{0\}$, $DV(x) f(t,x) \qquad W(x) \quad 0$ for all t.

With $u = Kx = LS^{\ 1}x$, the closed loop system has the form

$$\dot{x} = f(t,x) \int \left[A(t,x) + B(t,x)LS^{\ 1} \right] x$$

in which case

$$\begin{aligned} DV(x)\,f(t,x) &= 2x^T S^{\ 1}[A(t,x)S + B(t,x)L]S^{\ 1}x \\ &= x^T S^{\ 1}\left[A(t,x)S + B(t,x)L + \left(A(t,x)S + B(t,x)L\right)^T \right] S^{\ 1}x \end{aligned} \qquad (3.13)$$

Using equation 3.4, let

$$N(t,x) = A(t,x)S + B(t,x)L + SA^T(t,x) + L^T B^T(t,x) \int \ N_0 + \quad (t,x) \quad N\ .$$

where $N_0 = A_0 S + B_0 L + SA_0^T + L^T B_0^T$ and $\quad N = \quad AS + \quad BL + S \quad A^T + L^T \quad B^T$. By assumption, x implies $\|Cx\|$ as per equation 3.12 which implies that $a \quad (t,x) \quad b$. Using equations 3.6 and 3.7,

$$N(t,x) = N_0 + \quad (t,x) \quad N \qquad Q \qquad (3.14)$$

From 3.12 and 3.13, it follows that

$$DV(x)\,f(t,x) = x^T S^{\ 1}\left[N_0 + \quad (t,x) \quad N \right] S^{\ 1}x \qquad x^T S^{\ 1} Q S^{\ 1} x$$

It remains to show that this expression is less than

$$x^T S^{\ 1} Q S^{\ 1} x \qquad {}_{\min}\ \sqrt{S^{\ 1}}\,Q\sqrt{S^{\ 1}} \quad \left(x^T S^{\ 1} x \right) = \quad {}_{\min}\ \sqrt{S^{\ 1}}\,Q\sqrt{S^{\ 1}}\ V(x) < 0$$

where ${}_{\min}(M)$ is the smallest eigenvalue value of the positive definite symmetric matrix, M, and $\sqrt{S^{\ 1}}$ is the unique symmetric positive definite square root of $S^{\ 1} = \quad \sqrt{S^{\ 1}}^{\ 2}$. With this result,

$$\dot{V}(x) \qquad {}_{\min}\ \sqrt{S^{\ 1}}\,Q\sqrt{S^{\ 1}}\ V(x) < 0.$$

By the classical Lyapunov Theorem 1, the origin is a uniformly asymptotically stable equilibrium point with as an invariant region of attraction.

To verify that

$$x^T S^{\ 1} Q S^{\ 1} x \qquad {}_{\min}\ \sqrt{S^{\ 1}}\,Q\sqrt{S^{\ 1}} \quad \left(x^T S^{\ 1} x \right) = \quad {}_{\min}\ \sqrt{S^{\ 1}}\,Q\sqrt{S^{\ 1}}\ V(x) < 0$$

observe that

$$x^T S^{-1} Q S^{-1} x = x^T \left(\sqrt{S^{-1}}\sqrt{S^{-1}}\right) Q \left(\sqrt{S^{-1}}\sqrt{S^{-1}}\right) x$$

$$= \left(\sqrt{S^{-1}}x\right)^T \left(\sqrt{S^{-1}}Q\sqrt{S^{-1}}\right) \left(\sqrt{S^{-1}}x\right) \geq \lambda_{\min}\left(\sqrt{S^{-1}}Q\sqrt{S^{-1}}\right) \left(x^T S^{-1} x\right)$$

as was to be shown.

As a side note, observe that the matrices: $\sqrt{S^{-1}}Q\sqrt{S^{-1}}$ and $\left(QS^{-1}\right)$ are similar, because

$$QS^{-1} = \left(\sqrt{S^{-1}}\right)^{-1} \left(\sqrt{S^{-1}}Q\sqrt{S^{-1}}\right) \sqrt{S^{-1}}.$$

Therefore, these matrices have the same eigenvalues. Using this fact, it also follows that

$$\dot{V}(x(t)) \leq -\lambda_{\min}\left(QS^{-1}\right) V(x(t))$$

Furthermore, $S^{-1}Q = \left(QS^{-1}\right)^T$ because Q and S^{-1} are symmetric, i.e., $S^{-1}Q$ and QS^{-1} have the same eigenvalues. Hence, we have the third relationship

$$\dot{V}(x(t)) \leq -\lambda_{\min}\left(S^{-1}Q\right) V(x(t))$$

This completes the proof.

Theorem 2 can be generalized to systems described by 3.3 where the time/state dependent matrices $A(t,x)$ and $B(t,x)$ have the following more general structure:

$$A(t,x) = A_0 + \sum_{i=1}^{l} \theta_i(t,x) A_i \tag{3.15a}$$

and

$$B(t,x) = B_0 + \sum_{i=1}^{l} \theta_i(t,x) B_i \tag{3.15b}$$

where the $\theta_i(\circ,\circ)$ are scalar valued functions of t and x, $A_0, A_1, \ldots, A_l$ are constant $n \times n$ matrices and $B_0, B_1, \ldots, B_l$ are $n \times m$ matrices. Additionally we require that, whenever $\|C_i x\| \leq \gamma$, $a_i \leq \theta_i(t,x) \leq b_i$ for constant matrices C_i, a positive scalar γ, and constants a_i and b_i, for $i = 1, \ldots, l$.

For the origin to be an exponentially stable equilibrium point of the closed loop system with the region of attraction $\Omega = \{x \in R^n : x^T S^{-1} x < \gamma^2\}$, the matrices L and S and the scalar γ must satisfy

$$AS + BL + SA^T + L^T B^T < 0$$
$$CSC^T \quad 1 \tag{3.16a}$$

for all

$$C \quad \{C_i \mid i = 1,\ldots,l\} \tag{3.16b}$$

and all matrix pairs

$$(A,B) \qquad A_0 + \sum_{i=1}^{l} \quad {}_i \; A_i,\; B_0 + \sum_{i=1}^{l} \quad {}_i \; B_i \quad | \quad {}_i = a_i \text{ or } = b_i,\; i = 1,\ldots,l \;, \tag{3.16c}$$

In this paper, this more general formulation of Theorem 2, represented as a family of LMI's by the relations of 3.16, will be used to derive a linear state feedback for the reduced order model of the Diesel engine. The existence of L, S, and in the LMI 3.16 provides state feedback that is sufficient for stability. Sufficient conditions (theoretical) in terms of the system structure guaranteeing the existence of solution to the family of LMIs is an open question.

We must further impose two more constraints on the variables S and L for a practical solution to our control problem. The first constraint permits inclusion of a starting point x_o in the invariant ellipsoid centered at the origin that can be expressed as an LMI using the Schur complement:

$$\left(x_o \quad x_1^d\right)^T S \;\; ^1\left(x_o \quad x_1^d\right) < \;\; ^2 \qquad \begin{bmatrix} ^2 & \left(x_o \quad x_1^d\right)^T \\ \left(x_o \quad x_1^d\right) & S \end{bmatrix} > 0 \tag{3.17}$$

where the difference $\left(x_o \quad x_1^d\right)$ is to be driven to zero.

A second constraint upper bounds the 2-norm of $u(t)$ by . The value of depends on the saturation characteristics of the actuators involved in the implementing the control. Additionally, can be chosen to impose a maximum energy constraint on $u(t)$ over a control interval [0,T]; here

$$\int_0^T \|u(t)\|_2 \, dt \qquad \int_0^T \quad dt = \quad T \qquad E_{\max} \qquad\qquad \frac{E_{\max}}{T}$$

To impose this bound, for $x(t)$, we require that for all t

$$\|u(t)\|^2 = \|Kx(t)\|^2 = x^T(t) K^T K x(t) \qquad ^2 \tag{3.18}$$

But if

$$K^T K \qquad \frac{^2}{\;\;^2} S \;\; ^1 \tag{3.19}$$

holds, then $\|u(t)\|_2$ as desired. To convert 3.19 to LMI form, observe that

$$K^T K = S\ ^{1} L^T LS\ ^{1} \quad \frac{\ 2}{\ 2} S\ ^{1} \qquad\qquad L^T L \quad \frac{\ 2}{\ 2} S \tag{3.20}$$

Using the Schur complement 3.20 can be written as the LMI:

$$\begin{matrix} S & L^T \\ L & \frac{\ 2}{\ 2} \end{matrix} \geq 0 \tag{3.21}$$

In summary, the LMI formed by inequalities 3.16, 3.17 and 3.21 will be used in the next section to develop a gain scheduled controller for the error system set forth in the next section.

4. CONTROL DESIGN

This section details the application of the general version of Theorem 2 to the design of a control law for the reduced order model of the Diesel engine. The controller drives the system state through a sequence of operating points. About each operating point we generate a nonlinear error system amenable to polytopic form. For each such system, an LMI is formulated so that the previous equilibrium is included in the region of attraction of the current error system. A constraint on the gain is also imposed. The solution of the each LMI generates the needed control as per section 3.

4.1 Derivation of the error system

The reduced order model of the EGR-VGT Diesel engine [8] is given in the set of equations 2.4.

Assuming that the desired operating point is $\left[p_1^d, p_2^d, P_C^d\right]^T$ (see section 2.3), the equilibrium equations are obtained by setting the derivative in equation 2.4 to zero, i.e.,

$$0 = \frac{\ _C}{T_a C_p} \frac{P_C^d}{\left(\frac{p_1^d}{p_a}\right) \quad 1} \quad k_e p_1^d + \ W_{egr}$$

$$0 = k_e p_1^d + W_f^d \quad W_{egr}^d \quad u_{2t}^d \tag{4.1}$$

$$0 = P_C^d \quad\ _m\ _t T_2 C_p\ 1 \quad \frac{p_a}{p_2^d} \quad u_{2t}^d$$

where the superscript d has the meaning of the desired operating point. Since the control inputs for the model (2.4) are W_{egr} and u_{2t} we denote them by u_1 and u_2, respectively. We define $x = [\ p_1,\ p_2,\ P_C]^T = [p_1 \quad p_1^d, p_2 \quad p_2^d, P_C \quad P_C^d]^T$ as the error relative to the desired operating point and from equations 2.4 and 4.1 we obtain the nonlinear error dynamics:

$$\overset{\Sigma}{p}_1 = k_1 \frac{{}_C P_C^d}{T_a C_p} \frac{1}{\frac{p_1^d + \ p_1}{p_a} \quad 1} \quad \frac{1}{\frac{p_1^d}{p_a} \quad 1} \quad k_e \ p_1 + \frac{C}{T_a C_p} \frac{P_C}{\frac{p_1^d + \ p_1}{p_a} \quad 1} + \ u_1$$

$$\overset{\Sigma}{p}_2 = k_2 (k_e « p_1 \quad u_1 \quad u_2) \tag{4.2}$$

$$\overset{\Sigma}{P}_C = \ \frac{1}{} \ {}_m \ {}_t T_2 C_p u_2^d \ \frac{p_a}{p_2^d + \ p_2} \quad \frac{p_a}{p_2^d} \quad \frac{1}{} \ P_C + \frac{1}{} \ {}_m \ {}_t T_2 C_p \ 1 \quad \frac{p_a}{p_2^d + \ p_2} \quad u_2$$

4.2 Polytopic form of the error system

Let $x = [\ p_1 \quad p_2 \quad P_C]^T$. The error system (4.2) can be written in the following form:

$$\overset{\Sigma}{x} = A(\ x)\ x + B(\ x)\ u \tag{4.4}$$

where $A(\ x) = \begin{matrix} {}_{11} & 0 & {}_{13} \\ k_2 k_e & 0 & 0 \\ 0 & {}_{32} & \frac{1}{} \end{matrix}$, $\quad {}_{11} = \frac{k_1 \ {}_C P_C^d}{T_a C_p \ p_1} \frac{1}{\frac{p_1^d + \ p_1}{p_a} \quad 1} \quad \frac{1}{\frac{p_1^d}{p_a} \quad 1} \quad k_e,$

$${}_{13} = \frac{C}{T_a C_p} \frac{1}{\frac{p_1^d + \ p_1}{p_a} \quad 1}, \qquad {}_{32} = \ \frac{1}{} \frac{{}_m \ {}_t T_2 C_p u_2^d}{p_2} \quad \frac{p_a}{p_2^d + \ p_2} \quad \frac{p_a}{p_2^d}$$, and

$$B(\ x) = \begin{matrix} k_1 & 0 \\ k_2 & k_2 \\ 0 & B_{32} \end{matrix} \text{ with } B_{32} = \frac{1}{} \ {}_m \ {}_t T_2 C_p \ 1 \quad \frac{p_a}{p_2^d + \ p_2}$$

We remark that the entry $(1,1)$ of the matrix $A(\ x)$ is a differentiable function of p_1 and is defined for $p_1 = 0$. A similar discussion is valid for the $(3,2)$ entry with respect to p_2.

Let us define the following functions

$$ {}_1(\ x) = \frac{k_1\ {}_C P_C^d}{T_a C_p\ \ p_1} \ \frac{1}{\frac{p_1^d\ +\ \ p_1}{p_a}\quad 1} \quad \frac{1}{\frac{p_1^d}{p_a}\quad 1}\ , \quad {}_2(\ x) = \frac{{}_C}{T_a C_p} \frac{1}{\frac{p_1^d\ +\ \ p_1}{p_a}\quad 1} \tag{4.5} $$

$$ {}_3(\ x) = \ \frac{1}{} \frac{{}_m\ \ {}_t T_2 C_p u_2^d}{p_2} \ \frac{p_a}{p_2^d +\ \ p_2} \quad \frac{p_a}{p_2^d}\ , \quad {}_4(\ x) = \ \frac{1}{}\ \ {}_m\ \ {}_t T_2 C_p \ \frac{p_a}{p_2^d +\ \ p_2} \tag{4.6} $$

Using the above defined functions the matrix $A(\circ)$ can be written as in 3.15a:

$$ A(\ x) = A_0 + \sum_{i=1}^{4}\ {}_i(\ x)\ \ A_i \tag{4.7} $$

with $A_4 = [0]$,

$$ A_0 = \begin{matrix} k_e & 0 & 0 \\ k_2 k_e & 0 & 0 \\ 0 & 0 & \frac{1}{} \end{matrix}\ , \quad A_1 = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\ , \quad A_2 = \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\ , \quad A_3 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}\ . \tag{4.8} $$

Similarly, the matrix $B(\circ)$ can be written as in 3.15b :

$$ B(\ x) = B_0 + \sum_{i=1}^{4}\ {}_i(\ x)\ \ B_i \tag{4.9} $$

with $B_1 = \ \ B_2 = \ \ B_3 = [0]$,

$$ B_0 = \begin{matrix} k_1 & 0 \\ k_2 & k_2 \\ 0 & \frac{1}{}\ \ {}_m\ \ {}_t T_2 C_p \end{matrix}\ , \text{ and } \ B_4 = \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix} \tag{4.10} $$

The relations (4.7)-(4.10) represent the polytopic form of the reduced order model of the Diesel engine. From (4.5) and (4.6) we observe that there exist functions ${}_i(\circ), i = 1,2,3,4$ such that

$_i(\ \ x) = \ \ _i(C_i\ \ x), i = 1,2,3,4$ where $C_1 = C_2 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$ and $C_3 = C_4 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}$. Since in a sufficiently small region about the origin the functions $_i(\circ), i = 1,2,3,4$ are monotone then, for a suitably chosen small $\ \ > 0$, the functions $_i(C_i\ \ x)$ can be included in the interval defined by $_i(\ \ )$ and $_i(\ \ )$ whenever $\|C_i\ \ x\|$. It follows that the constants a_i and b_i can now be explicitly chosen such that $a_i\ \ \ _i(\ \ x)\ \ b_i$ whenever $\|C_i\ \ x\|$ for $i = 1,2,3,4$. Thus given the polytopic form of equation 3.15, the LMI formed by inequalities 3.16, 3.17 and 3.21 is now completely specified and the gain K of the linear state feedback controller $u = K\ \ \ x$ is then obtained as a function of the solution of this system of LMI's. Details of the computations in MATLAB of the gain K for a specific operating point can be found in Appendix 3.

5. GAIN SCHEDULED CONTROL LAW FOR DRIVING THE STATE OF THE DIESEL ENGINE FROM AN INITIAL STATE TO A DESIRED VALUE.

In an ordinary drive cycle, the diesel engine transitions through different reference states. These states are computed by the electronic control unit according to the driver demands and road conditions, and the pre-computed exhaust gas emissions constraints. We saw that to each reference state $x_i^d\ \ D$ associated with the triple $(L_i, S_i,\ \ _i)$, $i = 1,\ldots,n_D$, where D is the (finite) set of all reference states which may occur during the operation of the engine and n_D is the number of elements in D. Assuming that the engine is in state x_k^d and the next desired state is x_{k+1}^d the following situation may occur: $x_k^d\ \ \ \ _{k+1}$. In this case the engine cannot be driven from x_k^d to x_{k+1}^d using the control law $u_k = L_k S_k^{\ 1} x$. It is necessary for us to compute additional intermediate reference states $x_k^1, x_k^2, \ldots, x_k^{i_k}$ such that the engine will pass through the whole chain of regions of attractions $\ _k^1,\ \ _k^2, \ldots,\ \ _k^{i_k}$ associated with these states until it will reach the $\ _{k+1}$. By applying the control $u_k = L_k S_k^{\ 1} x$ the engine will be driven asymptotically to x_{k+1}^d; it will remain in this state as long as the reference state remains unchanged. The idea is illustrated by the following figure.

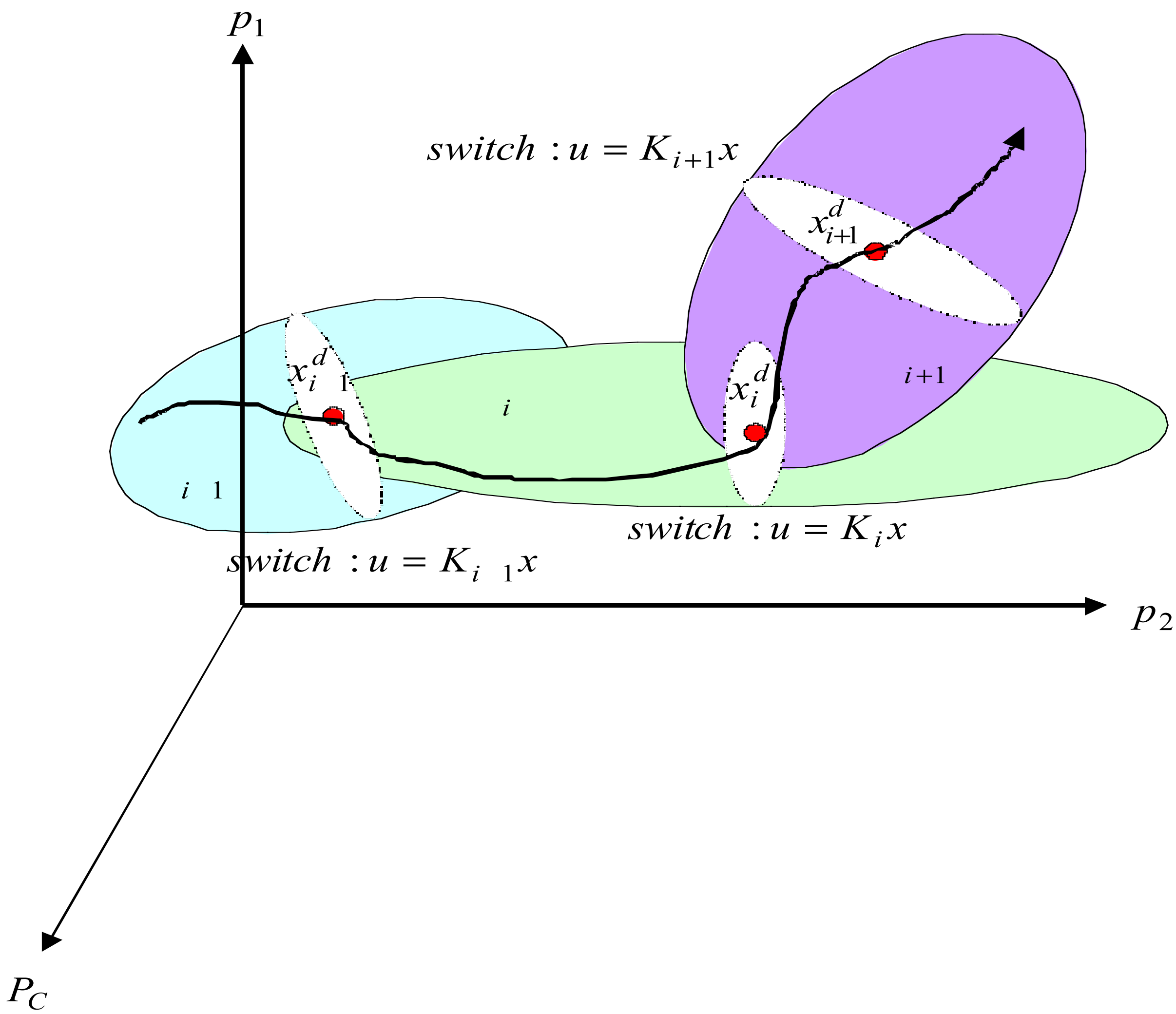

Figure 2. Sketch showing the concept of polytopic controller design for hybrid systems.

The above techniques were applied to the 3rd order engine model using a total of 12 equilibrium states including the initial and final points. The resulting control was then applied to the 7th order model and simulated. The results of the simulation are presented below.

In order to show the flexibility in the controller design and the trade-offs that must be made when designing the controller strategy, four sets of simulations have been performed. The controllers have been designed based on the reduced order model (the control model) and they have been implemented on the full order model (the simulation model). The discussions of the simulations include characteristics, advantages and disadvantages of each set of simulations.

SET 1 OF SIMULATIONS:

Characteristics: For this set of simulations

- 10 intermediate points (thus there are 12 operating points in total, including the first and the last operating points) have been considered;

- a relative error of 10^{3} has been considered (the relative error represents the distance between the desired state and the actual state when the switching of controller takes occurs);

- the controllers have been designed such that they have a (L_2 -induced) norm in the interval $[7 \quad 10^{5}, 10^{4}]$.

The results of this set of simulations are plotted in figures 3 and 4.

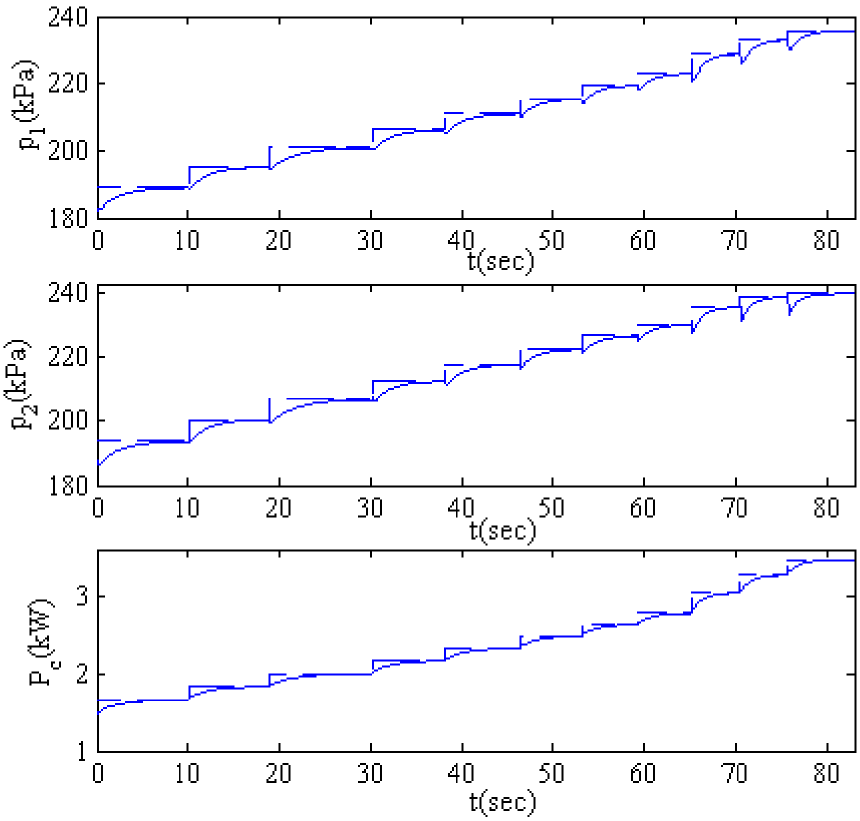

Figure 3. Plot of the (partial) state of the (full) system (the desired values are plotted with dashed line and the simulated values are plotted with solid line)

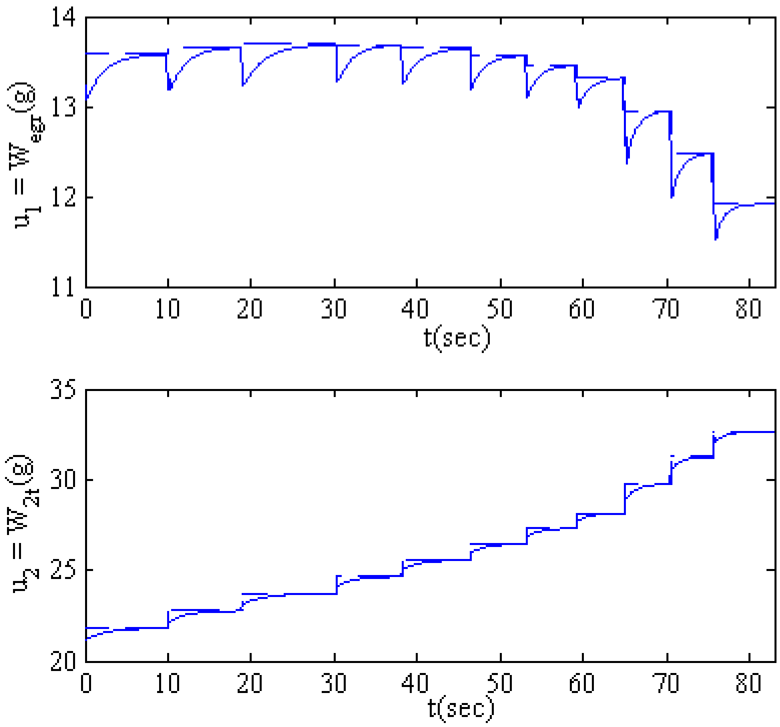

Figure 4. Plot of the control input (the desired values are plotted with dashed line and the simulated values are plotted with solid line)

Advantages:

- the trajectory of the state of the system is close to a desired trajectory(represented by a curve which passes through all intermediate desired operating points);

- the variations in the control input are relatively small (this is due to the fact the norm of the controllers have been chosen to be very small).

Disadvantages:

- the time of convergence from the first operating point to the last one is very large(from the practical point of view it is unacceptable large);

- there are (as expected) discontinuities in the control input when the switching of the controllers occurs.

SET 2 OF SIMULATIONS:

Characteristics: For this set of simulations

- 7 intermediate points (thus there are 9 operating points in total, including the first and the last operating points) have been considered;

- a relative error of 10^{3} has been considered (the relative error represents the distance between the desired state and the actual state when the switching of controller takes occurs)

- the controllers have been designed such that they have a (L_2 -induced) norm in the interval $[3 \quad 10^{4}, 3.45 \quad 10^{4}]$.

The results of this set of simulations are plotted in figures 5 and 6.

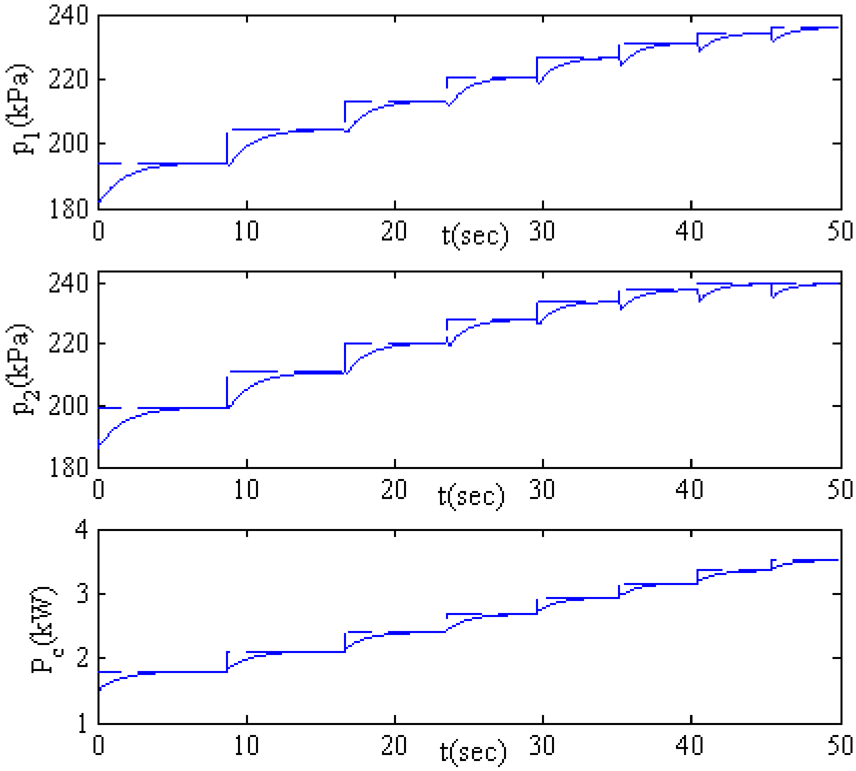

Figure 5. Plot of the (partial) state of the (full) system (the desired values are plotted with dashed line and the simulated values are plotted with solid line)

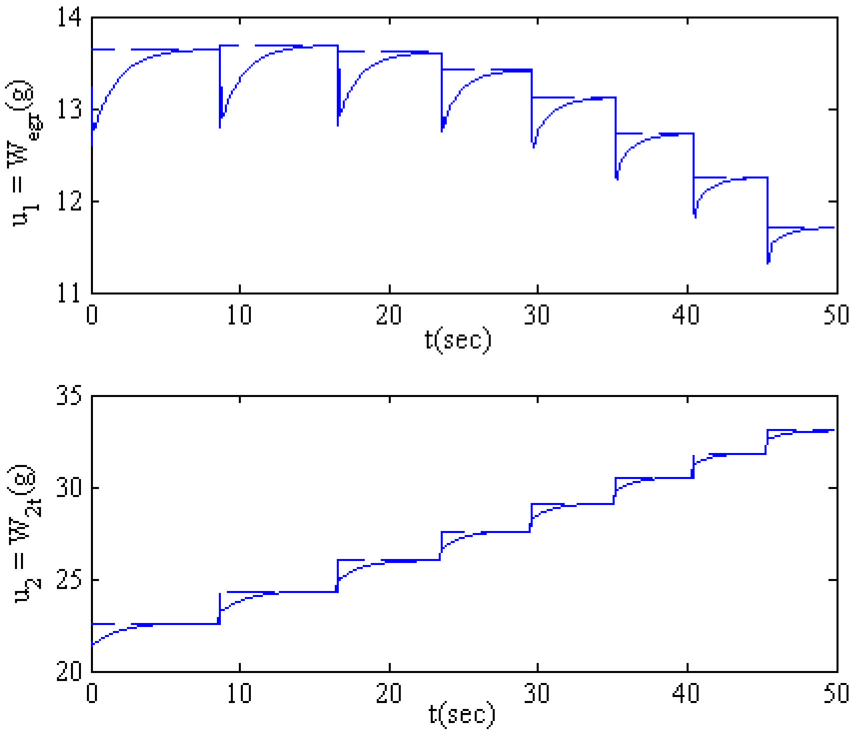

Figure 6. Plot of the control input (the desired values are plotted with dashed line and the simulated values are plotted with solid line)

Advantages:

- the trajectory of the state of the system is still relatively close to a desired trajectory;
- the variations in the control input are still relatively small (this is due to the fact the norm of the controllers have been chosen to be small).

Disadvantages:

- the time of convergence from the first operating point to the last one is large(from the practical point of view it is still unacceptable large);
- there are (as expected) discontinuities in the control input when the switching of the controllers occurs. For this set of simulations the variations in the control input at the discontinuities points are larger that in the previous set of simulations.

SET 3 OF SIMULATIONS:

Characteristics: For this set of simulations

- 4 intermediate points (thus there are 6 operating points in total, including the first and the last operating points) have been considered;

- a relative error of 0.5 has been considered (the relative error represents the distance between the desired state and the actual state when the switching of controller takes occurs)
- the controllers have been designed such that they have a (L_2-induced) norm in the interval $[0.0069;0.0090]$.

The results of this set of simulations are plotted figures 7 and 8

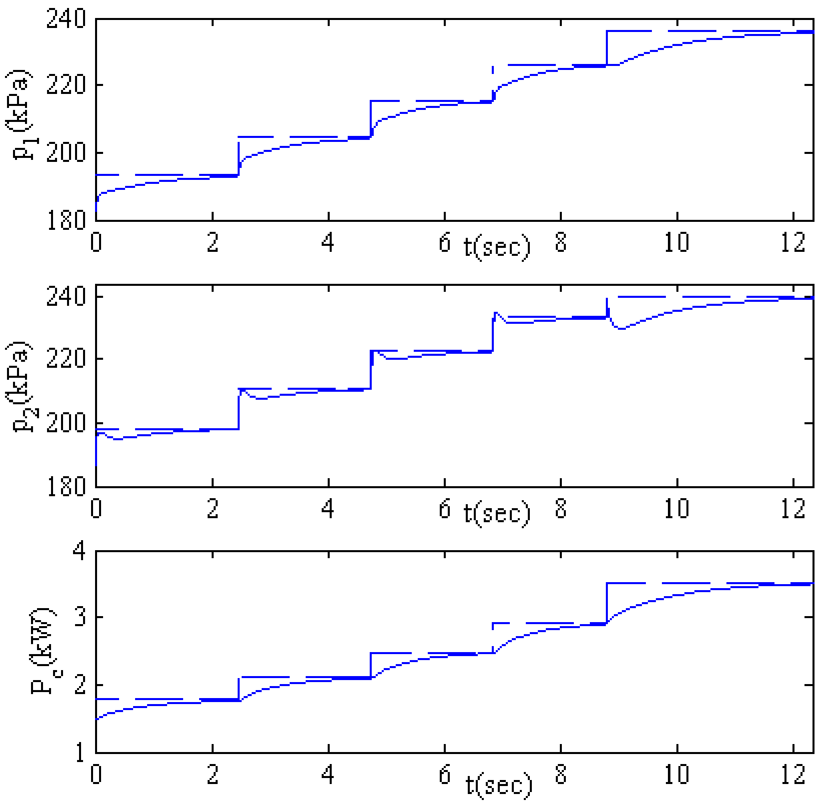

Figure 7. Plot of the (partial) state of the (full) system (the desired values are plotted with dashed line and the simulated values are plotted with solid line)

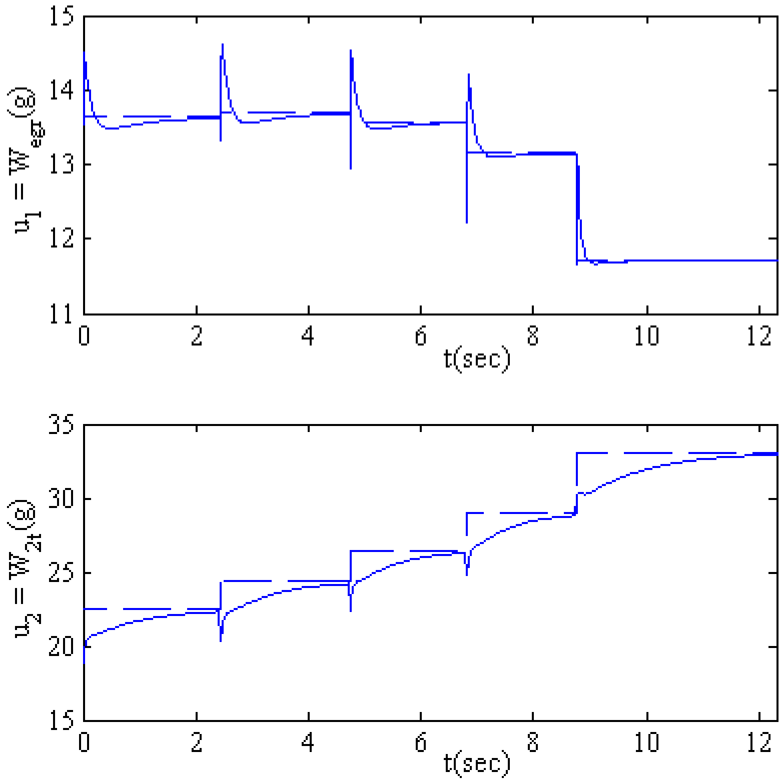

Figure 8. Plot of the control input (the desired values are plotted with dashed line and the simulated values are plotted with solid line)

Advantages:

- the trajectory of the state of the system is acceptable close to a desired trajectory, but the previous trajectories are closer than this trajectory;
- the variations in the control input are acceptable (this is due to the fact the norm of the controllers are lager for this set of simulations relatively to the previous two sets of simulations).
- the time of convergence from the first operating point to the last one is smaller in this case(from the practical point of view it is acceptable);

Disadvantages:

- there are discontinuities in the control input when the switching of the controllers occurs. For this set of simulations the variations in the control input at the discontinuities points are larger that in the previous sets of simulations.

In order to show what we mean when we say that a trajectory 2 is closer to the desired trajectory than trajectory 3, both trajectories from sets 2 and 3 of simulations are plotted in the next figure. The desired trajectory is also plotted on the same figure as per figure 9.

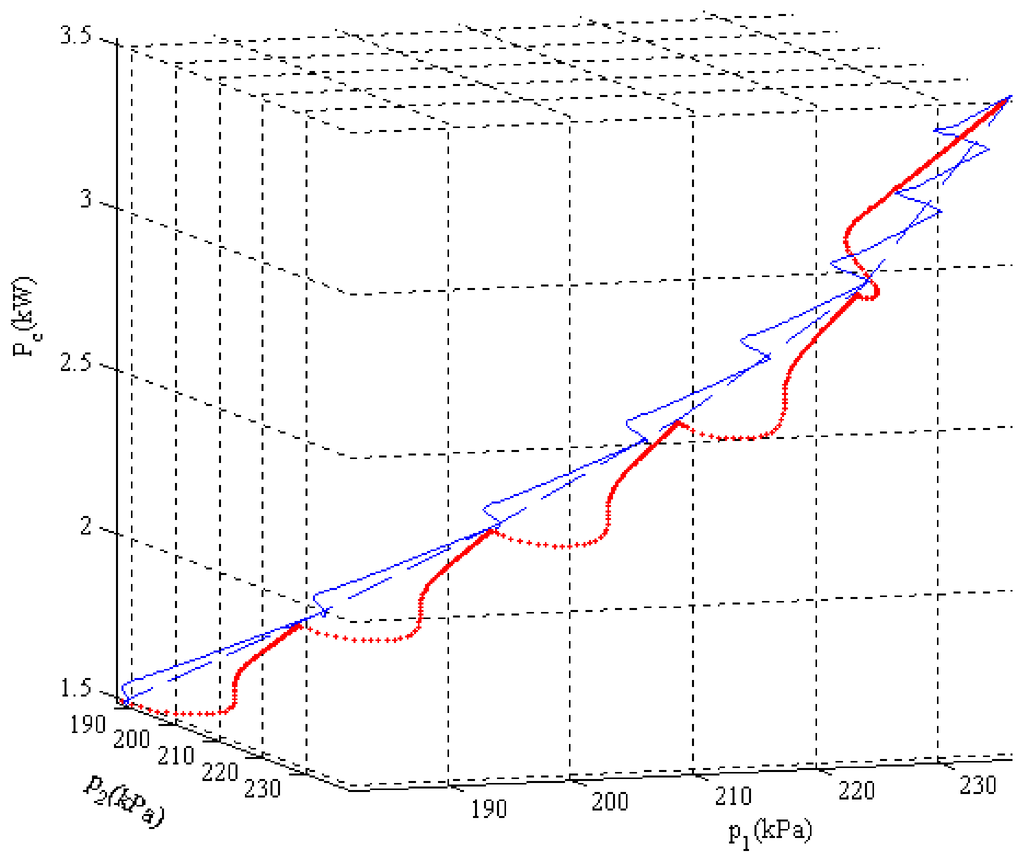

Figure 9. The desired trajectory (with dashed line), the trajectory from the 2nd set of simulations (with solid line), the trajectory from the 3rd set of simulations(with stars)

SET 4 OF SIMULATIONS

Characteristics: This is an extreme case when there is no intermediate point. The state of the system is driven directly from the first to the last operating point. The norm of the controller is 0.0069 and the relative error is 0.5.

The results of this set of simulations are plotted in figures 10 and 11.

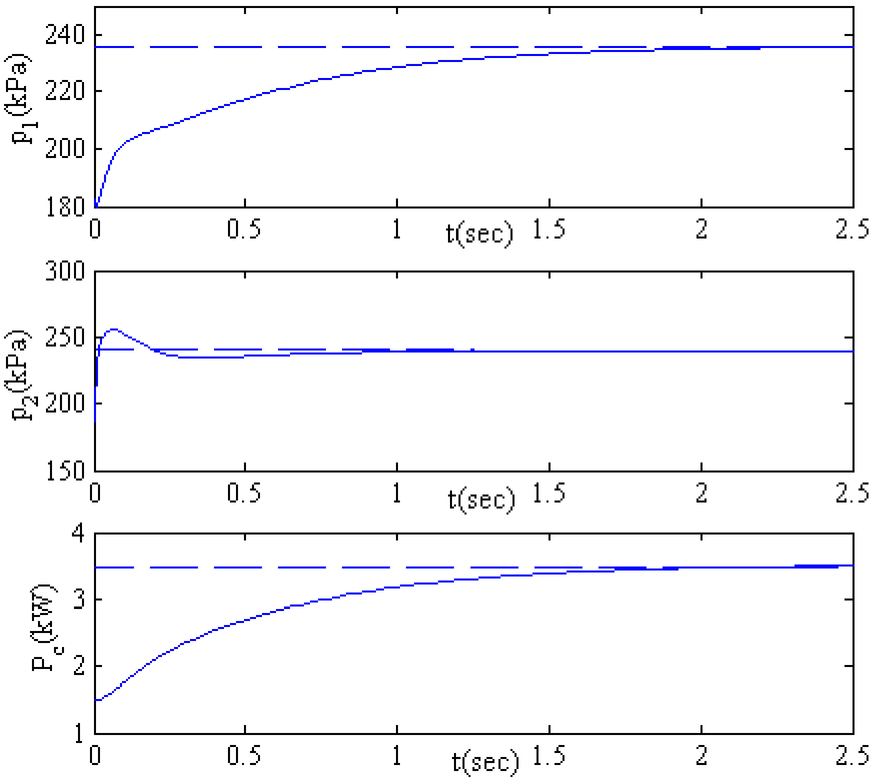

Figure 10. Plot of the (partial) state of the (full) system (the desired values are plotted with dashed line and the simulated values are plotted with solid line)

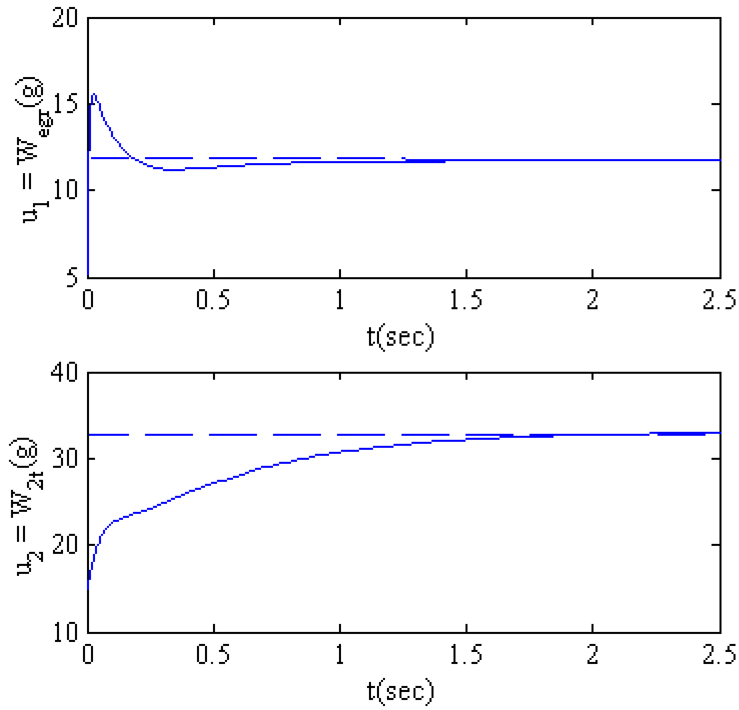

Figure 11. Plot of the control input (the desired values are plotted with dashed line and the simulated values are plotted with solid line)

Adavantages:

- the time of convergence is very small relatively with the previous sets of simulations.

Disadvantages:

- the trajectory is very far away form the desired trajectory;
- control input variations are too large and there may be possible saturations;
- the derivative of $u_1(t)$ is very large at the beginning of the trajectory and the bandwidth of the actuators may not be too large.

In conclusion, the fourth sets of simulations prove the flexibility of the control design approach. This flexibility may be used to design the sequence of the controllers so that an acceptable trade-off between the time of convergence, the closeness of the trajectory to the desired trajectory, and appropriate gain constraints on the control inputs can be achieved.

The results here are similar to those reported in [4] although we did not consider here a return to a lower load. An advantage of the approach herein is that we utilize the nonlinear error model as was done in [1] although in [1] a variable structure control was developed in contrast to the polytopic/LMI development here.

In implanting the control, a relative error of 10^{-3} for the norm of the difference between the desired state and x(t) was used before switching to the next controller. A larger relative error of 10^{-2} significantly decreased the time of convergence to approximately 45 seconds instead of 83 seconds. Also, we can reduce the time of convergence further by allowing a larger norm on the control. It is necessary to explore these tradeoffs relative to fuel economy and emissions between the intermediate equilibrium states.

6. CONCLUSIONS.

In this conclusions section, we first point out the overall design strategy as shown in figure 12.

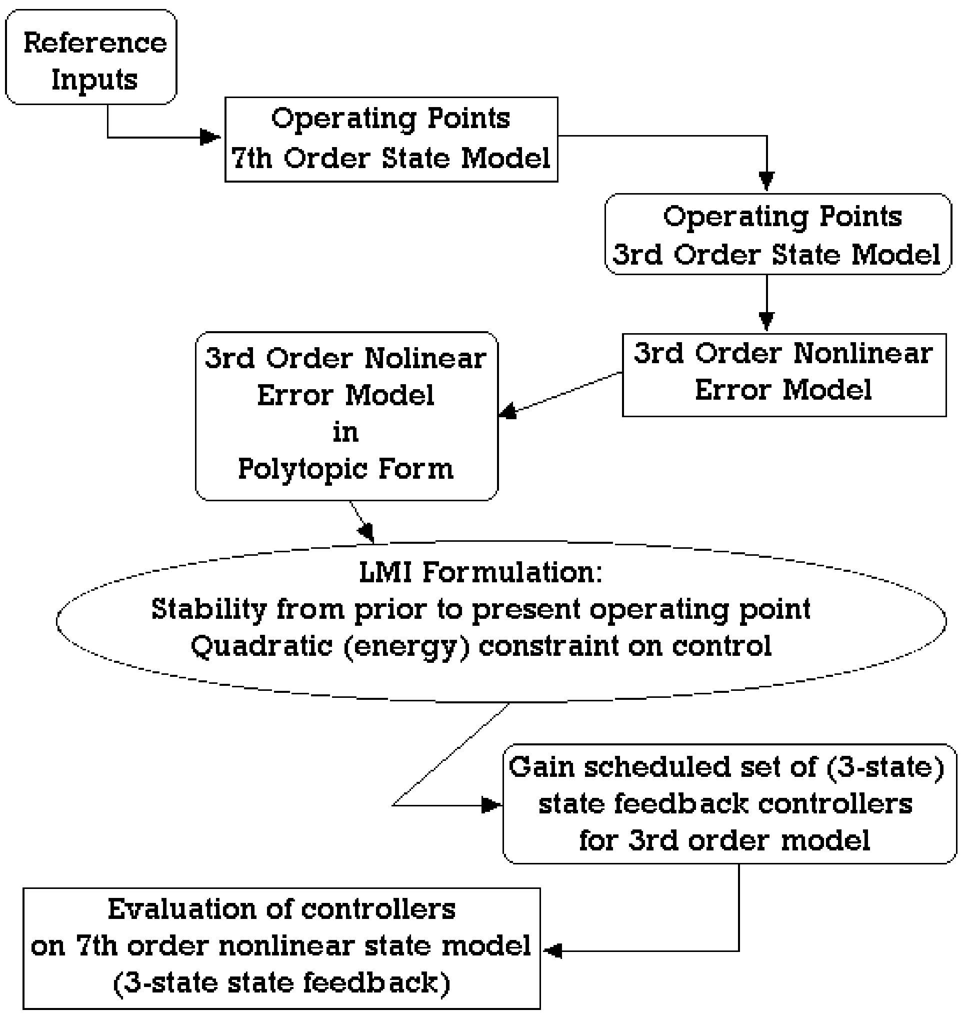

Figure 12. Flow diagram of polytopic control design strategy.

Additionally the controller stabilizes the engine about each operating point as none of the error systems is locally stable about any of the operating points. This occurs in the presence of a non-minimum phase behavior. Additioinally, the controller is robust with respect to parameter variations which in fact may vary over a wide range due to aging and use.

As with all work, this material needs to be extended. In terms of output, state equations are needed which relate the internal state variables and inputs (including engine speed) to the torque produced by the engine and to the NO_x produced. Further, actuator dynamics associated with the EGR valve and the VGT must be added. And most critically, the dynamics of the fueling process must be specified and utilized in the control design.

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APPENDIX 1: CALCULATIONS OF THE EQUILIBRIUM VALUES FOR THE STATE OF THE 7^{TH} ORDER DIESEL ENGINE MODEL BASED ON A *A PRIORI* SPECIFIED DRIVING PROFILE.

Section 1: Calculation of equilibrium values for a general Diesel engine model whose dynamics are given by equations (2.1), (2.2) and (2.3).

For the calculation of a set of operating points, it is assumed that a driving profile has been specified. The specified driving profile consists of a sequence of points in the space (W_f, N) (fueling rate and engine speed) along with a set of associated reference values (AFR_{ref}, EGR_{ref}) for (AFR, EGR) chosen so that a trade-off between minimum fuel consumption and minimum generated quantities of NO_X and transient smoke is achieved. Therefore one obtains a sequence of points $(W_f^d, AFR_{ref}, EGR_{ref}, N_{ref})$, in the space determined by W_f, EGR, AFR, and N. For this work, N = 2000 rpm is constant. Assuming that the values of W_f, EGR, AFR, and N are known for each operating point, the corresponding values of W_{egr}^d and W_C^d are computed using the relationships 4.15 and 4.3a in [8]:

$$W_C^d = \frac{W_f^d}{2} \quad + \sqrt{\quad {}^2 + 4(1 \quad EGR_{ref})AFR_{ref}}$$

$$W_{egr}^d = \frac{EGR_{ref}}{1 \quad EGR_{ref}} W_C^d$$

where

$$= AFR_{ref}(1 \quad EGR_{ref}) + \ (1 + \ AF_s)EGR_{ref} \quad 1.$$

In the above expression AF_s represents the stoichiometric value of the air-to-fuel ratio. At equilibrium the state, $\left[\ {}_1^d, F_1^d, p_1^d,\ {}_2^d, F_2^d, p_2^d, P_C^d\right]^T$, of the 7th order model of the Diesel engine must satisfy the following equations:

$$W_C^d + u_1^d \quad W_{1e}^d = 0 \tag{A1.1}$$

$$\left(F_2^d \quad F_1^d\right)u_1^d \quad F_1^d W_C^d = 0 \tag{A1.2}$$

$$W_C^d T_C^d + u_1^d T_2^d \quad W_{1e}^d T_1^d = 0 \tag{A1.3}$$

$$W_{e2}^d \quad u_2^d \quad u_1^d = \quad 0 \tag{A1.4}$$

$$\left(F_{e2}^d \quad F_2^d\right)W_{e2}^d = 0 \tag{A1.5}$$

$$W_{e2}^d T_{e2}^d \quad u_2^d T_2^d \quad u_1^d T_2^d = \quad 0 \tag{A1.6}$$

$$P_C^d = \ _m \ _t T_2^d C_p \ 1 \quad \frac{p_a}{p_2^d} \quad u_2^d \tag{A1.7}$$

where $u_1 = u_{21}$, represents the flow from the exhaust manifold to the intake manifold $\left(u_1^d = W_{egr}^d\right)$, and $u_2 = u_{2t}$ represents the turbine flow. From equation (A1.1) it follows that, at equilibrium,

$$W_C^d + u_1^d = W_{1e}^d$$

Hence

$$W_{1e}^d = W_C^d + u_1^d = W_C^d + W_{egr}^d \tag{A1.8}$$

The value of m_1^d (equilibrium value for the mixture mass in the intake manifold) can be found using the formula for W_{1e} (see relationship 3.11 in [8]). Therefore we have

$$m_1^d = \frac{120 \Diamond W_{1e}^d \Diamond V_1}{N \Diamond \ _{vol} \Diamond V_d} = \frac{120 \Diamond V_1}{N \Diamond \ _{vol} \Diamond V_d}\left(W_C^d + W_{egr}^d\right) \tag{A1.9}$$

From the balance equation in mass flow rates with respect to the exhaust manifold it follows that

$$W_{e2}^d = u_1^d + u_2^d$$

Using the above equation and equation (A1.6) it follows that, at equilibrium,

$$T_{e2}^d = T_2^d .$$

Using the above equation together with equation 3.20.6a in [8], which represents the temperature rise across the engine, it follows that

$$T_2^d \quad T_1^d = \quad a_1 + \frac{a_2}{N} + \frac{a_3}{N^2} \quad \frac{W_f^d}{W_f^d + W_C^d + W_{egr}^d} + \quad a_4 + \frac{a_5}{N} + \frac{a_6}{N^2} \ F_1^d + a_7 SOI$$

(A1.10)

where $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ are coefficients which depend on the type of the engine and SOI represents the start of injection in degrees. We assume that all these coefficients have been previously computed.

Using the relationship (A1.8), equation (A1.3) can be rewritten as

$$W_C^d T_C^d + W_{egr}^d T_2^d \quad (W_C^d + W_{egr}^d) T_1^d = 0$$

which is equivalent to

$$W_C^d T_C^d + W_{egr}^d (T_2^d \quad T_1^d) \quad W_C^d T_1^d = \quad 0 \tag{A1.11}$$

The above equation (A1.11) will be used to compute p_1^d. Note that in equation (A1.11) the values of W_C^d and W_{egr}^d have been previously computed and the temperature difference $T_2^d \quad T_1^d$ is given by the relationship (A1.10). The equilibrium value of the temperature in the intake manifold can be computed in terms of p_1^d from the universal gas law written for the intake manifold

$$T_1^d = \frac{p_1^d V_1}{m_1^d R}$$

Using the relationship (A1.9) the T_1^d can be written as

$$T_1^d = \frac{p_1^d N \quad {}_{vol} V_d}{120R(W_{egr}^d + W_C^d)} \tag{A1.12}$$

The equilibrium value of the compressor temperature can be computed using the relationship C.11 in the Appendix of [8].

$$T_C^d = T_a \quad 1 + \frac{1}{{}_C} \quad \frac{p_1^d}{p_a} \qquad 1 \tag{A1.13}$$

Replacing the relationships (12) and (13) into equation (11) we have

$$W_C^d T_a \quad 1 + \frac{1}{{}_C} \quad \frac{p_1^d}{p_a} \qquad 1 \quad + W_{egr}^d (T_2^d \quad T_1^d) \quad W_C^d \frac{p_1^d N \quad {}_{vol} V_d}{120R(W_{egr}^d + W_C^d)} = 0 \tag{A1.14}$$

Equation (14) and (10) form a system

$$\begin{aligned} &W_C^d T_a \quad 1 + \frac{1}{{}_C} \quad \frac{p_1^d}{p_a} \qquad 1 \quad + W_{egr}^d (T_2^d \quad T_1^d) \quad W_C^d \frac{p_1^d N \quad {}_{vol} V_d}{120R(W_{egr}^d + W_C^d)} = 0 \\ &T_2^d \quad T_1^d = \quad a_1 + \frac{a_2}{N} + \frac{a_3}{N^2} \quad \frac{W_f^d}{W_f^d + W_C^d + W_{egr}^d} + \quad a_4 + \frac{a_5}{N} + \frac{a_6}{N^2} \quad F_1^d + a_7 SOI \end{aligned} \tag{A1.15}$$

from which p_1^d can be computed if the value F_1^d is known. The next step toward computing p_1^d is to find F_1^d. From equation (5) it follows that, at equilibrium,

$$F_{e2}^d = F_2^d$$

The value of F_{e2}^d can be computed using the relationship 3.19.8 from [8].

$$F_{e2}^d = \frac{F_1^d W_{1e}^d + W_f^d (1 + AF_s)}{W_{1e}^d + W_f^d}$$

Therefore we have

$$F_2^d = \frac{F_1^d W_{1e}^d + W_f^d (1 + AF_s)}{W_{1e}^d + W_f^d}$$

The above equation together with equation (A1.2) form a system form which F_1^d can be computed

$$F_2^d = \frac{F_1^d W_{1e}^d + W_f^d (1 + AF_s)}{W_{1e}^d + W_f^d} \qquad \text{(A1.16)}$$
$$\left(F_2^d \quad F_1^d\right) u_1^d \quad F_1^d W_C^d = 0$$

Solving the above system of equations it follows that

$$F_1^d = \frac{W_f^d W_{egr}^d (1 + AF_s)}{W_{1e}^d (W_C^d + W_f^d)} \qquad \text{(A1.17)}$$

$$F_2^d = \frac{W_f^d (1 + AF_s)}{W_C^d + W_f^d} \qquad \text{(A1.18)}$$

The equations from the system (A1.15) together with equation (A1.17) form a new system as follows

$$W_C^d T_a \; 1 + \frac{1}{C} \quad \frac{p_1^d}{p_a} \qquad 1 \quad + W_{egr}^d (T_2^d \quad T_1^d) \quad W_C^d \, \frac{p_1^d N \;\; _{vol} V_d}{120 R (W_{egr}^d + W_C^d)} = 0 \qquad \text{(A1.19a)}$$

$$T_2^d \quad T_1^d = \quad a_1 + \frac{a_2}{N} + \frac{a_3}{N^2} \quad \frac{W_f^d}{W_f^d + W_C^d + W_{egr}^d} + \quad a_4 + \frac{a_5}{N} + \frac{a_6}{N^2} \;\; F_1^d + a_7 SOI \qquad \text{(A1.19b)}$$

$$F_1^d = \frac{W_f^d W_{egr}^d (1 + AF_s)}{W_{1e}^d (W_C^d + W_f^d)} \qquad \text{(A1.19c)}$$

A nonlinear equation with the unknown p_1^d can be obtained by, first, using the formula of F_1^d from (A1.19c), in (A1.19b), and, second, by replacing $T_2^d \quad T_1^d$, in (A1.19a), with the formula in (A1.19b). The obtained nonlinear equation can be solved using a specialized software and thus the value of p_1^d is obtained. Using the obtained value of p_1^d and the equilibrium equations, the desired values for state of the 7th order model of the Diesel engine can be obtained. For clarity and completeness we present below an algorithm with the main steps that needs to be followed in order to compute the desired values of the 7th order model state.

Algorithm for obtaining the equilibrium state values of the 7th order model:

STEP 1: *Input the value of the following parameters (which depend on the type of the engine and the operating conditions):* $W_f^d,\ N,\ AFR_{ref},\ EGR_{ref},\ AF_s,\ \ _m,\ \ _t,\ \ _C,\ p_a,\ T_a, C_p,\ R,\ \ ,\ \ _{vol},\ \ ,\ V_d,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ SOI$ (in degrees after TDC).

STEP 2: *Compute* W_C^d, W_{egr}^d *and* W_{1e}^d *using the following relations:*

$$= AFR_{ref}(1 \quad EGR_{ref}) + (1 + AF_s)EGR_{ref} \quad 1$$

$$W_C^d = \frac{W_f^d}{2} \quad + \sqrt{\ ^2 + 4(1 \quad EGR_{ref})AFR_{ref}}$$

$$W_{egr}^d = \frac{EGR_{ref}}{1 \quad EGR_{ref}} W_C^d$$

$$W_{1e}^d = W_C^d + W_{egr}^d$$

STEP 3: *Form a nonlinear equation whose unknown is* p_1^d *as follows:*

a) Use the below formula to compute F_1^d*:*

$$F_1^d = \frac{W_f^d W_{egr}^d (1 + AF_s)}{W_{1e}^d (W_C^d + W_f^d)}$$

b) Compute the temperature difference $T_2^d \quad T_1^d$ *using the following formula:*

$$T_2^d \quad T_1^d = \quad a_1 + \frac{a_2}{N} + \frac{a_3}{N^2} \ \frac{W_f^d}{W_f^d + W_C^d + W_{egr}^d} + \quad a_4 + \frac{a_5}{N} + \frac{a_6}{N^2} \ F_1^d + a_7 SOI$$

c) Form the nonlinear equation in p_1^d*:*

$$W_C^d T_a \ 1 + \frac{1}{\ _C} \quad \frac{p_1^d}{p_a} \qquad 1 \quad + W_{egr}^d (T_2^d \quad T_1^d) \quad W_C^d \ \frac{p_1^d N \ _{vol} V_d}{120 R (W_{egr}^d + W_C^d)} = 0$$

d) Solve the above equation in order to obtain p_1^d.

STEP 3: *Compute the equilibrium values for the remaining state as follows:*
a) Compute ${}_1^d$ *:*

$$ {}_1^d = \frac{120 \Diamond W_{1e}^d}{N \Diamond \ {}_{vol} \ \Diamond V_d} $$

b) Compute T_1^d *:*

$$ T_1^d = \frac{p_1^d}{{}_1^d R} $$

c) Compute T_2^d *:*

$$ T_2^d = T_1^d + \ a_1 + \frac{a_2}{N} + \frac{a_3}{N^2} \ \frac{W_f^d}{W_f^d + W_C^d + W_{egr}^d} + \ a_4 + \frac{a_5}{N} + \frac{a_6}{N^2} \ F_1^d + a_7 SOI $$

d) Compute P_C^d *:*

$$ P_C^d = \frac{W_C^d T_a C_p}{{}_C} \quad \frac{p_1^d}{p_a} \qquad 1 $$

e) Compute p_2^d *:*

$$ p_2^d = p_a \ 1 \quad \frac{P_C^d}{{}_m \ {}_t T_2^d C_p u_{2t}^d} \ ^{\frac{1}{}} $$

f) Compute ${}_2^d$ *:*

$$ {}_2^d = \frac{p_2^d}{T_2^d R} $$

g) Compute F_2^d *:*

$$ F_2^d = \frac{W_f^d (1 + AF_s)}{W_C^d + W_f^d}. $$

Section 2: A discussion on the use of the equilibrium value of the intake and exhaust manifold temperatures(T_1^d and T_2^d).

The reduced order model of the Diesel engine depends on T_1(intake manifold temperature) and on T_2(exhaust manifold temperature). Thus, in order to develop a controller such that the closed loop system will have certain desired properties, one needs to know the values of T_1 and T_2. The reduced order model assumes that these temperatures are constant. For our approach we have considered that T_1 and T_2 have constant values in a region around each desired operating point. Thus, for our controller development, we have considered that, locally, $T_1 = T_1^d$ and $T_2 = T_2^d$, where T_1^d and T_2^d represents the values of T_1 and T_2 at the desired operating point. These values can be computed as presented in the algorithm of Section 1, Step 3. If the mean values of T_1 and T_2 can be estimated then the algorithm from Section 1 must be replaced by the derivation of p_1^d, p_2^d and P_C^d presented in [1], Section 3.1.

Section 3 : A MATLAB code for computing the desired states of the 7th order model of the Diesel engine.

Our controller development is based on a Diesel engine whose simulator has been developed in SIMULINK by Devesh Upadhyay. For this model, equation (A1.10) is given by

$$T_2^d - T_1^d = -5.8940 + 0.00061 \cdot N + 4.56 \cdot 10^{-6} \cdot N^2 + \frac{W_f^d}{W_{1e}^d + W_f^d} \cdot 10000 \cdot \left(1.3722 + 0.0709\left(1 - \frac{10000}{N}\right)\right).$$

For the following MATLAB code we assume that the following values have been previously determined: W_f^d, N, AFR_{ref}, EGR_{ref}, η_m, η_t, η_C, p_a, T_a, R, C_p, μ, η_{vol}, V_1, V_d. The following MATLAB code can be used to implement the steps presented in the algorithm from Section 1.

```
>> beta1 = AFref*( 1-EGRref ) + 15.6*EGRref - 1;
>> Wcd = Wfd/2*( beta1+sqrt( beta1*beta1+4*( 1-EGRref )*AFref ) );
>> Wegrd = EGRref*Wcd/( 1-EGRref );
>> W1ed = Wegrd + Wcd;
>> m1d = (W1ed*V1*120)/(N*voleff*Vd);
>> u2td = Wcd + Wfd;
>> diffT = (-5.8940 + 0.00061*N + 4.56e-6*N*N) + Wfd/(Wfd+W1ed)*10000*( 1.3722
                                                        + 0.0709*(1-10000/N) ) ;
>> sol = fzero( @myfun,[1 3],[],(eta_c*pa*V1)/(m1d*R*Ta),eta_c-1+
                                                (eta_c*Wegrd*diffT)/(Wcd*Ta)) ;
>> p1d = sol*pa ;
>> T1d = p1d*V1/(m1d*R);
>> T2d = T1d + diffT;
>> Pcd = Wcd*Ta*Cp*( (p1d/pa)^miu-1 )/eta_c ;
>> p2d = pa*( 1 - Pcd/(eta_m*eta_t*T2d*Cp*u2td) )^(-1/miu) ;
>> F1d = 15.6*Wfd*Wegrd/( W1ed*(Wcd+Wfd) );
>> F2d = 15.6*Wfd/(Wcd+Wfd);
```

where *myfun* is used to define the nonlinear equation that appears in Step 3 of the algorithm from Section1:

```
>> function y = myfun(x,a,b)
>> y = x.^0.285715-x*a+b
```

APPENDIX 2: COMPUTATION OF THE LINEARIZED (REDUCED ORDER) MODEL AROUND THE EQUILIBRIUM POINT $\left(p_1^d, p_2^d, P_C^d\right)$

The 3rd order model can be written in the following form

$$
\begin{aligned}
\dot{p}_1 &= f_1(p_1, p_2, P_C, u_1, u_2) \\
\dot{p}_2 &= f_2(p_1, p_2, P_C, u_1, u_2) \\
\dot{P}_C &= f_3(p_1, p_2, P_C, u_1, u_2) \\
y_1 &= g_1(p_1, p_2, P_C, u_1, u_2) \\
y_2 &= g_2(p_1, p_2, P_C, u_1, u_2) \\
y_3 &= g_3(p_1, p_2, P_C, u_1, u_2)
\end{aligned}
$$

where $u_1 = W_{egr}$, $u_2 = u_{2t}$, $y_1 = p_1$, $y_2 = p_2$ and $y_3 = P_C$

$$f_1(p_1, p_2, P_C, u_1, u_2) = k_1 \quad \frac{C}{T_a C_p} \frac{P_C}{\frac{p_1}{p_a} \qquad 1} \quad k_e p_1 + \; u_1$$

$$f_2(p_1, p_2, P_C, u_1, u_2) = k_2\left(k_e p_1 + W_f^d \quad u_1 \quad u_2\right)$$

$$f_3(p_1, p_2, P_C, u_1, u_2) = \quad \frac{1}{\;} \; P_C \qquad m \quad {}_t T_2 C_p \; 1 \quad \frac{p_a}{p_2} \qquad u_2$$

$$
\begin{aligned}
g_1(p_1, p_2, P_C, u_1, u_2) &= p_1 \\
g_2(p_1, p_2, P_C, u_1, u_2) &= p_2 \\
g_2(p_1, p_2, P_C, u_1, u_2) &= P_C
\end{aligned}
$$

The linearized model has the form

$$\begin{matrix} \dot{p}_1 \\ p_2 \\ P_C \end{matrix} = A \begin{matrix} p_1 \\ p_2 \\ P_C \end{matrix} + B \begin{matrix} u_1 \\ u_2 \end{matrix}$$

$$\begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} = C \begin{matrix} p_1 \\ p_2 \\ P_C \end{matrix} + D \begin{matrix} u_1 \\ u_2 \end{matrix}$$

where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial p_2} & \frac{\partial f_1}{\partial P_C} \\ \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial p_2} & \frac{\partial f_2}{\partial P_C} \\ \frac{\partial f_3}{\partial p_1} & \frac{\partial f_3}{\partial p_2} & \frac{\partial f_3}{\partial P_C} \end{bmatrix}_{\substack{x = x^d \\ u = u^d}} \quad \text{and} \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \end{bmatrix}_{\substack{x = x^d \\ u = u^d}}$$

$$C = \begin{bmatrix} \frac{\partial g_1}{\partial p_1} & \frac{\partial g_1}{\partial p_2} & \frac{\partial g_1}{\partial P_C} \\ \frac{\partial g_2}{\partial p_1} & \frac{\partial g_2}{\partial p_2} & \frac{\partial g_2}{\partial P_C} \\ \frac{\partial g_3}{\partial p_1} & \frac{\partial g_3}{\partial p_2} & \frac{\partial g_3}{\partial P_C} \end{bmatrix}_{\substack{x = x^d \\ u = u^d}} \quad \text{and} \quad D = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} \\ \frac{\partial g_3}{\partial u_1} & \frac{\partial g_3}{\partial u_2} \end{bmatrix}_{\substack{x = x^d \\ u = u^d}}$$

In the above expression $x = \begin{bmatrix} p_1 & p_2 & P_C \end{bmatrix}^T$ and $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$. The superscript d has the meaning of the desired value (the value at the equilibrium point around which the linearization is computed). Using the above expressions for $f_1,\ f_2,\ f_3,\ g_1,\ g_2,\ g_3$ the following expressions are obtained for the matrices A and B:

$$A = \begin{bmatrix} \frac{k_1 \; {}_C P_C^d}{T_a C_p p_a} \frac{\left(\frac{p_1^d}{p_a}\right)^{1}}{\left(\left(\frac{p_1^d}{p_a}\right) \quad 1\right)^{2}} \quad k_1 k_e & 0 & \frac{k_1 \; {}_C}{T_a C_p} \frac{1}{\left(\frac{p_1^d}{p_a}\right) \quad 1} \\ k_2 k_e & 0 & 0 \\ 0 & \frac{m \quad {}_t T_2 C_p p_a u_2^d}{\left(p_2^d\right)^{\quad +1}} & \frac{1}{\;} \end{bmatrix}$$

$$B = \begin{matrix} k_1 & 0 \\ k_2 & k_2 \\ 0 & \frac{1}{\quad} \; m \; {}_t T_2 C_p \; 1 \quad \frac{p_a}{p_2^d} \end{matrix}, \; C = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}, \text{ and } D = \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix}.$$

In order to show the instability and the non-minimum phase behavior of the reduced order model the poles and the transmission zeros will be computed for one equilibrium point corresponding to:

$$W_f^d = 6.1111e \quad 004 \text{kg}/\sec; \; AFR_{ref} = 45; \; EGR_{ref} = 0.4$$

This operating point has been used in simulation in [1]. The equilibrium values corresponding to this equilibrium operating point are:

$$p_1^d = 182.4412 \text{kPa}$$
$$p_2^d = 186.4868 \text{kPa}$$
$$P_C^d = 1.4990 \text{kW}$$
$$T_1^d = 511.6083 \text{K}$$
$$T_2^d = 719.2392 \text{K}$$
$$u_{2t}^d = 0.0208 \text{kg/sec}$$

Using these values in the previous expressions of the matrices A,B,C,D the poles and the zeros can be computed. The poles are

$$13.7629 \pm \; 3.6782i, \; 5.0555.$$

The transmission zeros from the input $u_1 = W_{egr}$ to the output $y_1 = p_1$ are

$$12.4085 \text{ and } 9.0751.$$

The presence of a zero in the right half plane is consistent with our intuition, i.e. an increase in $u_1 = W_{egr}$ may result in a decrease in $y_1 = p_1$ since the turbine flow will decrease and so will do the compressor power.

The transmission zeros do not show a non-minimum phase behavior from the input to the output $y_2 = p_2$ for the third order model. However the non-minimum phase feature of the full(7th) order model of the Diesel engine has been proved in [8], page 172-180, by simulations and by computing the transmission zeros for different transfer functions.

The MATLAB code that we have used for computing the linearized model of the 3rd order model around the specified equilibrium point and computing the poles and the transmission zeros is added below.

```
>> p1d = 182.4412;
>> p2d = 186.4868;
>> Pcd = 1.4990;
>> T1d = 511.6083;
>> T2d = 719.2392;
>> u_2td = 0.0208;
>> k1 = R*T1d/V1;
>> k2 = R*T2d/V2;


>> A = zeros(3,3);
>> A(1,1) = k1*( -eta_c/(Ta*Cp)*Pcd*1/( ((p1d/pa)^miu-1)^2 )*miu*1/( pa^miu)
                                                    *p1d^(miu-1)-ke );
>> A(1,2) = 0;
>> A(1,3) = k1*eta_c/(Ta*Cp)*1/( (p1d/pa)^miu - 1 );
>> A(2,1) = k2*ke;
>> A(2,2) = 0;
>> A(2,3) = 0;
>> A(3,1) = 0;
>> A(3,2) = 1/tau*eta_m*eta_t*T2d*Cp2*miu*pa^miu*p2d^(-miu-1)*u_2td;
>> A(3,3) = -1/tau;
>> eig(A)
>> B = zeros(3,2);
>> B(1,1) = k1;
>> B(1,2) = 0;
>> B(2,1) = -k2;
>> B(2,2) = -k2;
>> B(3,1) = 0;
>> B(3,2) = 1/tau*eta_m*eta_t*T2d*Cp2*( 1-(pa/p2d)^miu );
>> C1 = [1 0 0];
>> C2 = [0 1 0];
>> D = zeros(1,1);
>> z1 = tzero(A,B(:,1),C1,D) ;
>> z2 = tzero(A,B(:,2),C2,D);
```

For deriving a linearized model, around a specified operating point, of the 7th order model of the Diesel engine a Mapple program has been used. The main part of the program is listed below:

Section 2. Program Description

This program is used to compute the linearized model of the Diesel engine around an equilibrium point. The input to the model considered in this program is : <u>u = [u1 u2]' = [Wegr u_2t]'</u>. The purpose of the linearization is to study the stability of the Diesel engine model at different equilibrium
points. The model has the standard form :

```
> Diff(<m[1], F[1], p[1], m[2], F[2], p[2], P[C]>,t)=f(<m[1],
F[1], p[1], m[2], F[2], p[2], P[C]>,<u[1], u[2]>);
```

$$\frac{\partial}{\partial t}\begin{bmatrix} m_1 \\ F_1 \\ p_1 \\ m_2 \\ F_2 \\ p_2 \\ P_C \end{bmatrix} = \mathrm{f}\left(\begin{bmatrix} m_1 \\ F_1 \\ p_1 \\ m_2 \\ F_2 \\ p_2 \\ P_C \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$$

Let x denote the state of the Diesel engine:

```
> x=<m[1], F[1], p[1], m[2], F[2], p[2], P[C]>;
```

$$x = \begin{bmatrix} m_1 \\ F_1 \\ p_1 \\ m_2 \\ F_2 \\ p_2 \\ P_C \end{bmatrix}$$

And let the input u be:

```
> u=<u[1], u[2]>;
```

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Then the model can be written as:

```
> Diff(x,t)=<f[1](x,u), f[2](x,t), f[3](x,u), f[4](x,u),
f[5](x,u), f[6](x,u), f[7](x,u) >;
```

$$\frac{\partial}{\partial t} x = \begin{matrix} f_1(x, u) \\ f_2(x, t) \\ f_3(x, u) \\ f_4(x, u) \\ f_5(x, u) \\ f_6(x, u) \\ f_7(x, u) \end{matrix}$$

Section 3. Definition of Wc1, W1e, Tc1, We2, Fe2, Te2, Pt in terms of the state of the system

In this section the above variables will be defined in terms of the state of the system and some constants. The reason for defining the above variables is that they will be used in the next section to compute the linearized model around an equilibrium operating point.

```
> Wc1:=(eta_c*Pc)/( Cp1*Ta*( (p1/pa)^miu-1) );
```

$$Wc1 := \frac{eta\_c\ Pc}{Cp1\ Ta\ \ \frac{p1}{pa}^{miu}\quad 1}$$

```
> W1e:=(Vd*eta_v*N*m1)/(120*V1);
```

$$W1e := \frac{1}{120}\frac{Vd\ eta\_v\ N\ m1}{V1}$$

```
> Tc1:=Ta+Ta/eta_c*( (p1/pa)^miu-1 );
```

$$Tc1 := Ta + \frac{Ta\ \ \frac{p1}{pa}^{miu}\quad 1}{eta\_c}$$

```
> We2:=Wf+W1e;
```

$$We2 := Wf + \frac{\frac{1}{120}\, Vd\ eta\_v\ N\ m1}{V1}$$

Fe2 represents the burnt gas ratio after the combustion has taken place and phi_s represents the stoichiometric value of FAR(fuel-to-air ratio=1/AFR).

```
> Fe2:=( W1e*F1+Wf*(1+1/phi_s) )/(Wf+W1e);
```

$$Fe2 := \frac{\frac{1}{120}\frac{Vd\ eta\_v\ N\ m1\ F1}{V1} + Wf\ \ 1 + \frac{1}{phi\_s}}{Wf + \frac{\frac{1}{120}\ Vd\ eta\_v\ N\ m1}{V1}}$$

```
> T1:=p1*V1/(m1*R);
```

$$T1 := \frac{p1\ V1}{m1\ R}$$

```
> T2:=p2*V2/(m2*R);
```

$$T2 := \frac{p2\ V2}{m2\ R}$$

```
> Te2:=T1+( -5.8940+0.00061*N+4.56*10^(-6)*N^2
)+Wf/(Wf+W1e)*(10000*(1.3722+0.0709*(1-10000/N)));
```

$$Te2 := \frac{p1\ V1}{m1\ R}\ \ 5.8940 + .00061\ N\ +.4560000000\ 10^{-5}\ N^2$$

$$+ \frac{Wf\ \ 14431.0000\ \ \frac{.7090000000\ 10^{7}}{N}}{Wf + \frac{\frac{1}{120}\ Vd\ eta\_v\ N\ m1}{V1}}$$

Pt represents the turbine power.

```
> Pt:=eta_t*T2*Cp2*( 1-(pa/p2)^miu )*u2;
```

$$Pt := \frac{eta\_t\ p2\ V2\ Cp2\ \ 1\ \ \ \frac{pa}{p2}^{miu}\ \ u2}{m2\ R}$$

Section 4. Computation of the linearized model

The linearized model has the following form:

```
> Diff(x,t)=A*x+B*u;
```

$$\frac{\partial}{\partial t} x = A\ x + B\ u$$

with the matrix A defined as below:

```
> A=<<Diff(f[1](x,u),m[1]) | Diff(f[1](x,u),F[1]) |
Diff(f[1](x,u),p[1]) | Diff(f[1](x,u),m[2]) |
Diff(f[1](x,u),F[2]) | Diff(f[1](x,u),p[2]) |
Diff(f[1](x,u),P[c]) > , <Diff(f[2](x,u),m[1]) |
Diff(f[2](x,u),F[1]) | Diff(f[2](x,u),p[1]) |
Diff(f[2](x,u),m[2]) | Diff(f[2](x,u),F[2]) |
Diff(f[2](x,u),p[2]) | Diff(f[2](x,u),P[c])> ,
```

```
<Diff(f[3](x,u),m[1]) | Diff(f[3](x,u),F[1]) |
Diff(f[3](x,u),p[1])| Diff(f[3](x,u),m[2]) |
Diff(f[3](x,u),F[2]) | Diff(f[3](x,u),p[2]) |
Diff(f[3](x,u),P[c])> , < Diff(f[4](x,u),m[1])|
Diff(f[4](x,u),F[1]) | Diff(f[4](x,u),p[1]) |
Diff(f[4](x,u),m[2]) | Diff(f[4](x,u),F[2]) |
Diff(f[4](x,u),p[2]) | Diff(f[4](x,u),P[c])> , <
Diff(f[5](x,u),m[1]) | Diff(f[5](x,u),F[1]) |
Diff(f[5](x,u),p[1]) | Diff(f[5](x,u),m[2]) |
Diff(f[5](x,u),F[2]) | Diff(f[5](x,u),p[2]) |
Diff(f[5](x,u),P[c])> , < Diff(f[6](x,u),m[1]) |
Diff(f[6](x,u),F[1]) | Diff(f[6](x,u),p[1]) |
Diff(f[6](x,u),m[2]) | Diff(f[6](x,u),F[2]) |
Diff(f[6](x,u),p[2]) | Diff(f[6](x,u),P[c])> , <
Diff(f[7](x,u),m[1]) | Diff(f[7](x,u),F[1]) |
Diff(f[7](x,u),p[1]) | Diff(f[7](x,u),m[2]) |
Diff(f[7](x,u),F[2]) | Diff(f[7](x,u),p[2]) |
Diff(f[7](x,u),P[c])>>;
>
```

$A =$

$$\frac{\partial}{\partial m_1} f_1(x,u), \frac{\partial}{\partial F_1} f_1(x,u), \frac{\partial}{\partial p_1} f_1(x,u), \frac{\partial}{\partial m_2} f_1(x,u), \frac{\partial}{\partial F_2} f_1(x,u), \frac{\partial}{\partial p_2} f_1(x,u),$$
$$\frac{\partial}{\partial P_c} f_1(x,u)$$

$$\frac{\partial}{\partial m_1} f_2(x,u), \frac{\partial}{\partial F_1} f_2(x,u), \frac{\partial}{\partial p_1} f_2(x,u), \frac{\partial}{\partial m_2} f_2(x,u), \frac{\partial}{\partial F_2} f_2(x,u), \frac{\partial}{\partial p_2} f_2(x,u),$$
$$\frac{\partial}{\partial P_c} f_2(x,u)$$

$$\frac{\partial}{\partial m_1} f_3(x,u), \frac{\partial}{\partial F_1} f_3(x,u), \frac{\partial}{\partial p_1} f_3(x,u), \frac{\partial}{\partial m_2} f_3(x,u), \frac{\partial}{\partial F_2} f_3(x,u), \frac{\partial}{\partial p_2} f_3(x,u),$$
$$\frac{\partial}{\partial P_c} f_3(x,u)$$

$$\frac{\partial}{\partial m_1} f_4(x,u), \frac{\partial}{\partial F_1} f_4(x,u), \frac{\partial}{\partial p_1} f_4(x,u), \frac{\partial}{\partial m_2} f_4(x,u), \frac{\partial}{\partial F_2} f_4(x,u), \frac{\partial}{\partial p_2} f_4(x,u),$$
$$\frac{\partial}{\partial P_c} f_4(x,u)$$

$$\frac{\partial}{\partial m_1} f_5(x,u), \frac{\partial}{\partial F_1} f_5(x,u), \frac{\partial}{\partial p_1} f_5(x,u), \frac{\partial}{\partial m_2} f_5(x,u), \frac{\partial}{\partial F_2} f_5(x,u), \frac{\partial}{\partial p_2} f_5(x,u),$$

$$\frac{\partial}{\partial P_c} f_5(x,u)$$

$$\frac{\partial}{\partial m_1} f_6(x,u), \frac{\partial}{\partial F_1} f_6(x,u), \frac{\partial}{\partial p_1} f_6(x,u), \frac{\partial}{\partial m_2} f_6(x,u), \frac{\partial}{\partial F_2} f_6(x,u), \frac{\partial}{\partial p_2} f_6(x,u),$$

$$\frac{\partial}{\partial P_c} f_6(x,u)$$

$$\frac{\partial}{\partial m_1} f_7(x,u), \frac{\partial}{\partial F_1} f_7(x,u), \frac{\partial}{\partial p_1} f_7(x,u), \frac{\partial}{\partial m_2} f_7(x,u), \frac{\partial}{\partial F_2} f_7(x,u), \frac{\partial}{\partial p_2} f_7(x,u),$$

$$\frac{\partial}{\partial P_c} f_7(x,u)$$

Computation of the 1st row of A

```
> Diff(m[1],t)=<Diff(f[1](x,u),m[1]) | Diff(f[1](x,u),F[1]) |
Diff(f[1](x,u),p[1]) | Diff(f[1](x,u),m[2]) |
Diff(f[1](x,u),F[2]) | Diff(f[1](x,u),p[2]) |
Diff(f[1](x,u),P[c]) >;
```

$$\frac{\partial}{\partial t} m_1 = \frac{\partial}{\partial m_1} f_1(x,u), \frac{\partial}{\partial F_1} f_1(x,u), \frac{\partial}{\partial p_1} f_1(x,u), \frac{\partial}{\partial m_2} f_1(x,u), \frac{\partial}{\partial F_2} f_1(x,u), \frac{\partial}{\partial p_2} f_1(x,u),$$

$$\frac{\partial}{\partial P_c} f_1(x,u)$$

```
> f1:=Wc1+u1-W1e;
```

$$f1 := \frac{eta\_c\; Pc}{Cp1\; Ta\; \frac{p1}{pa}^{miu} \quad 1} + u1 \quad \frac{1}{120} \frac{Vd\; eta\_v\; N\; m1}{V1}$$

```
> diff(f1,m1);
```

$$\frac{1}{120} \frac{Vd\; eta\_v\; N}{V1}$$

```
> diff(f1,F1);
```

$$0$$

```
> diff(f1,p1);
```

$$\frac{eta\_c\; Pc\; \frac{p1}{pa}^{miu}\; miu}{Cp1\; Ta\; \frac{p1}{pa}^{miu} \quad 1^{\;2}\; p1}$$

```
> diff(f1,m2);
```

$$0$$

```
> diff(f1,F2);
```

$$0$$

```
> diff(f1,p2);
```

$$0$$

```
> diff(f1,Pc);
```

$$\frac{eta\_c}{Cp1\ Ta\ \left(\left(\frac{p1}{pa}\right)^{miu}\ \ 1\right)}$$

Computation of the 2nd row of A

```
> Diff(F[1],t)=<Diff(f[2](x,u),m[1]) | Diff(f[2](x,u),F[1]) |
Diff(f[2](x,u),p[1]) | Diff(f[2](x,u),m[2]) |
Diff(f[2](x,u),F[2]) | Diff(f[2](x,u),p[2]) |
Diff(f[2](x,u),P[c])>;
```

$$\frac{\partial}{\partial t}F_1 = \frac{\partial}{\partial m_1}f_2(x, u), \frac{\partial}{\partial F_1}f_2(x, u), \frac{\partial}{\partial p_1}f_2(x, u), \frac{\partial}{\partial m_2}f_2(x, u), \frac{\partial}{\partial F_2}f_2(x, u), \frac{\partial}{\partial p_2}f_2(x, u),$$
$$\frac{\partial}{\partial P_c}f_2(x, u)$$

```
> f2:=1/m1*( (F2-F1)*u1-F1* Wc1 );
```

$$f2 := \frac{(F2\ \ F1)\ u1\ \ \frac{F1\ eta\_c\ Pc}{Cp1\ Ta\ \left(\left(\frac{p1}{pa}\right)^{miu}\ \ 1\right)}}{m1}$$

```
> diff(f2,m1);
```

$$\frac{(F2\ \ F1)\ u1\ \ \frac{F1\ eta\_c\ Pc}{Cp1\ Ta\ \left(\left(\frac{p1}{pa}\right)^{miu}\ \ 1\right)}}{m1^2}$$

```
> diff(f2,F1);
```

$$\frac{u1\ \ \frac{eta\_c\ Pc}{Cp1\ Ta\ \left(\left(\frac{p1}{pa}\right)^{miu}\ \ 1\right)}}{m1}$$

```
> diff(f2,p1);
```

$$\frac{F1\ eta\_c\ Pc\ \left(\frac{p1}{pa}\right)^{miu}\ miu}{m1\ Cp1\ Ta\ \left(\left(\frac{p1}{pa}\right)^{miu}\ \ 1\right)^2\ p1}$$

```
> diff(f2,m2);
```

$$0$$

```
> diff(f2,F2);
```

$$\frac{u1}{m1}$$

```
> diff(f2,p2);
```

$$0$$

```
> diff(f2,Pc);
```

$$\frac{F1\ eta\_c}{m1\ Cp1\ Ta\ \ \frac{p1}{pa}^{miu}\ \ \ 1}$$

Computation of the 3rd row of A

```
> Diff(p[1],t)=<Diff(f[3](x,u),m[1]) | Diff(f[3](x,u),F[1]) |
Diff(f[3](x,u),p[1])| Diff(f[3](x,u),m[2]) |
Diff(f[3](x,u),F[2]) | Diff(f[3](x,u),p[2]) |
Diff(f[3](x,u),P[c])> ;
```

$$\frac{\partial}{\partial t}p_1 = \frac{\partial}{\partial m_1}f_3(x,u), \frac{\partial}{\partial F_1}f_3(x,u), \frac{\partial}{\partial p_1}f_3(x,u), \frac{\partial}{\partial m_2}f_3(x,u), \frac{\partial}{\partial F_2}f_3(x,u), \frac{\partial}{\partial p_2}f_3(x,u),$$
$$\frac{\partial}{\partial P_c}f_3(x,u)$$

```
> f3:=gamma*R/V1*( Wc1*Tc1+u1*T2-W1e*T1 );
```

$$f3 := \frac{R\ \ \frac{eta\_c\ Pc\ \ \ Ta + \frac{Ta\ \ \frac{p1}{pa}^{miu}\ \ \ 1}{eta\_c}}{Cp1\ Ta\ \ \frac{p1}{pa}^{miu}\ \ \ 1} + \frac{u1\ p2\ V2}{m2\ R}\ \ \ \frac{1}{120}\ \frac{Vd\ eta\_v\ N\ p1}{R}}{V1}$$

```
> diff(f3,m1);
```

$$0$$

```
> diff(f3,F1);
```

$$0$$

```
> diff(f3,p1);
```

$$R\ \ \frac{eta\_c\ Pc\ \ \ Ta + \frac{Ta\ \ \frac{p1}{pa}^{miu}\ \ \ 1}{eta\_c}\ \ \frac{p1}{pa}^{miu}\ \ miu}{Cp1\ Ta\ \ \frac{p1}{pa}^{miu}\ \ \ 1^{\ 2}\ \ p1} + \frac{Pc\ \ \frac{p1}{pa}^{miu}\ \ miu}{Cp1\ \ \frac{p1}{pa}^{miu}\ \ \ 1\ \ p1}$$

$$\frac{1}{120}\ \frac{Vd\ eta\_v\ N}{R}\ \ /V1$$

```
> diff(f3,m2);
```

$$\frac{u1\ p2\ V2}{V1\ m2^2}$$

```
> diff(f3,F2);
```

$$0$$

```
> diff(f3,p2);
```

$$\frac{u1\ V2}{V1\ m2}$$

```
> diff(f3,Pc);
```

$$\frac{R\ eta\_c\ \ Ta + \dfrac{Ta\ \ \left(\dfrac{p1}{pa}\right)^{miu}\ \ 1}{eta\_c}}{V1\ Cp1\ Ta\ \ \left(\dfrac{p1}{pa}\right)^{miu}\ \ 1}$$

Computation of the 4th row of A

```
> Diff(m[2],t)=< Diff(f[4](x,u),m[1])| Diff(f[4](x,u),F[1]) |
Diff(f[4](x,u),p[1]) | Diff(f[4](x,u),m[2]) |
Diff(f[4](x,u),F[2]) | Diff(f[4](x,u),p[2]) |
Diff(f[4](x,u),P[c])>;
```

$$\frac{\partial}{\partial t} m_2 = \frac{\partial}{\partial m_1} f_4(x, u), \frac{\partial}{\partial F_1} f_4(x, u), \frac{\partial}{\partial p_1} f_4(x, u), \frac{\partial}{\partial m_2} f_4(x, u), \frac{\partial}{\partial F_2} f_4(x, u), \frac{\partial}{\partial p_2} f_4(x, u), \frac{\partial}{\partial P_c} f_4(x, u)$$

```
> f4:=(We2-u2-u1);
```

$$f4 := Wf + \frac{\frac{1}{120}\ Vd\ eta\_v\ N\ m1}{V1}\ \ u2\ \ u1$$

```
> diff(f4,m1);
```

$$\frac{1}{120}\ \frac{Vd\ eta\_v\ N}{V1}$$

```
> diff(f4,F1);
```

$$0$$

```
> diff(f4,p1);
```

$$0$$

```
> diff(f4,m2);
```

$$0$$

```
> diff(f4,F2);
```

$$0$$

```
> diff(f4,p2);
```

$$0$$

```
> diff(f4,Pc);
```

$$0$$

>

Computation of the 5th row of A

```
> Diff(F[2],t)=< Diff(f[5](x,u),m[1]) | Diff(f[5](x,u),F[1]) |
Diff(f[5](x,u),p[1]) | Diff(f[5](x,u),m[2]) |
Diff(f[5](x,u),F[2]) | Diff(f[5](x,u),p[2]) |
Diff(f[5](x,u),P[c])>;
```

$$\frac{\partial}{\partial t}F_2 = \frac{\partial}{\partial m_1}f_5(x,u), \frac{\partial}{\partial F_1}f_5(x,u), \frac{\partial}{\partial p_1}f_5(x,u), \frac{\partial}{\partial m_2}f_5(x,u), \frac{\partial}{\partial F_2}f_5(x,u), \frac{\partial}{\partial p_2}f_5(x,u), \frac{\partial}{\partial P_c}f_5(x,u)$$

```
> f5:=1/m2*(Fe2-F2)*We2;
```

$$f5 := \frac{\left(\dfrac{\frac{1}{120}\dfrac{Vd\ eta\_v\ N\ m1\ F1}{V1} + Wf\ \left(1 + \dfrac{1}{phi\_s}\right)}{Wf + \dfrac{\frac{1}{120}Vd\ eta\_v\ N\ m1}{V1}} \quad F2\right)\left(Wf + \dfrac{\frac{1}{120}Vd\ eta\_v\ N\ m1}{V1}\right)}{m2}$$

```
> diff(f5,m1);
```

$$\left(\frac{1}{120}\frac{Vd\ eta\_v\ N\ F1}{V1\ \left(Wf + \dfrac{\frac{1}{120}Vd\ eta\_v\ N\ m1}{V1}\right)} \quad \frac{1}{120}\frac{\left(\frac{1}{120}\dfrac{Vd\ eta\_v\ N\ m1\ F1}{V1} + Wf\ \left(1 + \dfrac{1}{phi\_s}\right)\right) Vd\ eta\_v\ N}{\left(Wf + \dfrac{\frac{1}{120}Vd\ eta\_v\ N\ m1}{V1}\right)^2 V1}\right)\left(Wf + \frac{\frac{1}{120}Vd\ eta\_v\ N\ m1}{V1}\right)/m2$$

$$+ \frac{\frac{1}{120}\left(\dfrac{\frac{1}{120}\dfrac{Vd\ eta\_v\ N\ m1\ F1}{V1} + Wf\ \left(1 + \dfrac{1}{phi\_s}\right)}{Wf + \dfrac{\frac{1}{120}Vd\ eta\_v\ N\ m1}{V1}} \quad F2\right) Vd\ eta\_v\ N}{m2\ V1}$$

```
> diff(f5,F1);
```

$$\frac{1}{120}\frac{Vd\ eta\_v\ N\ m1}{m2\ V1}$$

```
> diff(f5,p1);
```

$$0$$

```
> diff(f5,m2);
```

$$\frac{\frac{\left(\frac{1}{120}\frac{Vd\ eta\_v\ N\ m1\ F1}{V1}+Wf\right)\left(1+\frac{1}{phi\_s}\right)}{Wf+\frac{\frac{1}{120}Vd\ eta\_v\ N\ m1}{V1}}\ F2\ \left(Wf+\frac{\frac{1}{120}Vd\ eta\_v\ N\ m1}{V1}\right)}{m2^2}$$

```
> diff(f5,F2);
```

$$\frac{Wf+\frac{\frac{1}{120}Vd\ eta\_v\ N\ m1}{V1}}{m2}$$

```
> diff(f5,p2);
```

$$0$$

```
> diff(f5,Pc);
```

$$0$$

\>

Computation of the 6th row of A

```
> Diff(p[2],t)=< Diff(f[6](x,u),m[1]) | Diff(f[6](x,u),F[1]) |
Diff(f[6](x,u),p[1]) | Diff(f[6](x,u),m[2]) |
Diff(f[6](x,u),F[2]) | Diff(f[6](x,u),p[2]) |
Diff(f[6](x,u),P[c])> ;
```

$$\frac{\partial}{\partial t}p_2 = \frac{\partial}{\partial m_1}f_6(x,u), \frac{\partial}{\partial F_1}f_6(x,u), \frac{\partial}{\partial p_1}f_6(x,u), \frac{\partial}{\partial m_2}f_6(x,u), \frac{\partial}{\partial F_2}f_6(x,u), \frac{\partial}{\partial p_2}f_6(x,u),$$
$$\frac{\partial}{\partial P_c}f_6(x,u)$$

```
> f6:=gamma*R/V2*(We2*Te2-u2*T2-u1*T2);
```

$$f6 := \; R \quad Wf + \frac{\frac{1}{120}\, Vd\; eta\_v\; N\; m1}{V1} \quad \frac{p1\; V1}{m1\; R} \quad 5.8940 + .00061\; N$$

$$+ .4560000000\; 10^{-5}\; N^2 + \frac{Wf \quad 14431.0000 \quad \frac{.7090000000\; 10^7}{N}}{Wf + \frac{\frac{1}{120}\, Vd\; eta\_v\; N\; m1}{V1}} \quad \frac{u2\; p2\; V2}{m2\; R}$$

$$\frac{u1\; p2\; V2}{m2\; R} \quad / V2$$

> **diff(f6,m1);**

$$R \quad \frac{1}{120}\, Vd\; eta\_v\; N \quad \frac{p1\; V1}{m1\; R} \quad 5.8940 + .00061\; N \; + .4560000000\; 10^{-5}\; N^2$$

$$+ \frac{Wf \quad 14431.0000 \quad \frac{.7090000000\; 10^7}{N}}{Wf + \frac{\frac{1}{120}\, Vd\; eta\_v\; N\; m1}{V1}} \; / V1 + \quad Wf + \frac{\frac{1}{120}\, Vd\; eta\_v\; N\; m1}{V1}$$

$$\frac{p1\; V1}{m1^2\; R} \quad \frac{1}{120} \; \frac{Wf \quad 14431.0000 \quad \frac{.7090000000\; 10^7}{N} \quad Vd\; eta\_v\; N}{Wf + \frac{\frac{1}{120}\, Vd\; eta\_v\; N\; m1}{V1}^{\;2} \quad V1} \quad / V2$$

> **diff(f6,F1);**

$$0$$

> **diff(f6,p1);**

$$\frac{Wf + \frac{\frac{1}{120}\, Vd\; eta\_v\; N\; m1}{V1} \quad V1}{V2\; m1}$$

> **diff(f6,m2);**

$$\frac{R\left(\frac{u2\ p2\ V2}{m2^2\ R}+\frac{u1\ p2\ V2}{m2^2\ R}\right)}{V2}$$

```
> diff(f6,F2);
```

$$0$$

```
> diff(f6,p2);
```

$$\frac{R\left(\frac{u2\ V2}{m2\ R}\quad\frac{u1\ V2}{m2\ R}\right)}{V2}$$

```
> diff(F6,Pc);
```

$$0$$

```
>
```

Computation of the 7th row of the matrix A

```
> Diff(P[c],t)= < Diff(f[7](x,u),m[1]) | Diff(f[7](x,u),F[1]) |
Diff(f[7](x,u),p[1]) | Diff(f[7](x,u),m[2]) |
Diff(f[7](x,u),F[2]) | Diff(f[7](x,u),p[2]) |
Diff(f[7](x,u),P[c])>;
```

$$\frac{\partial}{\partial t}P_c=\frac{\partial}{\partial m_1}f_7(x,u),\frac{\partial}{\partial F_1}f_7(x,u),\frac{\partial}{\partial p_1}f_7(x,u),\frac{\partial}{\partial m_2}f_7(x,u),\frac{\partial}{\partial F_2}f_7(x,u),\frac{\partial}{\partial p_2}f_7(x,u),$$
$$\frac{\partial}{\partial P_c}f_7(x,u)$$

```
> f7:=1/tau*(-Pc+eta_m*Pt);
```

$$f7 := \frac{Pc+\frac{eta\_m\ eta\_t\ p2\ V2\ Cp2\ \left(1\quad\left(\frac{pa}{p2}\right)^{miu}\right)u2}{m2\ R}}{}$$

```
> diff(f7,m1);
```

$$0$$

```
> diff(f7,F1);
```

$$0$$

```
> diff(f7,p1);
```

$$0$$

```
> diff(f7,m2);
```

$$\frac{eta\_m\ eta\_t\ p2\ V2\ Cp2\ \left(1\quad\left(\frac{pa}{p2}\right)^{miu}\right)u2}{m2^2\ R}$$

```
> diff(f7,F2);
```

$$0$$

```
> diff(f7,p2);
```

$$\frac{\frac{eta\_m\ eta\_t\ V2\ Cp2\ \left(1\ \ \left(\frac{pa}{p2}\right)^{miu}\right) u2}{m2\ R} + \frac{eta\_m\ eta\_t\ V2\ Cp2\ \left(\frac{pa}{p2}\right)^{miu} miu\ u2}{m2\ R}}{}$$

```
> diff(f7,Pc);
```

$$\frac{1}{}$$

Section 4. Eigenvalue analysis.

The eigenvalues of the matrix A computed as above at the same operating point as the 3rd order model are:

59.2683, 42.3917, 5.1455, 15.9578, 11.4079, 5.5867, 4.9945.

Therefore at the same operating point in both models is unstable.

APPENDIX 3: SAMPLE MATLAB CODE FOR COMPUTING A CONTROLLER GAIN AS DESCRIBED IN SECTIONS 3. AND 4.

At the end of the Section 4., it was mentioned that the gain K of the linear state feedback controller $u = K \quad x$ is obtained as a function of the solution of a system of LMIs. The purpose of this Appendix is to present the MATLAB code that was used to form and solve the system of LMIs introduced in Section 3. This system is formed by equations 3.16, 3.17, and 3.21.

In the MATLAB code presented in this Appendix, the following notations are used:

- p1d, p2d, and Pcd stand for p_1^d , p_2^d , and P_C^d respectively;
- DelA1, DelA2, and DelA3 stand for A_1 , A_2, and A_3, respectively;
- “eta” stands for (different types of efficiencies);
- “mu” stands for ;
- u_2td stands for u_2^d .

The significance of the other symbols should be clear in view of the above explanations. In the following code, it is assumed that an equilibrium point has been chosen, and all the corresponding values for pressures, temperatures, and control inputs have been determined. Also, it is assumed that a starting point for the trajectory of the 3rd order model is known. The constants that appear in the code have been previously defined, in either the main body of the report or in the previous Appendices.

In order to make the MATLAB code more clear, it will be divided in several portions. Also short explanations are provided for each portion. Almost all the commands used in the code can be found in the LMI Toolbox of MATLAB.

STEP 1: *Set* d *(*d *places the role of from the end of the Section 4.) to a value.*

STEP 2: *Using the increasing, or decreasing, properties, in a neighborhood of the equilibrium point, of the functions* $_i(\cdot)$*'s, the matrices that define the corners of the polytope in the* (A, B)*-space are computed using the following code (see the relationships 4.5 – 4.10):*

```
>> Ao = [ -ke 0 0 ; k2*ke 0 0 ; 0 0 -1/tau ];

>> DelA1 = [ 1 0 0 ; 0 0 0 ; 0 0 0 ];  % the function  1(·) is locally increasing
>> a1 = k1*eta_c*Pcd/(Ta*Cp1*(-d))*( 1/( ( (p1d-d)/pa )^miu-1 ) - 1/( ( p1d/pa )^miu-1 ) );
>> b1 = k1*eta_c*Pcd/(Ta*Cp1*(+d))*( 1/( ( (p1d+d)/pa )^miu-1 ) - 1/( ( p1d/pa )^miu-1 ) );

>> DelA2 = [ 0 0 1 ; 0 0 0 ; 0 0 0 ]; % the function  2(·)  is locally decreasing
>> a2 = eta_c/(Ta*Cp1)*1/( ( (p1d+d)/pa )^miu1-1 );
>> b2 = eta_c/(Ta*Cp1)*1/( ( (p1d-d)/pa )^miu1-1 );

>> DelA3 = [ 0 0 0 ; 0 0 0 ; 0 1 0 ]; % the function  3(·)  is locally decreasing
```

```
>> a3 = -1/tau*eta_m*eta_t*T2*Cp2*u_2td/(-d)*( ( pa/(p2d-d) )^miu-( pa/p2d)^miu );
>> b3 = -1/tau*eta_m*eta_t*T2*Cp2*u_2td/(+d)*( ( pa/(p2d+d) )^miu-( pa/p2d)^miu );

>> A1 = Ao + a1*DelA1 + a2*DelA2 + a3*DelA3;
>> A2 = Ao + b1*DelA1 + a2*DelA2 + a3*DelA3;
>> A3 = Ao + a1*DelA1 + b2*DelA2 + a3*DelA3;
>> A4 = Ao + a1*DelA1 + a2*DelA2 + b3*DelA3;
>> A5 = Ao + b1*DelA1 + b2*DelA2 + a3*DelA3;
>> A6 = Ao + b1*DelA1 + a2*DelA2 + b3*DelA3;
>> A7 = Ao + a1*DelA1 + b2*DelA2 + b3*DelA3;
>> A8 = Ao + b1*DelA1 + b2*DelA2 + b3*DelA3;

>> Bo = [ k1 0 ; -k2 -k2; 0 1/tau*eta_m*eta_t*T2*Cp2 ];
>> DelB1 = [ 0 0 ; 0 0 ; 0 1 ]; % the function  4(◊) is increasing concave
>> a4 = -1/tau*eta_m*eta_t*T2*Cp2*( pa/(p2d+d ) )^miu;
>> b4 = -1/tau*eta_m*eta_t*T2*Cp2*( pa/(p2d-d ) )^miu;

>> B1 = Bo + a4*DelB1;
>> B2 = Bo + b4*DelB1;
```

STEP 3: *Form the LMIs.*

STEP 3.1:

```
>> setlmis([])  % the variables are : S, L and gamma
>> S = lmivar(1,[3,1]);
>> L = lmivar(2,[2,3]);
>> gamma = lmivar(1,[1,1]); % the upper bound on the input which needs to be minimized
```

STEP 3.2: *Form the LMIs represented by the relationships 3.16a (Remark that the polytope in the space* (A, B) *has 16 vertices)*

```
>> lmi1 = newlmi;
>> lmiterm([lmi1,1,1,S],A1,1,'s');      % A1*S+S*A1'
>> lmiterm([lmi1,1,1,L],B1,1,'s');      % B1*L+L'*B1'

>> lmi2 = newlmi;
>> lmiterm([lmi2,1,1,S],A1,1,'s');      % A1*S+S*A1'
>> lmiterm([lmi2,1,1,L],B2,1,'s');      % B2*L+L'*B2'

>> lmi3 = newlmi;
>> lmiterm([lmi3,1,1,S],A2,1,'s');      % A2*S+S*A2'
>> lmiterm([lmi3,1,1,L],B1,1,'s');      % B1*L+L'*B1'

>> lmi4 = newlmi;
>> lmiterm([lmi4,1,1,S],A2,1,'s');      % A2*S+S*A2'
>> lmiterm([lmi4,1,1,L],B2,1,'s');      % B2*L+L'*B2'
```

```
>> lmi5 = newlmi;
>> lmiterm([lmi5,1,1,S],A3,1,'s');      % A3*S+S*A3'
>> lmiterm([lmi5,1,1,L],B1,1,'s');      % B1*L+L'*B1'

>> lmi6 = newlmi;
>> lmiterm([lmi6,1,1,S],A3,1,'s');      % A3*S+S*A3'
>> lmiterm([lmi6,1,1,L],B2,1,'s');      % B2*L+L'*B2'

>> lmi7 = newlmi;
>> lmiterm([lmi7,1,1,S],A4,1,'s');      % A4*S+S*A4'
>> lmiterm([lmi7,1,1,L],B1,1,'s');      % B1*L+L'*B1'

>> lmi8 = newlmi;
>> lmiterm([lmi8,1,1,S],A4,1,'s');      % A4*S+S*A4'
>> lmiterm([lmi8,1,1,L],B2,1,'s');      % B2*L+L'*B2'

>> lmi9 = newlmi;
>> lmiterm([lmi9,1,1,S],A5,1,'s');      % A5*S+S*A5'
>> lmiterm([lmi9,1,1,L],B1,1,'s');      % B1*L+L'*B1'

>> lmi10 = newlmi;
>> lmiterm([lmi10,1,1,S],A5,1,'s');     % A5*S+S*A5'
>> lmiterm([lmi10,1,1,L],B2,1,'s');     % B2*L+L'*B2'

>> lmi11 = newlmi;
>> lmiterm([lmi11,1,1,S],A6,1,'s');     % A6*S+S*A6'
>> lmiterm([lmi11,1,1,L],B1,1,'s');     % B1*L+L'*B1'

>> lmi12 = newlmi;
>> lmiterm([lmi12,1,1,S],A6,1,'s');     % A6*S+S*A6'
>> lmiterm([lmi12,1,1,L],B2,1,'s');     % B2*L+L'*B2'

>> lmi13 = newlmi;
>> lmiterm([lmi13,1,1,S],A7,1,'s');     % A7*S+S*A7'
>> lmiterm([lmi13,1,1,L],B1,1,'s');     % B1*L+L'*B1'

>> lmi14 = newlmi;
>> lmiterm([lmi14,1,1,S],A7,1,'s');     % A7*S+S*A7'
>> lmiterm([lmi14,1,1,L],B2,1,'s');     % B2*L+L'*B2'

>> lmi15 = newlmi;
>> lmiterm([lmi15,1,1,S],A8,1,'s');     % A8*S+S*A8'
>> lmiterm([lmi15,1,1,L],B1,1,'s');     % B1*L+L'*B1'

>> lmi16 = newlmi;
```

```
>> lmiterm([lmi16,1,1,S],A8,1,'s');      % A8*S+S*A8'
>> lmiterm([lmi16,1,1,L],B2,1,'s');      % B2*L+L'*B2'

>> lmi17 = newlmi;
>> lmiterm([-lmi17,1,1,S],1,1);          % S
>> lmiterm([lmi17,1,1,0],0);             % this inequality requires that S be nonsingular
```

The invariant region is { x: x'*inv(S)*x < 1 }. LMI18 impose that the starting point be in this region. This LMI represents the inequality 3.16.

```
>> lmi18 = newlmi;
>> lmiterm([-lmi18,1,1,0],1);            % 1
>> lmiterm([-lmi18,1,2,0],[p1s-p1d p2s-p2d Pcs-Pcd]);  % (x0-x1)'
>> lmiterm([-lmi18,2,2,S],1,1);          % S
```

LMI19 and LMI20 impose that the invariant region(ellipsoid) be in the box(in the state space) where we are doing the approximation: C1*S*C1' < d^2 and C2*S*C2' < d^2

```
>> lmi19 = newlmi;
>> lmiterm([lmi19,1,1,S],C1,C1');        % C1*S*C1'
>> lmiterm([-lmi19,1,1,0],d*d);          % 1

>> lmi20 = newlmi;
>> lmiterm([lmi20,1,1,S],C2,C2');        % C2*S*C2'
>> lmiterm([-lmi20,1,1,0],d*d);          % 1
```

STEP 3.3: *Form the LMI representing the inequality 3.20*

```
>> lmi21 = newlmi;
>> lmiterm([-lmi21,1,1,S],1,1);
>> lmiterm([-lmi21,2,1,L],1,1);
>> lmiterm([-lmi21,2,2,gamma],1,1);
```

The LMI22 requires that gamma be positive.

```
>> lmi22 = newlmi;
>> lmiterm([-lmi22,1,1,gamma],1,1);
>> lmiterm([lmi22,1,1,0],0);

>> lmis = getlmis;
```

STEP 4: *The formulation of the optimization problem*

```
>> c = mat2dec(lmis,0,0,1);
>> options = [1e-5 0 0 0 0];
>> [copt,xopt] = mincx(lmis,c,options)
```

```
>> S = dec2mat(lmis,xopt,S)
>> L = dec2mat(lmis,xopt,L)
        >> K = L*inv(S)
```