Second order Pontryagin's principle for stochastic control problems

Joseph Frederic Bonnans

SECOND ORDER PONTRYAGIN'S PRINCIPLE FOR STOCHASTIC CONTROL PROBLEMS*

J. FRÉDÉRIC BONNANS†

Abstract. We discuss stochastic optimal control problems whose volatility does not depend on the control, and which have finitely many equality and inequality constraints on the expected value of functions of the final state, as well as control constraints. The main result is a proof of necessity of some second order optimality conditions involving Pontryagin multipliers.

Key words. Stochastic control, second order optimality conditions, Pontryagin's principle.

1. Introduction. In this paper we consider stochastic optimal control problems whose volatility does not depend on the control, and having two types of constraints: (i) bound constraints on the control, and (ii) finitely many equality and inequality constraints on the expected value of function of the final state. Such problems can be studied by the Hamilton-Jacobi-Bellman (or dynamic programming) approach, using the notion of viscosity solution, see [14, 19, 25]. This approach has the advantage to give characterizations of global optimality in some cases. However, it is not easy to apply in the presence of final state constraints. In this paper we will rely on the variational approach, which consists in obtaining necessary or sufficient optimality conditions by analyzing small perturbations of an optimal trajectory.

For deterministic control problems, a major result along this approach is Pontryagin's maximum principle, or PMP, which essentially says that with the solution of a deterministic optimal control problem, are associated some multipliers such that the optimal control minimizes the Hamiltonian of the problem. This has been extended to stochastic control problems, first by Kushner [18, 17], Bensoussan [2] Bismut [3, 4], and Haussmann [15, 16]. A major advance, due to Peng [22], was the extension of such results to the case when the volatility depends of the control. See also Cadenillas and Karatzas [9] and Yong and Zhou [25].

On the other hand, it is classical for abstract optimization problems to derive second order necessary conditions. These conditions typically say that the curvature of the Lagrangian of the problem is nonnegative over a set of critical directions, for some multiplier that may depend on the direction [7]. The only extension we know of such results for stochastic control problems is [8].

Finally, in some deterministic optimal control problems it is possible to obtain second order necessary conditions in Pontryagin form, i.e., where the involved multipliers satisfy the PMP, see [20]. A result of this form, for problems with state constraints and mixed state and control constraints, was recently obtained in [5]. Corresponding sufficient conditions were obtained in [6]. The aim of this paper is to obtain second order necessary conditions in Pontryagin form for stochastic control problems. We have to make the important restrictive hypothesis that the volatility does not depend on the control. Note, however, that in the second order optimality conditions obtained in [8], there were already important restrictions on the dependence of the volatility w.r.t. the control.

*Work supported by the Laboratoire de Finance des Marchés d'Energie (FIME), Dauphine-CREST-Polytechnique-EDF.
†INRIA-Saclay and Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau, France (Frederic.Bonnans@inria.fr).

As in [5] the analysis will be based on an auxiliary problem called "finite relaxation", that makes use of the notion of relaxed control. There is an important literature concerning the extension of relaxation to the stochastic case, see the early reference Becker and Mandrekar [1], and El Karoui, Nguyen and Jeanblanc-Picqué [11].

The analysis is simplified here for two reasons: (i) we use only finite relaxations, which can be viewed as classical controls for the auxiliary problem, and (ii) the volatility does not depend on the control. This simplifies, in particular, the construction of classical controls approximating relaxed ones.

The paper is organized as follows. The setting is presented in section 2, and the main results are stated in section 3. The proofs are given for weak minima in section 4, and for Pontryagin minima, using the idea of partial relaxation, in section 5. We recall some basic results on SDEs in the appendix.

2. Setting.

2.1. Some function spaces. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, $\mathbb{F} := \{\mathcal{F}_t\}$ being the augmented natural filtration associated with a Brownian motion $W(t) \in \mathbb{R}^d$ and $\mathbb{P}$ the corresponding probability law. We say that $x(\omega, t) \in \mathbb{R}$ is progressively measurable if, for all $t \in [0, T]$, the mapping: $\Omega \times [0, t] \to \mathbb{R}$, $(\omega, s) \mapsto x(\omega, s)$ is "$\mathcal{F}_t \times$ Borel" mesurable.

For $\beta \in [1, \infty]$ and $\gamma \in [1, \infty]$, we define the spaces $L^{\beta,\gamma}$ as the set of progressively measurable functions of (t, ω) such that the norm defined below, for $\beta \in [1, \infty)$ and $\gamma \in [1, \infty)$, is finite:

$$\|x\|_{\beta,\gamma} := \left(I\!\!E \left(\int_0^T |x(\omega, t)|^\gamma \mathrm{d}t \right)^{\beta/\gamma} \right)^{1/\beta}, \tag{2.1}$$

with obvious extensions if $\beta = \infty$ or $\gamma = \infty$, in particular

$$\begin{aligned} \|x\|_{\beta,\infty} \quad &:= \left(I\!\!E \left(\operatorname*{esssup}_{t\in(0,T)} |x(\omega, t)| \right)^\beta \right)^{1/\beta}. \\ \|x\|_{\infty,\infty} \quad &:= \operatorname*{esssup}_{\omega} \operatorname*{esssup}_{t\in(0,T)} |x(\omega, t)|. \end{aligned} \tag{2.2}$$

We define the control space by $\mathcal{U}^{\beta,\gamma} := (L^{\beta,\gamma})^m$ and the state space by $\mathcal{Y}^\beta := (L^\beta(\Omega, C([0, T])))^n$, endowed with the norm of $(L^{\beta,\infty})^n$. We set $\mathcal{U}^\beta := \mathcal{U}^{\beta,\beta}$ Both norms of $\mathcal{U}^\beta$ and $\mathcal{Y}^\beta$ will be denoted by $\| \cdot \|_\beta$, since no confusion should occur. We say that a property is valid a.e. if it holds for a.a. t, almost surely.

Let $F(t, u, y, \omega)$ be a finite dimensional function over $[0, T] \times \mathbb{R}^m \times \mathbb{R}^n \times \Omega$. Denote by $\mathcal{B}$ the Borelian σ field on $[0, T]$. We say that F is measurable if it is $\mathcal{B}([0, T]) \times \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{F}_T$ measurable. We say that a measurable function F is essentially bounded if it belongs to L^∞. By DF, D^2F, etc, we denote the partial derivative of F w.r.t. the variables (u, y). We adopt similar conventions for functions of (y, ω), such as the final cost.

2.2. Stochastic control problem. In the sequel we assume that (u, y) belongs to $\mathcal{U}^{\beta,\gamma} \times \mathcal{Y}^\beta$, for β and γ in $[1, \infty)$. We define the cost function

$$J(u, y) := I\!\!E\varphi(y(T), \omega), \tag{2.3}$$

the state equation

$$\text{(2.4)} \qquad \left\{ \begin{array}{l} \mathrm{d}y(t) = f(t, u(t,\omega), y(t,\omega), \omega)\mathrm{d}t + \sigma(t, y(t,\omega), \omega)\mathrm{d}W(t); \\ t \in (0,T); \quad y(0) = y_0, \end{array} \right.$$

the control constraints, where U_{ad} is a multimapping $[0,T] \times \Omega \to \mathbb{R}^m$:

$$\text{(2.5)} \qquad u(t,\omega) \in U_{ad}(t,\omega) \quad \text{a.e.},$$

the final state constraints

$$\text{(2.6)} \qquad I\!\!E\Phi(y(T),\omega) \in K_\Phi,$$

where $\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^{n_\Phi}$, and K_Φ stands for finitely many equalities and inequalities, i.e., for some n_E and n_I in $\mathbb{N}$ such that $n_E + n_I = n_\Phi$:

$$\text{(2.7)} \qquad K_\Phi := \{0\}_{\mathbb{R}^{n_E}} \times \mathbb{R}^{n_I}_-.$$

The stochastic control problem to be considered in this paper is

$$(P) \qquad \operatorname{Min} J(u,y) \ \text{ s.t. } (2.4)\text{-}(2.6).$$

We assume throughout the paper that the functions f, σ, ℓ, and φ belong to the class of "nice functions" defined below: F is said to be nice if

(2.8)
- (i) F is progressively measurable,
- (ii) The mapping $(u,y) \mapsto F(t,u,y,\omega)$ is a.e. C^2,
- (ii) The functions F, DF, and D^2F are essentially bounded, and Lipschitz with uniform constants,

with obvious adaptations for σ and φ that do not depend on u and (t,u), resp., φ being only measurable. Under these assumptions, for any control $u \in \mathcal{U}^{\beta,1}$, the state equation has a unique solution $y[u] \in \mathcal{Y}^\beta$, and so, the cost function and constraints are well defined.

We assume that $U_{ad}(t,\omega)$ has the following structure:

$$\text{(2.9)} \qquad U_{ad}(t,\omega) = \{u \in \mathbb{R}^m;\ a^i(t,\omega) \cdot u \le b_i(t,\omega), \quad i = 1,\dots,n_U\},$$

where the functions a^i and b_i are progressively measurable. We set

$$\text{(2.10)} \qquad \mathcal{U}_{ad} := \{u \in \mathcal{U}^2;\ u(t,\omega) \in U_{ad}(t,\omega) \text{ a.e.}\}.$$

We assume in the paper that

$$\text{(2.11)} \qquad \text{There exists } u^{00} \in \mathcal{U}_{ad} \cap \mathcal{U}^\infty.$$

The feasible set of problem (P), denoted by $F(P)$, satisfies therefore

$$\text{(2.12)} \qquad F(P) = \{u \in \mathcal{U}_{ad};\ I\!\!E\Phi(y[u](t)) \in K_\Phi\}.$$

We will consider a 'nominal' feasible trajectory $(\bar u, \bar y) \in \mathcal{U}^2 \times \mathcal{Y}^2$, and abbreviate for instance $f(t, \bar u(t,\omega), \bar y(t,\omega), \omega)$ into $\bar f(t)$, with similar conventions for the partial derivatives w.r.t. (u, y) denoted by $\bar{f}'(t)$ or $D\bar{f}(t)$, and the partial derivative w.r.t. to say y only, denoted by $\bar{f}_y(t)$ or $D_y\bar{f}(t)$.

We will assume throughout the paper that

$$
(2.13) \qquad \begin{cases} \text{(i)} & \text{The } a^i \text{ are essentially bounded,} \\ \text{(ii)} & \text{There exists } v^0 \in \mathcal{U}^\infty \text{ and } \varepsilon_0 > 0 \text{ such that} \\ & a^i \cdot (\bar{u} + v^0) + \varepsilon_0 \le b_i, \quad i = 1, \dots, n_U, \text{ a.e.} \end{cases}
$$

REMARK 2.1. *If $\mathcal{U}_{ad} \subset \mathcal{U}^\infty$, we may take $\mathcal{U}^\infty$ as control space. If in addition the b_i are essentially bounded, then* (2.13)*(ii) coincides with the inward condition for the mixed constraint in [5, Def. 2.5], therefore we will call it the* inward condition for the control constraint. *This hypothesis is convenient since in the sequel we will make the analysis of essentially bounded perturbations of the control $\bar{u}$.*

REMARK 2.2. *When the a^i and b_i (and therefore U_{ad}) do not depend on (t, ω), and U_{ad} is bounded, the inward condition for the control constraint is not restrictive. Indeed, consider the apparently slightly weaker hypothesis*

$$
(2.14) \qquad \begin{cases} \textit{There exists } \varepsilon_1 > 0 \textit{ such that, for any } i = 1, \dots, n_U, \\ \textit{there exists } u^i \in U_{ad} \textit{ such that } a^i \cdot u + \varepsilon_1 \le b_i. \end{cases}
$$

If (2.14) *holds,* (2.13)*(ii) also with $\varepsilon_0 := \varepsilon_1/n_U$ and $v^0 := -\bar{u} + (n_U)^{-1}\sum_{i=1}^{n_U} u^i$, and therefore both conditions are equivalent. On the other hand, if* (2.14) *does not hold, for at least one index $1 \le i \le n_U$, $a^i \cdot u = b_i$ for all $u \in U_{ad}$. If $a^i = 0$, the corresponding inequality can be removed. Otherwise, reindexing if necessary the components of the control, we may assume that for all $u \in \mathbb{R}^m$, $a^i \cdot u = b_i$ iff $u_m = \sum_{j=1}^{m-1} \alpha_j u_j$ for some coefficients $\alpha_1, \dots, \alpha_{m-1}$, so that we can reformulate the stochastic control problem with a control having $m - 1$ components. By induction we obtain a reformulation for which* (2.13)*(ii) holds.*

2.3. Expansion of the control to state mapping. When, as in this paper, the volatility does no depend on the control, as established in [8, Prop. 3.14] the mapping $u \mapsto y[u]$ happens to have a second order Taylor expansion, with the following restriction: we have to choose the pertubation of the control in a smaller space than the one of the control. Note that in general the control space should not be too small since it has to contain the optimal control.

For the first order expansion we give an analysis in section A.2 (see theorem A.2). For each $\beta \in [1, \infty)$, the mapping $v \mapsto y[\bar{u} + v]$ happens to be F-differentiable $\mathcal{U}^{2\beta,2} \to \mathcal{Y}^\beta$.

The *linearized state equation* is, skipping the arguments (t, ω) of v and z:

$$
(2.15) \qquad \begin{cases} \mathrm{d}z & = \bar{f}'(t)(v, z)\mathrm{d}t + \bar{\sigma}_y(t)z\mathrm{d}W(t), \quad t \in [0, T], \\ z(0) & = 0. \end{cases}
$$

The solution is denoted by $z[v]$, and we need to consider as well the *second order linearized state equation*, with the same conventions, and $z = z[v]$:

(2.16)

$$
\begin{cases} \mathrm{d}z_2 & = (\bar{f}_y(t)z_2 + \bar{f}''(t)(v, z)^2)\mathrm{d}t + (\sigma_y(t)z_2 + \sigma_{yy}(t)(z)^2)\mathrm{d}W(t), \ t \in [0, T]. \\ z_2(0) & = 0. \end{cases}
$$

LEMMA 2.3. (i) *For any $v \in \mathcal{U}^{\beta,1}$, with $\beta \in [1, \infty)$,* (2.15) *has a unique solution $z[v]$ in $\mathcal{Y}^\beta$, and $\|z\|_\beta = O(\|v\|_{\beta,1})$.* (ii) *For any $v \in \mathcal{U}^{\beta,2}$, with $\beta \in [2, \infty)$,* (2.16) *has a unique solution in $\mathcal{Y}^{\beta/2}$, denoted by $z_2[v]$, and $\|z_2\|_{\beta/2} = O(\|v\|_{\beta,2})$.*

Proof. Immediate consequence of lemma A.1. □

THEOREM 2.4. *Let $\bar u \in \mathcal{U}^2$ and $v \in \mathcal{U}^\infty$. Then we have the expansion*

$$y[\bar u + v] = y[u] + z + \tfrac{1}{2} z_2 + \rho_2, \tag{2.17}$$

with $\|\rho_2\|_{\beta,\infty} = O(\|v\|_{2\beta,2}\|v\|^2_{4\beta,4})$.

Proof. See [8, Prop. 3.14]. □

2.4. Hamiltonian and final Lagrangian. Let $(\bar u, \bar y)$ be a feasible control and associated trajectory. We define the Hamiltonian and final Lagrangian functions by (skipping the argument ω):

$$\left\{ \begin{array}{ll} H(\nu, t, u, y, p, q, \omega) & := p \cdot f(t, u, y, \omega) + \sum_{i=1}^d \sigma^i(t, y, \omega) \cdot q^i, \\ L^F(y, \nu, \Psi, \omega) & := \nu\varphi(y, \omega) + \Psi \cdot \Phi(y, \omega). \end{array} \right. \tag{2.18}$$

Here p and q^i, $i = 1$ to d, are elements of $\mathbb{R}^n$. The costate equation is

$$\left\{ \begin{array}{ll} -\mathrm{d}p(t) & = \nabla_y H(\nu, t, \bar u, \bar y, p, q, \omega)\mathrm{d}t + \sum_{i=1}^d q^i \mathrm{d}W_i(t) \\ p(T) & = \nabla_y L^F(\bar y(T), \nu, \Psi). \end{array} \right. \tag{2.19}$$

By [12, Thm 5.1] (see also [21, Proposition 3.1]), for each $\lambda := (\nu, \Psi)$, and $y = y[u]$, the costate equation has a unique solution (p^λ, q^λ) in $\mathcal{Y}^\beta \times (L^{\beta,2}_{\mathbb{F}})^{nd}$, for each $\beta \in (1, \infty)$, $\lambda \mapsto (p^\lambda, q^\lambda)$ is a linear mapping, and there exists $C_\beta > 0$ such that

$$\|p\|^\beta_{\beta,\infty} + \sum_{i=1}^d \|q^i\|_{\beta,2} \le C_\beta |\lambda|. \tag{2.20}$$

2.5. Lagrange and Pontryagin multipliers. Set

$$F(u) := J(u, y[u]). \tag{2.21}$$

The *reduced Lagrangian* of problem (P) is defined as

$$\begin{array}{ll} L(u, \lambda) & := \nu F(u) + \Psi \cdot \mathbb{E}\Phi(y[u](T)) \\ & = \mathbb{E} L^F(y[u](T), \nu, \Psi, \omega). \end{array} \tag{2.22}$$

We define the sets of *Lagrange* and *Pontryagin* multipliers, resp., as follows:

$$\Lambda_L(\bar u) := \left\{ \begin{array}{l} \lambda = (\nu, \Psi_E, \Psi_I) \in \mathbb{R}_+ \times \mathbb{R}^{n_E} \times \mathbb{R}^{n_I}_+; \\ \lambda \ne 0;\ \Psi \perp \mathbb{E}\Phi(\bar y(T)); \\ -\nabla_u H(\nu, t, \bar u, \bar y, p^\lambda, q^\lambda, \omega) \in N_{U_{ad}}(\bar u(t, \omega)) \text{ a.e.} \end{array} \right\},$$

$$\Lambda_P(\bar u) := \left\{ \begin{array}{l} \lambda = (\nu, \Psi_E, \Psi_I) \in \mathbb{R}_+ \times \mathbb{R}^{n_E} \times \mathbb{R}^{n_I}_+;\ \lambda \ne 0;\ \Psi \perp \mathbb{E}\Phi(\bar y(T)); \\ H(\nu, t, \bar u, \bar y, p^\lambda, q^\lambda, \omega) = \inf_u H(\nu, t, u, \bar y, p^\lambda, q^\lambda, \omega) \text{ a.e.} \end{array} \right\}.$$

We kown (see e.g. XXX) that, for any v in $\mathcal{U}^\beta$ and $\lambda \in \Lambda_L(\bar u)$:

$$L_u(\bar u, \lambda)v = \mathbb{E}\int_0^T H_u(\nu, t, \bar u, \bar y, p^\lambda, q^\lambda)v\mathrm{d}t. \tag{2.23}$$

REMARK 2.5. *When studying perturbations of the optimal control that belong to $\mathcal{U}^\infty$ we may reduce to the case when the b_i are bounded. Indeed we can locally rewrite the control constraints as*

$$a^i \cdot \bar u(t, \omega) \le b'_i(t, \omega) \tag{2.24}$$

where b' is the bounded vector defined by

$$b'_i := \min(b_i(t,\omega), 1 + a^i \cdot \bar{u}(t,\omega)). \tag{2.25}$$

The multipliers associated with the control constraints are defined as the set $M(\bar{u},\lambda)$ of $\mu \in (L^\infty)^{n_U}$ that satisfy the following relation:

$$\begin{cases} H_u(\bar{u},\bar{y},p^\lambda) + \sum_{i=1}^{n_U} \mu_i(t,\omega) a^i = 0, \\ \mu_i(t,\omega) \geq 0,\ \mu_i(t,\omega)(a^i \cdot u(t,\omega) - b^j) = 0 \text{ a.e.} \end{cases} \tag{2.26}$$

LEMMA 2.6. *Let $\lambda \in \Lambda_L(\bar{u})$. Then the set $M(\bar{u},\lambda)$ is nonempty and bounded.*

Proof. By remark 2.5, we may reduce to the case when b is essentially bounded. Set $\delta := \nabla_u H(\nu, t, \bar{u}, \bar{y}, p^\lambda, q^\lambda)$, and consider the problem

$$\underset{v \in \mathcal{U}^\infty}{\operatorname{Min}}\ I\!E \int_0^T \delta v; \quad v \in U_{ad} \text{ a.e.} \tag{2.27}$$

Clearly this problem has value zero, and is qualified in view of (2.13). Therefore, the set of dual solution, that we denote by $M'(\bar{u},\lambda)$, is a nonempty and bounded subset of $(\mathcal{U}^\infty)^*$. Let $\mu \in M'(\bar{u},\lambda)$. We need to show that

$$\langle \mu, g \rangle \leq c \|g\|_1, \quad \text{for all } g \in (L^\infty_{\mathbb{F}})^{n_U}. \tag{2.28}$$

Since $\mu \geq 0$, it suffices to obtain this inequality when $g \geq 0$. We may write

$$g(t,\omega) = \alpha(t,\omega)\bar{g}(t,\omega); \quad \alpha(t,\omega) \geq 0; \ |\bar{g}(t,\omega)| = 1 \text{ a.e.} \tag{2.29}$$

Let A be the matrix whose row i is $(a^i)^\top$. By (2.13), $h \in (L^\infty)^{n_U}$ defined by $h := b - A(\bar{u} + v^0)$ is such that $h \geq \varepsilon_0$ a.e. Since $g_i \leq \alpha \leq \alpha h_i/\varepsilon_0$, for $i = 1$ to n_U, we have that

$$\varepsilon_0 \langle \mu, g \rangle = \langle \mu, \varepsilon_0 g \rangle \leq \langle \mu, \alpha h \rangle. \tag{2.30}$$

On the other hand, since $\mu \geq 0$ and $h'_i := b_i - a^i \cdot \bar{u}$ is such that $\alpha h' \geq 0$:

$$0 \leq \langle \mu, \alpha h' \rangle \leq \sqrt{n_U} \|g\|_\infty \langle \mu, h' \rangle. \tag{2.31}$$

By the complementarity conditions, $\langle \mu, h' \rangle = 0$ and therefore the previous inequalities are equalities, so that $\langle \mu, \alpha h' \rangle = 0$. With (2.30), setting $h'' := -Av^0$ and $v^1 := \alpha v^0$, we deduce that

$$\begin{aligned} \varepsilon_0 \langle \mu, g \rangle &\leq \langle \mu, \alpha h \rangle = \langle \mu, \alpha h'' \rangle = -\langle \mu, A v^1 \rangle = L_u(\bar{u},\lambda) v^1 \\ &\leq \|L_u(\bar{u},\lambda)\|_\infty \|\alpha\|_1 |v^0| = \|L_u(\bar{u},\lambda)\|_\infty \|g\|_1 |v^0|. \end{aligned} \tag{2.32}$$

The result follows.□

The first order necessary optimality conditions, are as follows:

THEOREM 2.7. *Let $\bar{u}$ be a weak minimum (local minimum in L^∞). Then $\Lambda_L(\bar{u}) \neq \emptyset$.*

THEOREM 2.8. *Let $\bar{u}$ be a local minimum in $L^1_{\mathbb{F}}$. Then $\Lambda_P(\bar{u}) \neq \emptyset$.*

REMARK 2.9. *(i) Theorem 2.7 is established in [8, Prop. 4.3 and lemma 4.5] in the absence of final constraints, and in [8, Prop. 5.5] in the absence of control constraints.*
(ii) Theorem 2.8 is essentially the result of Peng [22], combined with the representation of the multiplier μ in lemma 4.2.

3. Second order optimality conditions.

3.1. Weak minima. Consider the following quadratic form over $\mathcal{U}^2$, that is well-defined for $v \in \mathcal{U}^2$, since, when $z[v]$ belongs to $\mathcal{Y}^2$:

$$Q[\lambda](v) := \mathbb{E}\left(\int_0^T D^2 \bar{H}(t)(v(t), z[v](t))^2 \mathrm{d}t + D^2 L^F(\bar{y}(T), \nu, \Psi)(z[v](T))^2 \right). \tag{3.1}$$

Set, for $\beta \in [1, \infty]$:

$$T^\beta_{\mathcal{U}_{ad}} := \{v \in \mathcal{U}^\beta; \quad v(t,\omega) \in T_{U_{ad}}(\bar{u}(t,\omega)) \text{ a.e.}\}. \tag{3.2}$$

By [8, Lemma 4.5], when $\beta \in [1, \infty)$, this is the set of control directions in $\mathcal{U}^\beta$ that are tangent directions (in the sense of convex analysis) to the control constraints set $\mathcal{U}_{ad}$ at the point $\bar{u}$. When $\beta = \infty$, it is easy to check that $T^\beta_{\mathcal{U}_{ad}}$ in general is smaller than this tangent set.

Similarly we define the corresponding set of strict tangent directions to the control constraints as (note that this is a closed vector space)

$$T^\beta_{\mathcal{U}_{ad},S} := \{v \in \mathcal{U}^\beta; \quad a^i \cdot v(t,\omega) = 0, \ \text{ if } a^i \cdot \bar{u}(t,\omega) = b_i, \ i = 1, \dots, n_U, \ \text{a.e.}\}. \tag{3.3}$$

The *critical cone* is, for $\beta \in [1, \infty]$, defined as:

$$C^\beta(\bar{u}) := \left\{ v \in T^\beta_{\mathcal{U}_{ad}}; \ D\Phi(\bar{y}(T)) z[v](T) \in T_K(\Phi(\bar{y}(T)) \cap \varphi'(\bar{y}(T))^- \right\}. \tag{3.4}$$

We also define the *strict critical cone* as

$$C^\beta_S(\bar{u}) := T^\beta_{\mathcal{U}_{ad},S} \cap C^\beta(\bar{u}). \tag{3.5}$$

The next result slightly improves [8, Thm 4.10 and thm 5.10], by considering simultaneously the control constraints and the final constraints. It will be proved in the next section.

THEOREM 3.1. *Let $(\bar{u}, \bar{y})$ a weak minimum and the associated state. Then, for any strict critical direction $v \in \mathcal{U}^\infty$:*

$$\max_{\Lambda_L(\bar{u})} Q[\lambda](v) \geq 0. \tag{3.6}$$

Note that, by [8, Prop. 4.3 and lemma 4.5], in the absence of final constraints, (3.6) also holds for critical directions.

3.2. Local minima in $L^1_{\mathbb{F}}$. This is the main result of the paper. Here we need an additional hypothesis of regularity w.r.t. time of the dynamics:

$$f \text{ and } \sigma \text{ are uniformly continuous w.r.t. } t. \tag{3.7}$$

as well as control constraints not dependig on time:

(3.8) The functions a and b do not depend on time.

THEOREM 3.2. *Let $(\bar{u}, \bar{y})$ be a local minimum in $L^1_{\mathbb{F}}$ and the associated state, and let* (3.7) *and* (3.8) *hold. Then, for any strict critical direction $v \in \mathcal{U}^\infty$:*

$$\max_{\Lambda_P(\bar{u})} \Omega[\lambda](v) \geq 0. \tag{3.9}$$

As mentioned in the introduction, we may view this result as partial extension to the stochastic setting of similar results in the deterministic setting, see [5, 20].

4. Proofs in the case of weak minima.

4.1. Degenerate equality constraints. We say that the equality constraints $I\!E\Phi_E(y[u](T)) = 0$ are *degenerate* at the trajectory $(\bar{u}, \bar{y})$ if the mapping

$$v \mapsto DI\!E\Phi_E(\bar{y}(T))z[v](T) \tag{4.1}$$

is not onto. This holds iff there exists a nonzero element of the orthogonal space, that we call *totally singular multiplier*, i.e.,

$$\begin{cases} \Psi_E \in \mathbb{R}^{n_E},\ \Psi_E \neq 0, \text{ such that} \\ v \mapsto \Psi_E \cdot I\!ED\Phi_E(\bar{y}(T))z[v](T) \text{ identically vanishes.} \end{cases} \tag{4.2}$$

If a totally singular multiplier $\hat{\Psi}_E$ exists, we may identify it with the multiplier λ^0 for problem (P) having $\nu = 0$, $\Psi_I = 0$ and $\hat{\Psi}_E$ as components. Then theorems 3.1 and 3.2 trivially holds, by taking either λ^0 or $-\lambda^0$ (the latter is also a totally singular multiplier). So in the sequel we may assume that the equality constraints are non degenerate.

4.2. Formulation with slack variables. We may assume w.l.o.g. that $J(\bar{u}, \bar{y})$ is equal to 0. Consider the following *slack problem* with slack variable $\theta \in \mathbb{R}$:

$$\begin{array}{rl} \displaystyle\operatorname*{Min}_{(v,\theta)\in\mathcal{U}^\infty\times\mathbb{R}} \ \theta & \text{s.t.} \\ \text{(i)} & I\!E\Phi_E(y[u](T)) = 0, \\ \text{(ii)} & F(u) \leq \theta, \\ \text{(iii)} & I\!E\Phi_I(y[u](T)) \leq \theta\mathbf{1}, \\ \text{(iv)} & A(t,\omega)u(t,\omega) \leq b'(t,\omega) + \theta \text{ a.e. in } (L^\infty)^{n_U}. \end{array} \tag{4.3}$$

Set $\bar{\theta} = 0$. The following result is easily obtained.

LEMMA 4.1. *If $\bar{u}$ is a weak minimum of problem (P), then $(\bar{u}, \bar{\theta})$ is a weak minimum of the slack problem* (4.3).

We denote by $\lambda = (\Psi_E, \nu, \Psi_I, \mu)$ the components of the multipliers associated with each of the four constraints of (4.3). We consider only *regular multipliers* for this problem, in the sense that there is no multiplier associated with the cost function. As in the proof of lemma 2.6 we can check that μ belongs to $(L^\infty)^{n_U}$. Of course $\nu \geq 0$ and $\Psi_I \geq 0$. A Lagrange multiplier is a nonzero λ for which we have stationarity w.r.t. the primal variables (u, θ) of the associated Lagrangian, whose expression is (compare to (2.22); here we have dualized also the control constraints), putting apart the coefficient of θ:

$$L'(u,\theta,\lambda) := L(u,\lambda) + \sum_{i=1}^{n_U}\langle \mu_i, A_i v - b'_i\rangle + \left(1 - \langle \nu, \mathbf{1}\rangle - \Psi_I\cdot\mathbf{1} - \sum_{i=1}^{n_U}\langle \mu_i, \mathbf{1}\rangle\right)\theta. \tag{4.4}$$

When $\mu \in (L^{\beta'})^{n_U}$, the above duality products can be written as $I\!E\sum_{i=1}^{n_U}\int_0^T \mu_i(t,\omega)(a^i\cdot u(t,\omega) - b'_i)\mathrm{d}t$ and $I\!E\sum_{i=1}^{n_U}\int_0^T \mu_i(t,\omega)\mathrm{d}t$ resp.

Taking (2.26) into account, we see that we recover the same set of Lagrange multipliers as for problem (P), up to the condition of stationarity of the Lagrangian w.r.t. θ, i.e.

$$\Lambda'_L(\bar{u}) := \left\{\lambda \in \Lambda_L(\bar{u});\quad \langle \nu, \mathbf{1}\rangle + \Psi_I\cdot\mathbf{1} + \sum_{i=1}^{n_U}\langle \mu_i, \mathbf{1}\rangle = 1\right\}. \tag{4.5}$$

LEMMA 4.2. *We have that*
(i) *The set* $\Lambda'_L(\bar{u})$ *is nonempty and bounded,* (ii) *for any* $\lambda = (\Psi_E, \nu, \Psi_I, \mu)$ *in* $\Lambda'_L(\bar{u})$ *we have that*

$$|\mu(t,\omega)| \leq \varepsilon_0^{-1} |v^0| |\bar{H}_u(t,\omega)| \quad a.e. \tag{4.6}$$

Proof. Remember that we assumed that the equality constraints are not degenerate. The constraints other than equalities correspond to a convex set with nonempty interior. In that case, we know [7, Corollary 2.101] that qualification holds iff there exists a direction $(v, \delta\theta)$ in the kernel of derivative of equality constraints, whose corresponding image in the constraint spaces points in the direction of the interior of the convex set. This condition holds: it suffices to take $v = 0$ and $\delta\theta = 1$. Therefore, the problem is qualified. By [23], the associated set of Lagrange multipliers is nonempty and bounded. We conclude with lemma 2.6. □

4.3. Second order necessary conditions. We need to approximate strict critical directions by a smaller set in order to be able to construct feasible paths, that will allow to obtain the second order necessary conditions. For this we need Dmitruk's *density lemma* [10].

LEMMA 4.3. *Consider a locally convex topological space* X*, a polyhedron (finite intersection of closed half spaces)* $C \subset X$*, and a linear subspace* L *of* X*, dense in* X*. Then* $C \cap L$ *is a dense subset of* C*.*

Set for any $\varepsilon > 0$:

$$T^{\beta}_{\mathcal{U}_{ad},\varepsilon} := \{v \in \mathcal{U}^{\beta};\ a^i \cdot v(t,\omega) = 0,\ \text{if } a^i \cdot \bar{u}(t,\omega) + \varepsilon \geq b_i, i = 1, \ldots, n_U,\ \text{a.e.}\}, \tag{4.7}$$

$$T^{\beta}_{\mathcal{U}_{ad},0} := \cup_{\varepsilon > 0} T^{\beta}_{\mathcal{U}_{ad},\varepsilon}. \tag{4.8}$$

and the related set of *totally strict* critical directions

$$C^{\beta}_0(\bar{u}) := T^{\beta}_{\mathcal{U}_{ad},0} \cap C^{\beta}(\bar{u}). \tag{4.9}$$

LEMMA 4.4. *The set* $C^{\beta}_0(\bar{u})$ *is a dense subset of* $C^2_S(\bar{u})$*.*

Proof. Obviously $C^{\beta}_0(\bar{u})$ is a subset of $C^2_S(\bar{u})$. In view of lemma 4.3, it suffices to prove that $T^{\beta}_{\mathcal{U}_{ad},0}$ is a dense subset of $T^{\beta}_{\mathcal{U}_{ad},S}$. Set for $\varepsilon > 0$

$$I(t,\omega) := \{1 \leq i \leq n_U;\ a^i(t,\omega) \cdot \bar{u}(t;\omega) < b_i(t,\omega)\}, \tag{4.10}$$

and

$$B(\varepsilon) := \{(t,\omega) \in [0,T] \times \Omega; \quad \max_{i \in I(t,\omega)} a^i(t,\omega)\bar{u}(t;\omega) - b_i(t,\omega) \geq -\varepsilon\}. \tag{4.11}$$

Given $v \in T^{\beta}_{\mathcal{U}_{ad},S}$ for $\varepsilon > 0$, define v^{ε} by

$$v^{\varepsilon}(t,\omega) := \begin{cases} \max(-1/\varepsilon, \min(1/\varepsilon, v(t,\omega))) & \text{if } (t,\omega) \notin B(\varepsilon), \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

Since $\cap_{\varepsilon} B(\varepsilon)$ is negligible, $v^{\varepsilon} \to v$ a.e., and since $|v^{\varepsilon}| \leq |v|$ a.e., the conclusion follows from the dominated convergence theorem. □

4.4. Proof of theorem 3.1. We say that a critical direction $v \in \mathcal{U}^\beta$ is *radial* if $\bar{u} + sv$ satisfies the control constraints for small enough $s > 0$. In that case the direction is also radial in the sense of [7]. We apply [7, Section 3.2] to the slack problem (4.3). The functions are not of class C^2, but as noticed in [8], we have the existence of a second order Taylor expansion, and that is sufficient. We obtain that (3.6) holds for any totally strict critical direction. Now let v be a strict critical direction. By lemma 4.4, there exists a sequence v^k of totally strict critical directions converging to v in $\mathcal{U}^2$; let $\lambda^k \in \Lambda'_L(\bar{u})$ be the associated sequence of multipliers such that (3.6) holds for the pair (v^k, λ^k). The set $\Lambda'_L(\bar{u})$ is bounded by lemma 4.2, and therefore, compact. So, extracting if necessay a subsequence, we may assume that λ^k converges to some $\lambda \in \Lambda'_L(\bar{u})$. Since $(\lambda, v) \mapsto \Omega[\lambda](v)$ is a continuous function, $\Omega[\lambda](v) = \lim_k \Omega[\lambda^k](v^k) \geq 0$ as was to be proved.

5. Proof of theorem 3.2.

5.1. Castaing representation of U_{ad}. Since the functions a and b do not depend on time we may write $U_{ad}(\omega)$ instead of $U_{ad}(t, \omega)$. A Castaing representation of U_{ad} in $\mathcal{U}^\beta$ is a sequence u^k in $\mathcal{U}^\beta$ such that $u^k(t,\omega) \in U_{ad}(\omega)$ a.e., and for fixed (t, ω), the sequence $u^k(t, \omega)$ is a dense subset of $U_{ad}(\omega)$ a.e. It is known that (see e.g. [24, Thm 1B, p. 161]) measurable multifunctions with closed values have a Castaing representation in L^p spaces. This result was extended to the case of adapted functions in [8, Prop. 6.1]. So, $U_{ad}(\omega)$ has a Castaing representation u^k in $\mathcal{U}^\infty$, where $k \in \mathbb{N}^*$. Here $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

5.2. Partial relaxation. We now introduce the *partially relaxed* (also called simply *relaxed* in the sequel) problem as follows. This is a family of problems indexed by $k \in \mathbb{N} \setminus \{0\}$. The control variables are the progressively measurable functions $u \in \mathcal{U}^\beta$ and $\alpha \in A_k := (L^\infty)^k$. The state equation is, omitting the argument ω for functions of (t, ω):

$$
\begin{aligned}
\mathrm{d}y(t) = \quad & \left(1 - \sum_{j=1}^{k} \alpha_j(t)\right) f(t, u(t), y(t))\mathrm{d}t + \sum_{j=1}^{k} \alpha_j(t) f(t, u^k(t), y(t))\mathrm{d}t \\
& +\sigma(t, y(t))\mathrm{d}W(t), \quad t \in (0, T); \quad y(0) = y_0,
\end{aligned} \tag{5.1}
$$

where the sequence $\{u^k\}$, $k \geq 1$, is a given Castaing representation of $U_{ad}(\omega)$ in u^k in $\mathcal{U}^\infty$, and the control variable is the pair $(u, \alpha) \in \mathcal{U} \times A_k$. This state equation has in $\mathcal{Y}^\beta$ a unique solution, denoted by $y[u, \alpha]$. For an *unrelaxed* control, i.e., when $\sum_{j=1}^{k} \alpha(t) \leq 1$ and $\alpha(t) \in \{0, 1\}^n$ a.e., we have that

$$
y[u, \alpha] = y\left[\left(1 - \sum_{j \leq k} \alpha_j\right) u + \sum_{j \leq k} \alpha_j u^j\right], \tag{5.2}
$$

where we remind that, by $y[\cdot]$ in the r.h.s., we denote the solution of the original state equation (2.4). The *nominal trajectory* in this setting is $u = \bar{u}$, $\bar{\alpha} = 0$, with associated state $y[\bar{u}, \bar{\alpha}] = \bar{y}$. The corresponding *linearized state equation* at the point $(\bar{u}, \bar{\alpha}, \bar{y})$, for $(v, \delta\alpha_j) \in \mathcal{U}^2 \times A_k$, has the form

$$
\begin{aligned}
\mathrm{d}z(t) = \quad & \bar{f}'(t)(v(t), z(t)) + \sum_{j \leq k} \delta\alpha_j(t)(f(t, u^k(t), \bar{y}(t)) - \bar{f}(t))\mathrm{d}t \\
& +\bar{\sigma}'(t)z(t)\mathrm{d}W(t), \quad t \in (0, T); \quad y(0) = y_0,
\end{aligned}
$$

We denote its unique solution in $\mathcal{Y}^2$ by $z[v, \delta\alpha]$, and introduce the family of *partially relaxed optimal control problems*, with slack variable $\theta \in \mathbb{R}$ (reminding that we may assume w.l.o.g. that the optimal value of the original problem is zero):

$$
(P_k) \qquad \left\{ \begin{array}{ll} \mathrm{Min}_{(u,\alpha,\theta)} \ \ \theta & I\!E\varphi(y[u,\alpha](T)) \leq \theta; \\ I\!E\Phi_E(y[u,\alpha](T)) = 0; & I\!E\Phi_I(y[u,\alpha](T)) \leq \theta \mathbf{1}; \\ u(t,\omega) \in U_{ad}(\omega) \text{ a.e.}, & \alpha(t,\omega) \geq -\theta \text{ a.e.} \end{array} \right.
$$

Set $\bar\theta := 0$. Clearly, $(\bar u, \bar\alpha, \bar\theta) \in F(P_k)$. We can identify $\alpha \in A_k$ with an element of $A_{k'}$, for $k' > k$, by setting $\alpha_j = 0$ when $j > k$. It follows that $F(P_k) \subset F(P_{k+1})$.

LEMMA 5.1. *Either the equality constraints of (P_k) are not qualified at the point $(\bar u, \bar\alpha, \bar\theta)$ for any $k \in \mathbb{N}$, or they are qualified for $k \geq k_0$.*

Proof. That these equality constraints are qualified for of (P_{k_0}) means that there exist $(v^i, \delta\alpha^i) \in \mathcal{U}^\infty \times A_{k_0}$, for $i = 1$ to n_E, such that the corresponding linearized states $z^i = z^i[v^i, \delta\alpha^i]$ are such that the family $\{I\!E\Phi'_E(\bar y(T))z^i\}$ has rank n_E. Since A_{k_0} is included in A_k for all $k > k_0$, we obtain the same rank property for any $k > k_0$, as was to be shown. □

In the next section we will prove the following:

PROPOSITION 5.2. *Let $\bar u$ be a local minimum in $L^1_{\mathbb{F}}$ of (P), the equality constraints of (P_k) be qualified, and (3.7) hold. Then $(\bar u, \bar\alpha, \bar\theta)$ is a weak minimum of (P_k).*

5.3. Set of Lagrange multipliers of the relaxed problem. First, by expressing the stationarity of the Lagrangian of the problem w.r.t. θ we obtain that each Lagrange multipliers of the relaxed problem satisfies (compare to (4.5)):

$$
\langle \nu, \mathbf{1} \rangle + \Psi_I \cdot \mathbf{1} + \sum_{i=1}^{n_U} \langle \mu_i, \mathbf{1} \rangle = 1. \tag{5.3}
$$

We denote the set of Lagrange multipliers of problem (P_k) at the nominal trajectory by Λ_k. By the above display, the components (ν, Ψ, μ) of elements of Λ_k are uniformly bounded.

The expression of the Hamiltonian of problem (P_k) is, dropping the argument ω:

$$
H^k(\nu, t, u, \alpha, y, p) := H(\nu, t, u, y, p) + p \cdot \sum_{j \leq k} \alpha_j (f(t, u^j, y) - f(t, u, y)). \tag{5.4}
$$

It is therefore easy to see that the costate equation at the nominal point $(\bar u, \bar\alpha, \bar\theta)$ coincides with the one for the original problem. So, the only difference in the expression of first-order optimality conditions w.r.t. the first formulation (4.3) with a slack variable is that we have the relation expressing the nonnegative variation of the Hamiltonian w.r.t. feasible (nonnegative) variations of $\bar\alpha$, and this means that for $j = 1$ to k:

$$
\bar p(t) \cdot f(t, \bar u(t), \bar y(t)) \leq \bar p(t) \cdot f(t, u^j(t), \bar y(t)) \text{ a.e.} \tag{5.5}
$$

that we can rewrite as, skipping the argument ω :

$$
H(t, \bar u(t), \bar y(t), \bar p(t), \bar q(t)) \leq H(t, u^j(t), \bar y(t), \bar p(t), \bar q(t)), \text{ a.e., for all } j \leq k. \tag{5.6}
$$

Observe that

$$
\Lambda_k = \{\lambda \in \Lambda'_L(\bar u);\ (5.6) \text{ holds}\}. \tag{5.7}
$$

By the definition of a Castaing representation, and in view of the continuity of the Hamiltonian $H(\cdot)$ w.r.t. the control variable, it follows that

$$\cap_k \Lambda_k = \{\lambda \in \Lambda_P(\bar{u});\ \ (4.5)\ \text{ holds}\}. \tag{5.8}$$

We have the following alternative. Either the equality constraints are qualified (at the nominal point) for k_0, and hence

$$\text{For } k > k_0,\ \Lambda_k \text{ is uniformly bounded,} \tag{5.9}$$

since Λ_k is included in Λ_{k_0} which is itself bounded. The other possibility is that the equality constraints are not qualified for any $k \in \mathbb{N}$.

Proof of theorem 3.2 in the case of never qualified equality constraints. This means that there exists some nonzero $\Psi_E \in \mathbb{R}^{n_E}$ in the orthogonal of the range space, i.e., such that

$$\Psi_E \cdot D\Phi_E(\bar{y}(T))z[v, \delta\alpha] = 0 \quad \text{for any } (v, \delta\alpha) \in \mathcal{U}^\infty \times A_k, \tag{5.10}$$

meaning that the associated costate say p satisfies the condition of stationarity of the Hamiltonian in (5.4) w.r.t. $(u, \alpha) \in \mathbb{R}^m \times \mathbb{R}^k$. The condition of stationarity w.r.t. α gives therefore

$$p \cdot (f(t, u^j, y) - f(t, \bar{u}, y)) = 0, \quad \text{for any } k \in \mathbb{N}. \tag{5.11}$$

Since u^k is a Castaing representation it follows that

$$p \cdot (f(t, u, y) - f(t, u, y)) = 0, \quad \text{for any } u \in U_{ad}. \tag{5.12}$$

In that degenerate case it appears that the conclusion of theorem 3.2 holds in a trivial way, with either Ψ_E or $-\Psi_E$.

Proof of theorem 3.2 in the case of of qualified equality constraints. Let $v \in \mathcal{U}^\infty$ be a strict critical direction of the original problem. For $k \in \mathbb{N}$, $k > 0$, set $\hat{v}^k := (v, 0, 0) \in \mathcal{U}^\infty \times A_k \times \mathbb{R}$. By proposition 5.2, $\hat{v}^k$ is a strict critical direction for the relaxed problem (P_k). So we can express optimality conditions based on theorem 3.1. We obtain the existence of a Lagrange multiplier (for problem (P_k)) say λ^k, such that

$$\Omega[\lambda^k](v) \geq 0. \tag{5.13}$$

By (5.9), λ^k is bounded. For some subsequence, Ψ^k and therefore also the associated costate (p^k, q^k) converge (the latter in $\mathcal{Y}^\beta \times (L^{\beta,2}_{\mathbb{F}})^{nd}$) and has therefore weak limit λ for a subsequence that by (5.8) belongs to $\Lambda_P(\bar{u})$ (note that we do not care about the convergence of the multiplier associated with the nonnegativity constraint on α). Since $\Omega[\lambda](v)$ is a continuous function of λ, it follows that $\Omega[\lambda](v) \geq 0$. The conclusion follows.

6. Proof of proposition 5.2. This is the most technical part for proving the main result. The approach is an adaptation of the technique of [5] to the stochastic setting. We start with a metric regularity result.

6.1. Metric regularity. Note that in the proposition below we do not use the same norms for the neighborhood of the control and the correction of the control.

PROPOSITION 6.1. *Let the equality constraints of* (P_k) *be qualified for* $k \geq k_0$. *Then there exists* $c_E > 0$ *such that, when* $k \geq k_0$, *for any relaxed trajectory* (u, α, y) *with* (u, y) *close enough in* $L^{2\beta,1} \times \mathcal{Y}^\beta$ *and* α *close enough to 0 in* A_k, *there exists a relaxed trajectory* (u', α', y') *such that* $\mathbb{E}\Phi^E(y(T)) = 0$ *and*

$$\|u' - u\|_\infty + \|\alpha' - \alpha\|_\infty + \|y' - y\|_\beta \leq c_E |\mathbb{E}\Phi^E(y(T))|, \tag{6.1}$$

where α' *satisfies* $\alpha'_j = \alpha_j$, *for all* $j > k_0$.

Proof. By [8], the mapping $(u, \alpha) \mapsto y$ is $C^1 : L^{2\beta,1} \to \mathcal{Y}^\beta$. Also, the function $y \to \mathbb{E}\Phi(y)$ is C^1: $L^{2\beta} \to \mathbb{R}$.

Since the equality constraints are qualified when $k = k_0$, there exists n_{Φ_E} elements (v^i, α^i) of $L^{2\beta,1} \times L^\infty$, with α having nonzero components only for $i \leq k_0$. By the Lyusternik theorem, (6.1) holds if we put the $L^{2\beta,1}$ instead of the L^∞ norm for the control. However in the proof of this theorem based on a Netwon type 'algorithm', we can choose the increments to belong to the vector space spanned by the (v^i, α^i), $1 \leq i \leq n_{\Phi_E}$, that will also therefore contain $u' - u$. But in this finite dimensional space, the norms are equivalent. The result follows. □

6.2. Derelaxation. Let $u \in \mathcal{U}^\beta$. It is convenient to set

$$u^0 := u; \quad \alpha_0(t, \omega) := 1 - \sum_{j=1}^{k} \alpha_j(t, \omega), \tag{6.2}$$

so that the state equation of the relaxed problem can be written as

$$\mathrm{d}y(t) = \sum_{j=0}^{k} \alpha_j(t) f(t, u^j(t), y(t))\mathrm{d}t + \sigma(t, y(t))\mathrm{d}W(t), \quad t \in (0, T); \quad y(0) = y_0. \tag{6.3}$$

THEOREM 6.2. *There exists* $c_0 > 0$ *such that, if* $k \geq k_0$, *and* (u, α) *is a feasible relaxed control for problem* (P_k), *with* $\theta < 0$, *sufficiently close to* $(\bar{u}, \bar{\alpha})$ *in the* L^∞ *norm, then, for any* $\varepsilon > 0$, *there exists a 'classical' control* $u^\varepsilon \in \mathcal{U}$ *with associated state* y^ε, *that satisfies*

$$\|u^\varepsilon - \bar{u}\|_{\beta,1} \leq c_0 \sum_{j=1}^{k} \|\alpha_j\|_{\beta,1} \|u^i - \bar{u}\|_\infty; \quad \|y^\varepsilon - \bar{y}\|_\beta \leq \varepsilon. \tag{6.4}$$

The proof has several steps involving the lemma below. Let (u, y) be a relaxed control for problem (P_k), with associated state y. Set

$$\eta(t) := 1 - \tfrac{1}{2} \sum_{j=1}^{N} \alpha_j(t), \quad \tilde{\alpha}_i(t) := \tfrac{1}{2}\alpha_j(t)/\eta(t), \ i = 1, \ldots, N; \tag{6.5}$$

and

$$\tilde{\alpha}_0(t) := \alpha_0(t)/\eta(t); \quad G(t, y) := \tfrac{1}{2} \sum_{j=1}^{N} \alpha_j(t) f(t, u^j, y). \tag{6.6}$$

Observe that

$$\sum_{j=0}^{N} \tilde{\alpha}_i(t) = 1. \tag{6.7}$$

We can write the state equation (5.1) in the form

$$\mathrm{d}y(t) = \eta(t) \sum_{j=0}^{N} \tilde{\alpha}_j(t) f(t, u^j, y) + G(t, y)\mathrm{d}t + \sigma(t, y)\mathrm{d}W(t). \tag{6.8}$$

We denote by $y[u, \tilde{\alpha}, \eta]$ the solution of this equation, to be compared to the 'derelaxed' one

$$\mathrm{d}y(t) = \eta(t) f(t, u, y) + G(t, y)\mathrm{d}t + \sigma(t, y)\mathrm{d}W(t). \tag{6.9}$$

whose solution is denoted by $y_G[u, \eta]$.

LEMMA 6.3. *There exists $c_0 > 0$ such that, given $k \geq k_0$, the following holds. If (u, α) is a feasible relaxed control for problem (P_k), sufficiently close to $(\bar{u}, \bar{\alpha})$ in the L^∞ norm, for any $\varepsilon > 0$, there exists $u^\varepsilon \in \mathcal{U}^\beta$ with associated 'state' $y^\varepsilon := y_G[u^\varepsilon, \eta]$ such that:*

$$\|u^\varepsilon - u\|_{\beta,1} \leq c_0 \sum_{j=1}^{k} \|\tilde{\alpha}_j\|_{\beta,1} \|u^i - u\|_\infty; \quad \|y^\varepsilon - \bar{y}\|_\beta \leq \varepsilon. \tag{6.10}$$

Proof. We borrow from Fleming [13] the idea of time averaging.
(i) 'Truncation' and 'quantization': set, for $j = 0$ to k:

$$\hat{u}^j(t, \omega) := \begin{cases} u^j(t, \omega) & \text{if } |u^j(t, \omega)| < 1/\varepsilon, \\ u^{00}(t, \omega) & \text{otherwise,} \end{cases} \tag{6.11}$$

where $u^{00} \in \mathcal{U}_{ad} \cap \mathcal{U}^\infty$ comes from hypothesis (2.11). By the dominated convergence theorem, given $\varepsilon' > 0$, we have that

$$\|u^j - \hat{u}^j\|_{\beta,1} \leq \varepsilon' \quad \text{if } \varepsilon \text{ is small enough.} \tag{6.12}$$

Let $\tilde{u}_i$ be a dense sequence in $B(0, 1/\varepsilon)$ (ball in $\mathbb{R}^m$). For some index i_ε we have that any element of $B(0, 1/\varepsilon)$ is at distance less than ε of $\{\tilde{u}^1, \dots, \tilde{u}^{i_\varepsilon}\}$. Set

$$\hat{w}^j(t, \omega) := \tilde{u}^i; \;\; i \text{ the smallest index such that } |\hat{u}^j(t, \omega) - \tilde{u}^i| < \varepsilon. \tag{6.13}$$

The $\hat{w}^j$ are adapted functions, and satisfy $\|\hat{w}^j - \hat{u}^j\|_\infty \leq \varepsilon$, so that

$$\|\hat{w}^j - \hat{u}^j\|_{\beta,1} \leq \varepsilon' \quad \text{if } \varepsilon > 0 \text{ is small enough.} \tag{6.14}$$

We can write,

$$\hat{w}^j := \sum_{i \leq i_\varepsilon} \mathbf{1}_{A_{ij}} \tilde{u}^i, \tag{6.15}$$

the measurable sets A_{ij} being such that $A_{ij} \cap A_{i'j}$ is negligible whenever $i \neq i'$. So we have an ε-uniform approximation (up to the truncation process) of the relaxed control by

$$\sum_{j=0}^{k} \alpha_j(t, \omega) \sum_{i \leq i_\varepsilon} \mathbf{1}_{A_{ij}} \delta_{\tilde{u}^i} = \sum_{i=0}^{k'} \alpha'_i(t, \omega) \delta_{\tilde{u}^i}, \tag{6.16}$$

where

$$\alpha'_i(t,\omega) := \sum_{j=0}^{k} \alpha_j(t,\omega)\mathbf{1}_{A_{ij}}. \tag{6.17}$$

Note that

$$\sum_{i=0}^{k} \alpha'_i(t,\omega) = \sum_{j=0}^{k} \alpha_j(t,\omega) = 1. \tag{6.18}$$

(ii) Time translation (in order to have a progressively measurable control):

$$\tilde{w}(t,\omega) := \begin{cases} u(t,\omega) & \text{if } t \in (0,\varepsilon), \\ \hat{w}(t-\varepsilon,\omega) & \text{for } t \in (t_{q,i}, t_{q,i}),\ q = 1,\dots,N-1. \end{cases} \tag{6.19}$$

(iii) Time averaging and reduction to a classical control: We may assume that $N := T/\varepsilon$ is an integer. For $q = 0$ to N, set $t_q := q\varepsilon$, and consider the *averaged coefficients* over each time step (t_{q-1}, t_q), for $q \geq 1$:

$$\bar{\alpha}_{iq}(\omega) := \int_{t_{q+1}}^{t_{q+2}} \alpha'_i(s,\omega)\mathrm{d}s. \tag{6.20}$$

Roughly speaking, this represents the amount of time in (t_{q+1}, t_{q+2}), spent by the relaxed control in the vicinity of $\tilde{u}^i$. Now define the stopping times

$$t_{q,0} = t_q;\ t_{q,i}(\omega) = t_{q,i-1}(\omega) + \varepsilon\bar{\alpha}_{iq}(\omega),\ i = 1,\dots,i_\varepsilon, \tag{6.21}$$

so that $t_{q,i_\varepsilon} = t_{q+1}$, and the corresponding *classical control* by

$$u_c(t,\omega) := \begin{cases} u(t,\omega) & \text{if } t \in (0,\varepsilon), \\ \tilde{u}^i & \text{for } t \in (t_{q,i-1}, t_{q,i}),\ q = 1,\dots,N-1. \end{cases} \tag{6.22}$$

Equivalently, set

$$\alpha''_i(t,\omega) = \begin{cases} 1 & \text{if } t \in (t_{q,i-1}, t_{q,i}),\ q = 1,\dots,N-1,\ i = 1,\dots,i_\varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{6.23}$$

Then we can write

$$u_c(t,\omega) = \sum_{i=1}^{i_\varepsilon} \alpha''_i \tilde{u}^i. \tag{6.24}$$

Let $y := y[u,\alpha]$, $y_c := y[u_c]$. By the mean value theorem, there exists a progressively measurable, bounded function $\hat{\sigma}$ such that $z := y - y_c$ satisfies $z(0) = 0$ and

$$\mathrm{d}z = \sum_{q=1}^{4} B_q \mathrm{d}t + \hat{\sigma} z \mathrm{d}W(t), \tag{6.25}$$

with

$$
(6.26)\quad \begin{cases} B_1 & := & \eta(t)\displaystyle\sum_{j=0}^{k}\alpha_j\left(f(t,u^j,y)-f(t,u^j,y_c)\right) \\ & & +G(t,y)-G(t,y_c) \qquad \text{(state variation)} \\ B_2 & := & \eta(t)\displaystyle\sum_{j=0}^{k}\alpha_j\left(f(t,u^j,y_c)-f(t,\hat{w}^j,y_c)\right) \quad \text{(quantization)} \\ B_3 & := & \eta(t)\displaystyle\sum_{j=0}^{k}\alpha_j\left(f(t,\hat{w}^j,y_c)-f(t,\tilde{w}^j,y_c)\right) \quad \text{(time translation)} \\ B_4 & := & \eta(t)\displaystyle\sum_{j=0}^{k}\alpha_j\left(f(t,\tilde{w}^j,y_c)-f(t,u_c,y_c)\right) \quad \text{(derelaxation)} \end{cases}
$$

By the mean value theorem and the previous relations:

$$
(6.27)\qquad |B_1(t,\omega)|\le c_1|z(t,\omega)|;\quad \|B_2(t,\omega)\|_{\beta,1}\to 0,\ \text{ when } \varepsilon\downarrow 0.
$$

From an easy variant of [21, Prop. 2.1], we deduce that

$$
(6.28)\qquad \begin{aligned} & I\!\!E\left(\max_{t\le t'\le t+\varepsilon}|y_c(t')-y_c(t)|\right)^\beta \\ & \qquad \le c\left[\left(\int_t^{t+\varepsilon}|f(s,u,0)|\right)^\beta \mathrm{d}s+\left(\int_t^{t+\varepsilon}|\sigma(s,0)|^2\right)^{\beta/2}\mathrm{d}s\right]. \end{aligned}
$$

Combining with (3.7), we deduce that

$$
(6.29)\qquad \|B_3\|_{\beta,1}\to 0 \quad \text{when } \varepsilon\downarrow 0.
$$

We next estimate B_4. Set $\Delta_q := [t_q,t_{q+1}]$. We have that, for each ω:

$$
(6.30)\qquad \int_{\Delta_q}\eta(t)\sum_{j=0}^{k}\alpha_j f(t_q,\hat{u}^j,y_c(t_q))\mathrm{d}t=\int_{\Delta_q}\eta(t)\sum_{i=0}^{i_\varepsilon}\bar{\alpha}_i f(\tilde{u}^i,y_c(t_q))\mathrm{d}t.
$$

Since f is Lipschitz and uniformy continuous w.r.t. time, say

$$
(6.31)\qquad |f(t'',\cdot)-f(t',\cdot)|\le\nu(|t''-t'|)
$$

with ν non decreasing $\mathbb{R}_+\to\mathbb{R}_+$ with limit 0 at 0, it follows that

$$
(6.32)\qquad \left|\int_{\Delta_q}B_4\mathrm{d}t\right|\le\varepsilon\nu(\varepsilon)+c\int_{\Delta_q}|y_c(t)-y_c(t_q)|\mathrm{d}t
$$

and therefore

$$
(6.33)\qquad \|B_4\|_\beta^\beta\le c(\varepsilon\nu(\varepsilon))^\beta+cI\!\!E\left(\int_{\Delta_q}|y_c(t)-y_c(t_q)|\mathrm{d}t\right)^\beta.
$$

From (6.28) we deduce that

$$
(6.34)\qquad I\!\!E(\max_{t\in\Delta_q}|y_c(t)-y_c(t_q)|)^\beta\le c\varepsilon^{\beta/2}.
$$

So, again by the estimates in [21], we deduce that

$$
(6.35)\qquad \|z\|_\beta^\beta\le c\left(\|B_2\|_{\beta,1}^\beta+\|B_3\|_{\beta,1}^\beta\right)
$$

can indeed be made as small as desired. □

6.2.1. Core of the proof of proposition 5.2 . (i) We follow the ideas detailed in the preprint version of [5, Appendix A.2]. Let $R := \mathrm{diam}_{L^\infty}(\hat{u}, u^1, \dots, u^N)$. We will next prove under the hypotheses of the proposition the existence of $c_0 > 0$ and of a sequence $w^i := (\hat{u}^i, \hat{y}^i, \hat{\alpha}_i, \hat{\theta}_i)$, satisfying all constraints of the 'slack' problem (P_k) other than the equalities, such that $w^0 = (\hat{u}, \hat{y}, \hat{\alpha}, \hat{\theta})$ and for $k \in \mathbb{N}$:

$$\text{(6.36)} \qquad \text{(i)} \ \|\hat{u}^{i+1} - \hat{u}^i\|_{1,\beta} \le \frac{c_0}{4}\left(\frac{3}{4}\right)^{i+1} \|\hat{\alpha}\|_\infty \quad \text{(ii)} \ \|\hat{y}^{i+1} - \hat{y}^i\|_\beta \le \left(\frac{3}{4}\right)^{i+1} \frac{\varepsilon}{4};$$

$$\text{(6.37)} \qquad \text{(i)} \ \|\alpha^{i+1}\|_\beta \le \left(\frac{3}{4}\right)^{i+1} \|\hat{\alpha}\|_\infty; \quad \text{(ii)} \ \hat{\theta}_{i+1} = \frac{1}{4}\hat{\theta}_i.$$

Clearly the sequence is converging to $\tilde{w} = (\tilde{u}, \tilde{y}, \tilde{\alpha}, \tilde{\theta})$ with $\tilde{\alpha} = 0$, $\tilde{\theta} = 0$, and

$$\text{(6.38)} \qquad \|\tilde{u} - \hat{u}\|_{1,\beta} \le 3c; \quad \|\tilde{y} - \hat{y}\|_\beta \le \frac{3}{4} c\varepsilon.$$

(ii) We next prove the existence of the sequence by induction; assume that the existence holds up to index i. By theorem 6.3, for any $\varepsilon' > 0$, there exists $(u, y) \in \mathcal{U}^\beta \times \mathcal{Y}^\beta$ such that $u_t \in \{\hat{u}^k_t, u^1_t, \dots, u^N_t)$ a.e., $(u_t, \frac{1}{2}\hat{\alpha}^k, y)$ is a relaxed trajectory, and

$$\text{(6.39)} \qquad \text{(i)} \ \|\hat{u}^{i+1} - \hat{u}^i\|_{1,\beta} \le c_0 \sum_{j=1}^{i} \|\tilde{\alpha}_j\|_{\beta,1}; \quad \text{(ii)} \ \|\hat{y}^{i+1} - \hat{y}^i\|_\beta \le \varepsilon'$$

(iii) Applying proposition 6.1, taking $\varepsilon > 0$ small enough, we deduce the existence of w^{i+1}.

Appendix A. SDE estimates.

A.1. Linear SDEs. The next lemma follows from [21, Prop. 2.1].

Lemma A.1. *Consider the linear SDE*

$$\text{(A.1)} \qquad \begin{aligned} &\mathrm{d}z(t) = (A'z + C')\mathrm{d}t + (A"z + C")\mathrm{d}W(t); \\ &t \in (0, T); \quad z(0) = z_0. \end{aligned}$$

with $z_0 \in \mathbb{R}^n$, A', B', B'' *in* $L^\infty_{\mathbb{F}}$, $C' \in L^{\beta,1}$, *and* $C' \in L^{\beta,2}$, *where* $\beta \in [1, \infty)$. *Then* (A.1) *has in a unique solution in* $L^{\beta,\infty}$ *such that*

$$\text{(A.2)} \qquad \|z\|_{\beta,\infty} = O\left(|z_0| + \|C'\|_{\beta,1} + \|C''\|_{\beta,2}\right).$$

A.2. Perturbed SDEs. Given u^1, u^2 in $\mathcal{U}^{\beta,1}$, and associated states y^1, y^2, setting $v := u^2 - u^1$ and $\delta y := y^2 - y^1$ for some bounded A, B, C:

$$\text{(A.3)} \qquad \mathrm{d}\delta y(t) = (Av + B\delta y)\mathrm{d}t + C\delta y \mathrm{d}W(t);$$

so that, by lemma A.1:

$$\text{(A.4)} \qquad \|\delta y\|_{\beta,\infty} = O(\|v\|_{\beta,1}).$$

Differentiability of $y[u]$. Set $\bar{y} = y[\bar{u}]$, $\bar{f}[t] = f(\bar{u}, \bar{y})$, etc, $\delta y := y[\bar{u} + \tau v] - y[\bar{u}]$. Then

$$\mathrm{d}\delta y(t) = \left(\bar{f}'[t](v, \delta y) + f^R[t]\right) \mathrm{d}t + \left(\bar{\sigma}_y[t]\delta y + \sigma^R[t]\right) \mathrm{d}W(t)$$

where for $\Psi = f, \sigma$; $\Psi^R(\omega, t) =$

$$\int_0^1 \left(\Psi'(\bar{u}(t) + \theta v(t), \bar{y}(t) + \theta \delta y(t)) - \Psi'(\bar{u}(t), \bar{y}(t))\right)(v(t), \delta y(t))\mathrm{d}\theta$$

so that

$$\begin{cases} |f^R(\omega, t)| = O(|v(\omega, t)|^2 + |\delta y(\omega, t)|^2) \\ |\sigma^R(\omega, t)| = O(|\delta y(\omega, t)|^2) \end{cases}$$

We denote by $z = z[v]$ the solution of the linearized state equation (2.15).

Linerization error $\rho = \delta y - z$, solution of $\rho(0) = 0$ and

$$\mathrm{d}\rho(t) = \left(\bar{f}_y[t]\rho\mathrm{d}t + f^R[t]\right) \mathrm{d}t + \left(\bar{\sigma}_y[t]\rho + \sigma^R[t]\right) \mathrm{d}W(t)$$

and so

$$\|\rho\|_{\beta,\infty} = O(\|f^R\|_{\beta,1} + \|\sigma^R\|_{\beta,2})$$

Linearization error again.

$$\|v^2\|_{\beta,1}^{\beta} = \mathbb{E}\left(\int_0^T v^2 \mathrm{d}t\right)^{2\beta/2} = \|v\|_{2\beta,2}^{2\beta}$$

and

$$\|\delta y^2\|_{\beta,2}^{\beta} = \mathbb{E}\left(\int_0^T (\delta y)^4 \mathrm{d}t\right)^{2\beta/4} = \|\delta y\|_{2\beta,4}^{2\beta} = O(\|v\|_{2\beta,1}^{2\beta})$$

and finally

$$\|\rho\|_{\beta,\infty} = O(\|v\|_{2\beta,2}^2)$$

Fréchet differentiability. Remember that z solution of the linearized equation is such that

$$y[\bar{u} + v] = y[\bar{u}] + z + \rho.$$

and

$$\|\rho\|_{\beta,\infty} = O(\|v\|_{2\beta,2}^2); \quad \|z\|_{\beta,\infty} = O(\|v\|_{\beta,1}).$$

THEOREM A.2. *The mapping* $v \mapsto y[\bar{u} + v]$ *is F-differentiable* $\mathcal{U}^{2\beta,2} \to \mathcal{Y}^{\beta}$, *for each* $\beta \in [1, \infty)$, *and in particular* $\mathcal{U}^{2,2} \to \mathcal{Y}^1$.

REMARK A.3. *If* $u \in \mathcal{U}^{2\beta,2}$, *then* $y[u] \in \mathcal{Y}^{2\beta}$.
But the differentiability holds only in $\mathcal{Y}^{\beta}$.

REFERENCES

[1] H. Becker and V. Mandrekar. On the existence of optimal random controls. *J. Math. Mech.*, 18:1151–1166, 1968/1969.

[2] A. Bensoussan. Stochastic maximum principle for distributed parameter systems. *J. Franklin Inst.*, 315(5-6):387–406, 1983.

[3] J.-M. Bismut. Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.*, 44:384–404, 1973.

[4] J.-M. Bismut. An introductory approach to duality in optimal stochastic control. *SIAM Rev.*, 20(1):62–78, 1978.

[5] J. F. Bonnans, X. Dupuis, and L. Pfeiffer. Second-Order Necessary Conditions in Pontryagin Form for Optimal Control Problems. *SIAM J. Control Optim.*, 52(6):3887–3916, 2014. Preprint: https://hal.inria.fr/hal-00825273.

[6] J.F. Bonnans, X. Dupuis, and L. Pfeiffer. Second-order sufficient conditions for strong solutions to optimal control problems. *ESAIM-COCV*, 20(3):704–724, 2013.

[7] J.F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems*. Springer-Verlag, New York, 2000.

[8] J.F. Bonnans and F.J. Silva. First and second order necessary conditions for stochastic optimal control problems. *Appl. Math. Optim.*, 65(3):403–439, 2012.

[9] A. Cadenillas and I. Karatzas. The stochastic maximum principle for linear convex optimal control with random coefficients. *SIAM J. Control Optim.*, 33(2):590–624, 1995.

[10] A.V. Dmitruk. Jacobi type conditions for singular extremals. *Control & Cybernetics*, 37(2):285–306, 2008.

[11] N. El Karoui, D. Hủù Nguyen, and M. Jeanblanc-Picqué. Compactification methods in the control of degenerate diffusions: existence of an optimal control. *Stochastics*, 20(3):169–219, 1987.

[12] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1–71, 1997.

[13] W. H. Fleming. Generalized solutions in optimal stochastic control. In *Differential games and control theory, II (Proc. 2nd Conf., Univ. Rhode Island, Kingston, R.I., 1976)*, pages 147–165. Lecture Notes in Pure and Appl. Math., 30. Dekker, New York, 1977.

[14] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition, 2006.

[15] U. G. Haussmann. General necessary conditions for optimal control of stochastic systems. *Math. Programming Stud.*, 6:30–48, 1976. Stochastic systems: modeling, identification and optimization, II (Proc. Sympos., Univ. Kentucky, Lexington, Ky., 1975).

[16] U. G. Haussmann. Some examples of optimal stochastic controls or: the stochastic maximum principle at work. *SIAM Rev.*, 23(3):292–307, 1981.

[17] H. J. Kushner. Necessary conditions for continuous parameter stochastic optimization problems. *SIAM J. Control*, 10:550–565, 1972.

[18] H. J. Kushner and F. C. Schweppe. A maximum principle for stochastic control systems. *J. Math. Anal. Appl.*, 8:287–302, 1964.

[19] P.-L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. I. The dynamic programming principle and applications. *Comm. Partial Differential Equations*, 8(10):1101–1174, 1983.

[20] A.A. Milyutin and N. N. Osmolovskii. *Calculus of Variations and Optimal Control*. American Mathematical Society, Providence, 1998.

[21] L. Mou and J. Yong. A variational formula for stochastic controls and some applications. *Pure Appl. Math. Q.*, 3(2, Special Issue: In honor of Leon Simon. Part 1):539–567, 2007.

[22] S. G. Peng. A general stochastic maximum principle for optimal control problems. *SIAM J. Control Optim.*, 28(4):966–979, 1990.

[23] S.M. Robinson. First order conditions for general nonlinear optimization. *SIAM Journal on Applied Mathematics*, 30:597–607, 1976.

[24] R.T. Rockafellar. Integral functionals, normal integrands and measurable selections. In *Nonlinear operators and the calculus of variations (Summer School, Univ. Libre Bruxelles, Brussels, 1975)*, pages 157–207. Lecture Notes in Math., Vol. 543. Springer, Berlin, 1976.

[25] J. Yong and X.Y. Zhou. *Stochastic controls, Hamiltonian systems and HJB equations*. Springer-Verlag, New York, Berlin, 1999.