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DETERMINING THE POTENTIAL IN A WAVE EQUATION WITHOUT A GEOMETRIC CONDITION. EXTENSION TO THE HEAT EQUATION

KAÏS AMMARI, MOURAD CHOULLI, AND FAOUZI TRIKI†

ABSTRACT. We prove a logarithmic stability estimate for the inverse problem of determining the potential in a wave equation from boundary measurements obtained by varying the first component of the initial condition. The novelty of the present work is that no geometric condition is imposed to the sub-boundary where the measurements are made. Our results improve those obtained by the first and second authors in [2]. We also show how the analysis for the wave equation can be adapted to an inverse coefficient problem for the heat equation.

1. INTRODUCTION

Let Ω be a C^3-smooth bounded domain of $\mathbb{R}^n$, $n \geq 2$, with boundary Γ and consider the following initial-boundary value problem, abbreviated to IBVP in the sequel, for the wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = 0 & \text{in } Q = \Omega \times (0,\tau), \\ u = 0 & \text{on } \Sigma = \Gamma \times (0,\tau), \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = u_1. \end{cases} \tag{1.1}$$

From here on

$$E_0 = H_0^1(\Omega) \oplus L^2(\Omega).$$

The unit ball of a Banach space X will denoted in the sequel by B_X.

By [4, Theorem A.3, page 493], for any $(u_0,u_1) \in E_0$ and $q \in L^\infty(\Omega)$, the IBVP (1.1) has a unique solution

$$u := \mathscr{S}_q^\tau(u_0,u_1) \in C([0,\tau]; H_0^1(\Omega))$$

so that

$$\partial_t u \in C([0,\tau]; L^2(\Omega)) \text{ and } \partial_\nu u \in L^2(\Sigma).$$

Additionally, for any $m > 0$, there exists a constant $C = C(m,\Omega) > 0$ so that, for each $q \in mB_{L^\infty(\Omega)}$ and $(u_0,u_1) \in E_0$,

$$\|\partial_\nu \mathscr{S}_q^\tau(u_0,u_1)\|_{L^2(\Sigma)} \leq C\|(u_0,u_1)\|_{E_0}.$$

Fix Υ a non empty open subset of Γ and set $\Lambda = \Upsilon \times (0,\tau)$. The inequality above says that the operator

$$\mathscr{C}_q^\tau : (u_0,u_1) \in E_0 \mapsto \partial_\nu \mathscr{S}_q^\tau(u_0,u_1)_{|\Lambda} \in L^2(\Lambda)$$

is bounded and

$$\|\mathscr{C}_q^\tau\|_{\mathscr{B}(E_0,L^2(\Lambda))} \leq C, \tag{1.2}$$

uniformly in $q \in mB_{L^\infty(\Omega)}$.

Define the operator $\widetilde{\mathscr{C}}_q^\tau$ by $\widetilde{\mathscr{C}}_q^\tau(u_0) = \mathscr{C}_q^\tau(u_0,0)$, $u_0 \in H_0^1(\Omega)$. Clearly $\widetilde{\mathscr{C}}_q^\tau \in \mathscr{B}(H_0^1(\Omega), L^2(\Lambda))$ and

$$\|\widetilde{\mathscr{C}}_q^\tau\|_{\mathscr{B}(H_0^1(\Omega),L^2(\Lambda))} \leq C, \tag{1.3}$$

again uniformly in $q \in mB_{L^\infty(\Omega)}$.

2010 *Mathematics Subject Classification.* 35R30.
Key words and phrases. inverse problem, wave equation, potential, boundary measurements.
†FT is partially supported by Labex PERSYVAL-Lab (ANR-11-LABX-0025-01).

Recall that the space $H_\Delta(\Omega)$ is given by

$$H_\Delta(\Omega)=\{\varphi\in L^2(\Omega);\ \Delta\varphi\in L^2(\Omega)\}.$$

With reference to Poincaré's inequality, $\mathcal{H}=H_0^1(\Omega)\cap H_\Delta(\Omega)$ is a Banach space for the norm

$$\|\varphi\|_{\mathcal{H}}=\|\nabla\varphi\|_{L^2(\Omega)^n}+\|\Delta\varphi\|_{L^2(\Omega)}.$$

When $u_0\in\mathcal{H}$ we easily see that $\partial_t\mathscr{S}_q^\tau(u_0,0)=\mathscr{S}_q^\tau(0,\Delta u_0-qu_0)$. Then proceeding similarly as before we conclude that $\widetilde{\mathscr{C}}_q^\tau$ restricted to $\mathcal{H}$, still denoted by $\widetilde{\mathscr{C}}_q^\tau$, define a bounded operator from $\mathcal{H}$ into $H^1((0,\tau);L^2(\Upsilon))$ and

$$\|\widetilde{\mathscr{C}}_q^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau);L^2(\Upsilon))}\le C, \tag{1.4}$$

uniformly in $q\in mB_{L^\infty(\Omega)}$.

Let

$$\Psi(\gamma)=|\ln\gamma|^{-\frac{1}{8+2n}}+\gamma,\ \ \gamma>0,$$

extended by continuity at $\gamma=0$ by setting $\Psi(0)=0$.

Let $m>0$ be fixed. In the rest of this text, unless otherwise stated, C, c, τ_0 and μ denote generic constants that can depend only on n, Ω, Υ and m.

We aim to prove the following theorem.

Theorem 1.1. *There exist two constants $\tau_0>0$ and $C>0$ so that, for any $\tau\ge\tau_0$, $q_0,q\in mB_{L^\infty(\Omega)}$ satisfying $q_0\ge 0$ and $q-q_0\in mB_{W^{1,\infty}(\Omega)}$,*

$$C\|q_0-q\|_{L^2(\Omega)}\le\Psi\left(\|\widetilde{\mathscr{C}}_q^\tau-\widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau);L^2(\Upsilon))}\right).$$

The proof of Theorem 1.1 we present here follows the method initiated by the first and second authors in [2]. This method is mainly based on a spectral decomposition combined with an observability inequality. Due to the fact that we do not assume any geometric condition on the domain, the classical observability inequality is no longer valid in our case. We substitute it by an interpolation inequality established by Robbiano in [9]. It is worthwhile to mention that a spectral decomposition combined with an observability inequality was also used in [3] to establish a logarithmic stability estimate for the problem of determining a boundary coefficient in a wave equation from boundary measurements.

To our knowledge, using observability inequalities to solve inverse problems related to the wave equation goes back to Puel and Yamamoto [8]. Later, Komornik and Yamamoto [7] applied this method to an inverse point source problem for a wave equation. A general framework of this method is due to Alves, Silvestre, Takahashi and Tucsnak [1] and extended recently to singular sources by Tucsnak and Weiss [10].

The rest of this paper consists in two sections. Section 2 is devoted to the proof of Theorem 1.1. In Section 3, we adapt our approach to an inverse problem for the heat equation.

2. Proof of Theorem 1.1

We firstly observe that a careful examination of the proof of [9, Theorem 1, page 98] allows us to deduce the following result.

Theorem 2.1. *There exist three constants $\tau_0>0$, $C>0$ and $\mu>0$ so that, for all $\tau\ge\tau_0$, $(u_0,u_1)\in E_0$, $q\in mB_{L^\infty(\Omega)}$ and $\epsilon>0$,*

$$C\|(u_0,u_1)\|_{E_{-1}}\le\frac{1}{\sqrt{\epsilon}}\|(u_0,u_1)\|_{E_0}+e^{\mu\epsilon}\|\mathscr{C}_q^\tau(u_0,u_1)\|_{L^2(\Lambda)},$$

where $E_{-1}=L^2(\Omega)\oplus H^{-1}(\Omega)$.

From now on, $\tau \geq \tau_0$ is fixed, where τ_0 is as in the preceding theorem.

Let $g \in H^1((0,\tau))$ satisfying $g(0) \neq 0$ and consider the IBVP

$$\begin{cases} \partial_t^2 v - \Delta v + q(x)v = g(t)f(x) & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(\cdot,0) = 0,\ \partial_t v(\cdot,0) = 0, \end{cases} \tag{2.1}$$

From [4, Theorem A.3, page 493], the IBVP (2.1) has a unique solution

$$v := \mathcal{S}_q^\tau(f,g) \in C([0,\tau]; H_0^1(\Omega))$$

so that

$$\partial_t v \in C([0,\tau]; L^2(\Omega)) \ \text{ and } \ \partial_\nu v \in L^2(\Sigma).$$

Applying Duhamel's formula we get in a straightforward manner

$$\mathcal{S}_q^\tau(f,g)(\cdot,t) = \int_0^t g(t-s)\mathscr{S}_q^\tau(0,f)(\cdot,s)ds.$$

Therefore

$$\mathcal{C}_q^\tau(f,g)(\cdot,t) := \partial_\nu \mathcal{S}_q^\tau(f,g)(\cdot,t) = \int_0^t g(t-s)\mathscr{C}_q^\tau(0,f)(\cdot,s)ds.$$

Let

$$H_\ell^1((0,\tau), L^2(\Upsilon)) = \left\{u \in H^1((0,\tau), L^2(\Upsilon));\ u(0) = 0\right\}$$

and define the operator $S : L^2(\Lambda) \longrightarrow H_\ell^1((0,\tau), L^2(\Upsilon))$ by

$$(Sh)(t) = \int_0^t g(t-s)h(s)ds.$$

From [2, Theorem 2.1] and its proof, S is an isomorphism and

$$\|h\|_{L^2(\Lambda)} \leq \frac{\sqrt{2}}{|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}} \|Sh\|_{H^1((0,\tau),L^2(\Upsilon))}.$$

Consequently

$$\|\mathscr{C}_q^\tau(0,f)\|_{L^2(\Lambda)} \leq \frac{\sqrt{2}}{|\lambda(0)|} e^{\tau \frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}} \|\mathcal{C}_q^\tau(f,g)\|_{H^1((0,\tau),L^2(\Upsilon))}.$$

In combination with the estimate in Theorem 2.1, this inequality yields, where $\epsilon > 0$ is arbitrary,

$$C\|f\|_{H^{-1}(\Omega)} \leq \frac{1}{\sqrt{\epsilon}}\|f\|_{L^2(\Omega)} + \frac{\sqrt{2}}{|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}} e^{\mu\epsilon} \|\mathcal{C}_q^\tau(f,g)\|_{H^1((0,\tau),L^2(\Upsilon))}. \tag{2.2}$$

Let $q_0, q \in mB_{L^\infty(\Omega)}$ satisfying $q_0 \geq 0$ and $q - q_0 \in mB_{W^{1,\infty}(\Omega)}$.

Consider the unbounded operator $A_0 : L^2(\Omega) \to L^2(\Omega)$ given by

$$A_0 = -\Delta + q_0, \quad D(A_0) = H_0^1(\Omega) \cap H^2(\Omega).$$

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \ldots \to +\infty$ be the sequence of eigenvalues of the operator A_0 and (ϕ_k) a sequence of the corresponding eigenfunctions so that (ϕ_k) form an orthonormal basis of $L^2(\Omega)$.

It is a simple exercise to check that $\mathscr{S}_{q_0}^\tau(\phi_k, 0) = g_k(t)\phi_k$ with $g_k(t) = \cos(\sqrt{\lambda_k}t)$.

Observing that

$$\mathscr{S}_q^\tau(\phi_k,0) - \mathscr{S}_{q_0}(\phi_k,0) = \mathcal{S}_q^\tau((q-q_0)\phi_k, g_k),$$

we get

$$\widetilde{\mathscr{C}}_q^\tau(\phi_k) - \widetilde{\mathscr{C}}_{q_0}^\tau(\phi_k) = \mathcal{C}_q^\tau((q-q_0)\phi_k, g_k).$$

Hence (2.2) gives

$$C\|(q-q_0)\phi_k\|_{H^{-1}(\Omega)} \leq \frac{1}{\sqrt{\epsilon}}\|(q-q_0)\phi_k\|_{L^2(\Omega)} + e^{\tau^2\lambda_k} e^{\mu\epsilon} \|\widetilde{\mathscr{C}}_q^\tau(\phi_k) - \widetilde{\mathscr{C}}_{q_0}^\tau(\phi_k)\|_{H_\ell^1((0,\tau),L^2(\Upsilon))}.$$

This and the fact that $\|\phi_k\|_{\mathcal{H}} \le \sqrt{\lambda_k} + m + \lambda_k$ imply

$$C\|(q-q_0)\phi_k\|_{H^{-1}(\Omega)} \le \frac{1}{\sqrt{\epsilon}}\|(q-q_0)\phi_k\|_{L^2(\Omega)} + (\sqrt{\lambda_k}+m+\lambda_k)e^{\tau^2\lambda_k}e^{\mu\epsilon}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}.$$

But $(\sqrt{\lambda_k}+m+\lambda_k) \le (\mu_1^{-1/2}+m\mu^{-1}+1)\lambda_k \le e^{(\mu_1^{-1/2}+m\mu^{-1}+1)\lambda_k}$, where μ_1 is the first eigenvalue of the Laplace operator under Dirichlet boundary condition. Whence

$$C\|(q-q_0)\phi_k\|_{H^{-1}(\Omega)} \le \frac{1}{\sqrt{\epsilon}}\|(q-q_0)\phi_k\|_{L^2(\Omega)} + e^{\kappa\lambda_k}e^{\mu\epsilon}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}.$$

Here $\kappa = \tau^2 + \mu_1^{-1/2} + m\mu^{-1} + 1$.

Since $\|(q-q_0)\phi_k\|_{L^2(\Omega)} \le m$, we have

$$C\|(q-q_0)\phi_k\|_{H^{-1}(\Omega)} \le \frac{1}{\sqrt{\epsilon}} + e^{\kappa\lambda_k}e^{\mu\epsilon}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}.$$

Using the usual interpolation inequality

$$\|h\|_{L^2(\Omega)} \le c\|h\|_{H_0^1(\Omega)}^{1/2}\|h\|_{H^{-1}(\Omega)}^{1/2} \quad h \in H_0^1(\Omega),$$

we obtain

$$C\|(q-q_0)\phi_k\|_{L^2(\Omega)}^2 \le \|(q-q_0)\phi_k\|_{H_0^1(\Omega)}\left(\frac{1}{\sqrt{\epsilon}} + e^{\kappa\lambda_k}e^{\mu\epsilon}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}\right).$$

Bearing in mind that $\|q-q_0\|_{W^{1,\infty}(\Omega)} \le m$, we get

$$\begin{aligned}\|(q-q_0)\phi_k\|_{H_0^1(\Omega)} &\le \|(q-q_0)\nabla\phi_k\|_{L^2(\Omega)^n} + \|\phi_k\nabla(q-q_0)\|_{L^2(\Omega)^n}\\ &\le m\sqrt{\lambda_k} + m\\ &\le m(1+\mu_1^{-1/2})\sqrt{\lambda_k}\\ &\le m\mu_1^{-1/2}(1+\mu_1^{-1/2})\lambda_k.\end{aligned}$$

Consequently

$$C\|(q-q_0)\phi_k\|_{L^2(\Omega)} \le \frac{\lambda_k}{\sqrt{\epsilon}} + e^{(\kappa+1)\lambda_k}e^{\mu\epsilon}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}.$$

Denote the scalar product of $L^2(\Omega)$ by $(\cdot,\cdot)_{L^2(\Omega)}$. By Cauchy-Schwarz's inequality

$$\left|(q-q_0,\phi_k)_{L^2(\Omega)}\right| \le |\Omega|^{1/2}\|(q-q_0)\phi_k\|_{L^2(\Omega)}.$$

Therefore

$$C(q-q_0,\phi_k)_{L^2(\Omega)}^2 \le \frac{\lambda_k}{\sqrt{\epsilon}} + e^{(\kappa+1)\lambda_k}e^{\mu\epsilon}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}. \tag{2.3}$$

According to the min-max principle, there exists $\widetilde{c} > 1$ (depending on m but not on q) so that

$$\widetilde{c}^{-1}k^{2/n} \le \lambda_k \le \widetilde{c}k^{2/n}. \tag{2.4}$$

We refer to [6] for a proof.

Estimates (2.3) and (2.4) entail

$$C(q-q_0,\phi_k)_{L^2(\Omega)}^2 \le \frac{k^{2/n}}{\sqrt{\epsilon}} + e^{\varrho k^{2/n}}e^{\mu\epsilon}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}, \tag{2.5}$$

with $\varrho = \widetilde{c}(\kappa+1)$.

Let $N \ge 1$ be an integer. Using that $\left(\sum_{k\ge1}(1+\lambda_k)(\cdot,\phi_k)_{L^2(\Omega)}^2\right)^{1/2}$ is an equivalent norm on $H^1(\Omega)$,

$$\|q-q_0\|_{H^1(\Omega)}^2 \le |\Omega|\left(\|q-q_0\|_{L^\infty(\Omega)}^2 + \|\nabla(q-q_0)\|_{L^\infty(\Omega)^n}^2\right) \le c_\Omega\|q-q_0\|_{W^{1,\infty}(\Omega)}^2$$

and (2.4), we get

$$\begin{aligned}\|q-q_0\|_{L^2(\Omega)}^2 &= \sum_{k\le N}(q-q_0,\phi_k)_{L^2(\Omega)}^2 + \sum_{k>N}(q-q_0,\phi_k)_{L^2(\Omega)}^2\\ &\le \sum_{k\le N}(q-q_0,\phi_k)_{L^2(\Omega)}^2 + \frac{1}{\lambda_{N+1}}\sum_{k>N}\lambda_k(q-q_0,\phi_k)_{L^2(\Omega)}^2\\ &\le \sum_{k\le N}(q-q_0,\phi_k)_{L^2(\Omega)}^2 + \frac{c_\Omega m^2}{\widetilde{c}(N+1)^{2/n}}.\end{aligned}$$

In combination with (2.5), this estimate yields

$$C\|q-q_0\|_{L^2(\Omega)}^2 \le \frac{N^{1+2/n}}{\sqrt{\epsilon}} + \frac{1}{(N+1)^{2/n}} + Ne^{\varrho N^{2/n}}e^{\mu\epsilon}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}. \tag{2.6}$$

For $s\ge 1$ a real number, let N be the unique integer so that $N\le s<N+1$. Then (2.6) implies

$$C\|q-q_0\|_{L^2(\Omega)}^2 \le \frac{s^{1+2/n}}{\sqrt{\epsilon}} + \frac{1}{s^{2/n}} + se^{\varrho s^{2/n}}e^{\mu\epsilon}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}.$$

Taking $\epsilon = s^{8/n+2}$ in this inequality, we find

$$C\|q-q_0\|_{L^2(\Omega)}^2 \le \frac{1}{s^{2/n}} + se^{\varrho s^{2/n}}e^{\mu s^{8/n+2}}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))},$$

and then

$$C\|q-q_0\|_{L^2(\Omega)}^2 \le \frac{1}{s^{2/n}} + e^{\theta s^{8/n+2}}\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}, \tag{2.7}$$

with $\theta = 1+\varrho+\mu$.

We use the temporary notation $\gamma = \|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}$. We consider the function $\chi(s) = s^{2/n}e^{\theta s^{8/n+2}}$, $s\ge 1$. Under the condition $\gamma\le\gamma^* = e^{-\theta}$, there exist $s^*\ge 1$ so that $\chi(s^*) = \gamma^{-1}$. In that case $s=s^*$ in (2.7) gives in a straightforward manner

$$C\|q-q_0\|_{L^2(\Omega)} \le |\ln\gamma|^{-\frac{1}{8+2n}}. \tag{2.8}$$

When $\gamma\ge\gamma^*$, we have trivially

$$\|q-q_0\|_{L^2(\Omega)} \le m|\Omega|^{1/2} \le m|\Omega|^{1/2}\frac{\gamma}{\gamma^*}. \tag{2.9}$$

In light of (2.8) and (2.9), we end up getting

$$C\|q-q_0\|_{L^2(\Omega)} \le \Psi\left(\|\widetilde{\mathscr{C}}_q^\tau - \widetilde{\mathscr{C}}_{q_0}^\tau\|_{\mathscr{B}(\mathcal{H},H^1((0,\tau),L^2(\Upsilon)))}\right)$$

as it is expected.

Remark 2.1. Fix g and q in (2.1), and let $\mathcal{C}^\tau(f) := \mathcal{C}_q^\tau(f,g)$. We can then use (2.2) to derive a stability estimate for inverse source problem consisting in the determination of f from $\mathcal{C}^\tau(f)$. A minimization argument in ϵ leads to the following result: there exist two constants $\tau_0>0$ and $C>0$ so that, for any $\tau\ge\tau_0$ and $f\in mB_{L^2(\Omega)}$,

$$C\|f\|_{H^{-1}(\Omega)} \le \Phi\left(\|\mathcal{C}^\tau(f)\|_{H^1((0,\tau),L^2(\Upsilon))}\right),$$

where $\Phi(\gamma) = |\ln\gamma|^{-1/2} + \gamma$, $\gamma>0$.

Remark 2.2. Let us explain briefly how one can get a stability estimate for the inverse problem of determining the damping coefficient or the damping coefficient together with the potential, from boundary measurements, again by varying the initial conditions. Let us substitute in the first equation of the IBVP (1.1) the operator $W = \partial_t^2 - \Delta$ by $W_a = \partial_t^2 - \Delta + a(x)\partial_t$. The function a is usually taken bounded and non negative. It is called the damping coefficient. In that case Theorem 2.1 holds if the operator W is substituted by W_a. This can be seen by examining the proof in [9]. The only difference between the two cases is that in the present case there is an additional term in the quantity between the brackets in [9, formula (8), page 109]. But this supplementary term does not modify the estimate [9, formula (9), page 109]. The rest of the analysis remains unchanged. These observations at hand, we can extend [2, Theorem 1.1, (1.3) and (1.4)] and [2, Theorem 4.2] to the case where no geometric condition is imposed to the sub-boundary where the measurements are made. We leave to the interested reader to write down the details.

3. Extension to the heat equation

We begin with an inverse source problem associated to the following IBVP for the heat equation.

$$\begin{cases} \partial_t u - \Delta u + q(x)u = g(t)f(x) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot,0) = 0. \end{cases} \tag{3.1}$$

From now on $\tau > 0$ is arbitrary but fixed.

Recall that the anisotropic Sobolev space $H^{2,1}(Q)$ is given as follows

$$H^{2,1}(Q) = L^2((0,\tau), H^2(\Omega)) \cap H^1((0,\tau), L^2(\Omega)).$$

It is well known that for any $f \in L^2(\Omega)$, $g \in L^2(0,\tau)$ and $q \in L^\infty(\Omega)$, the IBVP (3.1) has a unique solution

$$u := \mathscr{S}_q(f,g) \in H^{2,1}(Q).$$

Moreover, there exist a constant $C' = C'(\Omega, m) > 0$ so that, for any $q \in mB_{L^\infty(\Omega)}$,

$$\|\mathscr{S}_q(f,g)\|_{H^{2,1}(Q)} \le C'\|g\|_{L^2((0,\tau))}\|f\|_{L^2(\Omega)}. \tag{3.2}$$

We refer to [5, Theorem 1.43, page 27] and references therein for the statement of these results in the case of a general parabolic IBVP with non zero initial and boundary conditions.

Under the additional assumption that $g \in H^1(0,\tau)$, it is not hard to check that $\partial_t u$ is the solution of the IBVP (3.1) with g substituted by g'. Hence $\partial_t \mathscr{S}_q(f,g) \in H^{2,1}(Q)$ and

$$\|\partial_t \mathscr{S}_q(f,g)\|_{H^{2,1}(Q)} \le C'\|g'\|_{L^2((0,\tau))}\|f\|_{L^2(\Omega)}, \tag{3.3}$$

uniformly in $q \in mB_{L^\infty(\Omega)}$, where C' is the same constant as in (3.2).

As in the introduction, Υ is a non empty open subset of Γ and $\Lambda = \Upsilon \times (0,\tau)$. In light of the preceding analysis $\partial_\nu \mathscr{S}_q(f,g)$ is well defined as an element of $H^1((0,\tau); L^2(\Upsilon))$ and

$$\|\partial_\nu \mathscr{S}_q(f,g)\|_{H^1((0,\tau);L^2(\Upsilon))} \le C''\|g\|_{H^1((0,\tau))}\|f\|_{L^2(\Omega)},$$

for some constant $C'' = C''(m,\Omega)$ uniformly in $q \in mB_{L^\infty(\Omega)}$.

The following interpolation inequality will be useful in the sequel.

Theorem 3.1. *There exist two constants $c > 0$ and $C > 0$ so that, for any $q \in mB_{L^\infty(\Omega)}$, $f \in H_0^1(\Omega)$ and $g \in H^1((0,\tau))$ with $g(0) \ne 0$,*

$$C\|f\|_{L^2(\Omega)} \le \frac{1}{\sqrt{\epsilon}}\|f\|_{H_0^1(\Omega)} + \frac{1}{|g(0)|}e^{\tau\frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}}e^{c\epsilon}\|\partial_\nu \mathscr{S}_q(f,g)\|_{H^1((0,\tau);L^2(\Upsilon))}, \quad \epsilon \ge 1. \tag{3.4}$$

Proof. Pick $f \in H_0^1(\Omega)$ and $q \in mB_{L^\infty(\Omega)}$. Without loss of generality, we may assume that $q \ge 0$. Indeed, we have only to substitute u by ue^{-mt}, which is the solution of the IBVP (3.1) when q is replaced by $q + m \in 2mB_{L^\infty(\Omega)}$.

Let $v := \mathcal{S}_q(f) \in H^{2,1}(Q)$ be the unique solution of the IBVP

$$\begin{cases} \partial_t v - \Delta v + q(x)v = 0 & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(\cdot,0) = f. \end{cases}$$

Then $\partial_\nu \mathcal{S}_q(f)$ is well defined as an element of $L^2(\Lambda)$. As for the wave equation

$$\partial_\nu \mathscr{S}_q(f,g)_{|\Lambda}(\cdot,t) = \int_0^t g(t-s)\partial_\nu \mathcal{S}_q(f)_{|\Lambda}(\cdot,s)ds.$$

Therefore

$$\|\partial_\nu \mathcal{S}_q(f)\|_{L^2(\Lambda)} \le \frac{\sqrt{2}}{|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}} \|\partial_\nu \mathscr{S}_q(f,g)\|_{H^1((0,\tau);L^2(\Upsilon))}. \tag{3.5}$$

On the other hand, as it is shown in [2], the following final time observability inequality holds

$$\|\mathcal{S}_q(f)(\cdot,\tau)\|_{L^2(\Omega)} \le K\|\partial_\nu \mathcal{S}_q(f)\|_{L^2(\Lambda)}, \tag{3.6}$$

for some constant $K > 0$, independent on q and f.

A combination of (3.5) and (3.6) implies

$$C\|\mathcal{S}_q(f)(\cdot,\tau)\|_{L^2(\Omega)} \le \frac{1}{|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}} \|\partial_\nu \mathscr{S}_q(f,g)\|_{H^1((0,\tau);L^2(\Upsilon))}. \tag{3.7}$$

Denote by $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \ldots \to +\infty$ the sequence of eigenvalues of the unbounded operator $A : L^2(\Omega) \to L^2(\Omega)$ given by

$$A = -\Delta + q, \quad D(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

Let (ϕ_k) be a sequence of eigenfunctions, each ϕ_k corresponds to λ_k, so that (ϕ_k) form an orthonormal basis of $L^2(\Omega)$.

A usual spectral decomposition yields

$$\mathcal{S}_q(f)(\cdot,\tau) = \sum_{\ell\ge 1} e^{-\lambda_k \tau}(f,\phi_\ell)_{L^2(\Omega)}\phi_\ell.$$

Here $(\cdot,\cdot)_{L^2(\Omega)}$ is the usual scalar product on $L^2(\Omega)$. In particular

$$(f,\phi_\ell)^2_{L^2(\Omega)} \le e^{2\lambda_\ell \tau}\|\mathcal{S}_q(f)(\cdot,\tau)\|^2_{L^2(\Omega)}, \quad \ell \ge 1.$$

Whence, for any integer $N \ge 1$,

$$\sum_{\ell=1}^N (f,\phi_\ell)^2_{L^2(\Omega)} \le Ne^{2\lambda_N \tau}\|\mathcal{S}_q(f)(\cdot,\tau)\|^2_{L^2(\Omega)}.$$

This and the fact that $\left(\sum_{\ell\ge 1}\lambda_\ell(\cdot,\phi_\ell)^2_{L^2(\Omega)}\right)^{1/2}$ is an equivalent norm on $H_0^1(\Omega)$ lead

$$\begin{aligned}\|f\|^2_{L^2(\Omega)} &= \sum_{\ell=1}^N (f,\phi_\ell)^2_{L^2(\Omega)} + \sum_{\ell\ge N+1}^N (f,\phi_\ell)^2_{L^2(\Omega)} \\ &\le \sum_{\ell=1}^N (f,\phi_\ell)^2_{L^2(\Omega)} + \frac{1}{\lambda_{N+1}}\sum_{\ell\ge N+1}^N \lambda_\ell (f,\phi_\ell)^2_{L^2(\Omega)} \\ &\le Ne^{2\lambda_N \tau}\|\mathcal{S}_q(f)(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{1}{\lambda_{N+1}}\|f\|_{H^1(\Omega)}.\end{aligned}$$

In light of (2.4), this estimate gives

$$\|f\|^2_{L^2(\Omega)} \le Ne^{2\widetilde{c}\lambda_N^{2/n}\tau}\|\mathcal{S}_q(f)(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{\widetilde{c}}{(N+1)^{2/n}}\|f\|_{H^1(\Omega)}. \tag{3.8}$$

Let $\epsilon \ge 1$ and $N \ge 1$ be the unique integer so that $N \le \epsilon^{n/2} < N+1$. We obtain in a straightforward manner from (3.8)

$$\|f\|^2_{L^2(\Omega)} \le e^{(2\widetilde{c}\tau+1)\epsilon}\|\mathcal{S}_q(f)(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{\widetilde{c}}{\epsilon}\|f\|^2_{H^1(\Omega)}.$$

The proof is completed by using the elementary inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, $a, b \ge 0$ and (3.7). $\square$

When $q \in L^\infty(\Omega)$ and $g \in H^1((0,\tau))$ satisfying $g(0) \neq 0$ are fixed, we set $\mathscr{S}(f) := \mathscr{S}_q(f,g)$. In that case (3.4) takes the simple form,

$$\widetilde{C}\|f\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\epsilon}}\|f\|_{H_0^1(\Omega)} + e^{c\epsilon}\|\partial_\nu \mathscr{S}(f)\|_{H^1((0,\tau);L^2(\Upsilon))}, \;\; \epsilon \geq 1,$$

for any $f \in H_0^1(\Omega)$. Of course the constant $\widetilde{C}$ depends on q and g.

The last estimate enables us to get a logarithmic stability estimate for the inverse source problem consisting in determining f from $\partial_\nu \mathscr{S}(f)_{|\Lambda}$.

Corollary 3.1. *Fix $q \in L^\infty(\Omega)$, $g \in H^1((0,\tau))$ satisfying $g(0) \neq 0$. There exists a constant $\widehat{C} = \widehat{C}(n,\Omega,q,g,\Upsilon,m) > 0$ so that, for any $f \in mB_{H_0^1(\Omega)}$,*

$$\widehat{C}\|f\|_{L^2(\Omega)} \leq \Phi\left(\|\partial_\nu \mathscr{S}(f)\|_{H^1((0,\tau);L^2(\Upsilon))}\right),$$

where $\Phi(\gamma) = |\ln\gamma|^{-1/2} + \gamma$, $\gamma > 0$.

Next, we consider the IBVP

$$\begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot,0) = u_0. \end{cases} \tag{3.9}$$

To any $q \in L^\infty(\Omega)$ and $u_0 \in H_0^1(\Omega)$ corresponds a unique solution $u := \mathbf{S}_q(u_0) \in H^{2,1}(Q)$ and

$$\|\mathbf{S}_q(u_0)\|_{H^{2,1}(Q)} \leq C'\|u_0\|_{H_0^1(\Omega)}, \tag{3.10}$$

uniformly in $q \in mB_{L^\infty(\Omega)}$, where C' is the same constant as in (3.2).

Let $\mathcal{H}_0 = \{w \in H_0^1(\Omega);\ \Delta w \in H_0^1(\Omega)\}$ that we equip with its natural norm

$$\|u\|_{\mathcal{H}_0} = \|u\|_{H_0^1(\Omega)} + \|\Delta u\|_{H_0^1(\Omega)}.$$

When $q \in W^{1,\infty}(\Omega)$ and $u_0 \in \mathcal{H}_0$ then it is straightforward to check that $\partial_t \mathbf{S}_q(u_0) = \mathbf{S}_q(\Delta u_0 + qu_0)$. So applying (3.10), with u_0 substituted by $\Delta u_0 - qu_0$, we get

$$\|\partial_t \mathbf{S}_q(u_0)\|_{H^{2,1}(Q)} \leq C'\|u_0\|_{\mathcal{H}_0}, \tag{3.11}$$

uniformly in $q \in mB_{W^{1,\infty}(\Omega)}$.

Bearing in mind that the trace operator $w \in H^{2,1}(Q) \mapsto \partial_\nu w \in L^2(\Lambda)$ is bounded, we obtain that $\partial_\nu \mathbf{S}_q(u_0) \in H^1((0,\tau);L^2(\Upsilon))$ if $u_0 \in \mathcal{H}_0$ and $q \in W^{1,\infty}(\Omega)$, and using (3.10) and (3.11), we get

$$\|\partial_\nu \mathbf{S}_q(u_0)\|_{H^1((0,\tau);L^2(\Upsilon))} \leq C_0\|u_0\|_{\mathcal{H}_0},$$

uniformly in $q \in mB_{W^{1,\infty}(\Omega)}$, for some constant $C_0 = C_0(n,\Omega,\tau,m)$.

In other words, we proved that the operator $\mathcal{N}_q : u_0 \in \mathcal{H}_0 \mapsto \partial_\nu \mathbf{S}_q(u_0) \in H^1((0,\tau);L^2(\Upsilon))$ is bounded and

$$\|\mathcal{N}_q\|_{\mathscr{B}(\mathcal{H}_0,H^1((0,\tau);L^2(\Upsilon)))} \leq C_0,$$

uniformly in $q \in mB_{W^{1,\infty}(\Omega)}$.

From here on, for sake of simplicity, the norm of $\mathcal{N}_q - \mathcal{N}_{q_0}$ in $\mathscr{B}(\mathcal{H}_0;H^1((0,\tau);L^2(\Upsilon))$ is simply denoted by $\|\mathcal{N}_q - \mathcal{N}_{q_0}\|$.

Theorem 3.2. *There exists a constant $C > 0$ so that, for any q_0, $q \in mB_{W^{1,\infty}(\Omega)}$,*

$$C\|q - q_0\|_{L^2(\Omega)} \leq \Theta\left(\|\mathcal{N}_q - \mathcal{N}_{q_0}\|\right).$$

Here $\Theta(\gamma) = |\ln\gamma|^{-\frac{1}{1+4n}} + \gamma$.

Proof. Let q_0, $q \in mB_{W^{1,\infty}(\Omega)}$. As before, without loss of generality, we assume that $q_0 \geq 0$.

Let $A_0 : L^2(\Omega) \to L2(\Omega)$ be the unbounded operator given by $A_0 = -\Delta + q_0$ and $D(A_0) = H_0^1(\Omega) \cap H^2(\Omega)$. Denote by $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \ldots \to +\infty$ the sequence of eigenvalues of the operator A_0, and (ϕ_k) a sequence of the corresponding eigenfunctions so that (ϕ_k) form an orthonormal basis of $L^2(\Omega)$.

Taking into account that $\mathbf{S}_{q_0}(\phi_k) = e^{-\lambda_k t}\phi_k$, we obtain

$$\mathbf{S}_q(\phi_k) - \mathbf{S}_{q_0}(\phi_k) = \mathscr{S}_q((q-q_0)\phi_k, e^{-\lambda_k t}).$$

Therefore

$$\mathcal{N}_q(\phi_k) - \mathcal{N}_{q_0}(\phi_k) = \partial_\nu \mathscr{S}_q((q-q_0)\phi_k, e^{-\lambda_k t}).$$

Hence, a similar argument as in the preceding section yields

$$\|\partial_\nu \mathscr{S}_q((q-q_0)\phi_k, e^{-\lambda_k t})\|_{H^1((0,\tau);L^2(\Upsilon))} \leq C\lambda_k^{3/2}\|\mathcal{N}_q - \mathcal{N}_{q_0}\|$$

which, in combination with estimate (3.4), implies, with $\epsilon \geq 1$ is arbitrary,

$$C|\Omega|^{-1/2}|(q-q_0, \phi_k)_{L^2(\Omega)} \leq C\|(q-q_0)\phi_k\|_{L^2(\Omega)} \leq \frac{\sqrt{\lambda_k}}{\sqrt{\epsilon}} + e^{\tau\lambda_k^2}e^{c\epsilon}\lambda_k^2\|\mathcal{N}_q - \mathcal{N}_{q_0}\|, \tag{3.12}$$

where we used the estimate $\|(q-q_0)\phi_k\|_{H_0^1(\Omega)} \leq C\sqrt{\lambda_k}$.

A straightforward consequence of estimate (3.12) is

$$C\sum_{k=1}^{N}|(q-q_0,\phi_k)^2_{L^2(\Omega)} \leq \frac{N\lambda_N}{\epsilon} + Ne^{(2\tau+1)\lambda_N^2}e^{c\epsilon}\|\mathcal{N}_q - \mathcal{N}_{q_0}\|^2, \tag{3.13}$$

for any arbitrary integer $N \geq 1$.

We pursue similarly to the proof of Theorem 3.1 in order to get, for arbitrary $s \geq 1$,

$$C\|q-q_0\|^2_{L^2(\Omega)} \leq \frac{s^{1+2/n}}{\epsilon} + \frac{1}{s^{2/n}} + e^{\varrho s^{1+4/n}}e^{c\epsilon}\|\mathcal{N}_q - \mathcal{N}_{q_0}\|^2.$$

The proof is then completed in the same manner to that of Theorem 3.1. □

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