A traffic flow model with non-smooth metric interaction: Well-posedness and micro-macro limit

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Abstract

We prove existence and uniqueness of solutions to a transport equation modelling vehicular traffic in which the velocity field depends non-locally on the downstream traffic density via a discontinuous anisotropic kernel. The result is obtained recasting the problem in the space of probability measures equipped with the ∞-Wasserstein distance. We also show convergence of solutions of a finite dimensional system, which provide a particle method to approximate the solutions to the original problem.

Keywords: Transport equations, non-local velocity, Wasserstein distance, macroscopic traffic flow models, micro-macro limits.

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1   Introduction

In this paper, we are interested in studying the macroscopic traffic flow model introduced in [7] from the point of view of measure transport equations in Wasserstein spaces.

Transport equations with non-local velocities have drawn a growing attention in the mathematical community, starting from the Vlasov equation and other models in kinetic theory, see e.g. [9, 14, 31]. In this context, *non-local* means that the velocity at a given point of the space depends not only on the density at that point, but on the density in a whole neighborhood. The first general results of existence and uniqueness for such equations are given by Ambrosio-Ganbo [3]. There, the authors show that Wasserstein distances are key tools to deal with these equations, since vector fields resulting from non-local interactions are Lipschitz continuous with respect to such distances. Several extensions have been proposed since then, including definition of gradient flows [5], numerical schemes [26, 28], generalizations to domains with boundary [16] and to transport equations with sources [25, 27].

Non-local conservation laws have been introduced recently to model a variety of evolution dynamics: besides road traffic models [7, 17, 19, 21, 30] and crowd motion

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models [11, 12, 23, 24, 32], they are used to describe granular flows [1], sedimenata-
tion [6], conveyor belts [18] and aggregation phenomena [20].

As far as road traffic is concerned, an advantage in considering non-local mean ve-
locity depending on a weighted mean of the downstream traffic density is represented
by the consequent finite acceleration, whose unboundedness is one of the drawbacks
of Lighthill-Whitham-Richards [22, 29] model and other classical macroscopic models,
which allow for speed jumps. This limits their application in connection with consump-
tion and pollution models, which heavily rely on acceleration estimation, see for example
comments in [8].

In our case, the evolution equation for the density $\rho = \rho(t,x)$ of cars on a (infinite)
road is given by the following transport partial differential equation (PDE)

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho v \left(\int_x^{x+\eta} \rho(t,y) w(y-x) \, dy \right) \right) = 0, & x \in \mathbb{R}, \ t > 0, \\ \rho(0,x) = \rho_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where v is the mean traffic velocity, $\eta > 0$ a given parameter, and $w : [0,\eta] \to \mathbb{R}^+$ is a non-
increasing Lipschitz weight with $\int_0^\eta w(x) \, dx = 1$. This is intended to model the fact that
drivers adapt their velocity depending on the downstream traffic condition, eventually
giving more attention to what happens close to them than to cars far beyond. In this
respect we speak of *metric* interaction, opposite to *topological* interaction which take
into account the ordering of vehicles: in this case the influence of preceding vehicles
takes into account the presence of other vehicles in between [13, Section 1.1.1.7].

Solutions of (1.1) will be defined in the space $\mathcal{M}(\mathbb{R})$ of non-negative measures
equipped with the Wasserstein distance, and they are to be intended in the weak sense:

Definition 1 *A measure $\mu \in C^0([0,T], \mathcal{M}(\mathbb{R}))$ is a weak solution of (1.1) if for all $\varphi \in C_c^\infty((0,T) \times \mathbb{R})$ there holds*

$$\int_0^T \int_{\mathbb{R}} \left(\partial_t \varphi + \partial_x \phi v \left(\int_x^{x+\eta} w(y-x) \, d\mu(t,y) \right) \right) d\mu(t,x) = 0.$$

The main result of the paper guarantees existence and uniqueness of weak solutions.

Theorem 2 *Let $v : [0,1] \to \mathbb{R}^+$ be a Lipschitz and non-increasing function with $v(1) = 0$
and $\rho_0 \in BV_c(\mathbb{R}, [0,1])$. Then the Cauchy problem (1.1) admits a unique solution in the
weak sense, which satisfies*

$$0 \le \rho(t,x) \le \max\{\rho_0\} \qquad \text{for a.e.} \ \ x \in \mathbb{R}, t > 0.$$

Above, BV_c denotes the set of functions of bounded variation with compact sup-
port. The proof will be given in Section 4.2. Remark that, unlike [7, 17], the proof of
uniqueness of solutions does not rely on any entropy condition. On the other side, we
must restrict the problem to initial data with compact support.

Observe that the result is based on the study of (1.1) in a more general setting,
with v not necessarily decreasing. Assuming that v is Lipschitz and w is a (possibly
discontinuous) BV interaction kernel of bounded variation, we prove in Proposition 18
existence and uniqueness of the solution of (1.1) for small times. It is interesting to
observe that the density can blow-up in a finite time T, leading to non-existence of the
solution for times larger than T. Such phenomenon cannot appear for more smooth
interaction kernels w, see Proposition 16 and Remark 27.

The structure of the article is the following. A short review on Wasserstein distances and general transport equations properties is given in Section 2. Transport equations with BV interaction kernels are studied in Section 3, where we prove existence and uniqueness of solutions locally in time in Proposition 18. In Section 4, we also define a particle approximation for the density, that provides a finite-dimensional numerical scheme for the solution of the transport PDE, and we prove convergence of the micro-macro limit.

2 Wasserstein distances and transport equations

In this section, we recall the main definitions related to Wasserstein distances, and in particular the definition and properties of the ∞-Wasserstein distance. We then recall results about transport equations with non-local velocities, in the case of smooth interaction kernels. For more details, see the monographs [34, 33] and the articles [26, 27, 25].

We consider non-negative measures with a given mass $m>0$, which is conserved by the solutions of (1.1). Therefore, without loss of generality, in the following we will deal with probability measures, i.e. $m=1$.

We denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ and with $\mathcal{P}_c(\mathbb{R}^d)$ the subset of probability measures with compact support. We also denote with $\mathcal{P}^{ac}(\mathbb{R}^d)$ the subset of probability measures that are absolutely continuous with respect to the Lebesgue measure, and we identify the measure with its density with respect to the Lebesgue measure, e.g. by writing both $\int f(x)\,d\rho(x)$ and $\int f(x)\rho(x)\,dx$ for $\rho\in\mathcal{P}^{ac}(\mathbb{R}^d)$. We use the letters $\mu,\nu,\dots$ for general measures in $\mathcal{P}(\mathbb{R}^d)$, keeping the notation $\rho,\rho',\dots$ for measures in $\mathcal{P}^{ac}(\mathbb{R}^d)$. When not specified, we consider $\mathcal{P}(\mathbb{R}^d),\mathcal{P}_c(\mathbb{R}^d),\mathcal{P}^{ac}(\mathbb{R}^d)$ and $\mathcal{P}_c^{ac}(\mathbb{R}^d):=\mathcal{P}_c(\mathbb{R}^d)\cap\mathcal{P}^{ac}(\mathbb{R}^d)$ endowed with the ∞-Wasserstein distance, defined below.

Given a probability measure π on $\mathbb{R}^d\times\mathbb{R}^d$, one can interpret π as a path to transfer a probability measure μ on $\mathbb{R}^d$ to another probability measure ν on $\mathbb{R}^d$ as follows: each infinitesimal mass on a location x is sent to a location y with a probability given by $\pi(x,y)$. Formally, μ is sent to ν if the following property holds:

$$\int_{\mathbb{R}^d} d\pi(x,\cdot)=d\mu(x), \qquad \int_{\mathbb{R}^d} d\pi(\cdot,y)=d\nu(y),$$

or, equivantly, for all $f,g\in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(f(x)+g(y))\,d\pi(x,y)=\int_{\mathbb{R}^d}f(x)\,d\mu(x)+\int_{\mathbb{R}^d}g(y)\,d\nu(y)$$

In this case, π is called a transference plan from μ to ν. We denote by $\Pi(\mu,\nu)$ the set of such transference plans.

Fix now $p\in[1,+\infty)$. One can define a cost for π as follows

$$J[\pi]:=\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\,d\pi(x,y)$$

and look for a minimizer of J in $\Pi(\mu,\nu)$. Such problem is called the Monge-Kantorovich problem. A minimizer of J in $\Pi(\mu,\nu)$ always exists, see [33]. A natural space on which J is finite is the space of Borel probability measures with finite p-moment, that is

$$\mathcal{P}_p(\mathbb{R}^d):=\left\{\mu\in\mathcal{P}(\mathbb{R}^d)\ \Big|\ \int|x|^p\,d\mu(x)<\infty\right\}. \tag{2.2}$$

The p-**Wasserstein distance** is defined on $\mathcal{P}_p(\mathbb{R}^n) \times \mathcal{P}_p(\mathbb{R}^n)$ as

$$W_p(\mu,\nu) = \min_{\pi \in \Pi(\mu,\nu)} J[\pi]^{1/p}.$$

It is indeed a distance on $\mathcal{P}_p(\mathbb{R}^d)$, see [33]. Several topological properties are of interest for the space $\mathcal{P}_p(\mathbb{R}^d)$ endowed with the Wasserstein distance W_p, see [34, 33]. For future use, here we recall the following.

Proposition 3 *The Wasserstein distance metrizes weak convergence in $\mathcal{P}_p(\mathbb{R}^d)$, i.e.*

$$W_p(\mu_n,\mu) \to 0$$

if and only if

$$\mu_n \rightharpoonup \mu \qquad and \qquad \lim_{R \to +\infty} \limsup_{n \to \infty} \int_{|x| > R} |x|^p \, d\mu_n(x) = 0.$$

In particular, $W_p(\mu_n,\mu) \to 0$ if and only if $\mu_n \rightharpoonup \mu$ in $\mathcal{P}(K)$ with K compact.

Proposition 4 *The space $\mathcal{P}_p(\mathbb{R}^d)$ endowed with the Wasserstein distance W_p is complete.*

Proposition 5 *The Wasserstein distances are ordered, i.e. $p \leq q$ implies*

$$W_p(\mu,\nu) \leq W_q(\mu,\nu).$$

We finally recall the Kantorovich-Rubinstein duality formula for the 1-Wasserstein distance, see e.g. [33, Ch. 1].

Proposition 6 *Let $\mu,\nu \in \mathcal{P}_1(\mathbb{R}^d)$. Then it holds*

$$W_1(\mu,\nu) = \sup \left\{ \int f(x) \, d(\mu - \nu)(x) \mid \mathrm{Lip}(f) \leq 1 \right\}. \tag{2.3}$$

2.1   The ∞-Wassersein distance

In this section, we recall the definition of the ∞-Wasserstein distance and prove some useful properties. We remark that the use of the ∞-Wasserstein distance allows to recover the necessary estimates for the convolution $\int \rho(y) w(y - x) \, dy$ even for BV kernels w, see e.g. Proposition 23 below.

Given two probability measures μ,ν and the space of transference plans $\Pi(\mu,\nu)$ with marginal probabilities μ,ν, we denote with $C_\infty(\pi)$ the following cost of a transference plan $\pi \in \Pi(\mu,\nu)$:

$$C_\infty(\pi) = \{\pi - \mathrm{esssup}(|x - y|)\}.$$

The ∞-Wasserstein distance is then defined as

$$W_\infty(\mu,\nu) = \inf \{C(\pi) \mid \pi \in \Pi(\mu,\nu)\}.$$

We first recall the existence of an optimal transference plan for probability measures with compact support, see [10, Prop. 2.1]. Observe that, in analogy with (2.2), we have $\mathcal{P}_\infty(\mathbb{R}^d) = \mathcal{P}_c(\mathbb{R}^d)$.

Proposition 7 *Let $\mu,\nu \in \mathcal{P}_c(\mathbb{R}^d)$. Then there exists $\pi_* \in \Pi(\mu,\nu)$ realizing W_∞, i.e.*

$$W_\infty(\mu,\nu) = C_\infty(\pi_*).$$

It is easy to prove that ordering of the Wasserstein distances is preserved even with the ∞-Wasserstein distance.

Proposition 8 *For any $p \in [1,+\infty)$ it holds*

$$W_p(\mu,\nu) \leq W_\infty(\mu,\nu).$$

Proof. Let π be a transference plan realizing $W_\infty(\mu,\nu)$. Then it holds

$$W_p^p(\mu,\nu) \leq \int |x-y|^p \, d\pi(x,y) \leq \int C_\infty(\pi)^p \, d\pi(x,y) = W_\infty^p(\mu,\nu).$$

$\square$

We now prove lower semicontinuity of W_∞ with respect to the weak convergence of measures in a compact space.

Proposition 9 *Let K be a compact set in $\mathbb{R}^d$, and μ_n a sequence in $\mathcal{P}(K)$. If $\mu_n \rightharpoonup \mu$ and $\nu \in \mathcal{P}_c(\mathbb{R}^d)$, then*

$$W_\infty(\nu,\mu) \leq \liminf_{n \to \infty} W_\infty(\nu,\mu_n).$$

Proof. First observe that $\mathrm{supp}(\mu) \subset K$. Since $\mathrm{supp}(\nu)$ is compact, eventually replacing K by $K \cup \mathrm{supp}(\nu)$, one has that all measures have compact support in K.

We prove the result by passing to a minimizing subsequence (that we do not relabel) and prove that $W_\infty(\nu,\mu) \leq \lim_{n \to \infty} W_\infty(\nu,\mu_n)$. By Proposition 7, for each n there exists a transference plan π_n realizing $W_\infty(\nu,\mu_n)$. Since $\pi_n \in \mathcal{P}(K \times K)$ and $K \times K$ is compact, then Prokhorov's Theorem ensures the existence of a subsequence (that we do not relabel) for which it holds $\pi_n \rightharpoonup \pi_*$ for some $\pi_* \in \mathcal{P}(K \times K)$. It is easy to prove that $\pi_* \in \Pi(\nu,\mu)$. Since C_∞ is lower semicontinuous with respect to the weak topology (see e.g. [10, Lemma 2.3]), then $C_\infty(\pi_*) \leq \lim_n C_\infty(\pi_n)$. By recalling that $W_\infty(\nu,\mu) \leq C_\infty(\pi_*)$, the result is proved. $\square$

We now prove results related to the topology of $\mathcal{P}_c(\mathbb{R}^d)$ endowed with the ∞-Wasserstein distance.

Proposition 10 *The space $\mathcal{P}_c(\mathbb{R}^d)$ is complete with respect to the metric W_∞.*

Proposition 11 *Let μ_n a sequence in $\mathcal{P}_c(\mathbb{R}^d)$. If $W_\infty(\mu_n,\mu) \to 0$ for some $\mu \in \mathcal{P}_c(\mathbb{R}^d)$, then $\mu_n \rightharpoonup \mu$.*

Proof. Let μ_n be a Cauchy sequence in $\mathcal{P}_c(\mathbb{R}^d)$ endowed with the metric W_∞. For a given $\varepsilon > 0$, consider N such that for all $n \geq N$, $k > 0$ it holds $W_\infty(\mu_n,\mu_{n+k}) < \varepsilon$. In particular it holds $W_\infty(\mu_N,\mu_n) < \varepsilon$, that in turn implies that $\mathrm{supp}(\mu_n) \subset \cup \{B(x,\varepsilon) \mid x \in \mathrm{supp}(\mu_N)\}$ for all $n \geq N$. Since such set is bounded, and the supports of μ_k for $k < N$ are bounded too, then there exists a compact set K containing the support of all μ_n.

Since the μ_n have uniformly bounded support, then they also have uniformly bounded p-th moment for each $p \in [1,\infty)$. Recall that each space $\mathcal{P}_p(\mathbb{R}^d)$ is complete with respect to W_p, see Proposition 4. Observe that it holds $W_p(\mu_n,\mu_m) \leq W_\infty(\mu_n,\mu_m)$, then μ_n is a Cauchy sequence in $\mathcal{P}_p(\mathbb{R}^d)$, hence there exists μ_* for which $W_p(\mu_n,\mu_*) \to 0$ for all $p \in [1,\infty)$. Then, by Proposition 3, we have $\mu_n \rightharpoonup \mu_*$, proving Proposition 11.

We are now left to prove that $W_\infty(\mu_n,\mu_*) \to 0$, that is a direct consequence of lower semicontinuity of W_∞ with respect to the weak topology of measures, proved in Proposition 9. $\square$

Remark 12 *It is false that weak convergence of measures implies convergence with respect to the metric W_∞, even in $\mathcal{P}(K)$ with K compact. For example, consider the following sequence $\mu_n := \frac{n-1}{n}\delta_0 + \frac{1}{n}\delta_1$, that weakly converges to $\mu_* := \delta_0$, but for which it holds $W_\infty(\mu_n,\mu_*) = 1$ for all n. This is in sharp contrast with the W_p metric with $p \in [1,+\infty)$, as recalled in Proposition 3.*

2.2   Transport equations with smooth non-local interactions

In this section, we study the transport equation with non-local interactions, i.e. the following Cauchy problem:

$$\begin{cases} \partial_t\mu + \nabla_x \cdot (V[\mu]\mu) = 0, & x \in \mathbb{R}^d,\ t > 0, \\ \mu(0,x) = \mu_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (2.4)$$

where V is a function that associates to each measure μ a vector field $V[\mu]$. For the simplest case of V actually not depending on μ, the solution of (2.4) is given by the push-forward of the flow of V. We recall its definition here.

Definition 13 *Given a Borel map $\gamma : \mathbb{R}^d \to \mathbb{R}^d$, the push-forward of a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ is defined by:*

$$\gamma\#\mu(A) := \mu(\gamma^{-1}(A))$$

for every subset A such that $\gamma^{-1}(A)$ is μ-measurable.

If $v : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is a vector field uniformly Lipschitz with respect to the space variable and continuous with respect to time, then we denote by $\gamma_t = \phi_t^v$ is the flow generated by v, where $\phi_t^v(x_0)$ is the unique solution at time t of

$$\begin{cases} \dot{x} = v(t,x), \\ x(0) = x_0. \end{cases}$$

Then, we have the following result, see [33, Thm 5.34] and [26][1].

Proposition 14 *Let $v : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ be a uniformly Lipschitz vector field. Then the equation*

$$\begin{cases} \partial_t\mu + \nabla_x \cdot (v\mu) = 0, & x \in \mathbb{R}^d,\ t > 0, \\ \mu(0,x) = \mu_0(x), & x \in \mathbb{R}^d, \end{cases}$$

with $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$ admits a unique solution $\mu \in C^0([0,T],\mathcal{P}_c(\mathbb{R}^d))$. Such solution satisfies $\mu(t) = \phi_t^v\#\mu_0$. In particular, if $\mu_0 \in \mathcal{P}^{ac}(\mathbb{R}^d)$, then $\mu(t) \in \mathcal{P}^{ac}(\mathbb{R}^d)$ for all times $t > 0$.

Moreover, let v,w be two Lipschitz vector fields with Lipschitz constant L and bounded, and ϕ_t^v, ϕ_t^w the corresponding flows. Let $\mu,\nu \in \mathcal{P}_c(\mathbb{R}^d)$ be two probability measures. Then, for $p = [1,+\infty]$ it holds

$$W_p(\phi_t^v\#\mu, \phi_t^w\#\nu) \le e^{\frac{p+1}{p}Lt}W_p(\mu,\nu) + \frac{e^{Lt/p}(e^{Lt}-1)}{L}\sup_{t\in[0,T]}\|v(.,t) - w(.,t)\|_{C^0}. \quad (2.5)$$

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[1]The proof of (2.5) is given in [26] for $p < +\infty$, but it can be easily adapted to $p = +\infty$.

For V actually depending on μ, we have the following theorem, generalizing the results in [4, 26, 27, 25].

Theorem 15 *Let $p \in [1, +\infty]$. Let the function*

$$V[\mu] : \begin{cases} \mathcal{P}_c(\mathbb{R}) \to (C^1 \cap L^\infty)(\mathbb{R}^d; \mathbb{R}^d) \\ \mu \quad \mapsto V[\mu] \end{cases}$$

satisfy

- *$V[\mu]$ is uniformly Lipschitz and uniformly bounded, i.e. there exist L, M not depending on μ, such that for all $\mu \in \mathcal{P}_c(\mathbb{R}^d), x, y \in \mathbb{R}^d$,*

$$|V[\mu](x) - V[\mu](y)| \le L|x - y|, \qquad |V[\mu](x)| \le M.$$

- *V is a Lipschitz function, i.e. there exists K such that*

$$\|V[\mu] - V[\nu]\|_{C^0} \le K W_p(\mu, \nu).$$

Then the Cauchy problem (2.4) admits a unique solution $\mu \in C^0([0, T], \mathcal{P}_c(\mathbb{R}^d))$ for all times $T > 0$. Moreover, if the initial data μ_0 satisfies $\mu_0 \in \mathcal{P}^{ac}(\mathbb{R}^d)$, then $\mu(t) \in \mathcal{P}^{ac}(\mathbb{R}^d)$ for all times $t \in [0, T]$.

Finally, if μ, μ' are solutions of (2.4) with initial data μ_0, ν_0 respectively, there holds

$$W_p(\mu(t), \nu(t)) \le e^{(4L + 4K)t} W_p(\mu_0, \nu_0). \tag{2.6}$$

Proof. The proof is given for $p < +\infty$ in [26, Prop. 4 and Thm. 2] and in [27, 25]. The key estimate is (2.5), that holds also for $p = +\infty$. Then, the original proofs can be easily adapted to $p = +\infty$. □

We now adapt such result to our setting, in which $d = 1$ and $V[\mu](x) := v(\int_\mathbb{R} w(y - x) d\mu(y))$ with w Lipschitz.

Proposition 16 *Let $v : \mathbb{R} \to \mathbb{R}$ be a Lipschitz and bounded function, and $w : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with bounded support. If $\mu_0 \in \mathcal{P}_c(\mathbb{R})$, then the Cauchy problem*

$$\begin{cases} \partial_t \mu + \partial_x \left(\mu v \left(\int_\mathbb{R} w(y - x) d\mu(t, y) \right) \right) = 0, & x \in \mathbb{R}, \ t > 0, \\ \mu(0, x) = \mu_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.7}$$

admits a unique solution $\mu \in C^0([0, T], \mathcal{P}_c(\mathbb{R}))$ for all times $T > 0$.

Moreover, if μ, ν are solutions of (2.7) with initial data μ_0, ν_0 respectively, there holds

$$W_p(\mu(t), \nu(t)) \le e^{8 \mathrm{Lip}(v) \mathrm{Lip}(w) t} W_p(\mu_0, \nu_0), \tag{2.8}$$

for any $p \in [1, +\infty]$.

Proof. It is sufficient to prove that $V[\mu]$ defined by $V[\mu](x) := v(\int_\mathbb{R} w(z - x) d\mu(z))$ satisfies the hypotheses of Theorem 15. We have

$$|V[\mu](x) - V[\mu](y)| \le \mathrm{Lip}(v) \int |w(z - x) - w(z - y)| \, d\mu(z) \le \mathrm{Lip}(v) \mathrm{Lip}(w) |x - y|,$$

$$|V[\mu](x)| \le \sup(|v|),$$

$$|V[\mu](x) - V[\nu](x)| \le \mathrm{Lip}(v) \left| \int w(z - x) \, d(\mu - \nu)(z) \right| \le \mathrm{Lip}(v) \mathrm{Lip}(w) W_1(\mu, \nu) \le$$

$$\le \mathrm{Lip}(v) \mathrm{Lip}(w) W_p(\mu, \nu),$$

where the last estimate is based on the Kantorovich-Rubinstein duality (2.3) and ordering of Wasserstein distances in Propositions 5 and 8. By identifying L, K, we find (2.8). □

3 Transport equations with BV interactions kernels

In this section, we study transport equations in one space dimension with non-local velocities given by interaction kernels of the form (1.1). Unlike Proposition 16, here we do not assume that the kernel interaction w is Lipschitz continuous, but only BV. This prevents to use the results in Section 2.2. Moreover, the interaction $\int w(z-x)\,d\mu(z)$ itself is not well-defined for general probability measures, but for measures that are absolutely continuous with respect to the Lebesgue measure, i.e. for $\mu \in \mathcal{P}^{ac}(\mathbb{R})$ only.

In particular, we will often deal with measures in the space $L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$, endowed with the W_∞ distance. As stated above, the space $\mathcal{P}_c(\mathbb{R})$ is complete with respect to the W_∞ distance, but this does not hold anymore for $L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$. Nevertheless, we have convergence if both the support and the L^∞ norm are uniformly bounded, as proved in the following proposition.

Lemma 17 *Let $K \subset \mathbb{R}$ be a compact set, and $\mu_n \in L^\infty \cap \mathcal{P}^{ac}(K)$ be a sequence of measures with uniformly bounded L^∞ norm and weakly converging to μ. Then $\mu \in L^\infty \cap \mathcal{P}^{ac}(K)$ and $\|\mu\|_{L^\infty} \leq \limsup_{n\to\infty} \|\mu_n\|_{L^\infty}$.*

Proof. It is clear that $\mu \in \mathcal{P}(K)$. We now prove that μ is absolutely continuous with respect to the Lebesgue measure λ, by proving that, for each set A that is λ-measurable with $\lambda(A)=0$, it holds $\mu(A)=0$.

We first prove $\mu(A)=0$ in the case of A open. Fix $(a,b) \supset K$ and define $A_n := \left\{ x \in [a,b] \mid d(x,\mathbb{R}\setminus A) \geq \frac{1}{n} \right\}$ with $d(x,B)=\inf_{y\in B}|x-y|$. Observe that the A_n are compact, they satisfy $A_n \subset A_{n+1}$ and it holds $\lim_n A_n = A$. Define a sequence of continuous functions f_n with support in $[a-1,b+1]$ satisfying $\chi_{A_n} \leq f_n \leq \chi_A$, increasing in the sense that $f_n \leq f_{n+1}$. This implies $f_n \rightharpoonup \chi_A$. Hence, by the monotone convergence theorem and weak convergence, we have

$$\mu(A)=\sup_{n\in\mathbb{N}}\int f_n\,d\mu \leq \sup_{n\in\mathbb{N}}\limsup_{k\to\infty}\int f_n\,d\mu_k \leq \sup_{n\in\mathbb{N}}\int f_n\,dM\lambda \leq M\lambda(A)=0 \quad (3.9)$$

with $M \geq \sup_{n\in\mathbb{N}}\|\mu_n\|_{L^\infty}$.

By regularity of finite measures (see [15]), the result holds for any λ-measurable set A, then $\mu \in \mathcal{P}^{ac}(\mathbb{R})$. The estimate $\|\mu\|_{L^\infty} \leq \limsup_{n\to\infty}\|\mu_n\|_{L^\infty}$ is again a consequence of (3.9). $\qquad\square$

The goal of this section is to prove the following result of existence and uniqueness for small times.

Proposition 18 *Let the following hypotheses hold:*

(H): *The function $v:[0,1]\to\mathbb{R}$ is Lipschitz and bounded. The interaction kernel w satisfies $w \in BV([\alpha,\beta],\mathbb{R}^+)$ for fixed $\alpha,\beta \in \mathbb{R}$, extended with zero in $\mathbb{R}\setminus[\alpha,\beta]$, and $\int_\alpha^\beta w(x)\,dx = 1$. The initial density ρ_0 satisfies $\rho_0 \in L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$.*

Then, there exists $T>0$ such that for all $t \in (0,T)$ there exists a unique weak solution $\rho \in C^0([0,t],L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$ of the Cauchy problem

$$\begin{cases} \partial_t \rho(t,x) + \partial_x(V[\rho(t)](x)\rho(t,x))=0, & x\in\mathbb{R},\ t>0, \\ \rho(0,x)=\rho_0(x), & x\in\mathbb{R}. \end{cases} \quad (3.10)$$

where

$$V[\rho(t)](x) = v\left(\int \rho(t,y)w(y-x)\,dy\right).$$

To prove this proposition, we first study in the following sections a set of useful technical lemmas related to BV kernels and the corresponding transport equations. The proof of Proposition 18 will be then given in Section 3.3.

3.1 Estimates for convolutions with BV kernels

In this section, we prove some useful estimates for BV functions and functions defined by convolutions with BV kernels.

Lemma 19 *Let $f \in BV([\alpha,\beta],\mathbb{R})$. Let $[a,b] \subset [\alpha,\beta]$ and h such that $\alpha \le a-h \le b+h \le \beta$. Then it holds*

$$\int_a^b |f(x) - f(x-h)|\,dx \le |h|TV(f).$$

Above, $TV(f)$ denotes the (bounded) total variation of f.

Proof. See also [2, Remark 3.25]. Assume that f is non-decreasing and $h > 0$. It holds

$$\begin{aligned}
0 &\le \int_a^b f(x) - f(x-h)\,dx \\
&= \int_a^b (f(x) - f(a-h))\,dx - \int_a^b (f(x-h) - f(a-h))\,dx \\
&= \int_a^b (f(x) - f(a-h))\,dx - \int_{a-h}^{b-h} (f(x) - f(a-h))\,dx \\
&\le \int_{b-h}^b (f(x) - f(a-h))\,dx \\
&\le \int_{b-h}^b (f(b) - f(a-h))\,dx = h(f(b) - f(a-h)) \\
&\le hTV(f).
\end{aligned}$$

The proof for $h < 0$ is identical.

Let $f = g - l$ with g,l non-decreasing and $TV(f) = TV(g) + TV(l)$. Then it holds

$$\begin{aligned}
\int_a^b |f(x) - f(x-h)|\,dx &= \int_a^b |(g(x) - g(x-h)) - (l(x) - l(x-h))|\,dx \\
&\le \int_a^b |g(x) - g(x-h)|\,dx + \int_a^b |l(x) - l(x-h)|\,dx \\
&\le |h|(TV(g) + TV(l)) \\
&= |h|TV(f).
\end{aligned}$$

$\square$

Lemma 20 *Let $\mu,\mu' \in L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$ and $w \in BV([\alpha,\beta],\mathbb{R})$. Then it holds*

$$\left|\int w(x)\,d(\mu(x) - \mu'(x))\,dx\right| \le W_\infty(\mu,\mu')TV(w)\min\{\|\mu\|_{L^\infty}, \|\mu'\|_{L^\infty}\} \qquad (3.11)$$

Proof. Let $\pi \in \Pi(\mu, \mu')$ be a transference plan realizing $h := W_\infty(\mu, \mu')$, that exists due to [10, Prop. 2.1]. Decompose $w = f - g$ with f, g non-decreasing functions via the Jordan decomposition on the interval $[\alpha, \beta]$. We have

$$\int w(x) d(\mu(x) - \mu'(x)) = \int (f(x) - g(x)) d\pi(x, y) - \int (f(y) - g(y)) d\pi(x, y)$$

$$= \int (f(x) - f(y)) d\pi(x, y) + \int (g(y) - g(x)) d\pi(x, y)$$

$$\leq \int (f(x) - f(x - h)) d\pi(x, y) + \int (g(x + h) - g(x)) d\pi(x, y)$$

$$= \int (f(x) - f(x - h)) d\mu(x) + \int (g(x + h) - g(x)) d\mu(x)$$

$$\leq \|\mu\|_{L^\infty} \left(\int (f(x) - f(x - h)) dx + \int (g(x + h) - g(x)) dx \right)$$

$$\leq \|\mu\|_{L^\infty} h \, TV(w),$$

where we used that points (x, y) in the support of π satisfy $|x - y| \leq h$ except for a set of zero measure. By replacing w with $-w$, we have the absolute value on the left hand side of (3.11). Since the estimate is symmetric with respect to μ, μ', one has the result. $\square$

Remark 21 *The main reason for which Lemma 20 holds only for measures μ, μ' that are absolutely continuous with respect to the Lebesgue measure is that one needs to give sense to the integral $\int w(x) d\mu(x)$ when w is a BV function. The proposition holds for real measures since we need to use the Jordan decomposition of the BV functions.*

Proposition 22 *Let $w \in BV([\alpha, \beta], \mathbb{R})$ and $\rho \in L^\infty(\mathbb{R})$. Then the function $f(x) := \int \rho(y) w(y - x) dy$ is Lipschitz, with Lipschitz constant $L \leq \|\rho\|_{L^\infty} TV(w)$.*

Proof. By using Lemma 19, we have

$$|f(x_1) - f(x_2)| \leq \int \rho(y) |w(y - x_1) - w(y - x_2)| dy$$

$$\leq \|\rho\|_{L^\infty} \int |w(y - x_1) - w(y - x_1 + (x_1 - x_2))| dy$$

$$\leq \|\rho\|_{L^\infty} |x_1 - x_2| TV(w).$$

$\square$

Proposition 23 *Let $w, w' \in BV([\alpha, \beta], \mathbb{R})$ and $\rho, \rho' \in L^\infty(\mathbb{R})$. Then the functions*

$$f(x) := \int \rho(y) w(y - x) dy, \qquad f'(x) := \int \rho'(y) w'(y - x) dy$$

satisfy

$$\|f - f'\|_{C^0} \leq W_\infty(\rho, \rho') TV(w) \min\{\|\rho\|_{L^\infty}, \|\rho'\|_{L^\infty}\} + \|\rho'\|_{L^\infty} \|w - w'\|_{L^1}.$$

Proof. We have

$$|f(x) - f'(x)| \leq \left| \int \rho(y) w(y - x) - \rho'(y) w(y - x) dy \right| + \left| \int \rho'(y) w(y - x) - \rho'(y) w'(y - x) dy \right|$$

$$\leq W_\infty(\rho, \rho') TV(w) \min\{\|\rho\|_{L^\infty}, \|\rho'\|_{L^\infty}\} + \|\rho'\|_{L^\infty} \|w - w'\|_{L^1},$$

where we used Lemma 20 for the first integral and the L^1-L^∞ duality for the second integral. $\qquad\square$

3.2 Solution of transport equations with time-dependent interactions

In this section, we give estimates for solutions of the transport equations in which the vector field is time-dependent, but not depending on the solution itself.

Proposition 24 *Let* **(H)** *hold. Fix $\bar\rho \in C^0([0,T], L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$ and define the time-dependent vector field V_t as follows:*

$$V_t(x) := v\left(\int \bar\rho(t,y)w(y-x)\,dy\right).$$

Define $\rho(t)$ as the unique solution of

$$\partial_t \rho(t,x) + \partial_x(V_t(x)\rho(t,x)) = 0 \tag{3.12}$$

with the given initial data $\rho_0 \in L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$. Then it holds

$$W_\infty(\rho(t,\cdot), \rho(t+s,\cdot)) \le s\|v\|_\infty, \tag{3.13}$$

and

$$e^{-Lt}\|\rho_0\|_{L^\infty} \le \|\rho(t,\cdot)\|_{L^\infty} \le e^{Lt}\|\rho_0\|_{L^\infty}, \tag{3.14}$$

with $L := Lip(v)\,TV(w)\sup_{s\in[0,t]}\{\|\bar\rho_s\|_{L^\infty}\}$.

Proof. Let ϕ_t^V be the flow generated by V_t in the time interval $[t, t+s]$. Then the solution ρ of (3.12) is unique, and given by the push-forward $\rho(t,\cdot) = \phi_t^V \# \rho_0$, see Proposition 14.

We first prove (3.13). Since it holds $\rho(t+s,\cdot) = \phi_s^V \# \rho(t,\cdot)$, then the transference plan $\pi(x,y) = (\text{Id} \times \phi_s^V)\#\rho(t,\cdot)$ satisfies $\pi \in \Pi(\rho(t,\cdot), \rho(t+s,\cdot))$. Then it holds

$$W_\infty(\rho(t,\cdot), \rho(t+s,\cdot)) \le C_\infty(\pi) = \sup_{x\in\text{supp}(\rho(t,\cdot))}\left\{|x - \phi_s^V(x)|\right\} \le s\sup(|v|).$$

We now prove (3.14). Observe that the Gronwall lemma gives

$$e^{-L|t|}|b-a| \le |\phi_t^V(b) - \phi_t^V(a)| \le e^{L|t|}|b-a|, \tag{3.15}$$

where L is the Lipschitz constant of V_s in the interval $[0,t]$, that is $L = Lip(v)\,TV(w)\sup_{s\in[0,t]}\{\|\bar\rho_s\|_{L^\infty}\}$ by Proposition 22. This implies that for any interval (a,b) there holds

$$\int_a^b \rho(t,x)\,dx = \int_a^b \phi_t^V\#\rho_0(x)\,dx = \int_{\phi_t^V(a)}^{\phi_t^V(b)}\rho_0(x)\,dx \le \|\rho_0\|_{L^\infty}|\phi_t^V(b) - \phi_t^V(a)| \le e^{L|t|}\|\rho_0\|_{L^\infty}|b-a|,$$

that implies $\|\rho(t)\|_{L^\infty} \le e^{Lt}\|\rho_0\|_{L^\infty}$. By reversing time, we have the reverse inequality. $\square$

We now have the following comparison result.

Proposition 25 *Let $\bar{\rho}_t, \bar{\rho}'_t \in C^0([0,T], L^\infty \cap \mathcal{P}^{ac}_c(\mathbb{R}))$ be given and w, w' satisfy* **(H)**. *Define the time-dependent vector fields V_t, V'_t as follows:*

$$V_t(x) := v\left(\int \bar{\rho}(t,y) w(y-x) \, dy\right), \qquad V'_t(x) := v\left(\int \bar{\rho}'(t,y) w'(y-x) \, dy\right).$$

Define $\rho(t), \rho'(t)$ as the unique solutions of

$$\partial_t \rho(t,x) + \partial_x(V_t(x)\rho(t,x)) = 0, \qquad \partial_t \rho'(t,x) + \partial_x(V'_t(x)\rho'(t,x)) = 0,$$

with initial data ρ_0, ρ'_0, respectively. Then it holds

$$W_\infty(\rho(t,\cdot), \rho'(t,\cdot)) \le e^{L|t|} W_\infty(\rho_0, \rho'_0) +$$
$$+ (e^{L|t|} - 1)\left(\sup_{s \in [0,t]} W_\infty(\bar{\rho}(s,\cdot), \bar{\rho}'(s,\cdot)) + \frac{\|w - w'\|_{L^1}}{TV(w')}\right) \quad (3.16)$$

with $L := Lip(v) TV(w) \sup_{s \in [0,t]} \max\{\|\bar{\rho}(s,\cdot)\|_{L^\infty}, \|\bar{\rho}'(s,\cdot)\|_{L^\infty} TV(w')\}$.

Proof. By Proposition 22, we have that both V_t and V'_t are Lipschitz vector fields with Lipschitz constant L. Denote with ϕ_t, ϕ'_t the flows of the two vector fields, that are defined for every t with $0 \le t \le T$, since the vector fields are globally Lipschitz. Then, we have the following estimate by the Gronwall lemma:

$$|\phi_t(x) - \phi'_t(y)| \le e^{L|t|}|x-y| + \frac{e^{L|t|} - 1}{L} \sup_{s \in [0,t]} \|V_s - V'_s\|_{C^0}.$$

By definition of V_s and V'_s, Proposition 23 gives $\|V_s - V'_s\|_{C^0} \le$

$$Lip(v)\left(W_\infty(\bar{\rho}(s,\cdot), \bar{\rho}'(s,\cdot) TV(w) \min\{\|\bar{\rho}(s,\cdot)\|_\infty, \|\bar{\rho}'(s,\cdot)\|_\infty\} + \|\bar{\rho}'(s,\cdot)\|_{L^\infty} \|w - w'\|_{L^1}\right).$$

By plugging the explicit expression of L, and with a careful choice in the min and max terms, we have

$$|\phi_t(x) - \phi'_t(y)| \le e^{L|t|}|x-y| + (e^{L|t|} - 1)\left(\sup_{s \in [0,t]} W_\infty(\bar{\rho}(s,\cdot), \bar{\rho}'(s,\cdot)) + \frac{\|w - w'\|_{L^1}}{TV(w')}\right).$$

By reversing time, we also have

$$|\phi_t(x) - \phi'_t(y)| \ge e^{-L|t|}|x-y| - (1 - e^{-L|t|})\left(\sup_{s \in [0,t]} W_\infty(\bar{\rho}(s,\cdot), \bar{\rho}'(s,\cdot)) + \frac{\|w - w'\|_{L^1}}{TV(w')}\right).$$

Take now $\pi_0 \in \Pi(\rho_0, \rho'_0)$ and observe that $\pi_t := (\phi_t \times \phi'_t) \# \pi_0$ satisfies $\pi_t \in \Pi(\rho(t,\cdot), \rho'(t,\cdot))$. Consider the set $E_k := \{(x,y) \in \mathbb{R}^2 \mid |x-y| > k\}$ and observe that it holds

$$\pi_t(E_k) = \pi_0\left(\{(\phi_t(x), \phi'_t(y) \mid |x-y| > k\}\right) \le \pi_0\left(\left\{(x,y) \mid |x-y| > \tilde{k}\right\}\right),$$

with $\tilde{k} := e^{-L|t|} k - (1 - e^{-L|t|})\left(\sup_{s \in [0,t]} W_\infty(\bar{\rho}(s,\cdot), \bar{\rho}'(s,\cdot)) + \frac{\|w-w'\|_{L^1}}{TV(w')}\right)$.

If $\tilde{k} > C_\infty(\pi_0)$, one has $\pi_0\left(\left\{(x,y) \mid |x-y| > \tilde{k}\right\}\right) = 0$ by definition of $C_\infty(\pi_0)$. This implies that $\pi_t(E_k) = 0$ for all k satisfying

$$k > e^{L|t|} C_\infty(\pi_0) + (e^{L|t|} - 1)\left(\sup_{s \in [0,t]} W_\infty(\bar{\rho}(s,\cdot), \bar{\rho}'(s,\cdot)) + \frac{\|w - w'\|_{L^1}}{TV(w')}\right),$$

hence

$$C_\infty(\pi_t) \le e^{L|t|} C_\infty(\pi_0) + (e^{L|t|} - 1) \left(\sup_{s \in [0,t]} W_\infty(\bar\rho(s,\cdot), \bar\rho'(s,\cdot)) + \frac{\|w - w'\|_{L^1}}{TV(w')} \right). \quad (3.17)$$

By recalling that $W_\infty(\rho(t,\cdot), \rho'(t,\cdot)) \le C_\infty(\pi_t)$ by definition, and passing to the infimum among all transference plans $\pi_0 \in \Pi(\rho_0, \rho_0')$ on the right hand side of (3.17), we find (3.16). $\qquad\square$

3.3   Proof of Proposition 18

We first prove existence of a solution of (3.10) for small times, via convergence of an explicit Euler scheme. We then prove uniqueness of the solution.

Proposition 26 *Let **(H)** hold, and $T > 0$ be fixed, with $T < (e \mathrm{Lip}(v) TV(w) \|\rho_0\|_{L^\infty})^{-1}$. For each $n \in \mathbb{N}$, consider the following trajectory $\rho^n \in C^0([0,T], L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$:*

- *$\rho^n(0,\cdot) := \rho_0$;*

- *$\rho^n(k 2^{-n} T + t, \cdot) := \phi_t^{n,k} \# \rho^n(k 2^{-n} T, \cdot)$ for $t \in [0, 2^{-n} T]$ and $k = 0, 1, \ldots, 2^n - 1$, where $\phi_t^{n,k}$ is the flow generated by the vector field*

$$V_{n,k}(x) := v \left(\int \rho^n(k 2^{-n} T, y) w(y - x) \, dy \right).$$

Then, there exists a subsequence converging to $\rho^* \in C^0([0,T], L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$ that satisfies $\rho^*(0) = \rho_0$ and is a weak solution of

$$\partial_t \rho(t,x) + \partial_x (V[\rho(t)](x) \rho(t,x)) = 0, \quad (3.18)$$

where

$$V[\rho(t)](x) = v \left(\int \rho(t,y) w(y - x) \, dy \right).$$

Moreover, ρ^* satisfies $\|\rho^*(t)\|_{L^\infty} \le e \|\rho_0\|_{L^\infty}$ for all $t \in [0,T]$.

Proof. Observe that (3.13) together with boundedness of v implies that the sequence ρ^n is equibounded and equi-Lipschitz in $C^0([0,T], \mathcal{P}_c(\mathbb{R}))$. Then the Ascoli-Arzelà theorem implies the existence of a converging subsequence, that we denote again with ρ^n. We denote the limit with ρ^*, that satisfies $\rho^*(0) = \rho_0$ and $\rho^* \in C^0([0,T], \mathcal{P}_c(\mathbb{R}))$.

Observe that $\mathcal{P}_c(\mathbb{R})$ is complete with respect to the W_∞ distance, but this is not the case for $L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$. Then, we now prove that $\rho^*(t) \in L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$ for $t \in [0,T]$, by proving that $\sup_{t \in [0,T], n \in \mathbb{N}} \|\rho^n(t)\|_{L^\infty} < +\infty$ for $T < (e \mathrm{Lip}(v) TV(w) \|\rho_0\|_{L^\infty})^{-1}$ and applying Lemma 17. Denote $a_k^n := \|\rho^n(k 2^{-n} T, \cdot)\|_{L^\infty}$ and observe that (3.14), together with the definition of $\rho^n((k+1) 2^{-n} T, \cdot)$ as function of $\rho^n(k 2^{-n} T, \cdot)$, gives the following recurrence rule

$$a_{k+1}^n \le a_k^n e^{2^{-n} T \mathrm{Lip}(v) TV(w) a_k^n},$$

where we have set $a_0^n = \|\rho_0\|_{L^\infty}$. We now prove by induction that $a_k^n \le e \|\rho_0\|_{L^\infty}$ for all $k \le 2^n$. We clearly have $a_0^n \le \|\rho_0\|_{L^\infty}$. Assume that a_j^n satisfies $a_j^n \le e \|\rho_0\|_{L^\infty}$ for all $j \le k$. Then we have

$$a_{j+1}^n \le a_j^n e^{2^{-n} T \mathrm{Lip}(v) TV(w) a_j^n} \le a_j^n e^{2^{-n}} \qquad \text{for}\ \ j = 0, \ldots, k,$$

that in turn implies $a_{k+1}^n \leq a_0^n e^{k2^{-n}} \leq e a_0^n$ for $k \leq 2^n$. This implies $\sup_{n \in \mathbb{N}, k=\{0,1,\dots,2^k\}} \|\rho^n(k2^{-n}T, \cdot)\|_{L^\infty} \leq e\|\rho_0\|_{L^\infty}$.

Due to Lipschitzianity of the L^∞ norm given by (3.14), we have

$$\|\rho^n(t,\cdot)\|_{L^\infty} \leq e^{\mathrm{Lip}(v)e\|\rho_0\|_{L^\infty} TV(w)2^{-n}T} e\|\rho_0\|_{L^\infty},$$

that in turn implies $\limsup_{n \to \infty} \|\rho^n(t,\cdot)\|_{L^\infty} \leq e\|\rho_0\|_{L^\infty}$ for all $t \in [0,T]$. Then, by Lemma 17, we have $\rho^* \in C^0([0,T], L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$ and $\|\rho^*(t,\cdot)\|_{L^\infty} \leq e\|\rho_0\|_{L^\infty}$ for all $t \in [0,T]$.

We now prove that the limit ρ^* is a weak solution of (3.18). Let $\phi \in C_c^\infty((0,T) \times \mathbb{R}, \mathbb{R})$: we need to prove that

$$\int_0^T \int_{\mathbb{R}} \left(\rho^*(t,x)\partial_t\phi(t,x) + v\left(\int \rho^*(t,y)w(y-x)\,dy \right) \rho^*(t,x)\partial_x\phi(t,x) \right) dx\,dt = 0. \quad (3.19)$$

Since by construction of ρ^n the following identity holds

$$\int_0^T \int_{\mathbb{R}} \left(\rho^n(t,x)\partial_t\phi(t,x) + V_{n,k}(x)\rho^n(t,x)\partial_x\phi(t,x) \right) dx\,dt = 0,$$

we prove (3.19) by proving the following three limits:

$$\lim_{n \to \infty} \int_0^T \int_{\mathbb{R}} \left(\rho^*(t,x) - \rho^n(t,x) \right) \partial_t\phi(t,x)\,dx\,dt = 0; \quad (3.20)$$

$$\lim_{n \to \infty} \int_0^T \int_{\mathbb{R}} \left(v\left(\int \rho^*(t,y)w(y-x)\,dy \right) - V_{n,k}(x) \right) \rho^*(t,x)\partial_x\phi(t,x)\,dx\,dt = 0; \quad (3.21)$$

$$\lim_{n \to \infty} \int_0^T \int_{\mathbb{R}} V_{n,k}(x) \left(\rho^*(t,x) - \rho^n(t,x) \right) \partial_x\phi(t,x)\,dx\,dt = 0. \quad (3.22)$$

Observe that convergence of ρ^n to ρ^* in $C^0([0,T], L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$ implies the existence of a sequence $\varepsilon_n \to 0$ such that $\sup_{t \in [0,T]} W_\infty(\rho^n(t,\cdot), \rho^*(t,\cdot)) < \varepsilon_n$. Also recall that $\|\rho^*(t,.)\|_{L^\infty} < e\|\rho_0\|_{L^\infty}$ for all $t \in [0,T]$.

To prove (3.20), observe that for each $t \in [0,T]$ the function $\partial_t\phi(t,.)$ is a BV function, hence there exists $C_1 := \sup_{t \in [0,T]} TV(\partial_t\phi(t,.)) < +\infty$. Then, by Lemma 20, it holds

$$\int_0^T \left| \int_{\mathbb{R}} \left(\rho^*(t,x) - \rho^n(t,x) \right) \partial_t\phi(t,x)\,dx \right| dt \leq \int_0^T \varepsilon_n C_1 e\|\rho_0\|_{L^\infty},$$

that provides (3.20). To prove (3.21), observe that for each n there holds

$$\sup_{t \in [0,T], x \in \mathbb{R}} \left| v\left(\int \rho^*(t,y)w(y-x)\,dy \right) - V_{n,k}(x) \right|$$

$$\leq \mathrm{Lip}(v) \sup_{k \in \{0,\dots,2^n-1\}, t \in [0,2^{-n}T], x \in \mathbb{R}} \left| \int \left(\rho^*(k2^{-n}T+t,y) - \rho^n(k2^{-n}T,y) \right) w(y-x)\,dy \right|$$

$$\leq \mathrm{Lip}(v) \sup_{k \in \{0,\dots,2^n-1\}, t \in [0,2^{-n}T]} W_\infty\left(\rho^*(k2^{-n}T+t,\cdot), \rho^n(k2^{-n}T,\cdot) \right) TV(w) e\|\rho_0\|_{L^\infty}$$

$$\leq \mathrm{Lip}(v) \left(2^{-n}T \sup(|v|) + \varepsilon_n \right) TV(w) e\|\rho_0\|_{L^\infty}, \quad (3.23)$$

where we used Lemma 20 and the triangular inequality

$$W_\infty\left(\rho^*(k2^{-n}T+t,\cdot), \rho^n(k2^{-n}T,\cdot) \right) \leq W_\infty\left(\rho^*(k2^{-n}T+t,\cdot), \rho^*(k2^{-n}T,\cdot) \right) +$$
$$W_\infty\left(\rho^n(k2^{-n}T,\cdot), \rho^n(k2^{-n}T,\cdot) \right)$$

and (3.13). Then, going back to the left hand side of (3.21), observe that $\|\rho^*(t,\cdot)\|_{L^\infty} \leq e\|\rho_0\|_{L^\infty}$ and integrate on the bounded support of $\partial_x \phi$. By passing to the limit in (3.23), we have (3.21).

Finally, to prove (3.22), observe that Proposition 22 provides Lipschitzianity of $V_{n,k}$, with uniform Lipschitz constant $\text{Lip}(|v|)TV(w)e\|\rho_0\|_{L^\infty}$. Also, $\|V_{n,k}\|_{C_0}$ is uniformly bounded by $\|v\|_\infty$. Since one has Lipschitzianity and boundedness of $\partial_x \phi$ too, we have uniform Lipschitzianity of $V_{n,k}\partial_x \phi$. This property, together with boundedness of the support, implies uniform bounded variation, i.e. the existence of C_2 such that for all n,k it holds $TV(V_{n,k}\partial_x \phi) < C_2$. Then, Lemma 20 provides

$$\int_{\mathbb{R}} |V_{n,k}(x)(\rho^*(t,x) - \rho^n(t,x))\partial_x\phi(t,x)|\, dx \leq W_\infty(\rho^*(t,x),\rho^n(t,x))C_2 e\|\rho_0\|_{L^\infty} \leq \varepsilon_n C_2 e\|\rho_0\|_{L^\infty}.$$

By integrating with respect to time and passing to the limit, we have (3.22). $\qquad\square$

Remark 27 *It is interesting to observe that one can have blow-up of the density for solutions of (3.18). A simple example is given by $w = \chi_{[0,1]}$, $v(x) = x$ and $\rho_0 = \chi_{[-1,0]}$. Indeed, observe that the vector field $V[\rho_0]$ is a non-negative and non-increasing function satisfying $V[\rho_0](0) = 0$. The evolution $\rho(t)$ has then support contained in $[a(t);0]$ for all times in which it is defined, with $a(0) = -1$ and $a(t) \geq -1$. Observe that $V[\rho(t)](a(t)) = v(\int_{a(t)}^{a(t)+1} \rho(t)\, dx) = v(1) = 1$. Then $a(t) = -1 + t$. This implies that $\|\rho(t)\|_{L^\infty} \geq (1-t)^{-1}$, hence $\|\rho(t)\|_{L^\infty} \to \infty$ for $t \to 1$.*

This is in sharp contrast with the case of solutions of (3.18) with Lipschitz kernels w. Indeed, in this case hypotheses of Theorem 15 are satisfied, then one has existence and uniqueness of the solution for all times. Moreover, when the initial data μ_0 is a probability density in $\mathcal{P}_c^{ac}(\mathbb{R}^d)$, then it keeps being in $\mathcal{P}_c^{ac}(\mathbb{R}^d)$.

We are now ready to prove existence and uniqueness of the solution, i.e. Proposition 18.

Proof of Proposition 18. Choose $T < (e\text{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$ and apply Proposition 26 to have existence of a solution. Assume now to have two solutions $\bar{\rho}, \bar{\rho}' \in C^0([0,T], L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$ and define the time-dependent vector fields V_t, V_t' as follows:

$$V_t(x) := v\left(\int \bar{\rho}(t,y)w(y-x)\, dy\right), \qquad V_t'(x) := v\left(\int \bar{\rho}'(t,y)w(y-x)\, dy\right).$$

Then $\bar{\rho}, \bar{\rho}'$ clearly coincide with the solutions ρ, ρ' of

$$\partial_t \rho(t,x) + \partial_x(V_t(x)\rho(t,x)) = 0, \qquad \partial_t \rho'(t,x) + \partial_x(V_t'(x)\rho'(t,x)) = 0,$$

respectively, with initial data ρ_0.

We prove uniqueness by contradiction. Let $t_0 := \inf_{t \in [0,T]}\{W_\infty(\rho(t,\cdot),\rho'(t,\cdot)) > 0\}$ be the first time in which ρ, ρ' do not coincide. By applying Proposition 25 starting from t_0, we have

$$W_\infty(\rho(t_0+s,\cdot),\rho'(t_0+s,\cdot)) \leq (e^{Ls}-1)\sup_{s' \in [0,s]} W_\infty(\bar{\rho}(t_0+s',\cdot),\bar{\rho}'(t_0+s',\cdot)), \quad (3.24)$$

that implies $\sup_{s \in [0,t]} W_\infty(\rho(t_0+s,\cdot),\rho'(t_0+s,\cdot)) \leq (e^{Lt} - 1)\sup_{s \in [0,t]} W_\infty(\bar{\rho}(t_0+s,\cdot),\bar{\rho}'(t_0+s,\cdot))$. By choosing $t < \frac{\ln(2)}{L}$ we have $W_\infty(\rho(t_0+s,\cdot),\rho'(t_0+s,\cdot)) = 0$ for all $s \in [0,t]$, that implies

$$\inf_{t \in [0,T]}\{W_\infty(\rho(t,\cdot),\rho'(t,\cdot)) > 0\} > t_0,$$

which gives a contradiction. $\qquad\square$

Finally, we prove continuous dependence on the interaction kernel and on initial data for the solution of (3.18).

Proposition 28 *Let* **(H)** *hold. Let*

$$T < (e\mathrm{Lip}(v)TV(w)\max\{\|\rho_0\|_{L^\infty}, \|\rho_0\|_{L^\infty}\})^{-1}$$

and ρ,ρ' be the unique solutions of the Cauchy problems

$$\begin{cases} \partial_t\rho(t,x) + \partial_x(V\,[\rho(t)]\,(x)\rho(t,x)) = 0, \\ \rho(0,\cdot) = \rho_0 \end{cases} \qquad \begin{cases} \partial_t\rho'(t,x) + \partial_x(V'\,[\rho'(t)]\,(x)\rho'(t,x)) = 0, \\ \rho'(0,\cdot) = \rho_0' \end{cases}$$

where

$$V\,[\rho(t)]\,(x) = v\left(\int \rho(t,y)w(y-x)\,dy\right), \qquad V'\,[\rho'(t)]\,(x) = v\left(\int \rho'(t,y)w'(y-x)\,dy\right).$$

Then it holds

$$W_\infty(\rho(t,\cdot),\rho'(t,\cdot)) \le e^{4eLt}W_\infty(\rho_0,\rho_0') + (e^{4eLt}-1)\frac{\|w-w'\|_{L^1}}{TV(w)}, \qquad (3.25)$$

with $L = \mathrm{Lip}(v)\max\{\|\rho_0\|_{L^\infty}TV(w), \|\rho_0'\|_{L^\infty}TV(w')\}$.

Proof. We apply Proposition 25 with $\rho = \bar{\rho}$ and $\rho' = \bar{\rho}'$. Moreover, since $T < (e\mathrm{Lip}(v)TV(w)\max\{\|\rho_0\|_{L^\infty}, \|\rho_0\|_{L^\infty}\})^{-1}$, there holds $\|\rho(t,\cdot)\|_{L^\infty} \le e\|\rho_0\|_{L^\infty}$, $\|\rho'(t,\cdot)\|_{L^\infty} \le e\|\rho_0'\|_{L^\infty}$ for all $t \in [0,T]$. Then (3.16) reads as

$$W_\infty(\rho(t,\cdot),\rho'(t,\cdot)) \le e^{eLt}W_\infty(\rho_0,\rho_0') + (e^{eLt}-1)\left(\sup_{s\in[0,t]}W_\infty(\rho(t,\cdot),\rho'(t,\cdot)) + \frac{\|w-w'\|_{L^1}}{TV(w)}\right)$$

with $L = \mathrm{Lip}(v)\max\{\|\rho_0\|_{L^\infty}TV(w), \|\rho_0'\|_{L^\infty}TV(w')\}$. By applying the Gronwall lemma, we have (3.25). $\qquad\square$

4 Finite dimensional approximation: the micro-macro limit

In this section, we describe a finite-dimensional approximation for the solution of (1.1), that may serve as a numerical scheme to compute such solution.

We first define a Lipschitz approximation w_ℓ for the interaction kernel w.

Definition 29 *Let $\eta > 0$ be fixed, and $w:[0,\eta] \to \mathbb{R}^+$ be a non-increasing C^1 function such that $\int_0^\eta w(x)\,dx = 1$. We define $w_\ell:\mathbb{R} \to \mathbb{R}^+$ as follows*

$$w_\ell(x) := \begin{cases} w(0)\frac{\ell+2x}{\ell} & \text{if } x \in \left[-\frac{\ell}{2}, 0\right] \\ w(x) & \text{if } x \in [0,\eta] \\ w(\eta)\frac{2\eta+\ell-2x}{\ell} & \text{if } x \in \left[\eta, \eta+\frac{\ell}{2}\right] \\ 0 & \text{elsewhere.} \end{cases} \qquad (4.26)$$

We also define a discretization $[\mu]^n$ of a given probability density $\mu \in \mathcal{P}^{ac}(\mathbb{R})$. As stated at the beginning, the same idea can be adapted to any density $\mu \in \mathcal{M}^{ac}(\mathbb{R})$, not necessarily of mass one.

Definition 30 *Let $\mu \in \mathcal{P}_c^{ac}(\mathbb{R})$ and $n \in \mathbb{N}$ be fixed. Define $\{x_1, \ldots, x_n\}$ as follows*

$$\begin{cases} x_1 = \sup\left\{ x \in \mathbb{R} \mid \int_{-\infty}^x d\mu < \frac{1}{n} \right\}, \\ x_{i+1} = \sup\left\{ x \in \mathbb{R} \mid \int_{x_i}^x d\mu < \frac{1}{n} \right\}, \qquad \text{for } i = 1, \ldots, n-1, \end{cases} \tag{4.27}$$

and

$$[\mu]^n := \frac{1}{n} \sum_{i=1}^n \delta_{x_i}. \tag{4.28}$$

Observe that $[\mu]^n \notin \mathcal{P}^{ac}(\mathbb{R})$, hence convolutions with discontinuous kernels are not well defined. We now prove some simple properties of the discretization $[\mu]^n$.

Proposition 31 *Let $\mu \in \mathcal{P}_c^{ac}([a,b])$ and $n \in \mathbb{N}$ be fixed, and the x_i and $[\mu]^n$ defined as in (4.28). Then it holds*

$$|x_i - x_j| \geq |i - j| (n \|\mu\|_{L^\infty})^{-1}, \tag{4.29}$$

and

$$W_1(\mu, [\mu]^n) \leq \frac{b-a}{n}. \tag{4.30}$$

Proof. Assume $i > j$ and observe that it holds

$$\frac{|i-j|}{n} = \int_{x_j}^{x_i} d\mu \leq (x_i - x_j) \|\mu\|_{L^\infty},$$

that gives (4.29). To prove (4.30), define $x_0 := a$ and divide the interval $[x_0, x_n] \subset [a,b]$ in the n intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$. Observing that each of the intervals contains a mass of $\frac{1}{n}$, we have

$$W_1(\mu, [\mu]^n) \leq \sum_{i=1}^n W_1\left(\mu \chi_{[x_{i-1}, x_i]}, \frac{1}{n} \delta_{x_i} \right) \leq \sum_{i=1}^n \frac{1}{n}(x_i - x_{i-1}) = \frac{1}{n}(x_n - x_0) \leq \frac{1}{n}(b-a).$$

$\square$

We are now ready to define the approximation $[\rho]^n$ of the solution ρ of (1.1).

Definition 32 *Let $v : [0,1] \to \mathbb{R}^+$ be a Lipschitz and non-increasing function with[2] $v(1) = 0$. Let $\eta > 0$ be fixed, and $w : [0, \eta] \to \mathbb{R}^+$ be a non-increasing Lipschitz function with $\int_0^\eta w(x)\, dx = 1$. Let $\rho_0 \in L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$ be given.*
Fix $n \in \mathbb{N}$ and choose the constant $\ell_n := (n \|\rho_0\|_{L^\infty})^{-1}$. Define $[\rho(t)]^n$ as

$$[\rho(t)]^n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i(t)}, \tag{4.31}$$

---

[2] We extend v with the convention $v(x) = 0$ for $x > 1$.

where $\{x_1(t),\dots,x_n(t)\}$ is the unique solution of the finite-dimension dynamical system

$$\begin{cases} \dot{x}_n = v(0), \\ \dot{x}_i = v\left(\frac{1}{n}\sum_{j=1}^n w_{\ell_n}(x_j - x_i)\right) & \text{for } i = 1,\dots,n-1, \\ x_i(0) = x_{i,0} & \text{for } i = 1,\dots,n, \end{cases} \tag{4.32}$$

where the $x_{i,0}$ are given by the discretization (4.27) of ρ_0, i.e. $[\rho_0]^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_{i,0}}$.

It is clear that the solution of (4.32) exists and is unique for all times, due to the Lipschitzianity of w_{ℓ_n}. Also remark that $[\rho(t)]^n$ is the unique solution of the following transport equation

$$\begin{cases} \partial_t [\rho(t)]^n (x) + \partial_x \left([\rho(t)]^n (x) v\left(\int w_{\ell_n}(y - x) d[\rho(t)]^n (y)\right)\right) = 0, \\ [\rho(0)]^n = [\rho_0]^n. \end{cases} \tag{4.33}$$

The proof of such identity is direct, by rewriting the interaction kernel $\int w_{\ell_n}(y - x)\,d\mu^n(y)$ in the case of μ^n given by (4.31).

The main result of this section is the following convergence result, that will be proved in Section 4.2.

Theorem 33 *Let $v:[0,1]\to\mathbb{R}^+$ be a Lipschitz and non-increasing function with $v(1) = 0$. Let $\eta > 0$ be fixed, and $w:[0,\eta]\to\mathbb{R}^+$ be a non-increasing Lipschitz function with $\int_0^\eta w(x)\,dx = 1$. Let $\rho_0 \in BV(\mathbb{R},[0,1])$ with compact support and $\int \rho_0(x)\,dx = 1$.*

Fix any $T > 0$. Then, for each $n \in N$, the approximation μ^n given by (4.31) is defined in $C^0([0,T],\mathcal{P}_c(\mathbb{R}))$ and it satisfies $[\rho(t)]^n \rightharpoonup \rho(t)$, where $\rho \in C^0([0,T],\mathcal{P}_c(\mathbb{R}))$ is the unique solution of the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho v\left(\int_x^{x+\eta} \rho(t,y) w(y - x)\,dy\right)\right) = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{4.34}$$

In particular, ρ is defined for all times $t > 0$, and it satisfies

$$0 \le \rho(t,x) \le \max\{\rho_0\} \qquad \text{for a.e. } x \in \mathbb{R}, t > 0. \tag{4.35}$$

Before proving Theorem 33, we address some properties of the finite-dimensional problem (4.32).

4.1 Preservation of the minimal and maximal distances

In this section, we prove that, when the initial minimal distance among agents for the model (4.32) is larger than or equal to ℓ, then it keeps being larger than or equal to ℓ for all times.

Proposition 34 *Let $\ell \ge \ell_n$ be fixed. Consider a sequence $x_1^0 < x_2^0 < \dots < x_n^0$ and denote with $x(t) = (x_1(t),\dots,x_n(t))$ the unique solution of (4.32). If $x_i^0 - x_{i-1}^0 \ge \ell$ for all $i = 1,\dots,n$, then it holds $x_i(t) - x_{i-1}(t) \ge \ell$ for all times $t > 0$.*

Proof. We prove that $x_i - x_{i-1} = \ell$ implies $\dot{x}_i - \dot{x}_{i-1} \ge 0$, that clearly gives the result.
If $i = n$, we have $\dot{x}_n = v(0) = \max_{\sigma \in [0,1]} v(\sigma) \ge \dot{x}_{n-1}$. For $i < n$, we have

$$\dot{x}_i - \dot{x}_{i-1} = v\left(\frac{1}{n}\sum_{j=1}^n w_{\ell_n}(x_j - x_i)\right) - v\left(\frac{1}{n}\sum_{j=1}^n w_{\ell_n}(x_j - x_{i-1})\right).$$

Since v is a non-increasing function, we have $\dot{x}_i - \dot{x}_{i-1} \geq 0$ if and only if

$$\frac{1}{n}\sum_{j=1}^{n} w_{\ell_n}(x_j - x_i) \leq \frac{1}{n}\sum_{j=1}^{n} w_{\ell_n}(x_j - x_{i-1}) = \frac{1}{n}\sum_{j=2}^{n+1} w_\ell(x_{j-1} - x_{i-1}). \qquad (4.36)$$

For $j < i$, we have both $x_j - x_i \leq -\ell$ and $x_{j-1} - x_{i-1} \leq -\ell$, thus $w_{\ell_n}(x_j - x_i) = w(x_j - x_i) = 0$ and $w_{\ell_n}(x_{j-1} - x_{i-1}) = w(x_{j-1} - x_{i-1}) = 0$.

For $j = i$ we have $w_{\ell_n}(x_i - x_i) = w_{\ell_n}(x_{i-1} - x_{i-1}) = w_{\ell_n}(0) = w(0)$. Then, we can restrict ourselves to $j > i$. Condition (4.36) is verified if for each $j = i+1, \ldots, n$ we have

$$w_{\ell_n}(x_j - x_i) \leq w_{\ell_n}(x_{j-1} - x_{i-1}). \qquad (4.37)$$

Since $x_j - x_i > 0$ and $x_{j-1} - x_{i-1} > 0$ for $j > i$ and w_{ℓ_n} is non-increasing in $[0, +\infty]$ as a consequence of the fact that w is non-increasing in $[0, \eta]$, condition (4.37) is equivalent to $x_j - x_i \geq x_{j-1} - x_{i-1}$, i.e. $x_j - x_{j-1} \geq x_i - x_{i-1} = \ell$, which is true by hypothesis. $\qquad \square$

The above property is the discrete counterpart of the maximum principle (4.35). The symmetric result is also true, with a similar proof.

Proposition 35 *Let $L \geq \ell_n$ be fixed. Consider a sequence $x_1^0 < x_2^0 < \ldots < x_n^0$ and denote with $x(t) = (x_1(t), \ldots, x_n(t))$ the unique solution of (4.32). If $x_i^0 - x_{i-1}^0 \leq L$ for all $i = 1, \ldots, n$, then it holds $x_i(t) - x_{i-1}(t) \leq L$ for all times $t > 0$.*

Observe that, due to the compact support of the solution, the minimum value of the density is always zero, and Proposition 35 does not allow to recover a finer minimum principle as in [7, 17].

4.2   Convergence to the solution of (1.1)

We now prove Theorem 33. We combine estimates for solutions of the finite-dimensional problem (4.32) with estimates for solutions of the transport equation with a Lipschitz interaction kernel (2.7) and with estimate for solutions of the transport equation with a BV interaction kernel (1.1).

Proof of Theorem 33.   The idea of the proof is to prove convergence of the approximate solution $[\rho(t)]^n$ to the solution ρ of (1.1) by proving intermediate convergence results, divided in four steps. In the first step, we restrict ourselves to a subsequence of $[\rho(t)]^n$ admitting a limit ρ^* in $C^0([0,T], L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$. In the second step, we define a finite-dimensional approximation $[\tilde{\rho}(t)]^n$ and prove that $[\rho(t)]^n$ and $[\tilde{\rho}(t)]^n$ have the same limit ρ^*. In the third step, we define an approximation ρ^n in $\mathcal{P}^{ac}(\mathbb{R})$ and prove that $[\tilde{\rho}(t)]^n$ and ρ^n have the same limit ρ^*. Finally, in the fourth step we prove that the limit of ρ^n is exactly ρ, first for small times and then for any time.

Step 1. Fix any $T > 0$. We prove that the sequence $[\rho(t)]^n \in C^0([0,T], \mathcal{P}_c(\mathbb{R}))$ admits a subsequence (that we do not relabel) with a limit $\rho^* \in C^0([0,T], L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$, that moreover satisfies $\|\rho^*(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$.

Fix $[a,b]$ an interval containing the compact support of ρ_0. Then, by construction of $[\rho(0)]^n = [\rho_0]^n$, we have $\mathrm{supp}([\rho(0)]^n) \subset [a,b]$. Due to boundedness of v, we have both $\mathrm{supp}([\rho(t)]^n) \subset [a - T\sup(v), b + T\sup(v)]$ and $W_\infty([\rho(t+s)]^n, [\rho(t)]^n) \leq s\sup(v)$ for all $t \in [0,T]$ and $n \in \mathbb{N}$, i.e. equiboundedness and equi-Lipschitzianity of the sequence $[\rho(t)]^n$ with respect to the W_∞-distance. Then, eventually passing to a subsequence, there exists a limit $\rho^* \in C^0([0,T], \mathcal{P}_c(\mathbb{R}))$, that has both uniformly bounded support and uniform Lipschitz constant. From now on, the sequence of indexes n is always replaced by the subsequence with limit ρ^*.

We now prove that $\|\rho^*(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$. This is equivalent to prove that for each interval $[\alpha,\beta] \subset \mathbb{R}$ it holds $\int_\alpha^\beta \rho^*(t)\,dx \leq \|\rho_0\|_{L^\infty}(\beta-\alpha)$. Observe that convergence in W_∞ implies weak convergence of measures, then $[\rho(t)]^n \rightharpoonup \rho^*(t)$. For each $\varepsilon > 0$, consider a function $\phi_\varepsilon \in C^\infty(\mathbb{R},\mathbb{R})$ with support in $[\alpha-\varepsilon,\beta+\varepsilon]$ such that $\phi_\varepsilon(x) \in [0,1]$ for all x and $\phi_\varepsilon(x) = 1$ for $x \in [\alpha,\beta]$. Observe that, by definition of $[\rho(t)]^n$, for each fixed t it holds

$$\int \phi_\varepsilon(x)\,d[\rho(t)]^n(x) = \frac{1}{n} \sum_{x_i(t) \in (\alpha-\varepsilon,\beta+\varepsilon)} \phi_\varepsilon(x_i(t)) \leq \frac{1}{n}\frac{\beta-\alpha+2\varepsilon}{\ell_n},$$

where we used the fact that the number of $x_i(t)$ in the interval $(\alpha-\varepsilon,\beta+\varepsilon)$ is bounded from above, by preservation of the minimal distance $\ell_n = (n\|\rho_0\|_{L^\infty})^{-1}$, see Proposition 34. By replacing ℓ_n in the last term, by recalling that $[\rho(t)]^n \rightharpoonup \rho^*(t)$ and observing that $\phi_\varepsilon \geq \chi_{[\alpha,\beta]}$, for each fixed t we have

$$\int_\alpha^\beta d\rho^*(t,x) \leq \int \phi_\varepsilon(x)\,d\rho^*(t,x) \leq \|\rho_0\|_{L^\infty}(\beta-\alpha+2\varepsilon).$$

By passing to the limit for $\varepsilon \to 0$, we have that $\rho^* \in L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$ and it satisfies

$$\|\rho^*(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} \tag{4.38}$$

Step 2. We now define $[\tilde{\rho}(t)]^n$. For each n, consider the approximated kernel w_{m_n} with $m_n := \ln(n)^{-1}$. Then, define $[\tilde{\rho}(t)]^n$ similarly to $[\rho(t)]^n$, as follows:

$$[\tilde{\rho}(t)]^n = \frac{1}{n} \sum_{i=1}^n \delta_{y_i(t)},$$

where $\{y_1(t),\ldots,y_n(t)\}$ is the unique solution of the finite-dimension dynamical system

$$\begin{cases} \dot{y}_i = v\left(\frac{1}{n}\sum_{j=1}^n w_{m_n}(y_j-y_i)\right) & \text{for } i=1,\ldots,n \\ y_i(0) = x_{i,0} & \text{for } i=1,\ldots,n, \end{cases} \tag{4.39}$$

and the $x_{i,0}$ are given by the discretization (4.27) of ρ_0, i.e. $[\rho_0]^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_{i,0}}$. Remark that, similarly to (4.33), ν^n is the unique solution of the following transport equation

$$\begin{cases} \partial_t [\rho(t)]^n(x) + \partial_x\left([\rho(t)]^n(x)\,v\left(\int w_{m_n}(x-y)\,d[\rho(t)]^n(y)\right)\right) = 0, \\ [\rho(0)]^n = [\rho_0]^n. \end{cases} \tag{4.40}$$

We now prove that, for each $T > 0$ it holds $\lim_{n\to\infty} \sup_{t\in[0,T]} W_\infty([\rho(t)]^n,[\tilde{\rho}(t)]^n) = 0$. This implies that the limit of both $[\rho(t)]^n$ and $[\tilde{\rho}(t)]^n$ is ρ^*. For each n, we define $\varepsilon_i^n(t) := \sup_{s\in[0,t]} |x_i(s) - y_i(s)|$ and $\varepsilon^n(t) := \max_{i=1,\ldots,n} \varepsilon_i^n(t)$. Observe that we have $W_\infty([\rho(t)]^n,[\tilde{\rho}(t)]^n) \leq \varepsilon^n(t)$, by choosing the transference plan sending $x_i(t)$ to $y_i(t)$. Also observe that $\varepsilon_i^n(0) = \varepsilon^n(0) = 0$, since the initial data coincide.

Observe that the initial data satisfies $x_{i+1}^0 - x_i^0 \geq \ell_n$, then Proposition 34 for (4.32) with kernel w_{ℓ_n} gives $x_{i+1}(t) - x_i(t) \geq \ell_n$. The property does not hold for y_i for big n, since $m_n > \ell_n$.

We now estimate the evolution of ε_i^n. By the definition, we have $\varepsilon_n^n = 0$. By boundedness of v, we have $\varepsilon_i^n(t+s) - \varepsilon_i^n(t) \leq 2\sup(v)s$, hence the ε_i^n are Lipschitz function. By definition of the dynamics (4.32), we have

$$\dot{\varepsilon}_i^n \leq \text{Lip}(v)\frac{1}{n}\sum_{j=1}^n |w_{\ell_n}(x_j(t)-x_i(t)) - w_{m_n}(y_j(t)-y_i(t))|. \tag{4.41}$$

It is clear that $y_j(t) - y_i(t) = \varepsilon_j^n(t) - \varepsilon_i^n(t) + x_j(t) - x_i(t)$, hence $y_j(t) - y_i(t) \in [x_j(t) - x_i(t) - 2\varepsilon^n(t), x_j(t) - x_i(t) + 2\varepsilon^n(t)]$. For any fixed index i and for each time t, we divide the indexes j in the following five sets:

- j is such that $x_j(t) - x_i(t) \in \left] -\infty, -\frac{m_n}{2} - 2\varepsilon^n(t) \right]$. In this case, both $x_j(t) - x_i(t)$ and $y_j(t) - y_i(t)$ are in the interval $\left] -\infty, -\frac{m_n}{2} \right]$, for which it holds $w_{\ell_n}(x_j(t) - x_i(t)) = w_{m_n}(y_j(t) - y_i(t)) = 0$.

- j is such that $x_j(t) - x_i(t) \in \left(-\frac{m_n}{2} - 2\varepsilon^n(t), 2\varepsilon^n(t) \right)$. In this case, we simply estimate
$$|w_{\ell_n}(x_j(t) - x_i(t)) - w_{m_n}(y_j(t) - y_i(t))| \leq w(0).$$
Observe that $|x_j(t) - x_i(t)| \geq |j - i| \ell_n$, hence the number of indexes j in this set is smaller or equal than $\frac{\frac{m_n}{2} + 4\varepsilon^n(t)}{\ell_n}$.

- j is such that $x_j(t) - x_i(t) \in [2\varepsilon^n(t), \eta - 2\varepsilon^n(t)]$: in this case both $x_j(t) - x_i(t)$ and $y_j(t) - y_i(t)$ belong to the interval $[0, \eta]$, on which w_{ℓ_n} and w_{m_n} coincide with w, that is Lipschitz. Hence $|w_{\ell_n}(x_j(t) - x_i(t)) - w_{m_n}(y_j(t) - y_i(t))| \leq 2\mathrm{Lip}(w)\varepsilon^n(t)$. In this case the number of indexes j in this set is strictly smaller than $\frac{\eta}{\ell_n}$.

- j is such that $x_j(t) - x_i(t) \in (\eta - 2\varepsilon^n(t), \eta + \frac{m_n}{2} + 2\varepsilon^n(t))$. Similarly to the second case, we estimate $|w_{\ell_n}(x_j(t) - x_i(t)) - w_{m_n}(y_j(t) - y_i(t))| \leq w(0)$ and observe that the number of indexes j in this set is smaller or equal than $\frac{\frac{m_n}{2} + 4\varepsilon^n(t)}{\ell_n}$.

- j is such that $x_j(t) - x_i(t) \in \left[\eta + \frac{m_n}{2} + 2\varepsilon^n(t), +\infty \right[$. Similarly to the first case, we have $w_{\ell_n}(x_j(t) - x_i(t)) = w_{m_n}(y_j(t) - y_i(t)) = 0$.

By using the previous decomposition of the indexes in (4.41), we have

$$\dot{\varepsilon}_i^n(t) \leq \mathrm{Lip}(v)\frac{1}{n}\left(2w(0)\frac{\frac{m_n}{2} + 4\varepsilon^n(t)}{\ell_n} + 2\mathrm{Lip}(w)\varepsilon^n(t)\frac{\eta}{\ell_n} \right).$$

Taking the supremum over ε_i^n, and recalling that $\ell_n = (n\|\rho_0\|_{L^\infty})^{-1}$, we have

$$\dot{\varepsilon}^n(t) \leq C_0 m_n + C_1 \varepsilon^n(t)$$

with $C_0 := \mathrm{Lip}(v)w(0)\|\rho_0\|_{L^\infty}$, $C_1 := \mathrm{Lip}(v)\|\rho_0\|_{L^\infty}(8w(0) + 2\mathrm{Lip}(w)\eta)$. Since $\varepsilon^n(0) = 0$ and the constants C_0, C_1 do not depend on t, the Gronwall estimate gives

$$\varepsilon^n(t) \leq \frac{C_0}{C_1}m_n e^{C_1 t} = \frac{C_0}{C_1 \ln(n)}e^{C_1 t}.$$

This implies $W_\infty([\rho(t)]^n, [\tilde{\rho}(t)]^n) \leq \frac{C_0}{C_1 \ln(n)}e^{C_1 T}$ for all times $t \in [0, T]$. Then, the limit of $[\tilde{\rho}(t)]^n$ is ρ^*.

Step 3. Choose again $[a, b]$ any interval containing the support of the initial datum ρ_0, and fix any $T > 0$. We define $\rho^n \in C^0([0, T], L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$ as the solution of (4.34) with initial data ρ_0 when replacing w with w_{m_n}. Since w_{m_n} is Lipschitz, we have existence and uniqueness of ρ^n for any $t \in [0, T]$ thanks to Proposition 16.

Fix now $T_1 < (32\mathrm{Lip}(v)w(0))^{-1}$. We now compare $[\tilde{\rho}(t)]^n$ and $\rho^n(t)$ on the interval $[0, T_1]$, by observing that they are the solution of the same equation

$$\partial_t \mu + \partial_x \left(\mu v \left(\int w_{m_n}(y - x) \, d\mu(y) \right) \right) = 0, \tag{4.42}$$

with different initial data, that are $[\tilde{\rho}(0)]^n = [\rho_0]^n$ and $\rho^n(0,\cdot) = \rho_0$. Since (4.42) satisfies the hypotheses of Proposition 16, we have

$$W_1([\tilde{\rho}(t)]^n, \rho^n(t)) \le e^{8\mathrm{Lip}(v)\mathrm{Lip}(w_{m_n})T_1} W_1([\rho_0]^n, \rho_0) \le e^{16\mathrm{Lip}(v)w(0)\ln(n)T_1} W_1([\rho_0]^n, \rho_0) \le$$
$$\le \sqrt{n}\frac{b-a}{n} = \frac{b-a}{\sqrt{n}}$$

for a sufficiently big n, for which it holds $\mathrm{Lip}(w_{m_n}) \le 2\frac{w(0)}{m_n} \le 2w(0)\ln(n)$. We also used (4.30) to estimate the initial distance. Then, $[\tilde{\rho}(t)]^n$ and ρ^n have the same limit in the interval $[0,T_1]$.

To prove the result for the initial interval $[0,T]$, subdivide it in intervals $[kT_1,(k+1)T_1]$ with $k=0,1,\dots,\frac{T}{T_1}-1$. Then, by induction one can prove $W_1([\tilde{\rho}(t)]^n, \rho^n(t)) \le \frac{32\mathrm{Lip}(v)w(0)T(b-a)}{\sqrt{n}}$. As a consequence, the limit of ρ^n is ρ^* on the whole interval $[0,T]$.

Step 4. We now fix $T < (e\mathrm{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$ and study $W_\infty(\rho^n(t), \rho(t))$. In this case, we observe that both ρ^n and ρ are solutions of an equation of the form (1.1) with different interaction kernels w_{m_n} and w respectively, and the same initial data ρ_0. Observe that, by the particular structure of w and of its approximation w_{m_n}, it holds $TV(w) = TV(w_{m_n}) = 2w(0)$ and $\|w_{m_n} - w\|_{L^1} \le \frac{m_n w(0)}{2}$. Then, by Proposition 28, we have

$$W_\infty(\rho^n(t), \rho(t)) \le (e^{4eLt}-1)\frac{\|w_{m_n}-w\|_{L^1}}{TV(w)} \le (e^{4eLT}-1)\frac{m_n}{4} \le \frac{e^4-1}{4\ln(n)},$$

with $L = 2w(0)\mathrm{Lip}(v)\|\rho_0\|_{L^\infty}$. This implies that $\rho = \rho^*$, i.e. convergence of $[\rho(t)]^n$ to ρ in the interval $[0,T]$.

Going back to Step 1, we recall that we chose a converging subsequence of $[\rho(t)]^n$, and we proved that the limit of such sub-sequence is ρ, which is the unique solution of (1.1). Then we have that the whole sequence $[\rho(t)]^n$ converges to ρ, in the time interval $[0,T]$ with $T < (e\mathrm{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$.

We now prove that $\rho^* = \rho$ in the time interval $[0,T]$ for any $T>0$, and in particular that a solution for (1.1) exists (and is unique) for all times. For $T>0$ fixed, divide the interval $[0,T]$ in intervals $[kT',(k+1)T']$ with $T' < (e\mathrm{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$ and $k=0,1,\dots,\frac{T}{T'}-1$. Then, the previous results show that $\rho^* = \rho$ in $[0,T']$. This also implies

$$\|\rho(T')\|_{L^\infty} = \|\rho^*(T')\|_{L^\infty} \le \|\rho_0\|_{L^\infty}, \tag{4.43}$$

where we used estimate (4.38). Observe that T' satisfies hypotheses of Proposition 18 taking $\rho(T')$ as initial datum: this gives existence and uniqueness of the solution of (1.1) on the time interval $[T',2T']$, that coincides with ρ^* on the same interval, and for which it holds $\|\rho(2T')\|_{L^\infty} \le \|\rho_0\|_{L^\infty}$. By induction, we have $\rho^* = \rho$ on the whole interval $[0,T]$, that also gives $\|\rho(t)\|_{L^\infty} \le \|\rho_0\|_{L^\infty}$ for all $t \in [0,T]$, i.e. condition $\rho^*(t,x) \le \max(\rho_0)$ a.e. for $t \in [0,T], x \in \mathbb{R}$. $\qquad\square$

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