Qualitative Symbolic Perturbation: a new geometry-based perturbation framework

Olivier Devillers, Menelaos Karavelas, Monique Teillaud

Qualitative Symbolic Perturbation: two applications of a new geometry-based perturbation framework

Olivier Devillers., Menelaos I. Karavelas, Monique Teillaud

RESEARCH REPORT
N° 8153
December 2012
Project-Team Vegas

ISSN 0249-6399 ISRN INRIA/RR--8153--FR+ENG

Qualitative Symbolic Perturbation: two applications of a new geometry-based perturbation framework

Olivier Devillers[*†‡]., Menelaos I. Karavelas[§¶], Monique Teillaud

Project-Team Vegas

Research Report n° 8153 — version 2 — initial version December 2012 — revised version October 2015 — 34 pages

Abstract: In a classical *Symbolic Perturbation* scheme, degeneracies are handled by substituting some polynomials in ε to the input of a predicate. Instead of a single perturbation, we propose to use a sequence of (simpler) perturbations. Moreover, we look at their effects geometrically instead of algebraically; this allows us to tackle cases that were not tractable with the classical algebraic approach.

Key-words: Degeneracies, Robustness issues, Apollonius diagram, Trapezoidal map.

The work in this paper has been partially supported by the FP7-REGPOT-2009-1 project "Archimedes Center for Modeling, Analysis and Computation". M. Karavelas acknowledges support by the European Union (European Social Fund – ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framerwork (NSRF) – Research Funding Program: THALIS – UOA (MIS 375891).

[*] Inria, Centre de recherche Nancy - Grand Est, France.
[†] CNRS, Loria, France.
[‡] Université de Lorraine, France
[§] Department of Applied Mathematics, University of Crete, Greece
[¶] Institute of Applied and Computational Mathematics, FORTH, Greece.

RESEARCH CENTRE
NANCY – GRAND EST

615 rue du Jardin Botanique
CS20101
54603 Villers-lès-Nancy Cedex

Perturbation symbolique qualitative: deux applications d'une nouvelle méthode de perturbation basée géométrie

Résumé : Avec les méthodes de *perturbations symboliques* classiques, les dégénérescences sont résolues en substituant certains polynômes en ε aux entrées du prédicat. Au lieu d'une seule perturbation compliquée, nous proposons d'utiliser une suite de perturbation plus simple. Et nous regardons les effets de ces perturbations géométriquement plutôt qu'algébriquement ce qui permet de traiter des cas inatteignables par les méthodes algébriques classiques.

Mots-clés : Dégénérescences, robustesse algorithmique, Diagramme d'Apollonius, carte des trapèzes

1 Introduction

In earlier computational geometry papers, the treatment of degenerate configurations was mainly ignored. However, degenerate situations actually do occur in practice. When data are highly degenerate by nature, a direct handling of special cases in a particular algorithm can be efficient [4]. But in many situations, degeneracies happen only occasionally, and perturbation schemes are an easy and efficient generic solution. Controlled perturbations [16] combine increasing arithmetic precision together with actual displacement of the data, and eventually compute a non-degenerate configuration. On the other hand, the use of a symbolic perturbation allows a geometric algorithm or data structure that was originally designed without addressing degeneracies, to still operate on degenerate cases, without concretely modifying the input [8, 19, 20]. Actually, similar strategies were often used by earlier implementors of simple geometric algorithms, without identifying them as symbolic perturbations: for instance when incrementally computing a convex hull, when the new inserted point was lying on a facet of the convex hull, the point was decided to be inside the convex hull.

Let $G(\boldsymbol{u})$ be a geometric structure defined when the input data $\boldsymbol{u}$ satisfies some non-degeneracy assumptions, and let $\boldsymbol{u}_0$ be some input that is degenerate for G. A *symbolic perturbation* consists in using, as input $\boldsymbol{u}$ for G, a continuous function $\pi(\boldsymbol{u}_0, \varepsilon)$ of a parameter ε. This is done in such a way that, for $\varepsilon = 0$ $\pi(\boldsymbol{u}_0, 0)$ is equal to $\boldsymbol{u}_0$, and $\pi(\boldsymbol{u}_0, \varepsilon)$ is non-degenerate for G for sufficiently small positive values of ε. In that case the structure $G(\boldsymbol{u}_0)$ is defined as the limit of $G(\pi(\boldsymbol{u}_0, \varepsilon))$ when $\varepsilon \to 0^+$.

A symbolic perturbation allows an algorithm that computes $G(\boldsymbol{u})$ in generic situations to compute $G(\boldsymbol{u}_0)$ for the degenerate input $\boldsymbol{u}_0$. Most decisions made by the algorithm are usually made by looking at *geometric predicates*, which are combination of elementary predicates. An elementary predicate is the sign of a continuous real function of the input. The original algorithm assumes that such a function p never returns 0. When applying a symbolic perturbation, a predicate $\text{sign}(p(\boldsymbol{u}))$ evaluated at $\boldsymbol{u}_0$ returns the limit of $\text{sign}(p(\pi(\boldsymbol{u}_0, \varepsilon)))$ as $\varepsilon \to 0^+$. The sign of $p(\boldsymbol{u}_0)$ can thus be evaluated, provided that $p(\pi(\boldsymbol{u}_0, \varepsilon))$ is not identically equal to 0 in a (right) neighborhood of $\varepsilon = 0$. A perturbation scheme is said to be *effective* for a predicate $\text{sign}(p(\boldsymbol{u}))$ if for any $\boldsymbol{u}_0$ the function $p(\pi(\boldsymbol{u}_0, \varepsilon))$ is never the null function in a neighborhood of the origin. The main difficulty when designing a perturbation scheme for $G(\boldsymbol{u})$ is to find a function $\pi(\boldsymbol{u}_0, \varepsilon)$, such that the perturbation scheme can be proved to be effective for all relevant functions $p(\boldsymbol{u})$, and the perturbed predicates are easy to evaluate, e.g., using as few as possible arithmetic operations. The work of designing and proving the effectiveness of a perturbation for G is typically tailored to a specific algorithm for computing the geometric structure G.

In previous works [1, 7, 8, 9, 17], a predicate is the sign of a polynomial P in some input $\boldsymbol{u} \in \mathbb{R}^m$. The input $\boldsymbol{u}$ is perturbed as an element $\pi(\boldsymbol{u}_0, \varepsilon)$ of $\mathbb{R}^m$ whose coordinates are polynomials in $\boldsymbol{u}_0$ and ε, such that $\pi(\boldsymbol{u}_0, \varepsilon)$ goes to $\boldsymbol{u}_0$ when $\varepsilon \to 0^+$, and the perturbed predicate needs to evaluate somehow the limit $\lim_{\varepsilon \to 0^+} \text{sign}(P(\pi(\boldsymbol{u}_0, \varepsilon)))$. Since P is a polynomial, $P(\pi(\boldsymbol{u}_0, \varepsilon))$ can be rewritten as a polynomial in ε whose monomials in ε are ordered in terms of increasing degree. The constant monomial is actually $P(\boldsymbol{u}_0)$, and the signs of the following coefficients can be viewed as secondary predicates on $\boldsymbol{u}_0$. The coefficients of $P(\pi(\boldsymbol{u}_0, \varepsilon))$ are evaluated in increasing degrees in ε, until a non-vanishing coefficient is found. The sign of this coefficient is then returned as the value of the predicate $\text{sign}(P(\boldsymbol{u}_0))$.

Contribution In this paper we propose QSP (Qualitative Symbolic Perturbation), a new framework for resolving degenerate configurations in geometric computing. Unlike classical symbolic perturbation techniques, QSP resolves degeneracies in a purely geometric manner, and independently of a specific algebraic formulation of the predicate. So, the technique is particularly suitable for predicates whose algebraic description is not unique or too complicated, such as the ones treated in this paper. In fact, QSP can even handle predicates that are signs of non-polynomial functions.

In addition, instead of having a single perturbation parameter that governs the way the input objects and/or predicates are modified, QSP allows for a sequence of perturbation parameters: conceptually, we symbolically perturb the input objects one-by-one, using a well-defined canonical ordering that corresponds to considering first the object that is perturbed most. To achieve termination, we must devise an appropriate sequence of perturbations which guarantees that eventually, i.e., after having perturbed sufficiently many input objects, the degenerate predicate is resolved in a non-degenerate manner. The number of objects that need to be perturbed depends on the specific predicate that we analyze. For example in the 2D Apollonius diagram, for a given predicate, perturbing a single object always suffices, whereas in its 3D counterpart, we may need to perturb two input objects.

Standard algebraic symbolic perturbation schemes [8, 9, 17] automatically provide us with the auxiliary predicates that we need to evaluate. These predicates are, by design, of at most the same algebraic degree as the original predicate, but evaluating them in an efficient manner (e.g., by factorizing the predicate) is far from being an obvious task. QSP schemes cannot guarantee that the auxiliary predicates are not more complicated algebraically (i.e., are of lower algebraic degree) from the original predicate; however, in principle, the auxiliary predicates that we have to deal with are expected to be more tractable, since their analysis is based on geometric considerations.

As for any perturbation scheme, QSP assumes exact arithmetic to detect degeneracies. Degeneracies are rare enough to allow high efficiency using the exact geometric computing paradigm [21].

In the next section of the paper we formally define the QSP framework. In Section 3 and 4 we describe QSP schemes for the main predicates of two geometric structures: (1) the 2D arrangement of circular arcs and (2) the 2D/3D Apollonius diagram. We end with Section 5, where we discuss the advantages and disadvantages of our framework, and indicate directions for future research.

2 General framework

Let us start with two easy observations about the limit of the sign of a function of two variables.

2.1 Preliminary observations

The first observation allows us to swap the order of evaluation of limits:

Observation 1. *Let f be a continuous fonction of two variables (a, b) defined in a neighborhood of the origin. Let $\lim_{b\to0^+} \mathrm{sign} f(0, b)$ be denoted as s. If $s \neq 0$ then*

$$\lim_{b\to0^+} \lim_{a\to0^+} \mathrm{sign} f(a, b) = s.$$

Proof. Let us assume that $s \neq 0$. There exists $\delta > 0$ such that $\forall b \in (0, \delta], \mathrm{sign} f(0, b) = s$. For any b fixed in $(0, \delta]$ the function $f(a, b)$ is a continuous function in variable a, thus $\lim_{a\to0^+} f(a, b) = f(0, b)$ and, since $s \neq 0$, f does not vanish when a is in a neighborhood of 0. We have $\lim_{a\to0^+} \mathrm{sign} f(a, b) = \mathrm{sign} f(0, b) = s$. For any $b \in (0, \delta]$ the function $\lim_{a\to0^+} \mathrm{sign} f(a, b)$ is the constant function of value s. So its limit is s when $b \to 0^+$. $\square$

The second observation formalizes a situation that is in fact trivial.

Observation 2. *Let f be a continuous fonction in two variables (a, b) defined in a neighborhood of the origin. Assume that $\forall b, b' \geq 0,\ \mathrm{sign} f(a, b) = \mathrm{sign} f(a, b')$. Then*

$$\lim_{b \to 0^+} \lim_{a \to 0^+} \mathrm{sign} f(a, b) = \lim_{a \to 0^+} \mathrm{sign} f(a, 0).$$

Proof. Let $s = \lim_{a \to 0^+} \mathrm{sign} f(a, 0)$. There exists $\delta > 0$ such that $\forall a \in (0, \delta],\ \mathrm{sign} f(a, 0) = s$. By the hypothesis in the observation we have $\forall a \in (0, \delta],\ \forall b \geq 0,\ \mathrm{sign} f(a, 0) = \mathrm{sign} f(a, b) = s$. The function has constant sign s on $(0, \delta] \times (0, \infty)$ so the limit is s. □

2.2 The QSP scheme

Let $G(\boldsymbol{u})$ be a geometric structure whose computation depends on a predicate $\mathrm{sign}(p(\boldsymbol{u}))$, where p is a continuous real function. In this formal presentation, p appears as a function of the whole input $\boldsymbol{u}$; however, in practice, a predicate depends only on a constant size subset of $\boldsymbol{u}$.

We design the perturbation scheme π as a sequence of successive perturbations π_i, $0 \leq i < N$, with

$$\pi(\boldsymbol{u}, \boldsymbol{\varepsilon}) = \pi_0(\pi_1(\pi_2(\dots \pi_{N-1}(\boldsymbol{u}, \varepsilon_{N-1}) \dots, \varepsilon_2), \varepsilon_1), \varepsilon_0),$$

where $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{N-1}) \in \mathbb{R}^N$. The number of perturbations N is part of the perturbation scheme and usually depends on the input size. The perturbations are numbered by increasing order of magnitude, i.e., ε_i is considered much bigger than ε_j if $i > j$. Since $\boldsymbol{\varepsilon}$ is no longer a single real number, we have to determine how the limit is taken; we thus define $G(\boldsymbol{u})$ to be the limit:

$$G(\boldsymbol{u}) = \lim_{\varepsilon_{N-1} \to 0^+} \lim_{\varepsilon_{N-2} \to 0^+} \cdots \lim_{\varepsilon_1 \to 0^+} \lim_{\varepsilon_0 \to 0^+} G(\pi(\boldsymbol{u}, \boldsymbol{\varepsilon})).$$

QSP implies an evaluation strategy of this limit, as follows. The perturbed predicate

$$\lim_{\varepsilon_{N-1} \to 0^+} \lim_{\varepsilon_{N-2} \to 0^+} \cdots \lim_{\varepsilon_1 \to 0^+} \lim_{\varepsilon_0 \to 0^+} \mathrm{sign}(p(\pi(\boldsymbol{u}, \boldsymbol{\varepsilon})))$$

is evaluated by first computing $p(\pi(\boldsymbol{u}, (0, 0, \dots, 0)))) = p(\boldsymbol{u})$, and returning its sign if it is non-zero. If $p(\boldsymbol{u}) = 0$, we look at the function $p(\pi(\boldsymbol{u}, (0, 0, \dots, \varepsilon_{N-1}))) = p(\pi_{N-1}(\boldsymbol{u}, \varepsilon_{N-1}))$; if this function is not vanishing when ε_{N-1} lies in a sufficiently small neighborhood to the right of 0, its sign can be returned. More formally, we compute the limit

$$\ell_1 = \lim_{\varepsilon_{N-1} \to 0^+} \mathrm{sign}(p(\pi_{N-1}(\boldsymbol{u}, \varepsilon_{N-1}))). \tag{1}$$

If ℓ_1 is non-zero, using Observation 1, it is returned as the value of the predicate $\mathrm{sign}(p(\boldsymbol{u}))$. Otherwise, we have to further perturb our geometric input; we examine the limit

$$\ell_2 = \lim_{\varepsilon_{N-1} \to 0^+} \lim_{\varepsilon_{N-2} \to 0^+} \mathrm{sign}(p(\pi_{N-2}(\pi_{N-1}(\boldsymbol{u}, \varepsilon_{N-1}), \varepsilon_{N-2}))). \tag{2}$$

The expression in Eq.(2) can be simplified in cases that actually often occur in applications: if $p(\pi_{N-1}(\boldsymbol{u}, \varepsilon_{N-1})))$ is zero on $[0, \eta)$, it is often because the sign of $p(\pi_{N-2}(\pi_{N-1}(\boldsymbol{u}, \varepsilon_{N-1}), \varepsilon_{N-2}))$ does not depend on ε_{N-1} in $[0, \eta)$. In such a case, and provided that this function is not also zero, we can evaluate its sign using Observation 2: by taking $\varepsilon_{N-1} = 0$, Eq.(2) boils down to

$$\ell_2 = \lim_{\varepsilon_{N-2} \to 0^+} \mathrm{sign}(p(\pi_{N-2}(\boldsymbol{u}, \varepsilon_{N-2}))). \tag{3}$$

The process is iterated until a non-zero limit is found. In very degenerate situations, when the $\mu - 1$ first limits evaluate to 0, i.e., $\ell_1 = \ell_2 = \ldots = \ell_{\mu-1} = 0$, we need to evaluate ℓ_μ. Its definition is

$$\ell_\mu = \lim_{\varepsilon_{N-1}\to 0^+} \lim_{\varepsilon_{N-2}\to 0^+} \ldots \lim_{\varepsilon_{N-\mu}\to 0^+} \operatorname{sign}(p(\pi(\boldsymbol{u}, (0, 0, \ldots, 0, \varepsilon_{N-\mu}, \varepsilon_{N-\mu+1}, \ldots, \varepsilon_{N-1})))).$$

Similarly to what we described for ℓ_2 above, it is frequently the case that the sign of

$$p(\pi(\boldsymbol{u}, (0, 0, \ldots, 0, \varepsilon_{N-\mu+1}, \ldots, \varepsilon_{N-1})))$$

does not depend on $\varepsilon_{N-\mu+1}, \ldots, \varepsilon_{N-1}$ in a neighborhood of 0 in $\mathbb{R}^{\mu-1}$; then the simplified evaluation allowed by Observation 2 gives:

$$\ell_\mu = \lim_{\varepsilon_{N-\mu}\to 0^+} \operatorname{sign}(p(\pi(\boldsymbol{u}, (0, 0, \ldots, 0, \varepsilon_{N-\mu}, \varepsilon_{N-\mu+1}, \ldots, \varepsilon_{N-1})))).$$

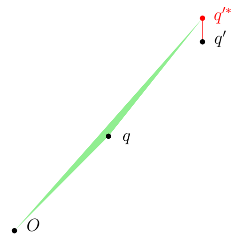
To assert that the perturbation scheme π is effective, we need to prove that one of these limits is indeed non-zero.

When the predicate is a polynomial, we get a sequence of successive evaluations as in algebraic symbolic perturbations; however, the expressions that need to be evaluated have been obtained in a different way and are *a priori* different. The main advantage of this approach is that we may use a very simple perturbation π_ν, since we do not need each perturbation π_ν to be effective, but rather the composed perturbation π. For geometric problems, the simplicity of π_ν allows us to look at the limit in a geometric manner, instead of algebraically computing some appropriate coefficient of $p(\pi(\boldsymbol{u}, \varepsilon))$.

2.3 Toy examples

We illustrate these principles by four toy examples. In all examples, we set $\boldsymbol{u} = (q, q') = ((x_0, x_1), (x_2, x_3))$, a pair of two 2D points and $\pi_i(\boldsymbol{u}, \varepsilon_i) = \boldsymbol{u} + \varepsilon_i \boldsymbol{e}_i$ where $\boldsymbol{e}_0 = ((1, 0), (0, 0))$, $\boldsymbol{e}_1 = ((0, 1), (0, 0))$, $\boldsymbol{e}_2 = ((0, 0), (1, 0))$, and $\boldsymbol{e}_3 = ((0, 0), (0, 1))$ form the canonical basis of $(\mathbb{R}^2)^2$. The differences between the examples below lie in the evaluated predicate $\operatorname{sign}(p(\boldsymbol{u}))$ and the degenerate position $\boldsymbol{u}_0$.

First example: orientation of a flat triangle
$p(\boldsymbol{u}) = x_0 x_3 - x_1 x_2$ and $\boldsymbol{u}_0 = (q, q') = ((1, 1), (2, 2))$.
QSP defines the result for $\operatorname{sign}(p(\boldsymbol{u}_0))$, the orientation of Oqq', as



$$\operatorname{sign}(p(\boldsymbol{u}_0)) = \lim_{\varepsilon_3\to 0^+} \lim_{\varepsilon_2\to 0^+} \lim_{\varepsilon_1\to 0^+} \lim_{\varepsilon_0\to 0^+} \operatorname{sign}((1+\varepsilon_0)(2+\varepsilon_3) - (2+\varepsilon_1)(1+\varepsilon_2)).$$

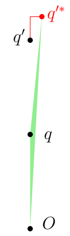
A standard evaluation of this expression would consist in taking the limits in order:

$$\begin{aligned}
\operatorname{sign}(p(\boldsymbol{u}_0)) &= \lim_{\varepsilon_3\to 0^+} \lim_{\varepsilon_2\to 0^+} \lim_{\varepsilon_1\to 0^+} \operatorname{sign}((2+\varepsilon_3) - (2+\varepsilon_1)(1+\varepsilon_2)) \\
&= \lim_{\varepsilon_3\to 0^+} \lim_{\varepsilon_2\to 0^+} \operatorname{sign}((2+\varepsilon_3) - 2(1+\varepsilon_2)) \\
&= \lim_{\varepsilon_3\to 0^+} \operatorname{sign}(\varepsilon_3) = 1.
\end{aligned}$$

Following the QSP evaluation strategy instead, in such a case, the biggest perturbation, i.e., the perturbation on x_3, allows to quickly conclude. The only computed limit is the one in Eq. (1):

$$\ell_1 = \lim_{\varepsilon_3 \to 0^+} \mathrm{sign}(p((1,1),(2,2+\varepsilon_3))) = \lim_{\varepsilon_3 \to 0^+} \mathrm{sign}((2+\varepsilon_3)-2) = \lim_{\varepsilon_3 \to 0^+} \mathrm{sign}(\varepsilon_3) = 1,$$

The geometric interpretation is that we get the orientation of a triangle Oqq'^* for a point q'^* slightly above q'.



Second example: orientation of a vertical flat triangle
$p(\boldsymbol{u}) = x_0x_3 - x_1x_2$ and $\boldsymbol{u}_0 = (q,q') = ((0,1),(0,2))$.
QSP defines the result for $\mathrm{sign}(p(\boldsymbol{u}_0))$, the orientation of Oqq', as

$$\mathrm{sign}(p(\boldsymbol{u}_0)) = \lim_{\varepsilon_3 \to 0^+} \lim_{\varepsilon_2 \to 0^+} \lim_{\varepsilon_1 \to 0^+} \lim_{\varepsilon_0 \to 0^+} \mathrm{sign}((0+\varepsilon_0)(2+\varepsilon_3)-(1+\varepsilon_1)(0+\varepsilon_2)).$$

Taking the limits in order leads to:

$$\begin{aligned}\mathrm{sign}(p(\boldsymbol{u}_0)) &= \lim_{\varepsilon_3 \to 0^+} \lim_{\varepsilon_2 \to 0^+} \lim_{\varepsilon_1 \to 0^+} \mathrm{sign}(-(1+\varepsilon_1)\varepsilon_2) \\ &= \lim_{\varepsilon_3 \to 0^+} \lim_{\varepsilon_2 \to 0^+} \mathrm{sign}(-\varepsilon_2)) \\ &= \lim_{\varepsilon_3 \to 0^+} \mathrm{sign}(-1) = -1.\end{aligned}$$

In this case, the QSP evaluation strategy is to first compute

$$\ell_1 = \lim_{\varepsilon_3 \to 0^+} \mathrm{sign}(p((0,1),(0,2+\varepsilon_3))) = \lim_{\varepsilon_3 \to 0^+} 0 = 0,$$

which does not allow us to resolve the degeneracy. Then we observe that

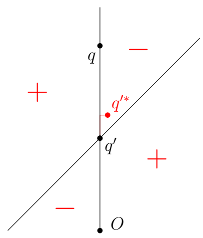
$$\mathrm{sign}(p((0,1),(x_2,x_3))) = \mathrm{sign}(-x_2)$$

does not depend on x_3, thus we can evaluate ℓ_2 using Eq. (3):

$$\ell_2 = \lim_{\varepsilon_2 \to 0^+} \mathrm{sign}(p((0,1),(\varepsilon_2,2))) = \lim_{\varepsilon_2 \to 0^+} \mathrm{sign}(-\varepsilon_2) = -1.$$

Two perturbations π_3 and π_2 must be used, but the simplified evaluation of ℓ_2 suffices.

The geometric interpretation is that we look at the orientation of a triangle Oqq'^* for a moved point q'^*. Since moving q'^* slightly above q' doesn't change anything to the degeneracy, the point is moved to the right, which resolves the degeneracy.



Third example: points and quadratic form
$p(\boldsymbol{u}) = x_0(x_1-1)-x_0^2-x_2(x_3-1)+x_2^2$ and $\boldsymbol{u}_0 = (q,q') = ((0,2),(0,1))$.
The predicate p stands for the difference of a degenerate quadratic form evaluated at q and q'. QSP defines the result for $\mathrm{sign}(p(\boldsymbol{u}_0))$ as

$$\mathrm{sign}(p(\boldsymbol{u}_0)) = \lim_{\varepsilon_3 \to 0^+} \lim_{\varepsilon_2 \to 0^+} \lim_{\varepsilon_1 \to 0^+} \lim_{\varepsilon_0 \to 0^+} \mathrm{sign}\left(\varepsilon_0(1-2+\varepsilon_1)-\varepsilon_0^2-\varepsilon_2(1-1+\varepsilon_3)+\varepsilon_2^2\right),$$

which could be evaluated as follows:

$$\begin{aligned}\mathrm{sign}(p(\boldsymbol{u}_0)) &= \lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\lim_{\varepsilon_1\to 0^+}\mathrm{sign}\left(-\varepsilon_2\varepsilon_3+\varepsilon_2^2\right)\\ &= \lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\mathrm{sign}\left(\varepsilon_2(\varepsilon_2-\varepsilon_3)\right)\\ &= \lim_{\varepsilon_3\to 0^+}\mathrm{sign}(-\varepsilon_3)=-1.\end{aligned}$$

Again the evaluation strategy first computes

$$\ell_1=\lim_{\varepsilon_3\to 0^+}\mathrm{sign}(p((0,2),(0,1+\varepsilon_3)))=\lim_{\varepsilon_3\to 0^+}0=0,$$

which does not allow us to resolve the degeneracy.

Then we observe that $\mathrm{sign}(p((0,2),(x_2,x_3)))=\mathrm{sign}(x_2(x_3-1)+x_2^2)$ actually depends on x_3, thus we must evaluate ℓ_2 using Eq. (2):

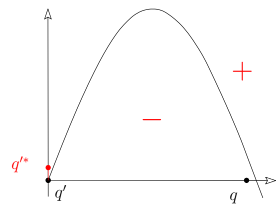
$$\begin{aligned}\ell_2 &= \lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\mathrm{sign}(p((0,2),(\varepsilon_2,1+\varepsilon_3)))\\ &= \lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\mathrm{sign}\left(\varepsilon_2(\varepsilon_2-\varepsilon_3)\right)\\ &= \lim_{\varepsilon_2\to 0^+}\mathrm{sign}(-\varepsilon_3)=-1.\end{aligned}$$

Notice that since $\mathrm{sign}(p((2,0),(x_2,x_3)))$ depends on x_3, the simplified evaluation of Eq. (3) would have given a wrong result:

$$\ell_2\neq\lim_{\varepsilon_2\to 0^+}\mathrm{sign}(p((2,0),(\varepsilon_2,1)))=\lim_{\varepsilon_2\to 0^+}\mathrm{sign}(\varepsilon_2^2)=1.$$

The geometric interpretation is that q and q' are both on one of the two lines defined by the quadratic equation $x(y-1)-x^2=0$. Point q' is first slightly moved above but this motion leaves it on that same line, then it is moved to the right, and the sign of the quadratic form depends on the vertical position of q' with respect to the other line.

Fourth example: side of a sinusoid
$p(\boldsymbol{u})=(x_1-\sin x_0)(x_3-\sin x_2)$ and $\boldsymbol{u}_0=(q,q')=((3,0),(0,0))$.
QSP works even if the predicate is non-polynomial. This predicate is positive if q and q' are on the same side of the sinusoid. QSP defines the result for $\mathrm{sign}(p(\boldsymbol{u}_0))$ as



$$\begin{aligned}\mathrm{sign}(p(\boldsymbol{u}_0)) &= \lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\lim_{\varepsilon_1\to 0^+}\lim_{\varepsilon_0\to 0^+}\mathrm{sign}(((\varepsilon_1-\sin(3+\varepsilon_0))(\varepsilon_3-\sin\varepsilon_2))\\ &= \lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\lim_{\varepsilon_1\to 0^+}\mathrm{sign}((\varepsilon_1-\sin 3)(\varepsilon_3-\sin\varepsilon_2))\\ &= \lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\mathrm{sign}(-(\sin 3)(\varepsilon_3-\sin\varepsilon_2))\\ &= \lim_{\varepsilon_3\to 0^+}\mathrm{sign}(-(\sin 3)\varepsilon_3)=-1\end{aligned}$$

In this case, the QSP evaluation strategy first computes

$$\ell_1=\lim_{\varepsilon_3\to 0^+}\mathrm{sign}(p((3,0),(0,\varepsilon_3)))=\mathrm{sign}(-(\sin 3)\varepsilon_3)=-1$$

and it allows us to resolve the degeneracy.

The geometric interpretation is that q'^* is moved slightly above q', so on the side of the sinusoid opposite to q.

2.4 Discussion

Multiple epsilons The idea of utilizing multiple perturbation parameters is already present in Yap's scheme [19], or very recently in Irving and Green's work [12], but without the geometric interpretation allowed by QSP. In other previous works, such as SoS [8], the algebraic symbolic perturbation framework was proved to be effective by a careful choice of the exponents for ε, depending on the choice of $G(\boldsymbol{u})$, so as to make some terms negligible. QSP can be forced to fit in such a traditional framework, with a single epsilon, by making all the variables ε_ν dependent on a single parameter κ that plays the traditional role of ε. For polynomial predicates, it is enough to take ε_ν exponentially increasing with respect to ν. For example one such choice can be to set $\varepsilon_{N-1} = \kappa$, and $\varepsilon_\nu = \left(\exp\left(\frac{1}{\varepsilon_{\nu+1}}\right)\right)^{-1}$, for $0 \le \nu < N-1$. The interest of QSP, however, is not to use this traditional view, but rather make the variables ε_ν independent.

Efficiency The aim of a perturbation scheme is to solve degeneracies, and a common assumption is that such degeneracies are rare enough so that some extra time can be spent to make a reliable decision when a degeneracy happens. Another implicit assumption is that degeneracies are actually detected, that is, it is implicitly assumed that the original predicates are computed exactly, possibly with some filtering mechanism to ensure efficiency [21].

Nevertheless, the actual additional complexity in case of degeneracy must be addressed. Since QSP is geometrically defined and addresses very general problems, such a complexity analysis cannot be done at the general level. For the two applications described in this paper, the extra predicates needed to resolve the degeneracy have the same complexity as the original ones, while the number of epsilons used to perturb is not bigger than two.

QSP, as many other perturbation schemes, relies on an indexing of the input. However, as mentioned earlier, a given predicate usually depends on a constant number of input objects. It is important to keep in mind that the comparison of indices is necessary only for the few objects involved in a given predicate; sorting the whole input with respect to indices is not required.

Generality In the first three toy examples above, SoS would have taken $\varepsilon_0 = \varepsilon^8$, $\varepsilon_1 = \varepsilon^4$, $\varepsilon_2 = \varepsilon^2$, and $\varepsilon_3 = \varepsilon$, which yields the same result as QSP. When it leads to a simple result, the classical algebraic view is a very good solution. However, if the original predicate is a bit intricate, the algebraic way will produce numerous extra predicates to resolve degeneracies. Moreover, as for any predicate, some specific work is often still needed on the polynomial to evaluate it efficiently, e.g., finding a good factorization.

QSP provides a very general approach that is able to handle various predicates, even non-polynomial as in the fourth example. Of course applying this scheme to a given problem requires some problem-specific work, but, as noted at the previous paragraph, this is also often the case for most algebraic approaches.

We would not advise the use of QSP for simple cases such as Delaunay triangulations of points where other approaches work well [7], but rather only in cases where the predicates are very complex or non-polynomial. The applications below use high degree polynomials and QSP is a good solution. As far as we know, there is no equivalent perturbation scheme for Apollonius diagrams. Regarding intersections of circles, Irving and Green [12] recently proposed a pseudo-random scheme, only handling a particular case of the predicate we address in this paper.

Meaningfulness According to a classification by Seidel [18], QSP is (geometrically) meaningful, that is we have some control on the direction (in input data space) used to move away from the degeneracies. For example, for the Apollonius diagram we will choose to minimize the number of Apollonius vertices (it is also possible to choose to maximize it). QSP is not independent

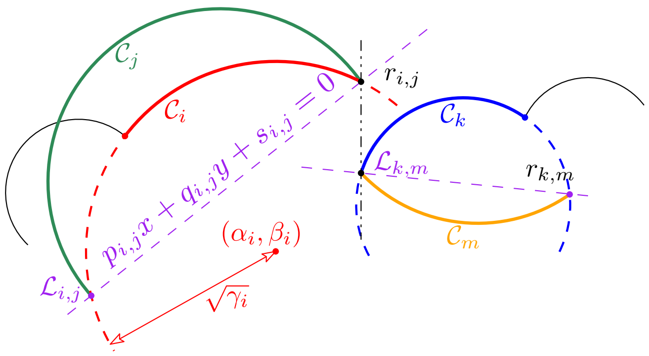


Figure 1: x-comparison of endpoints:
a degenerate case where $r_{i,j}$ and $l_{k,m}$ have the same abcissa.

of indexing, but if this indexing is geometrically meaningful, then we can ensure invariance with respect to some geometric transformations.

3 Predicates for circular arcs arrangements

To address the problem of computing arrangements of circular arcs by sweep-line algorithms, it is necessary to compare abscissae of endpoints of circular arcs. In the formalism of the previous section, $\boldsymbol{u}$ is a vector of parameters defining a set of circular arcs and $G(\boldsymbol{u})$ is the arrangement, but we will use notations more adapted to our application. This predicate was studied in a previous paper [5]. The arc endpoints are described as intersections of two circles, which leads us to consider the arrangement of all circles supporting arcs or defining their endpoints. Degeneracies occur if several vertices of the arrangement have the same abscissa or if more than two circles meet at a common point. For arrangements exhibiting a lot of degeneracies, it may be interesting to design an algorithm that directly handles special cases, while in other contexts where degeneracies are occasional, it would be preferable to keep the algorithm simple and handle degeneracies through a perturbation scheme.

An endpoint $z_{\nu,\mu}$, defined as an intersection of two circles $\mathcal{C}_\nu$ and $\mathcal{C}_\mu$, is determined by the centers (α_ν, β_ν) and (α_μ, β_μ) of the circles (see Figure 1), their squared radii γ_ν and γ_μ, and a Boolean $b_{\nu,\mu}$ encoding whether $z_{\nu,\mu}$ is the leftmost $l_{\nu,\mu}$ or rightmost intersection point $r_{\nu,\mu}$ (if they have the same abscissa, $r_{\nu,\mu}$ is the highest and $l_{\nu,\mu}$ the lowest intersection point).

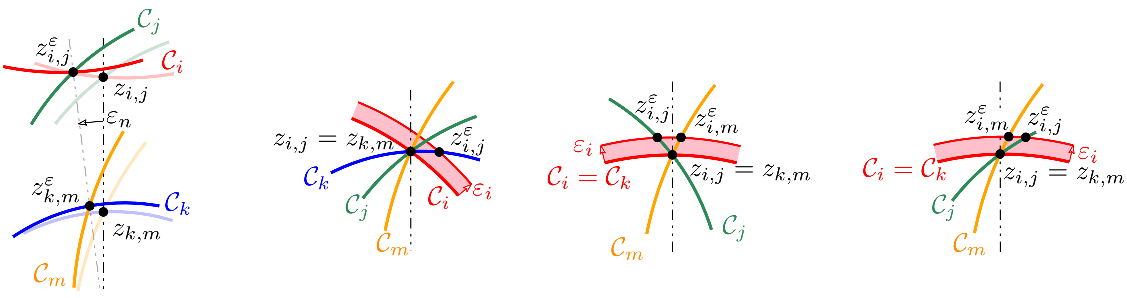
3.1 Algebraic formulation

Intersection points of $\mathcal{C}_\nu$ and $\mathcal{C}_\mu$ can also be seen as intersections between $\mathcal{C}_\nu$ and the line $\mathcal{L}_{\nu,\mu}$ whose equation $p_{\nu,\mu}x + q_{\nu,\mu}y + s_{\nu,\mu} = 0$ is obtained by subtracting the equations of $\mathcal{C}_\nu$ and $\mathcal{C}_\mu$:

$$p_{\nu,\mu} = 2(\alpha_\nu - \alpha_\mu), \quad q_{\nu,\mu} = 2(\beta_\nu - \beta_\mu), \quad \text{and} \quad s_{\nu,\mu} = \gamma_\nu - \gamma_\mu - \alpha_\nu^2 - \beta_\nu^2 + \alpha_\mu^2 + \beta_\mu^2.$$

The predicate $x\text{-}\mathsf{compare}(\mathcal{C}_i, \mathcal{C}_j, b_{i,j}, \mathcal{C}_k, \mathcal{C}_m, b_{k,m})$ compares the abscissae of two arc endpoints $z_{i,j}$ defined by $\mathcal{C}_i$, $\mathcal{C}_j$, and $b_{i,j}$ $(i \neq j)$ on the one hand, and $z_{k,m}$ defined by $\mathcal{C}_k$, $\mathcal{C}_m$, and $b_{k,m}$ $(k \neq m)$ on the other hand. The most complicated evaluation is the sign of the following degree 12 polynomial [5]:

$$(A_{i,j}\, C_{k,m} - A_{k,m}\, C_{i,j})^2 - 4(A_{i,j}\, B_{k,m} - A_{k,m}\, B_{i,j})(B_{i,j}\, C_{k,m} - B_{k,m}\, C_{i,j})$$



(a) $z_{i,j}$ left of $z_{k,m}$ (b) $z_{i,j}$ right of $z_{k,m}$ (c) $z_{i,j}$ left of $z_{i,m}$ (d) $z_{i,j}$ right of $z_{i,m}$

Figure 2: Perturbing $z_{i,j}$ and $z_{k,m}$

where:

$$\begin{aligned}
A_{\nu,\mu} &= p^2_{\nu,\mu} + q^2_{\nu,\mu},\\
B_{\nu,\mu} &= q^2_{\nu,\mu}\,\alpha_\nu - p_{\nu,\mu}(\,s_{\nu,\mu} + q_{\nu,\mu}\,\beta_\nu),\\
C_{\nu,\mu} &= (s_{\nu,\mu} + q_{\nu,\mu}\,\beta_\nu)^2 + q^2_{\nu,\mu}\,(\alpha^2_\nu - \gamma_\nu).
\end{aligned}$$

Introducing an algebraic symbolic perturbation yields quite a complicated polynomial in ε, not really suitable for efficient predicate evaluation. For the same problem, Irving and Green [12] use an algebraic perturbation with pseudo-random coefficients, but they only address the special case where $\mathcal{C}_i = \mathcal{C}_k$.

3.2 Qualitative symbolic perturbation

We construct a sequence of $n+1$ successive perturbations for an input $\boldsymbol{\mathcal{C}}$ consisting of n circles $\mathcal{C}_\nu$, $0 \le \nu < n$.

The first perturbation, $\pi_n(\boldsymbol{\mathcal{C}}, \varepsilon_n)$ is a rotation centered at the origin and with angle ε_n. This perturbation handles the cases where $z_{i,j}$ and $z_{k,m}$ are different points with the same abscissa. In other words, it just uses lexicographic comparisons: y-comparisons are used to break ties in x-comparisons (see Figure 2(a)). The rotation by a small angle has the same effect as a shear transform. A shear transform is often preferred in the literature because it is a rational transformation. We use a rotation instead because it transforms circles into circles, which is preferable in order to apply the remaining perturbations in the sequence. On top of that, we are not interested in the algebraic formulation of the transformation, since we look at the limit in a geometric way instead of algebraically.

We are now left with cases when $z_{i,j} = z_{k,m}$. The other perturbations in the sequence consist in inflating the circles: $\pi_\nu(\boldsymbol{\mathcal{C}}, \varepsilon_\nu)$ replaces γ_ν by $\gamma^\varepsilon_\nu = \gamma_\nu + \varepsilon_\nu$, $\varepsilon_\nu \ge 0$. Recall that the comparison of indices is necessary only for the four circular arcs involved in a given predicate; sorting the whole input with respect to indices is not required. We consider the circles by decreasing radii to get rid of pairs of tangent circles in a geometrically meaningful way: if two circles are tangent, the perturbation inflates the largest one by a larger amount, making the intersection point either disappear if the two circles are internally tangent, or split into two points if they are externally tangent.

Without loss of generality, we may assume that $i \ge j, k, m$, i.e., $\mathcal{C}_i$ is the most perturbed circle. If $i \notin \{k, m\}$, then $z_{i,j}$ moves (i.e., $z^\varepsilon_{i,j} \ne z_{i,j}$) while $z_{k,m}$ stays fixed. After determining the vertical order of $\mathcal{C}_i$ and $\mathcal{C}_j$ at the right of $z_{i,j}$, and whether $z_{i,j}$ is on the top or bottom part of $\mathcal{C}_i$ using the auxiliary predicates described below, it is easy to decide if $z^\varepsilon_{i,j}$ moves left or right when $\varepsilon_i > 0$ (see Figure 2(b)). If $i \in \{k, m\}$, assume, without loss of generality, that $i = k$. If $z^\varepsilon_{i,j}$ and $z^\varepsilon_{i,m}$ are perturbed in opposite x-directions, it is easy to decide which one is the leftmost (see Figure 2(c)). Otherwise, we determine the vertical order of the three circles $\mathcal{C}_i$, $\mathcal{C}_j$, and $\mathcal{C}_m$ at the right of $z_{i,j} = z_{i,m}$; we know that $\mathcal{C}_i$ is either the topmost or the bottommost circle in this vertical ordering. The point lying on the closest arc to $\mathcal{C}_i$ is more perturbed than the other, and the auxiliary predicates below allow us to decide which of $z^\varepsilon_{i,j}$ and $z^\varepsilon_{i,m}$ is to the left (see Figure 2(d)).

Auxiliary predicates A circle can be split in four parts: top-right, top-left, bottom-left, and bottom-right at its points with horizontal or vertical tangents. Knowing if a point $z_{i,j}$ is on the left or right part of $\mathcal{C}_i$ can be evaluated by $x\text{-}\mathsf{compare}(\mathcal{C}_i, \mathcal{C}_j, b_{i,j}, \mathcal{C}_i, \mathcal{C}_m, b_{i,m})$ for a suitable $\mathcal{C}_m$ such that $\mathcal{L}_{i,m}$ has equation $x - \alpha_i = 0$. Discriminating between the top and bottom parts is done in the same way, by exchanging the roles of the x- and y-coordinates. Another predicate consists in deciding if $\mathcal{C}_i$ is above or below $\mathcal{C}_j$ at the right of $r_{i,j}$; this can be done by elementary geometric computations.

4 The Apollonius diagram

4.1 Definition

The *Apollonius diagram*, also known as *additively weighted Voronoi diagram*, is defined on a set of weighted points in the Euclidean space $\mathbb{R}^d$. In the formalism of Section 2, $\boldsymbol{u}$ is a vector of coordinates and weights of a set of weighted points and $G(\boldsymbol{u})$ is the Apollonius diagram; as in Section 3, we will use notations more adapted to our application. The Euclidean norm is denoted as $|\cdot|$. The *weighted distance* from a query point q to a weighted point (p, w), where p is a point in the Euclidean space and $w \in \mathbb{R}$, is $|pq| - w$. The Apollonius diagram is the closest point diagram for this distance. It generalizes the Voronoi diagram, defined on non-weighted points.

Given a set of weighted points, also called *sites*, it is clear that adding the same constant to all weights does not change the Apollonius diagram. Thus, in the sequel, we may freely translate the weights to ensure, for example, that all weights are positive, or that a particular weight is zero. A site $(s, w), w \geq 0$, can be identified with the sphere S centered at s and of radius w. The distance from a query point r to a site $S = (s, w)$ is the Euclidean distance from r to S, with a negative sign if r lies inside S.

An Apollonius vertex v is a point at the same distance from $d + 1$ sites $S_0, S_1, \ldots, S_d$ in general position. We call the configuration *external* if v is outside sphere S_i, for all $i = 0, \ldots, d$, and *internal* if it is inside the spheres. If the configuration is external (resp., internal), v is the center of a sphere externally (resp., internally) tangent to the sites S_i (see green (resp., dark green) disks in Figure 3). It is always possible to ensure an external configuration by adding a suitable constant to the weights of all S_i, such that all weights are non-negative, while the smallest among them is equal to zero.

Let us show that $d + 1$ sites in general position define zero, one or two Apollonius vertices. Assume, without loss of generality, that all weights are non-negative for $i = 1, \ldots, d$ and $w_0 = 0$, so as to be in external configuration. Consider now the inversion with point s_0 as the pole. The point s_0 goes to infinity, while each sphere S_i, $i = 1, \ldots, d$ becomes a new sphere $Z_i = (z_i, \rho_i)$ (see Figure 4). Determining the balls B_α (where α indexes the different solutions) tangent to the spheres S_i, $i = 0, \ldots, d$ is equivalent to determining halfspaces delimited by the hyperplanes T_α tangent to the spheres Z_i, $i = 1, \ldots, d$, with all spheres on the same side of T_α. Requiring that a

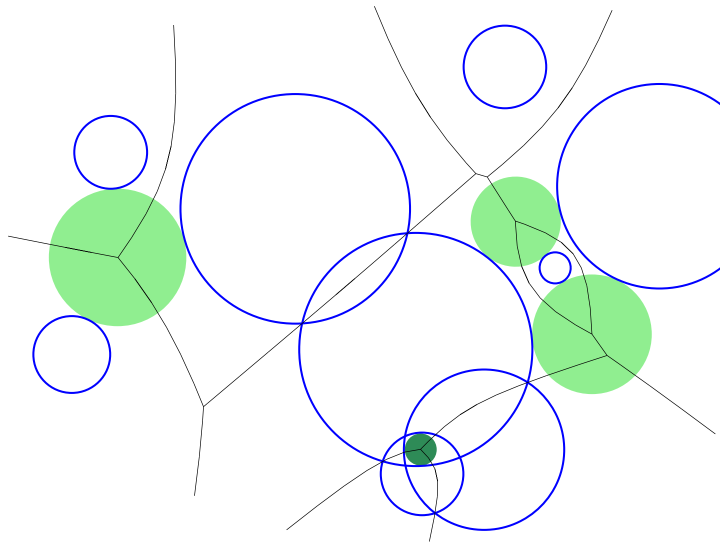


Figure 3: Planar Apollonius diagram. Weighted points are in light blue. A green disk is centered at an Apollonius vertex and its radius is the weighted distance of the center to its three closest sites. The darker green disk has a negative distance to its closest neighbors.

given B_α is externally tangent to the spheres S_i is equivalent to requiring that T_α separates the spheres Z_i from the origin. The normalized equation of T_α: $\lambda_\alpha \cdot x + \delta_\alpha = 0$, with $\lambda_\alpha \in \mathbb{R}^d$,$|\lambda_\alpha| = 1$ and $\delta_\alpha \in \mathbb{R}$, gives the signed distance of a point $x \in \mathbb{R}^d$ to T_α. We have

$$T_\alpha \text{ tangent to } Z_i,\ 1 \leq i \leq d \quad \Longleftrightarrow \quad \begin{cases} \lambda_\alpha \cdot z_i + \delta_\alpha = \rho_i,\ 1 \leq i \leq d \\ |\lambda_\alpha|^2 = 1 \end{cases} . \tag{4}$$

In the inverted space, the general position hypothesis means that the spheres Z_i do not have an infinity of tangent hyperplanes; the latter can occur only if the points z_i (and thus the points s_i) are affinely dependent. Therefore, the system (4) of one quadratic and d linear equations in $d+1$ unknowns ($\lambda_\alpha \in \mathbb{R}^d$ and $\delta_\alpha \in \mathbb{R}$) has at most two real solutions by Bézout's theorem, hence the first claim follows. Depending on the position of the origin with respect to T_α (or equivalently on the sign of δ_α), zero, one or the two solutions may correspond to external configurations.

An Apollonius vertex is actually defined by a sequence of $d + 1$ sites in general position, up to a positive permutation of the sequence. Indeed, in the previous paragraph, if there are two solutions T_α and $T_{\alpha'}$, we observe that they are symmetric with respect to the hyperplane spanned by the points z_i, thus the d-simplex formed by the tangency points and the origin has different orientations for the two solutions (see Figure 5). This implies that the two solutions can be distinguished by the signature of the permutation of the spheres S_i.

4.2 The VConflict predicate

Several predicates are necessary to compute an Apollonius diagram. We start with the *vertex conflict* predicate $\mathsf{VConflict}(\boldsymbol{S}^v, Q)$, which answers the following question:

> *Does an Apollonius vertex v defined, up to a positive permutation, by a $(d+1)$-tuple of sites $\boldsymbol{S}^v = (S_{i_0}, S_{i_1}, \ldots, S_{i_d})$ remain as a vertex of the diagram after another site Q is added?*

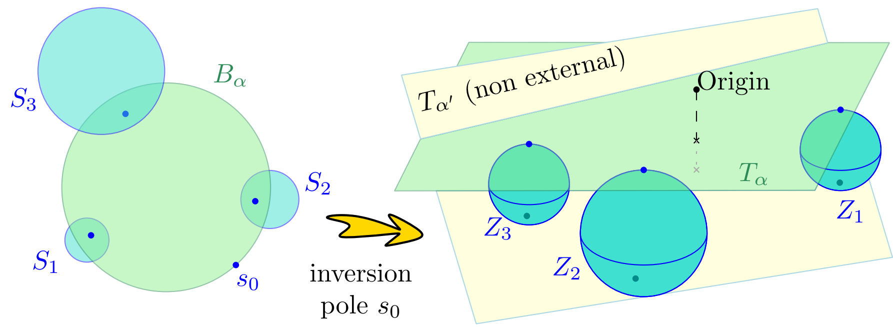


Figure 4: The spheres B_α externally tangent to the sites S_i, $i = 0, \ldots, d$, correspond, via the inversion transformation with s_0 as the pole, to hyperplanes T_α tangent to the spheres Z_i that separate them from the origin.

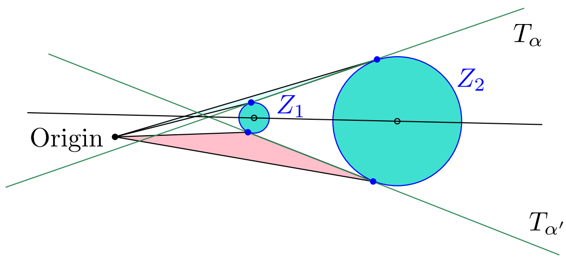


Figure 5: If T_α and $T_{\alpha'}$ are both external, the simplices formed by the tangency points and the origin have different orientations.

If the site centered at v and tangent to the sites of the tuple $\boldsymbol{S}^v$ is in internal configuration, we can add a negative constant to the radii of all spheres in $\boldsymbol{S}^v \cup \{Q\}$ so that the smallest site in $\boldsymbol{S}^v$ has zero radius. Then the configuration of the common tangent sphere becomes external. In this manner, we can always restrict our analysis to the case where the Apollonius vertex we consider is in external configuration. Note that this may lead to a negative weight w_q for Q, which was *a priori* excluded above, but is treated below.

We denote by $B_{i_0 i_1 \ldots i_d}$ the open ball whose closure $\overline{B}_{i_0 i_1 \ldots i_d}$ is tangent to the sites of $\boldsymbol{S}^v$. The contact points $t_{i_0}, t_{i_1}, \ldots, t_{i_d}$ define a positively oriented d-simplex.

If $w_q \geq 0$, the predicate $\mathsf{VConflict}(\boldsymbol{S}^v, Q)$ answers (Figure 6)

- "conflict" if Q intersects $B_{i_0 i_1 \ldots i_d}$,
- "no conflict" if Q and $\overline{B}_{i_0 i_1 \ldots i_d}$ are disjoint,
- "degenerate" if Q and $B_{i_0 i_1 \ldots i_d}$ do not intersect, while Q and $\overline{B}_{i_0 i_1 \ldots i_d}$ are tangent.

If $w_q < 0$, we define Q^- as the sphere with the same center s_q as Q and radius $-w_q$. Then $\mathsf{VConflict}(\boldsymbol{S}^v, Q)$ answers

- "conflict" if Q^- is included in $B_{i_0 i_1 \ldots i_d}$,
- "no conflict" if Q^- intersects the complement of $\overline{B}_{i_0 i_1 \ldots i_d}$,
- "degenerate" if Q^- is included in $\overline{B}_{i_0 i_1 \ldots i_d}$ and is tangent to its boundary.

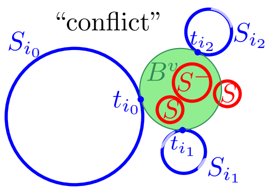


Figure 6: Examples of positions of Q or Q^- that are in "conflict", "degenerate" or "no conflict" configuration with an Apollonius vertex v.

Qualitative perturbation of the VConflict **predicate** QSP relies on some ordering of the sites. Each site $S_\nu = (s_\nu, w_\nu)$ is perturbed to $S_\nu^\varepsilon = (s_\nu, w_\nu + \varepsilon_\nu)$, $\varepsilon_\nu \geq 0$, with S_λ perturbed more than S_ν if $\lambda > \nu$. Following the QSP framework, if the configuration is still degenerate after we have enlarged the site of maximum index, then we enlarge the site with the second largest index, and so on. As mentioned in the general presentation (Section 2.4), we need only consider the sites involved in the predicate, and enlarge them one-by-one until the resulting configuration is non-degenerate, in which case the predicate is resolved. Sites are sorted internally in the predicate, among a constant number of objects; there is no need for globally sorting sites.

Any indexing can be used. We choose what we call the *max-weight* indexing that assigns a larger index to the site with larger weight. As a result, a site with larger weight is perturbed more, and in order to resolve the predicate we need to consider the sites in order of decreasing weights, until the degeneracy is resolved. To break ties between sites with the same weights, we use the lexicographic comparison of their centers: among two sites with the same weight, the site whose center is lexicographically smaller than the other is assigned a smaller max-weight index. The max-weight indexing has the strong advantage of being geometrically meaningful. It favors sites with larger weights, so, if two sites are internally tangent, then the site with the larger weight will be perturbed first, in which case the site with the smallest weight will be inside the interior of the other site, and its Apollonius region will disappear in the perturbed diagram. As a first consequence, this indexing minimizes the number of Apollonius regions in the diagram, or, equivalently it maximizes the number of hidden sites in the diagram. Secondly, and most importantly, the tangency points of the sites with the Apollonius sites that they define in the diagram are pairwise distinct. This property makes the analysis of the perturbed predicates much simpler, whereas the Apollonius diagram computed does not exhibit pathological cases, such as Apollonius regions with empty interiors. Some inevitable degenerate constructions, such as zero length Apollonius edges, are handled seamlessly by the method. As a final comment, the max-weight scheme can be used to resolve the degeneracies of all predicates described by Emiris and Karavelas for the 2D case [10].

4.3 Perturbing circles for the 2D Apollonius diagram

In two dimensions, the two main predicates for computing Apollonius diagrams are the VConflict predicate introduced in the previous section and the EdgeConflict predicate, which will be analyzed in Section 4.3.5.

4.3.1 Algebraic expression for the 2D VConflict predicate

Let S_i, S_j, S_k be the three sites that define an Apollonius circle in the Apollonius diagram and let $Q = S_q$ be the query site. In the algebraic formulation of the predicate by Emiris and

Karavelas [10], the evaluation of $\mathsf{VConflict}(S_i, S_j, S_k, Q)$ relies on the computation of the sign of the quantity:

$$I := E_{xw}E_x + E_{yw}E_y + E_{xy}\sqrt{\Delta}, \qquad \Delta = (E_x)^2 + (E_y)^2 - (E_w)^2,$$

where

$$E_s = \begin{vmatrix} s_j^* & p_j^* \\ s_k^* & p_k^* \end{vmatrix}, \qquad E_{st} = \begin{vmatrix} s_j^* & t_j^* & p_j^* \\ s_k^* & t_k^* & p_k^* \\ s_q^* & t_q^* & p_q^* \end{vmatrix}, \qquad s,t \in \{x,y,w\},$$

and

$$x_\nu^* = x_\nu - x_i, \quad y_\nu^* = y_\nu - y_i, \quad w_\nu^* = w_\nu - w_i, \quad p_\nu^* = (x_\nu^*)^2 + (y_\nu^*)^2 - (w_\nu^*)^2, \quad \nu \in \{j,k,q\}.$$

The quantity I is a quantity of the form $X_0 + X_1\sqrt{Y}$, where the algebraic degrees of X_0, X_1 and Y are 7, 4 and 6, respectively. Its sign can be computed by means of the formula

$$\operatorname{sign}(X_0 + X_1\sqrt{Y}) = \begin{cases} \operatorname{sign}(X_1) & \text{if } X_0 = 0 \\ \operatorname{sign}(X_0) & \text{if } X_1 = 0 \text{ or } Y = 0 \\ \operatorname{sign}(X_0) & \text{if } \operatorname{sign}(X_0) = \operatorname{sign}(X_1) \\ \operatorname{sign}(X_0)\operatorname{sign}(X_0^2 - X_1^2 Y) & \text{otherwise.} \end{cases} \tag{5}$$

It is thus concluded by Emiris and Karavelas that the algebraic degree of the predicate is 14 [10, Theorem 11].

If fact we can further decrease the algebraic degree of the predicate by observing that the quantity $X_0^2 - X_1^2 Y$ can be factorized as follows (see Appendix B for the details of this derivation):

$$X_0^2 - X_1^2 Y = [(E_x)^2 + (E_y)^2]\,[(E_{xw})^2 + (E_{yw})^2 - (E_{xy})^2],$$

where the first factor is a non-negative quantity of degree 6, whereas the second factor is of degree 8. In fact, when we compute the sign of the quantity $X_0^2 - X_1^2 Y$, we already know that the quantity $(E_x)^2 + (E_y)^2$ is strictly positive, since otherwise X_0 would have been zero (X_0 is a linear combination of E_x and E_y), which has already been ruled out according to the procedure in Eq. (5). Hence the algebraic degree of the predicate is 8.

Before using our qualitative symbolic perturbation framework to design the perturbed predicate, we briefly sketch how a standard algebraic perturbation framework could be applied.

4.3.2 Algebraic perturbation of the 2D VConflict predicate

If $S_\nu = (x_\nu, y_\nu, w_\nu)$ is perturbed in $S_\nu^\varepsilon = (x_\nu, y_\nu, w_\nu + \varepsilon_\nu)$ for $\nu \in \{i,j,k,q\}$, then developing an expression like $(E_{xw})^2$ will give a polynomial of degree 6 in ε_i, ε_j, ε_k and ε_q with 186 terms. Assigning ε_i, ε_j, ε_k and ε_q to be polynomial functions of a single variable ε (for example, we may set $\varepsilon_\nu = \varepsilon^{\alpha_\nu}$, $\nu \in \{i,j,k,q\}$) transforms the expression in a univariate polynomial in ε. When performing such an assignment, either some of the terms collapse making their geometric and algebraic interpretation difficult, or α_i, α_j, α_k and α_q have to be chosen carefully so that the coefficients of the various monomials of the variables ε_ν in the resulting polynomial do not collapse. Even if one can find an assignment that does not make the coefficients (of the originally different terms) collapse, we are still faced with the problem of analyzing the monomials, and, by employing algebraic and/or geometric arguments, showing that there is at least one coefficient of the polynomial that does not vanish.

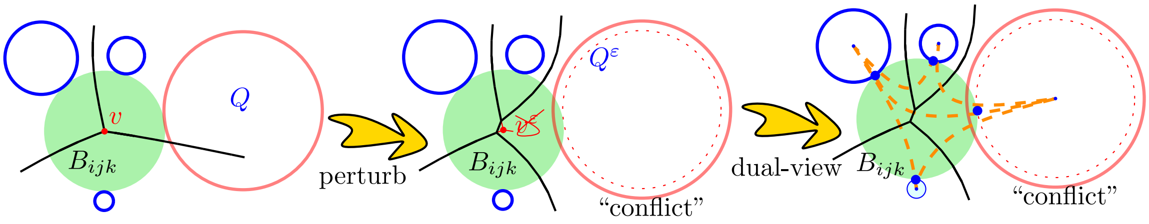


Figure 7: Perturbing a degenerate Apollonius vertex: the case $q > i, j, k$.

4.3.3 Qualitative perturbation of the 2D VConflict predicate

We now precisely describe how the perturbation works on the VConflict predicate in dimension 2. Let us denote by q the max-weight index of Q, i.e., $Q = S_q$. We denote with superscript ε the perturbed version of objects, that is B^ε_{ijk} is a shorthand for the ball tangent to S^ε_i, S^ε_j, and S^ε_k, and

$$\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, S_q) = \lim_{\varepsilon_{i_3}\to 0^+} \lim_{\varepsilon_{i_2}\to 0^+} \lim_{\varepsilon_{i_1}\to 0^+} \lim_{\varepsilon_{i_0}\to 0^+} \mathsf{VConflict}(S^\varepsilon_i, S^\varepsilon_j, S^\varepsilon_k, Q^\varepsilon)$$

with $i_0 < i_1 < i_2 < i_3$, $\{i_0, i_1, i_2, i_3\} = \{i, j, k, q\}$.

If $q > i, j, k$ and VConflict (S_i, S_j, S_k, Q) = "degenerate", we compute the limit given by Eq. (1)

$$\lim_{\varepsilon_q\to 0^+} \mathsf{VConflict}(S_i, S_j, S_k, Q^\varepsilon).$$

It is clear that this limit always evaluates to "conflict", since Q is growing while the open ball B_{ijk} whose closure is tangent to S_i, S_j, and S_k can be considered as fixed and we do not need to look at smaller perturbations (see Figure 7).

If q is not the largest index, then B_{ijk} can be viewed as defined by three other circles among S_i, S_j, S_k, and Q. Since $B_{ijk} = B_{jki} = B_{kij}$, we can assume, without loss of generality, that $i > j, k, q$. Moreover, B_{ijk} coincides with either B_{jkq} or B_{kjq}, depending on the orientation of the tangency points of S_j, S_k and Q with B_{ijk}.

In the perturbed setting, S^ε_i is in conflict with B^ε_{jkq} (or B^ε_{kjq}) since S^ε_i is growing, while B^ε_{jkq} can be considered as fixed. We simply need to determine if B^ε_{ijk} remains empty in the perturbed setting. Let t_ν (resp., t_q) be the tangency point of S_ν (resp., Q) with B_{ijk}, $\nu \in \{i, j, k\}$, and notice that $t_i t_j t_k$ is a ccw triangle. We consider three cases depending on the position of t_q on ∂B_{ijk}. If t_q is different from t_i, t_j, and t_k, the four points form a convex quadrilateral. When perturbing S_i to become S^ε_i, the Apollonius vertex is split in two, which, in the dual,[1] corresponds to a triangulation of the quadrilateral with vertices S_i, S_j, S_k, S_q. Since S_i is the most perturbed circle, the quadrilateral will be triangulated by linking S_i to the other three vertices. If t_q is on the same side as t_i with respect to the line $t_j t_k$, then the triangulation contains triangle $S_i S_j S_k$ and, therefore, Q is not in conflict with B_{ijk} (see Figure 8), otherwise $S_i S_j S_k$ is not in the triangulation and Q has to be in conflict with B_{ijk} (see Figure 9).

If t_q is equal to t_i then, since $i > q$, Q is internally tangent to S_i and there is no conflict (S^ε_i contains Q in its interior, and thus Q has empty Apollonius region in the diagram). If t_q is equal to t_ν with $\nu \in \{j, k\}$ then either Q is internally tangent to S_ν, or S_ν is internally tangent to Q. In the former case, Q does not intersect the perturbed Apollonius disk B^ε_{ijk} and thus

[1] The dual of the Apollonius diagram is called Apollonius graph. The Apollonius region of a site S_i is associated to a vertex of the dual graph, thus S_i can be used to refer to the corresponding vertex in the dual graph.

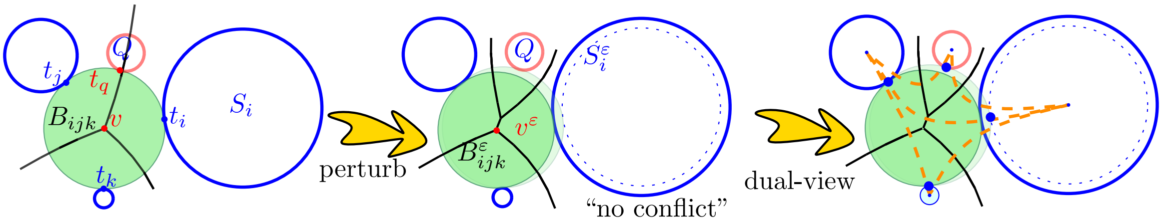


Figure 8: Perturbing a degenerate Apollonius vertex: the case $i > j, k, q$ and $t_j t_k t_q$ is a ccw triangle.

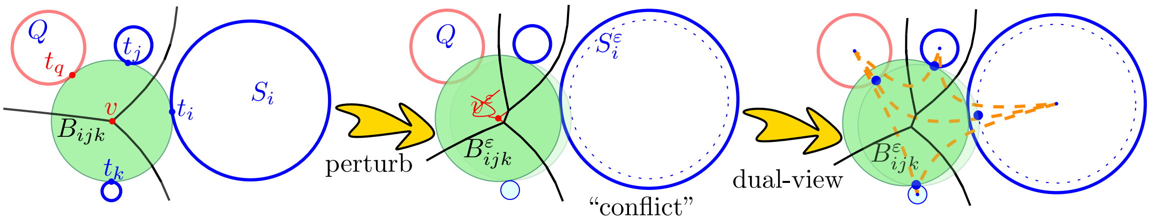


Figure 9: Perturbing a degenerate Apollonius vertex: the case $i > j, k, q$ and $t_j t_k t_q$ is a cw triangle.

the result of the perturbed predicate is “no conflict”; in the latter case, Q intersects B^ε_{ijk}, and the perturbed predicate returns “conflict”. Hence, in the case $t_q = t_\nu$, $\nu \in \{j, k\}$, the perturbed predicate returns “conflict” if an only if $q > \nu$.

4.3.4 Practical evaluation of the 2D $\mathsf{VConflict}^\varepsilon$ predicate

Following the analysis in the previous section, $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q)$ can be evaluated by the following procedure:

1. **if** $\mathsf{VConflict}(S_i, S_j, S_k, Q) \neq$ “degenerate” **then return** $\mathsf{VConflict}(S_i, S_j, S_k, Q)$;
2. **if** $q > \max\{i, j, k\}$ **then return** “conflict”;
3. ensure that $i > \max\{j, k\}$ by a cyclic permutation of (i, j, k);
4. **if** $t_q = t_i$ **then return** “no conflict”;
5. **if** $t_q = t_j$ **then** { **if** $q > j$ **then return** “conflict”; **else return** “no conflict”; };
6. **if** $t_q = t_k$ **then** { **if** $q > k$ **then return** “conflict”; **else return** “no conflict”; };
7. **if** $t_j t_k t_q$ is ccw **then return** “no conflict”; **else return** “conflict”;

Step 1 is evaluated as described in Section 4.3.1. Steps 2 and 3 amount to sorting the indices of the four sites and determining if q is the largest, or, if this is not the case, finding the largest index. At Step 4, we already know that $i > q$, which implies that $w_i \geq w_q$, and hence the only possibility is that Q is internally tangent to S_i. So, in order to perform Step 4, we simply look at $p^*_q = (x_q - x_i)^2 + (y_q - y_i)^2 - (w_q - w_i)^2$: if $p^*_q = 0$, return “no conflict”, otherwise continue with Step 5. Steps 5 and 6 can be resolved in a similar way: if $(x_q - x_\nu)^2 + (y_q - y_\nu)^2 - (w_q - w_\nu)^2 = 0$, then if $q > \nu$ (resp. $q < \nu$), we return “conflict” (resp. “no conflict”). Otherwise, we continue with the last step of the procedure.

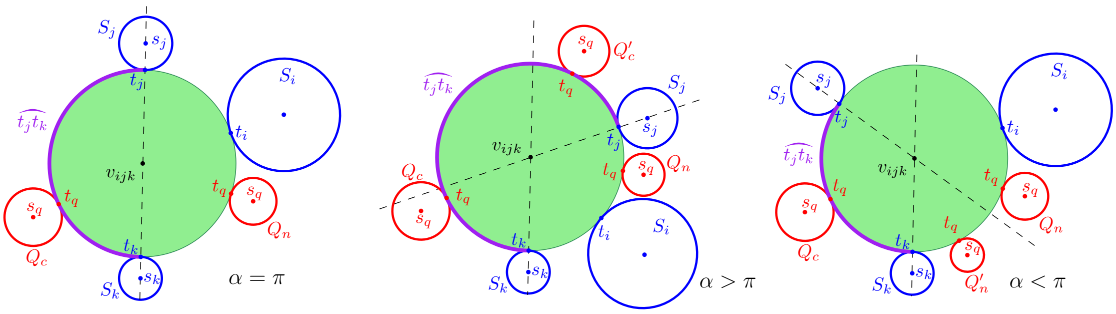


Figure 10: Computing the auxiliary predicate $\mathsf{Orientation}(t_j, t_k, t_q)$ using orientations involving the Apollonius vertex v_{ijk}. From left to right: the case $\alpha = \pi$, the case $\alpha > \pi$, and the case $\alpha < \pi$, where α is the angle of the ccw oriented arc $\widehat{t_j t_k}$.

We will now focus on this last step, Step 7, because it introduces a new geometric predicate, which is difficult to evaluate: $\mathsf{Orientation}(t_j, t_k, t_q)$, for three tangency points. Our aim is to reduce the complexity of the expressions to be evaluated, which is why we avoid computing the tangency points explicitly. The end of this section describes a method with algebraic degree 8, as in Step 1. This computation can be done in another way: in Appendix A we proposed an alternative method that requires very few additional arithmetic computations besides the quantities already computed in the unperturbed evaluation of Step 1, however these few extra computations have algebraic degree 12.

It has been shown in [10] that evaluating the orientation of three points where two are centers of sites and the third is an Apollonius vertex, can be performed using algebraic expressions of degree at most 14. In fact, this degree may be decreased to 8 (Appendix B), in which case we resolve $\mathsf{Orientation}(t_j, t_k, t_q)$, without resorting to a higher degree predicate, as described below.

Firstly, we evaluate $o_1 = \mathsf{Orientation}(s_j, v_{ijk}, s_k)$, where v_{ijk} is the center of the Apollonius circles B_{ijk} of the three sites S_i, S_j, S_k. We perform this evaluation in order to determine whether the angle α of the ccw arc $\widehat{t_j t_k}$ on B_{ijk} is more or less than π. Secondly, we distinguish between the following cases (see Figure 10):

o_1 = "collinear". In this case $\alpha = \pi$, and the line through t_j and t_k coincides with the line through s_j and s_k. Hence: $\mathsf{Orientation}(t_j, t_k, t_q) = \mathsf{Orientation}(s_j, s_k, s_q)$ (see Q_n (resp. Q_c) in Figure 10(left) to illustrate a position of Q not in conflict (resp. in conflict)).

o_1 = "ccw". In this case $\alpha > \pi$. We start by evaluating $o_2 = \mathsf{Orientation}(s_j, v_{ijk}, s_q)$. If $o_2 \neq$ "ccw" (see Q'_c in Figure 10(middle)), t_q lies to the right of the line through t_j and t_k, and thus $\mathsf{Orientation}(t_j, t_k, t_q)$ = "cw". Otherwise, we need to evaluate the orientation $o_3 = \mathsf{Orientation}(v_{ijk}, s_k, s_q)$; then $\mathsf{Orientation}(t_j, t_k, t_q)$ = "ccw" if and only if o_3 = "ccw" (see Q_n and Q_c in Figure 10(middle)).

o_1 = "cw". In this case $\alpha < \pi$. We start by evaluating $o_2 = \mathsf{Orientation}(s_j, v_{ijk}, s_q)$. If $o_2 \neq$ "cw", t_q lies to the left of the line through t_j and t_k, and thus $\mathsf{Orientation}(t_j, t_k, t_q)$ = "ccw" (see Q_n in Figure 10(right)). Otherwise, we need to evaluate the orientation $o_3 = \mathsf{Orientation}(v_{ijk}, s_k, s_q)$; then $\mathsf{Orientation}(t_j, t_k, t_q)$ = "cw" if and only if o_3 = "cw" (see Q_c and Q'_n in Figure 10(right)).

To summarize, the evaluation of Step 7 requires at most three orientation tests involving an Apollonius vertex and two sites; one may be obtained as a subproduct of Step 1, while the other two require work similar to the work performed for Step 1. Thus, the evaluation of Step 7 does not increase the algebraic degree of the VConflict predicate.

4.3.5 Qualitative perturbation for the 2D EdgeConflict predicate

The computation of the 2D Apollonius diagram requires another predicate. When a new site is added it is not enough to find which Apollonius vertices remain or disappear: we need a more complete analysis of the modification of the edges of the diagram. This is the subject of the EdgeConflict predicate. Given four sites S_i, S_j, S_k and S_l that define a Voronoi edge e in the diagram, and a query site Q, EdgeConflict determines the type of conflict of Q with the edge e. This predicate is the basis of the randomized incremental construction algorithm for computing abstract Voronoi diagrams by Klein, Mehlhorn, and Meiser [15], as well as one of the main predicates analyzed by Emiris and Karavelas [10]. In [10] this predicate is decomposed to a number of subpredicates, one of them being the VConflict predicate.

We assume below that e lies on the bisector of S_i and S_j, oriented so that S_i is lying to the right of the bisector (refer also to Figure 11). The edge e inherits the orientation from its supporting bisector. The origin vertex of e is the Apollonius vertex defined by the (oriented) triple S_i, S_j and S_k, while the target vertex of e is the Apollonius vertex defined by the triple S_j, S_i and S_l. The EdgeConflict predicate determines the type of the subset of e that is destroyed by the insertion of Q in the Apollonius diagram of the four sites, and has six possible outcomes:

- "conflict origin":

  a subsegment of e adjacent to its origin vertex disappears in the Apollonius diagram of the five sites. This case occurs if and only if $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q)$ = "conflict" and $\mathsf{VConflict}^\varepsilon(S_j, S_i, S_l, Q)$ = "no conflict". This case is illustrated by Q_{cn} in Figure 11.

- "conflict target": is the symmetric case that occurs iff $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q)$ = "no conflict" and $\mathsf{VConflict}^\varepsilon(S_j, S_i, S_l, Q)$ = "conflict". See Q_{nc} in Figure 11.

- "no conflict": no portion of e is destroyed by the insertion of Q in the Apollonius diagram of the four sites. This case can occur only when $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q) = \mathsf{VConflict}^\varepsilon(S_j, S_i, S_l, Q)$ = "no conflict". See Q_{nn} in Figure 11.

- "conflict interior": a subsegment in the interior of e disappears in the Apollonius diagram of the five sites. This case can occur only when $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q) = \mathsf{VConflict}^\varepsilon(S_j, S_i, S_l, Q)$ = "no conflict". See Q'_{nn} in Figure 11.

- "conflict entire edge":

  the entire edge e is destroyed by the addition of Q in the Apollonius diagram of the four sites. This case can occur only when $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q) = \mathsf{VConflict}^\varepsilon(S_j, S_i, S_l, Q)$ = "conflict". See Q_{cc} in Figure 11.

- "conflict both": subsegments of e adjacent to its two vertices disappear in the Apollonius diagram of the five sites. This case can occur only when $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q) = \mathsf{VConflict}^\varepsilon(S_j, S_i, S_l, Q)$ = "conflict". See Q'_{cc} in Figure 11.

Thus when the evaluations of predicate $\mathsf{VConflict}^\varepsilon$ on (S_i, S_j, S_k, Q) and (S_j, S_i, S_l, Q) are available, it only remains to distinguish between "no conflict" and "conflict interior", as well as between "conflict entire edge" and "conflict both". Assuming a non-degenerate configuration, this question is addressed in [10] using an auxiliary predicate of algebraic degree 16. The only situation where this auxiliary predicate has degeneracies is when $\mathsf{VConflict}(S_i, S_j, S_k, Q)$ or/and

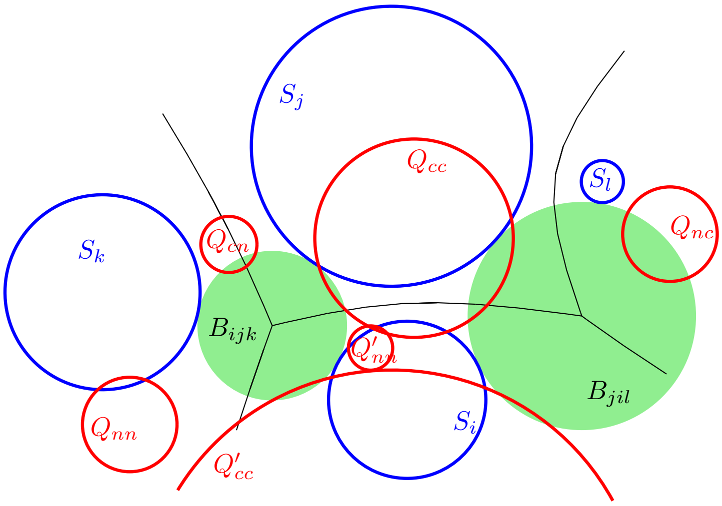


Figure 11: The different non-degenerate possible configurations for the EdgeConflict predicate.

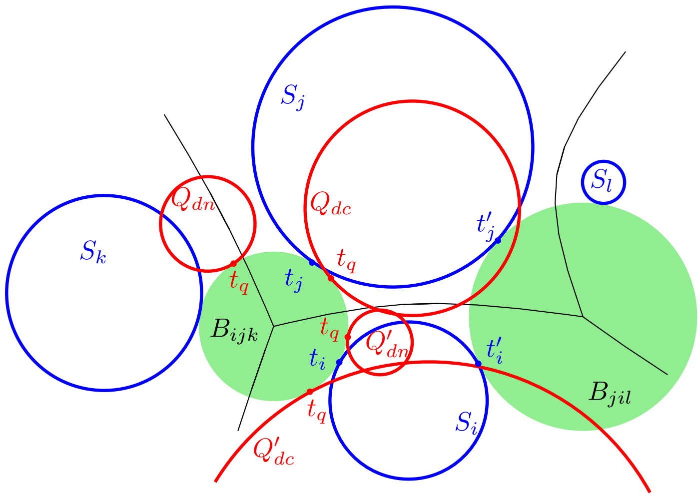


Figure 12: The different degenerate possible configurations for the EdgeConflict predicate.

$\mathsf{VConflict}(S_j, S_i, S_l, Q)$ are "degenerate" and $\mathsf{VConflict}^{\varepsilon}(S_i, S_j, S_k, Q) = \mathsf{VConflict}^{\varepsilon}(S_j, S_i, S_l, Q)$. Then the predicate can be answered by looking at the relative position of t_q with respect to t_i and t_j on ∂B_{ijk} (or the relative position of t'_q with respect to t'_i and t'_j on ∂B_{jil}).

More precisely $\mathsf{EdgeConflict}^{\varepsilon}(S_i, S_j, S_k, S_l, Q)$ can be evaluated by means of the following procedure:

1. **if** ($\mathsf{VConflict}^{\varepsilon}(S_i, S_j, S_k, Q) =$ "conflict"
   and $\mathsf{VConflict}^{\varepsilon}(S_j, S_i, S_l, Q) =$ "no conflict") **then**

return "conflict origin";

2. **if** ($\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q)$ = "no conflict"
   and $\mathsf{VConflict}^\varepsilon(S_j, S_i, S_l, Q)$ = "conflict") **then**
   return "conflict target";

3. **if** ($\mathsf{VConflict}(S_i, S_j, S_k, Q) = \mathsf{VConflict}(S_j, S_i, S_l, Q)$
   and $\mathsf{VConflict}(S_i, S_j, S_k, Q) \neq$ "degenerate") **then**
   return $\mathsf{EdgeConflict}(S_i, S_j, S_k, S_l, Q)$;

4. **if** $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q) = \mathsf{VConflict}^\varepsilon(S_j, S_i, S_l, Q)$ = "no conflict" **then**
   if $\mathsf{VConflict}(S_i, S_j, S_k, Q)$ = "degenerate" **then**
   if $t_i t_j t_q$ is ccw **then return** "no conflict"; [Q_{dn} in Figure 12]
   else return "conflict interior"; [Q'_{dn} in Figure 12]
   else
   [in this case: $\mathsf{VConflict}(S_j, S_i, S_l, Q)$ = "degenerate"]
   if $t'_j t'_i t'_q$ is ccw **then return** "no conflict";
   else return "conflict interior";

5. **if** $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q) = \mathsf{VConflict}^\varepsilon(S_j, S_i, S_l, Q)$ = "conflict" **then**
   if $\mathsf{VConflict}(S_i, S_j, S_k, Q)$ = "degenerate" **then**
   if $t_i t_j t_q$ is ccw **then return** "conflict both"; [Q_{dc} in Figure 12]
   else return "conflict entire edge"; [Q'_{dc} in Figure 12]
   else
   [in this case: $\mathsf{VConflict}(S_j, S_i, S_l, Q)$ = "degenerate"]
   if $t'_j t'_i t'_q$ is ccw **then return** "conflict both";
   else return"conflict entire edge";

The main observation from the above analysis is that the max-weight qualitative perturbation scheme described in Section 4.2 not only resolves the VConflict predicate, but also the EdgeConflict predicate (although not described in this paper, our perturbation scheme resolves, in fact, all degeneracies of all predicates used in [10] for the computation of the 2D Apollonius diagram). To resolve the EdgeConflict predicate, we need only evaluate $\mathsf{Orientation}(t_i, t_j, t_q)$ and/or $\mathsf{Orientation}(t'_j, t'_i, t'_q)$. As described in the previous section, this predicate is of algebraic degree 8, and thus does not increase the algebraic degree of the EdgeConflict predicate. Furthermore, with a careful implementation, it is possible to keep track of the intermediate results of the evaluation of the $\mathsf{VConflict}^\varepsilon$ predicate and resolve the EdgeConflict predicate using these intermediate results in a purely combinatorial manner.

As a final note, the above procedure for the evaluation of the EdgeConflict predicate works as is even when the Apollonius edge e is of zero length. This is the case when the Apollonius vertices v_{ijk} and v_{jil} coincide, or, equivalently, when the four sites S_i, S_l, S_j and S_k are all tangent (and in that ccw order) to the same Apollonius circle. The only difference with respect to the non-zero-length Apollonius edge case are the possible outcomes: the EdgeConflict predicate will never return "conflict interior" nor "conflict both"; this is, however, automatically handled by the procedure described above, that is without the need to handle any additional special cases.

4.4 Perturbing spheres for the 3D Apollonius diagram

In three dimensions, an Apollonius vertex v_{ijkl} is defined by four sites S_i, S_j, S_k, and S_l, while the predicate $\mathsf{VConflict}(S_i, S_j, S_k, S_l, Q)$ tests if after adding a fifth site $Q = S_q$, v_{ijkl} remains a valid Apollonius vertex or not. Let B_{ijkl} denote the ball tangent to S_i, S_j, S_k, and S_l, whose tangency points t_i, t_j, t_k, and t_l form a positively oriented tetrahedron $t_i t_j t_k t_l$.

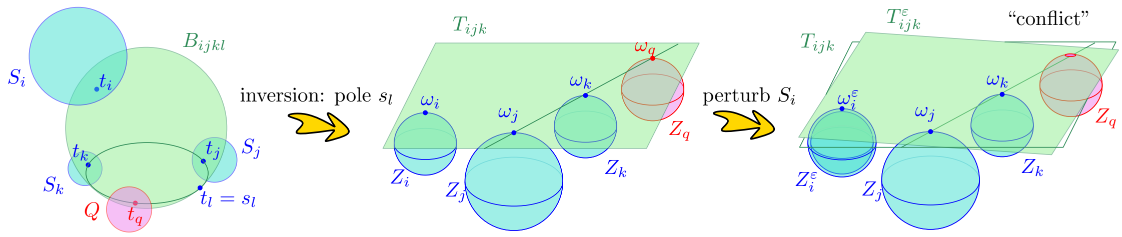


Figure 13: The case, for the VConflict predicate, where the tetrahedron $t_q t_j t_k t_l$ is flat.

The predicate in general position can be solved in different ways. One way is to do like Boissonnat and Delage [2]. Another way is to use inversion like in the 2D case by Emiris and Karavelas [10], and arrive at an alternative expression [11]. In degenerate configuration, Q and B_{ijkl} are tangent at t_q and we can obtain, as a side product, the orientation of the tetrahedra formed by 4 of the 5 tangency points t_i, t_j, t_k, t_l, and t_q.

As for the 2D case, we apply the max-weight QSP scheme. If the predicate is degenerate the effect of the perturbation is that the weight of the site with largest index increases, and thus intersects the ball tangent to the four other sites. In the neighborhood of the center of B_{ijkl} the Apollonius diagram of S_i, S_j, S_k, S_l, and Q has the same combinatorial structure as the Voronoi diagram of t_i, t_j, t_k, t_l, and t_q. We, thus, get an equivalent formulation for the predicate: given five co-spherical points t_i, t_j, t_k, t_l, and t_q, does the tetrahedron $t_i t_j t_k t_l$ remain in the Delaunay triangulation when the point of the largest index is moved inside the ball. Notice that it implies that the point with largest index is linked to all other points in this Delaunay triangulation.

Similarly to the two-dimensional case, we can conclude that Q is in conflict with B_{ijkl} if $q > i, j, k, l$. We can also take care of the cases where t_q is equal to one of the four points t_i, t_j, t_k, and t_l. Otherwise, we rename the indices so that i is the largest one. The definition of B_{ijkl} says that tetrahedron $t_i t_j t_k t_l$ is positively oriented. Notice that this can be true in two ways: either the tetrahedron is really positively oriented, or it is flat, as the limit of a positively oriented tetrahedron when $\varepsilon_i \to 0^+$.

If $t_q t_j t_k t_l$ is positively oriented, t_q and t_i are on the same side of $t_j t_k t_l$, which is a convex hull facet. Since the 3D Apollonius graph is "star-shaped" from S_i, S_i is linked to $S_j S_k S_l$ to create the tetrahedron $S_i S_j S_k S_l$, and thus there is no conflict for Q.

If $t_q t_j t_k t_l$ is negatively oriented, t_q and t_i are on opposite sides of $t_j t_k t_l$, which implies that $S_j S_k S_l$ ceases to be a facet of the Apollonius graph. Thus, the tetrahedron $S_i S_j S_k S_l$ disappears and Q is in conflict.

If $t_q t_j t_k t_l$ is flat, the question reduces to determining the orientation of $t^{\varepsilon}_q t^{\varepsilon}_j t^{\varepsilon}_k t^{\varepsilon}_l$, which are the points of tangency of S_q, S_j, S_k, S_l with B^{ε}_{ijkl} after the perturbation of S_i by ε_i. This orientation will be non-degenerate except in two very special cases where the centers of Q, S_j, S_k, and S_l are either co-circular or collinear.

We first address the case where s_q, s_j, s_k, and s_l are neither co-circular nor collinear. Let us assume that l is smaller than j and k, which means that $w_l \leq w_j, w_k$. By subtracting w_l from all weights, we can consider that S_l has zero weight, and then perform an inversion with pole s_l (see Figure 13). Let Z_i, Z_j, Z_k, and Z_q be the images of sites S_i, S_j, S_k, and Q, and ω_i, ω_j, ω_k, and ω_q be the images of t_i, t_j, t_k, and t_q under inversion, and denote by z_i, z_j, z_k, and z_q the centers of Z_i, Z_j, Z_k, and Z_q. Since $t_q t_j t_k t_l$ is a flat tetrahedron, the four points t_q, t_j, t_k, t_l are co-circular and thus $\omega_q, \omega_j, \omega_k$ are collinear. Their supporting line lies in the *unique* plane T_{ijk} that is commonly tangent to all three sites Z_q, Z_j, and Z_k. The uniqueness follows from the fact that z_q, z_j, and z_k are not collinear, since s_q, s_j, s_k, and $s_l = t_l$ have been assumed to be neither co-circular nor collinear. The planes tangent to both Z_j and Z_k are tangents to the cone C, whose axis is the line through z_j and z_k. When we perturb S_i to S_i^ε, the plane T_{ijk} moves a bit, in the set of planes tangent to C, to become T_{ijk}^ε.

Consider first the case $w_q \geq w_l$, which implies that the weight of Z_q is non-negative. Since z_q, z_j, and z_k are not collinear and Z_q is tangent to C at ω_q, Z_q either properly intersects C or is inside C. If Z_q is (tangent to and) inside C, then, for all values of ε_i, T_{ijk}^ε does not intersect Z_q, and the result of the perturbed predicate is "no conflict", otherwise, for all values of ε_i, T_{ijk}^ε intersects Z_q, and the result of the perturbed predicate is "conflict". The way to evaluate the $\mathsf{VConflict}^\varepsilon$ predicate, in this case, is by determining the value of $\mathsf{Orientation}(z_q, z_j, z_k)$, where this orientation is seen as a two-dimensional orientation in the plane that is perpendicular to T_{ijk} and passes through z_j and z_k. If $w_j = w_k = w_l$, the cone C degenerates to the line through z_j and z_k. In this case we have $w_q > w_l$ (since, otherwise, s_j, s_k, s_l and s_q would have been co-circular), and thus $\mathsf{VConflict}^\varepsilon$ returns "conflict". If at least two of w_j, w_k, w_l differ, then at least one of ω_j and ω_k differs from z_j and z_k, respectively. Denoting by $\omega_\star$ a/the point of tangency that differs from the corresponding center, it suffices to determine if $\mathsf{Orientation}(z_q, z_j, z_k) = \mathsf{Orientation}(\omega_\star, z_j, z_k)$, in which case the $\mathsf{VConflict}^\varepsilon$ predicate returns "no conflict", otherwise "conflict" is returned.

Finally, notice that when $w_q < w_l$, the site Z_q has negative weight. In this case, the sphere Z_q^- will properly intersect the plane T_{ijk}^ε for all values of ε_i, which implies that S_q does not intersect B_{ijk}^ε. Hence, in this case, the result of the perturbed predicate is "no conflict".

In all cases above, perturbing S_i was sufficient to remove the degeneracy. In the very degenerate cases where s_j, s_k, s_l, and s_q are co-circular or collinear, the unique edge of the (degenerate) Apollonius diagram of S_j, S_k, S_l, and Q is a circle or a line. In these cases, the position of S_i has no influence on the combinatorial structure of the diagram of S_i, S_j, S_k, S_l, and Q, and we need to perturb the second most perturbed site S_j or S_k or Q to remove the degeneracy. The resolution of the degeneracy is similar to the 2D case: first perform a positive permutation of j, k, l to ensure that $j > k, l$. If $q > j$ then Q will be in conflict, otherwise, if $q < j$ then Q will be in conflict if and only if $t_j t_k t_l$ and $t_q t_k t_l$ have different two-dimensional orientations.

Following the above analysis, $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, S_l, Q)$ can be evaluated as follows:

1. **if** $\mathsf{VConflict}(S_i, S_j, S_k, S_l, Q) \neq$ "degenerate" **then return** $\mathsf{VConflict}(S_i, S_j, S_k, S_l, Q)$;
2. **if** $q > \max\{i, j, k, l\}$ **then return** "conflict";
3. ensure that $i > \max\{j, k\} \geq \min\{j, k\} > l$ by a positive permutation of (i, j, k, l);
4. **if** $t_q = t_i$ **then return** "no conflict";
5. **if** $t_q = t_j$ **then** { **if** $q > j$ **then return** "conflict"; **else return** "no conflict"; };
6. **if** $t_q = t_k$ **then** { **if** $q > k$ **then return** "conflict"; **else return** "no conflict"; };
7. **if** $t_q = t_l$ **then** { **if** $q > l$ **then return** "conflict"; **else return** "no conflict"; };
8. **if** $t_q t_j t_k t_l$ is positively oriented **then return** "no conflict";
9. **if** $t_q t_j t_k t_l$ is negatively oriented **then return** "conflict";
10. **if** s_j, s_k, s_l, s_q are neither collinear nor co-circular **then**
    i. **if** $w_q < w_l$ **then return** "no conflict";
    ii. **if** $w_j = w_k = w_l$ **then return** "conflict";

iii. compute a/the tangency point $\omega_\star$;
if z_q, z_j, z_k and $\omega_\star, z_j, z_k$ have the same 2D orientation **then return** "no conflict";
else return "conflict";

11. ensure that $i > j > \max\{k, l\}$ by a positive permutation of (j, k, l);

12. **if** $q > j$ **then return** "conflict";

13. **if** $t_j t_k t_l$ and $t_q t_k t_l$ have the same 2D orientation **then return** "no conflict";
else return "conflict";

Steps 8 and 9 of the above algorithm rely on the auxiliary predicate $\mathsf{Orientation}(t_q, t_j, t_k, t_l)$ that, given five sites, computes the orientation of the four tangency points on the common tangent sphere to the fifth site. Since the tetrahedron $t_i t_j t_k t_l$ is, by definition positively oriented, $\mathsf{Orientation}(t_q, t_j, t_k, t_l)$ will be positive if and only if t_q lies on the upper *half* of B_{ijkl} as t_i, where the two halves of B_{ijkl} are delimited by the circle through t_j, t_k and t_l. We first reduce all the weights by w_l. If $w_j = w_k = w_l (= 0)$, computing the orientation of t_q, t_j, t_k, t_l amounts to evaluating the orientation of t_q, s_j, s_k, s_l. If w_j, w_k and w_l are not all equal, we consider again the inversion transformation with s_l as the pole. Then t_q lies on the same half of B_{ijkl} as t_i if and only if ω_q lies on the same half-plane of Π_{ijk}, with respect to the line through ω_j and ω_k, with ω_i. The equality of these 2D orientation tests is equivalent to testing the result of $\mathsf{Orientation}(z_q, z_j, z_k, \omega_\star)$; if $\mathsf{Orientation}(z_q, z_j, z_k, \omega_\star) =$ "ccw" return "no conflict", otherwise return "conflict".

For Step 10 we first need to test if the points s_j, s_k, s_l and s_q are either collinear or co-circular. The possible collinearity can easily be tested via the cross-products $(s_l - s_j) \times (s_k - s_j)$ and $(s_q - s_j) \times (s_k - s_j)$; if both are the zero vector, the four points are collinear. Co-circularity in the original space corresponds to collinearity in the inverted space (where the pole of inversion is s_l); to test if s_j, s_k, s_l and s_q are co-circular we simply need to test if z_j, z_k, z_q are collinear, which amounts to computing the cross-product $(z_q - z_j) \times (z_k - z_j)$. For the 2D orientations of z_q, z_j, z_k and $\omega_\star, z_j, z_k$, we need only choose a point $x \notin plane(z_q z_j z_k)$ and determine if the tetrahedra $z_q z_j z_k x$ and $\omega_\star z_j z_k x$ have the same orientation.

Finally, if the centers s_j, s_k, s_l, s_q are co-circular the 2D orientations of $t_j t_k t_l$ and $t_q t_k t_l$ in Step 13 are the same as those of $s_j s_k s_l$ and $s_q s_k s_l$. Choosing a point $x \notin plane(s_j s_k s_l)$, we may compute these 2D orientations via their 3D counterparts $s_j s_k s_l x$ and $s_q s_k s_l x$. If the centers s_j, s_k, s_l, s_q are collinear, notice that, due to the fact that the tetrahedron $t_i t_j t_k t_l$ is positively oriented, s_k lies inside the segment $s_j s_l$. Hence, determining if the 2D orientations of $t_j t_k t_l$ and $t_q t_k t_l$ are the same reduces to determining if s_q lies inside the segment $s_k s_l$ or not; in the former case we return "conflict", while in the latter case we return "no conflict".

We end this section by briefly discussing the algebraic degree of three-dimensional $\mathsf{VConflict}^\varepsilon$ predicate. The unperturbed predicate can be evaluated with algebraic expressions of degree at most 10 [11]; this accounts for Step 1 of the algorithm described above. Steps 2, 3, 11 and 12 all amount to comparing indices of sites, so they are all of degree 1. To resolve Steps 4 to 7 we need to test whether the points of tangency of two spheres with the common Voronoi sphere coincide; this amounts to testing whether one sphere is internally tangent to another one, which is a degree-2 predicate. Step 13 can easily be resolved using expressions of degree at most 6 [13], while Steps 10(i) and 10(ii) are clearly degree-1 operations. The most demanding parts of our evaluation procedure are Steps 8, 9 and 10(iii). Their algebraic degrees can be shown to be 28 and 20, respectively [13]. Hence, the $\mathsf{VConflict}^\varepsilon$ predicate can be evaluated, as described above, with expressions of algebraic degree at most 28.

5 Conclusion

In this paper, a new framework for dealing with geometric degeneracies has been proposed: QSP. Conversely to usual approaches for symbolic perturbation, the new framework does not rely on a particular algebraic description of the predicate, but rather directly on its geometric description.

A QSP scheme consists of a sequence of perturbations, but given a specific predicate only a few of these perturbations are really *active*. The number of active perturbations used to resolve a specific predicate depends on the problem at hand. For the 2D Apollonius diagram perturbing one site always suffices. In its 3D counterpart we may need to perturb two sites, whereas in the case of circular arcs we may need perform a rotation (perturb the axes) and perturb up to one supporting circle per predicate. Minimizing the number of active perturbations is not necessarily desirable, since it might result in a more complicated design of the perturbed predicate (for example, trying to resolve degeneracies for the trapezoidal map of circular arcs with a single active perturbation seems much more complicated).

Besides the number of active perturbations, another important issue is the ordering of the perturbations: for the Apollonius diagram we consider sites by decreasing weight, whereas for the trapezoidal map of circular arcs we first consider a (global) rotation and then the circles by means of decreasing radius. Different perturbation sequences than the ones described in this paper are definitely possible; the analysis, however, can become unnecessarily more complicated.

Our qualitative symbolic perturbation framework, and in particular the schemes described in this paper, can also be applied to a variety of other problems, such as the 2D Voronoi diagram of disjoint convex objects under any L_p metric, as well as the Euclidean Voronoi diagram of certain disjoint convex objects in 3D (the objects can be, for example, non-intersecting lines, line segments or rays). It suffices to replace a site S_i with its Minkowski sum with a ball of radius ε_i, and then consider the limits $\varepsilon_i \to 0^+$, for an appropriately defined ordering of the sites. Another type of geometric problem, involving complex predicates, for which the QSP framework is relevant, is the computation of lines tangent to four given lines in 3D [3, 6].

As a parallel goal, we plan to implement QSP schemes for the problems presented in this paper. In fact, the implementation of the max-weight perturbation scheme inside Package `Apollonius_graph_2` of CGAL [14] is under way, and is expected to become part of the package in the future.

References

[1] Pierre Alliez, Olivier Devillers, and Jack Snoeyink. Removing degeneracies by perturbing the problem or the world. *Reliable Computing*, 6:61–79, 2000.
http://hal.inria.fr/inria-00338566/.

[2] Jean-Daniel Boissonnat and Christophe Delage. Convex hull and Voronoi diagram of additively weighted points. In G. S. Brodal and S. Leonardi, editors, *Proceedings of 13th Annual European Symposium on Algorithms (ESA 2005)*, volume 3669 of *LNCS*, pages 367–378. Springer, 2005.
http://www.springerlink.com/index/ehwb4lu0h7kklj61.pdf.

[3] H. Brönnimann, O. Devillers, Vida Dujmović, H. Everett, M. Glisse, X. Goaoc, S. Lazard, H.-S. Na, and S. Whitesides. Lines and free line segments tangent to arbitrary three-dimensional convex polyhedra. *SIAM Journal on Computing*, 37:522–551, 2007.
http://hal.inria.fr/inria-00103916.

[4] C. Burnikel, K. Mehlhorn, and S. Schirra. On degeneracy in geometric computations. In *Proc. 5th ACM-SIAM Sympos. Discrete Algorithms*, pages 16–23, 1994.
http://dl.acm.org/citation.cfm?id=314474.

[5] Olivier Devillers, Alexandra Fronville, Bernard Mourrain, and Monique Teillaud. Algebraic methods and arithmetic filtering for exact predicates on circle arcs. *Comput. Geom. Theory Appl.*, 22:119–142, 2002.
http://dx.doi.org/10.1016/S0925-7721(01)00050-5.

[6] Olivier Devillers, Marc Glisse, and Sylvain Lazard. Predicates for line transversals to lines and line segments in three-dimensional space. In *Proc. 24th Annual Symposium on Computational Geometry*, pages 174–181, 2008.
http://hal.inria.fr/inria-00336256/.

[7] Olivier Devillers and Monique Teillaud. Perturbations for Delaunay and weighted Delaunay 3D triangulations. *Computational Geometry: Theory and Applications*, 44:160–168, 2011.
http://hal.archives-ouvertes.fr/inria-00560388/.

[8] H. Edelsbrunner and E. P. Mücke. Simulation of simplicity: A technique to cope with degenerate cases in geometric algorithms. *ACM Trans. Graph.*, 9(1):66–104, 1990.
http://dl.acm.org/citation.cfm?id=77639.

[9] I. Emiris and J. Canny. A general approach to removing degeneracies. *SIAM J. Comput.*, 24:650–664, 1995.
http://epubs.siam.org/sicomp/resource/1/smjcat/v24/i3/p650_s1.

[10] Ioannis Z. Emiris and Menelaos I. Karavelas. The predicates of the Apollonius diagram: algorithmic analysis and implementation. *Computational Geometry: Theory and Applications*, 33(1-2):18–57, January 2006. Special Issue on Robust Geometric Algorithms and their Implementations.
http://dx.doi.org/10.1016/j.comgeo.2004.02.006.

[11] Iordan Marinov Iordanov. The Euclidean *InSphere* Predicate. B.S. thesis, University of Crete, Department of Applied Mathematics, 2013.
http://www.tem.uoc.gr/~mkaravel/files/theses/Iordanov-BSthesis.pdf.

[12] Geoffrey Irving and Forrest Green. A deterministic pseudorandom perturbation scheme for arbitrary polynomial predicates. Technical Report 1308.1986v1, arXiv, 2013.
http://arxiv.org/abs/1308.1986.

[13] Emmanouil Kamarianakis. Personal communication, 2013.

[14] Menelaos Karavelas and Mariette Yvinec. 2D Apollonius graphs (Delaunay graphs of disks). In *CGAL User and Reference Manual*. CGAL Editorial Board, 4.2 edition, 2013.
http://www.cgal.org/Manual/4.2/doc_html/cgal_manual/packages.html#Pkg:ApolloniusGraph2.

[15] Rolf Klein, Kurt Mehlhorn, and Stefan Meiser. Randomized incremental construction of abstract Voronoi diagrams. *Comput. Geom. Theory Appl.*, 3(3):157–184, 1993.
http://dx.doi.org/10.1016/0925-7721(93)90033-3.

[16] K. Mehlhorn, R. Osbild, and M. Sagraloff. A general approach to the analysis of controlled perturbation algorithms. *Comput. Geom. Theory Appl.*, 44:507–528, 2011.
http://dx.doi.org/10.1016/j.comgeo.2011.06.001.

[17] R. Seidel. The nature and meaning of perturbations in geometric computing. *Discrete Comput. Geom.*, 19:1–17, 1998.
http://www.springerlink.com/content/px52xh005cxxvdku/.

[18] Raimund Seidel. Perturbations in geometric computing, 2013. Talk at the Workshop on Geometric Computing, Heraklion.
http://www.acmac.uoc.gr/GC2013/files/Seidel-slides.pdf.

[19] C. K. Yap. A geometric consistency theorem for a symbolic perturbation scheme. *J. Comput. Syst. Sci.*, 40(1):2–18, 1990.
http://www.sciencedirect.com/science/article/pii/002200009090016E.

[20] C. K. Yap. Symbolic treatment of geometric degeneracies. *J. Symbolic Comput.*, 10:349–370, 1990.
http://www.sciencedirect.com/science/article/pii/S0747717108800697.

[21] C. K. Yap and T. Dubé. The exact computation paradigm. In D.-Z. Du and F. K. Hwang, editors, *Computing in Euclidean Geometry*, volume 4 of *Lecture Notes Series on Computing*, pages 452–492. World Scientific, Singapore, 2nd edition, 1995.
http://www.cs.nyu.edu/~exact/doc/paradigm.ps.gz.

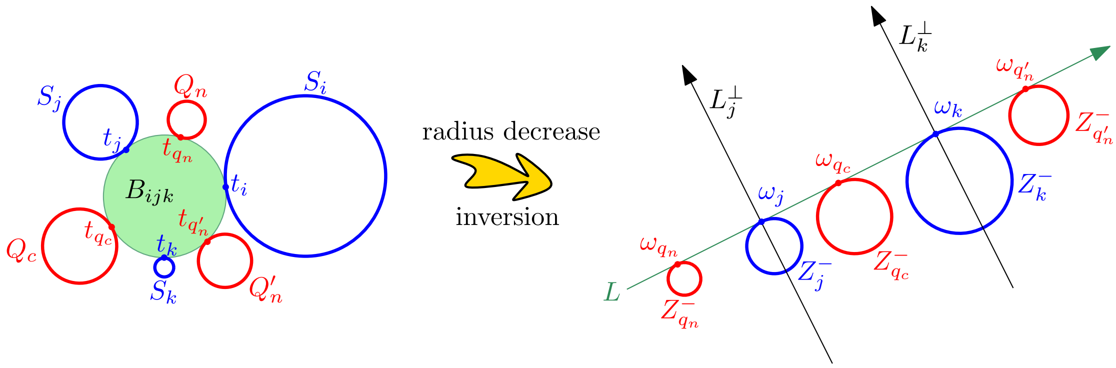


Figure 14: Computing the auxiliary predicate $\mathsf{Orientation}(t_j, t_k, t_q)$ using inversion. ω_{q_c} lies inside $\omega_j\omega_k$, which corresponds to $\mathsf{Orientation}(t_j, t_k, t_{q_c})$ = "cw". ω_{q_n} (resp., $\omega_{q'_n}$) lies outside $\omega_j\omega_k$, which corresponds to $\mathsf{Orientation}(t_j, t_k, t_{q_n})$ = "ccw" (resp., $\mathsf{Orientation}(t_j, t_k, t_{q'_n})$ = "ccw").

A Alternative evaluation of the $\mathsf{VConflict}^\varepsilon$ predicate

In this appendix, we propose a method to evaluate Step 7 of $\mathsf{VConflict}^\varepsilon$ predicate. Actually, this step requires the evaluation of the orientation of tangent points $\mathsf{Orientation}(t_j, t_k, t_q)$. The version proposed here is simpler than the one at Section 4.3.4 but has algebraic degree 12.

We essentially follow the same procedure that was used in [10] for evaluating the $\mathsf{VConflict}$ predicate. The various calculations described below may be found in Section B.2. We decrease the weights of S_i, S_j, S_k and S_q, which results in S_i becoming a point, while the weights of S_j, S_k and S_q become non-positive. We define $S_j^- = (s_j, -w_j)$, $S_k^- = (s_k, -w_k)$, $S_q^- = Q^- = (s_q, -w_q)$. Using s_i as the pole, we invert S_j^-, S_k^- and Q^-, and let $Z_j^- = (u_j, v_j, \rho_j)$, $Z_k^- = (u_k, v_k, \rho_k)$ and $Z_q^- = (u_q, v_q, \rho_q)$ be the corresponding inverted sites (see Figure 14). Let L denote the line that is tangent to Z_j^-, Z_k^- and Z_q^-, and call ω_j, ω_k and ω_q the points of tangency of Z_j^-, Z_k^- and Z_q^- with L. Determining $\mathsf{Orientation}(t_j, t_k, t_q)$ is equivalent to determining if ω_q lies inside or outside the segment $\omega_j\omega_k$ (the cases $\omega_q = \omega_j$ and $\omega_q = \omega_k$ have been ruled out by Steps 5 and 6 of the predicate evaluation procedure): $\mathsf{Orientation}(t_j, t_k, t_q)$ is ccw if and only if t_q lies on B_{ijk} and on the arc delimited by t_j and t_k that contains t_i, which, in turn, is the case if and only if $\omega_q \in L$ lies outside the segment $\omega_j\omega_k$. To determine this, we consider the oriented lines $L_j^\perp$ and $L_k^\perp$ that are perpendicular to L and pass through the centers of Z_j^- and Z_k^- respectively. We assume that the positive orientation of $L_j^\perp$ and $L_k^\perp$ is in the direction of the (open) half-plane delimited by L that does not contain Z_j^- and Z_k^-. Let o_j and o_k be the result of the orientation test of (u_q, v_q) with respect to the two lines. If $o_j = o_k$, then ω_q lies outside the segment $\omega_j\omega_k$, and the predicate returns "no conflict" (see $Z^-_{q_n}$ and $Z^-_{q'_n}$ in Figure 14); otherwise, $o_j \neq o_k$, in which case ω_q lies inside the segment $\omega_j\omega_k$, and the predicate returns "conflict" (see $Z^-_{q_c}$ in Figure 14).

Computing the orientation o_ν, $\nu \in \{j, k\}$ amounts to computing the sign of the quantity:

$$o'_\nu = p^*_\nu E_{xy} E_w + (E_x F_x + E_y F_y)\sqrt{\Delta}, \qquad F_s = \begin{vmatrix} s^*_q & p^*_q \\ s^*_\nu & p^*_\nu \end{vmatrix}, \qquad s \in \{x, y\}.$$

The sign can be resolved using the procedure in (5). At a first glance the algebraic degree of the predicate is 18. However, as for the $\mathsf{VConflict}$ predicate, we can factor the quantity $X_0^2 - X_1^2 Y$

as follows:

$$X_0^2 - X_1^2 Y = [(E_x)^2 + (E_y)^2]\, o''_\nu, \qquad o''_\nu = [(F_x)^2 + (F_y)^2]\,(E_w)^2 - (E_x F_x + E_y F_y)^2.$$

Since $(E_x)^2 + (E_y)^2$ cannot be zero (otherwise we would have resolved the sign of $X_0 + X_1\sqrt{Y}$ without resorting to the computing the sign of $X_0^2 + X_1^2 Y$), determining the sign of o'_ν reduces to determining the sign of $[(F_x)^2 + (F_y)^2]\,(E_w)^2 - (E_x F_x + E_y F_y)^2$, which is of algebraic degree 12. Notice that the way of evaluating $\mathsf{Orientation}(t_j, t_k, t_q)$ results in a perturbed predicate with higher algebraic degree than the unperturbed one. On the other hand, both o_j and o_k can be evaluated with extremely few operations in addition to those required for the $\mathsf{VConflict}$ predicate: observing that the quantities p^*_ν, E_x, E_y, E_w, $(E_w)^2$ and E_{xy} have already been computed when evaluating the $\mathsf{VConflict}$ predicate, and that F_x and F_y are minors of E_{xy}, and thus can be stored while evaluating E_{xy}, we need a maximum of 9 operations in order to compute the sign of o'_ν (3 ops for $E_x F_x + E_y F_y$, 3 ops for $(F_x)^2 + (F_y)^2$, 1 op for $(E_x F_x + E_y F_y)^2$, and another 2 ops for o''_ν).

B Analysis of the predicates for the 2D Apollonius diagram

In this section we introduce a slightly heavier, yet more general notation. In particular, we introduce the following determinant shorthands:

$$D^s_{\mu\nu} = \begin{vmatrix} s_\mu & 1 \\ s_\nu & 1 \end{vmatrix} = s_\mu - s_\nu, \qquad D^{st}_{\mu\nu} = \begin{vmatrix} s_\mu & t_\mu \\ s_\nu & t_\nu \end{vmatrix}, \qquad D^{st}_{\mu\nu\lambda} = \begin{vmatrix} s_\mu & t_\mu & 1 \\ s_\nu & t_\nu & 1 \\ s_\lambda & t_\lambda & 1 \end{vmatrix},$$

with $s, t \in \{x, y, u, v, \rho\}$ and $\mu, \nu, \lambda \in \{j, k, q\}$, and

$$E^s_{\mu\nu} = \begin{vmatrix} s^*_\mu & p^*_\mu \\ s^*_\nu & p^*_\nu \end{vmatrix}, \qquad E^{st}_{\mu\nu} = \begin{vmatrix} s^*_\mu & t^*_\mu \\ s^*_\nu & t^*_\nu \end{vmatrix}, \qquad E^{st}_{\mu\nu\lambda} = \begin{vmatrix} s^*_\mu & t^*_\mu & p^*_\mu \\ s^*_\nu & t^*_\nu & p^*_\nu \\ s^*_\lambda & t^*_\lambda & p^*_\lambda \end{vmatrix},$$

with $s, t \in \{x, y, w\}$ and $\mu, \nu, \lambda \in \{j, k, q\}$, and

$$x^*_\nu = x_\nu - x_i, \quad y^*_\nu = y_\nu - y_i, \quad w^*_\nu = w_\nu - w_i, \quad p^*_\nu = (x^*_\nu)^2 + (y^*_\nu)^2 - (w^*_\nu)^2, \quad \nu \in \{j, k, q\}.$$

B.1 The algebraic degree of the VConflict predicate

Recall from Section 4.3.1 that the $\mathsf{VConflict}$ predicate can be resolved by determining the sign of the quantity:

$$I := E^{xw}_{jkq} E^x_{jk} + E^{yw}_{jkq} E^y_{jk} + E^{xy}_{jkq}\sqrt{\Delta}, \qquad \Delta = (E^x_{jk})^2 + (E^y_{jk})^2 - (E^w_{jk})^2.$$

This is a quantity of the form $X_0 + X_1\sqrt{Y}$, which means that we might need to compute the sign of the quantity $Z = X_0^2 - X_1^2 Y$. In our case Z can be factorized as described below:

$$\begin{aligned}
Z &= (E^{xw}_{jkq}E^x_{jk} + E^{yw}_{jkq}E^y_{jk})^2 - (E^{xy}_{jkq})^2((E^x_{jk})^2 + (E^y_{jk})^2 - (E^w_{jk})^2) \\
&= (E^{xw}_{jkq}E^x_{jk})^2 + (E^{yw}_{jkq}E^y_{jk})^2 + 2E^x_{jk}E^y_{jk}E^{yw}_{jkq}E^{xw}_{jkq} - (E^{xy}_{jkq})^2[(E^x_{jk})^2 + (E^y_{jk})^2] + (E^{xy}_{jkq}E^w_{jk})^2 \\
&= (E^{xw}_{jkq})^2(E^x_{jk})^2 + (E^{yw}_{jkq})^2(E^y_{jk})^2 + (E^{yw}_{jkq})^2(E^x_{jk})^2 + (E^{xw}_{jkq})^2(E^y_{jk})^2 - (E^{xy}_{jkq})^2[(E^x_{jk})^2 + (E^y_{jk})^2] \\
&\quad - (E^{yw}_{jkq}E^x_{jk})^2 - (E^{xw}_{jkq}E^y_{jk})^2 + 2E^x_{jk}E^y_{jk}E^{yw}_{jkq}E^{xw}_{jkq} + (E^{xy}_{jkq}E^w_{jk})^2 \\
&= (E^{xw}_{jkq})^2[(E^x_{jk})^2 + (E^y_{jk})^2] + (E^{yw}_{jkq})^2[(E^x_{jk})^2 + (E^y_{jk})^2] - (E^{xy}_{jkq})^2[(E^x_{jk})^2 + (E^y_{jk})^2] \\
&\quad + (E^{xy}_{jkq}E^w_{jk})^2 - (E^{xw}_{jkq}E^y_{jk} - E^{yw}_{jkq}E^x_{jk})^2 \\
&= [(E^{xw}_{jkq})^2 + (E^{yw}_{jkq})^2 - (E^{xy}_{jkq})^2][(E^x_{jk})^2 + (E^y_{jk})^2] + (E^{xy}_{jkq}E^w_{jk})^2 - (E^{xw}_{jkq}E^y_{jk} - E^{yw}_{jkq}E^x_{jk})^2.
\end{aligned}$$

It easy to verify that the following identities hold:

$$\begin{aligned}
p_j^* E^{xw}_{jkq} &= E^x_{Qj}E^w_{jk} - E^x_{jk}E^w_{Qj}, \\
p_j^* E^{yw}_{jkq} &= E^y_{Qj}E^w_{jk} - E^y_{jk}E^w_{Qj}, \text{ and} \\
p_j^* E^{xy}_{jkq} &= E^x_{jk}E^y_{Qj} - E^y_{jk}E^x_{Qj},
\end{aligned}$$

which implies that

$$\begin{aligned}
p_j^*(E^{xw}_{jkq}E^y_{jk} - E^{yw}_{jkq}E^x_{jk}) &= (E^x_{Qj}E^w_{jk} - E^x_{jk}E^w_{Qj})E^y_{jk} - (E^y_{Qj}E^w_{jk} - E^y_{jk}E^w_{Qj})E^x_{jk} \\
&= E^x_{Qj}E^y_{jk}E^w_{jk} - E^x_{jk}E^y_{jk}E^w_{Qj} - E^x_{jk}E^y_{Qj}E^w_{jk} + E^x_{jk}E^y_{jk}E^w_{Qj} \\
&= E^x_{Qj}E^y_{jk}E^w_{jk} - E^x_{jk}E^y_{Qj}E^w_{jk} \\
&= (E^x_{Qj}E^y_{jk} - E^x_{jk}E^y_{Qj})E^w_{jk} \\
&= -p_j^* E^{xy}_{jkq}E^w_{jk},
\end{aligned}$$

Since $p_j^* \neq 0$, we have

$$E^{xw}_{jkq}E^y_{jk} - E^{yw}_{jkq}E^x_{jk} = -E^{xy}_{jkq}E^w_{jk}. \tag{6}$$

Hence $(E^{xy}_{jkq}E^w_{jk})^2 - (E^{xw}_{jkq}E^y_{jk} - E^{yw}_{jkq}E^x_{jk})^2 = 0$, and Z simplifies to:

$$Z = [(E^x_{jk})^2 + (E^y_{jk})^2][(E^{xw}_{jkq})^2 + (E^{yw}_{jkq})^2 - (E^{xy}_{jkq})^2]. \tag{7}$$

Clearly, the two factors of Z are of degree 6 and 8, respectively. Moreover, as discussed in Section 4.3.1, when we evaluate Z it cannot be the case that $E^x_{jk} = E^y_{jk} = 0$ (since then we would have resolved the sign of I without resorting to Z); as a result the degree of the $\mathsf{VConflict}$ predicate is 8.

B.2 The auxiliary predicate for $\mathsf{VConflict}^\varepsilon$ using inversion

In this section we assume that we are given three sites $S_\mu = (x_\mu, y_\mu, w_\mu)$, $\mu \in \{i, j, k\}$ that define an Apollonius vertex, and a query site $Q = (x_q, y_q, w_q)$ such that $\mathsf{VConflict}(S_i, S_j, S_k, Q) =$ "degenerate" (see Figure 14). We are going to compute the $\mathsf{VConflict}^\varepsilon(S_i, S_j, S_k, Q)$ predicate in the manner described in Appendix A.

Following the analysis in [10], the equation of the line L in the plane of inversion is $au+bv+c =$

0, where:

$$a = \frac{D^u_{jk} D^w_{jk} + D^v_{jk}\sqrt{\Gamma}}{(D^u_{jk})^2 + (D^u_{jk})^2} = \frac{E^x_{jk} E^w_{jk} + E^y_{jk}\sqrt{\Delta}}{(E^x_{jk})^2 + (E^y_{jk})^2},$$

$$b = \frac{D^v_{jk} D^w_{jk} - D^u_{jk}\sqrt{\Gamma}}{(D^u_{jk})^2 + (D^u_{jk})^2} = \frac{E^y_{jk} E^w_{jk} - E^x_{jk}\sqrt{\Delta}}{(E^x_{jk})^2 + (E^y_{jk})^2},$$

$$c = \frac{D^u_{jk} D^{u\rho}_{jkq} + D^v_{jk} D^{v\rho}_{jkq} + D^{uv}_{jkq}\sqrt{\Gamma}}{(D^u_{jk})^2 + (D^u_{jk})^2} = \frac{E^x_{jk} E^{xw}_{jkq} + E^y_{jk} E^{yw}_{jkq} + E^{xy}_{jkq}\sqrt{\Delta}}{(E^x_{jk})^2 + (E^y_{jk})^2}.$$

where $\Gamma = (D^u_{jk})^2 + (D^v_{jk})^2 - (D^\rho_{jk})^2$. Therefore, the line $L^\perp_\nu$ that is perpendicular to L and passes through c_ν has equation (written in the coordinate system of the plane of inversion):

$$\beta(u - u_\nu) - \alpha(v - v_\nu) = 0,$$

where $(u_\nu, v_\nu) = (x^*_\nu/p^*_\nu, y^*_\nu/p^*_\nu)$ is the image under the inversion transformation of the center c_ν of S_ν, $\nu \in \{j, k\}$. To evaluate the orientation of the center (u_q, v_q) of Z_q, we need to compute the signs of the quantities:

$$o_\nu = \beta(u_q - u_\nu) - \alpha(v_q - v_\nu),$$

which in the inverted coordinates gives:

$$\begin{aligned} o_\nu\,[(D^u_{jk})^2 + (D^v_{jk})^2] &= (D^v_{jk} D^\rho_{jk} - D^u_{jk}\sqrt{\Gamma})\, D^u_{q\nu} - (D^u_{jk} D^\rho_{jk} + D^v_{jk}\sqrt{\Gamma})\, D^v_{q\nu} \\ &= (D^v_{jk} D^u_{q\nu} - D^u_{jk} D^v_{q\nu})\, D^\rho_{jk} - (D^u_{jk} D^u_{q\nu} + D^v_{jk} D^v_{q\nu})\sqrt{\Gamma}. \end{aligned} \tag{8}$$

Since $\nu \in \{j, k\}$, it is straightforward to verify that:

$$D^v_{jk} D^u_{q\nu} - D^u_{jk} D^v_{q\nu} = D^{uv}_{qjk} = D^{uv}_{jkq}.$$

Substituting in terms of the original coordinates we have:

$$\begin{aligned} \Gamma &= (p^*_j p^*_k)^{-2}\,\Delta \\ D^\rho_{jk} &= (p^*_j p^*_k)^{-1}\, E^w_{jk} \\ D^{uv}_{jkq} &= (p^*_j p^*_k p^*_q)^{-1}\, E^{xy}_{jkq} \\ D^u_{jk} D^u_{q\nu} + D^v_{jk} D^v_{q\nu} &= (p^*_j p^*_k p^*_q p^*_\nu)^{-1}\,(E^x_{jk} E^x_{q\nu} + E^y_{jk} E^y_{q\nu}) \\ (D_u)^2 + (D_v)^2 &= (p^*_j p^*_k)^{-2}\,[(E^x_{jk})^2 + (E^y_{jk})^2]. \end{aligned}$$

We can thus rewrite (8) as follows, is terms of the original quantities:

$$o_\nu\,[(E^x_{jk})^2 + (E^y_{jk})^2] = p^*_\nu E^{xy}_{jkq} E^w_{jk} + (E^x_{jk} E^x_{q\nu} + E^y_{jk} E^y_{q\nu})\sqrt{\Delta}.$$

To determine the sign of o_ν, we must determine the sign of

$$o'_\nu = p^*_\nu E^{xy}_{jkq} E^w_{jk} + (E^x_{jk} E^x_{q\nu} + E^y_{jk} E^y_{q\nu})\sqrt{\Delta},$$

which is a quantity of the form $X_0 + X_1\sqrt{Y}$, where the algebraic degrees of X_0, X_1 and Y are 9, 6 and 6, respectively. In fact, X_0 is already factorized into p^*_ν, E^{xy}_{jkq} and E^w_{jk}, the algebraic degrees of which are 2, 4, and 3, respectively. Hence, determining the signs of X_0 and X_1 reduces to computing the signs of algebraic expressions of degree at most 6. To deduce the sign of o'_ν, however, we might need to compute the sign of $Z = X_0^2 - X_1^2 Y$, which, a priori, is of degree 18. Below, we will show that Z can be factorized appropriately, thus reducing the algebraic degree of the quantities we need to evaluate in order to determine its sign. Indeed,

$$\begin{aligned} Z &= (p_\nu^* E_{jkq}^{xy} E_{jk}^w)^2 - (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2 \Delta \\ &= (p_\nu^* E_{jkq}^{xy})^2 (E_{jk}^w)^2 - (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2 \left[(E_{jk}^x)^2 + (E_{jk}^y)^2\right] + (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2 (E_{jk}^w)^2 \\ &= \left[(p_\nu^* E_{jkq}^{xy})^2 + (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2\right] (E_{jk}^w)^2 - (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2 \left[(E_{jk}^x)^2 + (E_{jk}^y)^2\right]. \end{aligned}$$

But,

$$\begin{aligned} (p_\nu^* E_{jkq}^{xy})^2 + (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2 &= (E_{jk}^x E_{q\nu}^y - E_{jk}^y E_{q\nu}^x)^2 + (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2 \\ &= (E_{jk}^x)^2 (E_{q\nu}^y)^2 + (E_{jk}^y)^2 (E_{q\nu}^x)^2 \\ &\quad + (E_{jk}^x)^2 (E_{q\nu}^x)^2 + (E_{jk}^y)^2 (E_{q\nu}^y)^2 \\ &= \left[(E_{jk}^x)^2 + (E_{jk}^y)^2\right] \left[(E_{q\nu}^x)^2 + (E_{q\nu}^y)^2\right]. \end{aligned}$$

Hence, we have:

$$\begin{aligned} Z &= \left[(E_{jk}^x)^2 + (E_{jk}^y)^2\right] \left[(E_{q\nu}^x)^2 + (E_{q\nu}^y)^2\right] (E_{jk}^w)^2 - (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2 \left[(E_{jk}^x)^2 + (E_{jk}^y)^2\right] \\ &= \left[(E_{jk}^x)^2 + (E_{jk}^y)^2\right] \left\{\left[(E_{q\nu}^x)^2 + (E_{q\nu}^y)^2\right] (E_{jk}^w)^2 - (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2\right\} \end{aligned}$$

Notice that it cannot be the case that $E_{jk}^x = E_{jk}^y = 0$ (since otherwise X_0 would have been zero and we would have been able to compute the sign of o'_ν without resorting to Z), the sign of Z is the sign of the quantity $\left[(E_{q\nu}^x)^2 + (E_{q\nu}^y)^2\right] (E_{jk}^w)^2 - (E_{jk}^x E_{q\nu}^x + E_{jk}^y E_{q\nu}^y)^2$, which is of algebraic degree 12.

B.3 The algebraic degree of the Orientation predicate involving an Apollonius vertex

Suppose we are given three sites $S_\nu = (x_\nu, y_\nu, w_\nu)$, $\nu \in \{i, j, k\}$ and two points $S_\nu = (x_\nu, y_\nu, 0)$, $\nu \in \{l, m\}$, we are interested in computing the orientation $\mathsf{Orientation}(v_{ijk}, S_l, S_m)$, where v_{ijk} is the Apollonius vertex of S_i, S_j and S_k. Emiris and Karavelas [10] have shown that this predicate can be resolved by computing the sign of the quantity

$$\begin{aligned} O := {} & 2(E_{jk}^x E_{jk}^{xw} + E_{jk}^y E_{jk}^{yw}) E_{lm}^{xy} + (E_{jk}^y D_{lm}^x - E_{jk}^x D_{lm}^y) E_{jk}^w \\ & + (2E_{jk}^{xy} E_{lm}^{xy} - E_{jk}^x D_{lm}^x - E_{jk}^y D_{lm}^y)\sqrt{\Delta}, \\ = {} & (2E_{lm}^{xy} E_{jk}^{xw} - E_{jk}^w D_{lm}^y) E_{jk}^x + (2E_{lm}^{xy} E_{jk}^{yw} + E_{jk}^w D_{lm}^x) E_{jk}^y \\ & + (2E_{jk}^{xy} E_{lm}^{xy} - E_{jk}^x D_{lm}^x - E_{jk}^y D_{lm}^y)\sqrt{\Delta}. \end{aligned}$$

As for the VConflict predicate, it is of the form $X_0 + X_1\sqrt{Y}$, and in order to evaluate its sign we might need to compute the quantity $Z = X_0^2 - X_1^2 Y$. As discussed in [10], since the algebraic degrees of X_0, X_1 and Y are 7, 4 and 6 respectively, the Orientation predicate is of algebraic degree 14.

However, as for the VConflict predicate we can factorize Z:

$$
\begin{aligned}
Z &= [(2E^{xy}_{lm}E^{xw}_{jk} - E^{w}_{jk}D^{y}_{lm})E^{x}_{jk} + (2E^{xy}_{lm}E^{yw}_{jk} + E^{w}_{jk}D^{x}_{lm})E^{y}_{jk}]^2 \\
&\quad - (2E^{xy}_{jk}E^{xy}_{lm} - E^{x}_{jk}D^{x}_{lm} - E^{y}_{jk}D^{y}_{lm})^2[(E^{x}_{jk})^2 + (E^{y}_{jk})^2 - (E^{w}_{jk})^2] \\
&= (2E^{xy}_{lm}E^{xw}_{jk} - E^{w}_{jk}D^{y}_{lm})^2(E^{x}_{jk})^2 + (2E^{xy}_{lm}E^{yw}_{jk} + E^{w}_{jk}D^{x}_{lm})^2(E^{y}_{jk})^2 \\
&\quad + 2(2E^{xy}_{lm}E^{xw}_{jk} - E^{w}_{jk}D^{y}_{lm})(2E^{xy}_{lm}E^{yw}_{jk} + E^{w}_{jk}D^{x}_{lm})E^{x}_{jk}E^{y}_{jk} \\
&\quad - (2E^{xy}_{jk}E^{xy}_{lm} - E^{x}_{jk}D^{x}_{lm} - E^{y}_{jk}D^{y}_{lm})^2[(E^{x}_{jk})^2 + (E^{y}_{jk})^2] \\
&\quad + (2E^{xy}_{jk}E^{xy}_{lm} - E^{x}_{jk}D^{x}_{lm} - E^{y}_{jk}D^{y}_{lm})^2(E^{w}_{jk})^2 \\
&= (2E^{xy}_{lm}E^{xw}_{jk} - E^{w}_{jk}D^{y}_{lm})^2[(E^{x}_{jk})^2 + (E^{y}_{jk})^2] + (2E^{xy}_{lm}E^{yw}_{jk} + E^{w}_{jk}D^{x}_{lm})^2[(E^{x}_{jk})^2 + (E^{y}_{jk})^2] \\
&\quad - (2E^{xy}_{jk}E^{xy}_{lm} - E^{x}_{jk}D^{x}_{lm} - E^{y}_{jk}D^{y}_{lm})^2[(E^{x}_{jk})^2 + (E^{y}_{jk})^2] \\
&\quad - (2E^{xy}_{lm}E^{xw}_{jk} - E^{w}_{jk}D^{y}_{lm})^2(E^{y}_{jk})^2 - (2E^{xy}_{lm}E^{yw}_{jk} + E^{w}_{jk}D^{x}_{lm})^2(E^{x}_{jk})^2 \\
&\quad + 2(2E^{xy}_{lm}E^{xw}_{jk} - E^{w}_{jk}D^{y}_{lm})(2E^{xy}_{lm}E^{yw}_{jk} + E^{w}_{jk}D^{x}_{lm})E^{x}_{jk}E^{y}_{jk} \\
&\quad + [(2E^{xy}_{jk}E^{xy}_{lm} - E^{x}_{jk}D^{x}_{lm} - E^{y}_{jk}D^{y}_{lm})E^{w}_{jk}]^2 \\
&= [(E^{x}_{jk})^2 + (E^{y}_{jk})^2]\,O' + [(2E^{xy}_{jk}E^{xy}_{lm} - E^{x}_{jk}D^{x}_{lm} - E^{y}_{jk}D^{y}_{lm})E^{w}_{jk}]^2 \\
&\quad - [(2E^{xy}_{lm}E^{xw}_{jk} - E^{w}_{jk}D^{y}_{lm})E^{y}_{jk} - (2E^{xy}_{lm}E^{yw}_{jk} + E^{w}_{jk}D^{x}_{lm})E^{x}_{jk}]^2 \\
&= [(E^{x}_{jk})^2 + (E^{y}_{jk})^2]\,O' + [(2E^{xy}_{jk}E^{xy}_{lm} - E^{x}_{jk}D^{x}_{lm} - E^{y}_{jk}D^{y}_{lm})E^{w}_{jk}]^2 \\
&\quad - [2E^{xy}_{lm}(E^{xw}_{jk}E^{y}_{jk} - E^{yw}_{jk}E^{x}_{jk}) - (D^{x}_{lm}E^{x}_{jk} + D^{y}_{lm}E^{y}_{jk})E^{w}_{jk}]^2,
\end{aligned}
$$

where

$$O' = (2E^{xy}_{lm}E^{xw}_{jk} - E^{w}_{jk}D^{y}_{lm})^2 + (2E^{xy}_{lm}E^{yw}_{jk} + E^{w}_{jk}D^{x}_{lm})^2 - (2E^{xy}_{lm}E^{xy}_{jk} - E^{x}_{jk}D^{x}_{lm} - E^{y}_{jk}D^{y}_{lm})^2.$$

But

$$E^{xw}_{jk}E^{y}_{jk} - E^{yw}_{jk}E^{x}_{jk} = E^{xy}_{jk}E^{w}_{jk},$$

which implies that the last two terms in the last expression for Z above cancel out. Hence, $Z = [(E^{x}_{jk})^2 + (E^{y}_{jk})^2]\,O'$. Clearly, the algebraic degree of O' is 8. Moreover, the quantity $(E^{x}_{jk})^2 + (E^{y}_{jk})^2$ is strictly positive when we compute the sign of Z, since otherwise X_0 would have been zero (X_0 is a linear combination of $E^{x}_{jk}$and E^{y}_{jk}), a case which has already been ruled out according to the procedure in (5). Hence the algebraic degree of the Orientation(v_{ijk}, S_l, S_m) predicate is 8.

RESEARCH CENTRE
NANCY – GRAND EST

615 rue du Jardin Botanique
CS20101
54603 Villers-lès-Nancy Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-6399