On the approximation of electromagnetic fields by edge finite elements. Part 1: Sharp interpolation results for low-regularity fields

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Abstract

We propose sharp results on the numerical approximation of low-regularity electromagnetic fields by edge finite elements. We consider general geometrical settings, including topologically non-trivial domains or domains with a non-connected boundary. In the model, the electric permittivity and magnetic permeability are symmetric, tensor-valued, piecewise smooth coefficients. In all cases, the error can be bounded by h^δ times a constant, where h is the meshsize, for some exponent $\delta \in]0, 1]$ that depends both on the geometry and on the coefficients. It relies either on classical estimates when $\delta > 1/2$, or on a new combined interpolation operator when $\delta < 1/2$. The optimality of the value of δ is discussed with respect to abstract shift theorems. In some simple configurations, typically for scalar-valued permittivity and permeability, the value of δ can be further characterized. This paper is the first one in a series dealing with the approximation of electromagnetic fields by edge finite elements.

Keywords: Maxwell's equations, interface problem, edge elements, interpolation operators, error estimates

1. Introduction

The aim of this paper is to study the numerical approximation of electromagnetic fields, governed by Maxwell's equations with volume sources in bounded regions of $\mathbb{R}^3$. More precisely, we are interested in exhibiting the approximation capabilities of those fields with the help of edge element interpolation operators. Typically, the domain under scrutiny is bounded and enclosed in a perfect conductor, and it can be made of different materials. In particular, we shall provide interpolation results that depend on the geometry of the domain, on

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the electric permittivity and the magnetic permeability that describe the materials, and also on the regularity of the sources, that is the current and charge densities. Special attention will be devoted to cases where the regularity of the fields is minimal.

In the next section, we begin by recalling a model problem in electromagnetic theory, namely the time-harmonic Maxwell equations set in a bounded domain. We recall equivalent variational formulations and well-posedness results, and the approximation by edge elements. To obtain a priori convergence estimates, we then study the minimal regularity of those fields: this is the main topic of section 3. The regularity results are derived thanks to a splitting of the fields and their curls into a regular part and a gradient. In section 4, we study in detail the approximability by edge finite elements of the fields. We review the classical interpolation results, before we define a new, combined, interpolation operator which relies explicitly on the splitting of the fields, and not only on the minimal regularity. We conclude this section by a comparison with the more recent quasi-interpolation theory. As a result of the approximability properties, we finally derive in section 5 optimal error estimates.

Throughout the paper, C is used to denote a generic positive constant which is independent of the meshsize, the triangulation and the fields of interest. On the other hand, C may depend on the geometry of the domain, or on the coefficients defining the model. We also use the shorthand notation $A \lesssim B$ for the inequality $A \leq CB$, where A and B are two scalar fields, and C is a generic constant. Respectively, $A \approx B$ for the inequalities $A \lesssim B$ and $B \lesssim A$. We denote constant fields by the symbol *cst*. Vector-valued (resp. tensor-valued) function spaces are written in boldface character (resp. blackboard bold characters) ; for the latter, the index *sym* indicates symmetric fields. Given an open set $\mathcal{O}$ of $\mathbb{R}^3$, we use the notation $(\cdot|\cdot)_{0,\mathcal{O}}$ (respectively $\|\cdot\|_{0,\mathcal{O}}$) for the $L^2(\mathcal{O})$ and the $\boldsymbol{L}^2(\mathcal{O}) := (L^2(\mathcal{O}))^3$ hermitian scalar products (resp. norms). More generally, $(\cdot|\cdot)_{s,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$ (respectively $|\cdot|_{s,\mathcal{O}}$) denote the hermitian scalar product and the norm (resp. semi-norm) of the Sobolev spaces $H^s(\mathcal{O})$ and $\boldsymbol{H}^s(\mathcal{O}) := (H^s(\mathcal{O}))^3$ for $s \in \mathbb{R}$ (resp. for $s > 0$). The index *zmv* indicates zero-mean-value fields. If moreover the boundary $\partial\mathcal{O}$ is Lipschitz, $\boldsymbol{n}$ denotes the unit outward normal vector field to $\partial\mathcal{O}$. Finally, it is assumed that the reader is familiar with function spaces related to Maxwell's equations, such as $\boldsymbol{H}(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}_0(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}(\mathrm{div};\mathcal{O})$, $\boldsymbol{H}_0(\mathrm{div};\mathcal{O})$ etc. We refer to the monograph of Monk [28] for details. We will define more specialized function spaces later on.

2. Time-harmonic problems in electromagnetics

Let Ω be a domain in $\mathbb{R}^3$, ie. an open, connected and bounded subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. For a given pulsation $\omega > 0$, the time-harmonic Maxwell's equations (with time-dependence $\exp(-\imath\omega t)$) write

$$\mathbf{curl}\,\boldsymbol{h} + \imath\omega\varepsilon\boldsymbol{e} = \boldsymbol{j} \text{ in } \Omega, \tag{1}$$

$$\mathbf{curl}\,\boldsymbol{e} - \imath\omega\mu\boldsymbol{h} = 0 \text{ in } \Omega, \tag{2}$$

$$\operatorname{div}\varepsilon\boldsymbol{e} = \varrho \text{ in } \Omega, \tag{3}$$

$$\operatorname{div}\mu\boldsymbol{h} = 0 \text{ in } \Omega. \tag{4}$$

Above, the real-valued coefficient ε is the electric permittivity tensor and the real-valued coefficient μ is the magnetic permeability tensor, whereas $(\boldsymbol{e}, \boldsymbol{h})$ is the couple of electromagnetic fields, and the source terms $\boldsymbol{j}$ and ϱ are respectively the current density and the charge density. The latter are related by the charge conservation equation

$$-\imath\omega\varrho + \operatorname{div}\boldsymbol{j} = 0 \text{ in } \Omega. \tag{5}$$

The other two electromagnetic fields are the electric displacement $\boldsymbol{d}$ and the magnetic induction $\boldsymbol{b}$. They are related to $\boldsymbol{e}$ and $\boldsymbol{h}$ by the constitutive relations

$$\boldsymbol{d} = \varepsilon\boldsymbol{e},\ \ \boldsymbol{b} = \mu\boldsymbol{h} \text{ in } \Omega. \tag{6}$$

In what follows, we focus mainly on the couple of fields $(\boldsymbol{e}, \boldsymbol{h})$. However the results are easily extended to the couple of fields $(\boldsymbol{d}, \boldsymbol{b})$ thanks to the relations (6). We assume that the coefficients ε, μ, together with their inverses $\varepsilon^{-1}, \mu^{-1}$, belong to $\mathbb{L}^\infty_{sym}(\Omega)$. Classically([1]), to be able to define the electromagnetic energy, they are such that $\lambda_{min}(\varepsilon) > 0$ and $\lambda_{min}(\mu) > 0$ a.e. in Ω where λ_{min} stands for the smallest eigenvalue, and the couple of electromagnetic fields belongs to $\boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$. We choose source terms $\boldsymbol{j} \in \boldsymbol{L}^2(\Omega)$, and $\varrho \in H^{-1}(\Omega)$.
We assume that the medium Ω is surrounded by a perfect conductor, so that the boundary condition below holds:

$$\boldsymbol{e} \times \boldsymbol{n} = 0 \quad \text{ on } \partial\Omega. \tag{7}$$

Hence the couple of electromagnetic fields $(\boldsymbol{e}, \boldsymbol{h})$ belongs to $\boldsymbol{H}_0(\mathbf{curl}; \Omega) \times \boldsymbol{H}(\mathbf{curl}; \Omega)$.

2.1. Variational formulations

The Maxwell problem can be formulated in the electric field $\boldsymbol{e}$ only, namely

$$\left\{ \begin{array}{l} \textit{Find } \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \textit{ such that} \\ -\omega^2\varepsilon\boldsymbol{e} + \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) = \imath\omega\boldsymbol{j} \text{ in } \Omega \\ \operatorname{div}\varepsilon\boldsymbol{e} = \varrho \text{ in } \Omega. \end{array} \right. \tag{8}$$

Note that in (8), the equation $\operatorname{div}\varepsilon\boldsymbol{e} = \varrho$ is implied by the second-order equation $-\omega^2\varepsilon\boldsymbol{e} + \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) = \imath\omega\boldsymbol{j}$, together with the charge conservation

[1] For more "exotic" configurations of Maxwell's equations, in which ε or μ exhibit a sign-change of one or several eigenvalues across some interface, we refer to [10, 9, 20].

equation (5), so it can be omitted. Furthermore, the magnetic field can be recovered using Faraday's law (2). Moreover, one can check that the equivalent variational formulation in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ writes

$$\left\{\begin{array}{l} \textit{Find } \boldsymbol{e}\in\boldsymbol{H}_0(\mathbf{curl};\Omega) \textit{ such that} \\ (\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega}-\omega^2(\varepsilon\boldsymbol{e}|\boldsymbol{v})_{0,\Omega}= \\ \qquad\qquad\qquad\qquad \imath\omega(\boldsymbol{j}|\boldsymbol{v})_{0,\Omega},\ \forall\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega). \end{array}\right. \tag{9}$$

On the other hand, one can also write the time-harmonic Maxwell problem in the magnetic field $\boldsymbol{h}$ only. Note that as $\boldsymbol{e}$ belongs to $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, its curl belongs to $\boldsymbol{H}_0(\mathrm{div};\Omega)$: it follows from Faraday's law (2) that $\mu\boldsymbol{h}\cdot\boldsymbol{n}_{|\partial\Omega}=0$. In addition, the field $\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h}-\boldsymbol{j})$ actually belongs to $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ (cf. Ampère's law (1)). So the Maxwell problem formulated in $\boldsymbol{h}$ only is

$$\left\{\begin{array}{l} \textit{Find } \boldsymbol{h}\in\boldsymbol{H}(\mathbf{curl};\Omega) \textit{ such that} \\ -\omega^2\mu\boldsymbol{h}+\mathbf{curl}(\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h}-\boldsymbol{j}))=0 \text{ in } \Omega \\ \mathrm{div}\,\mu\boldsymbol{h}=0 \text{ in } \Omega \\ \mu\boldsymbol{h}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega \\ \varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h}-\boldsymbol{j})\times\boldsymbol{n}=0 \text{ on } \partial\Omega. \end{array}\right. \tag{10}$$

Again, the equation $\mathrm{div}\,\mu\boldsymbol{h}=0$ in (10) is implied by the second-order equation $-\omega^2\mu\boldsymbol{h}+\mathbf{curl}(\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h}-\boldsymbol{j}))=0$. Likewise the boundary condition $\mu\boldsymbol{h}\cdot\boldsymbol{n}_{|\partial\Omega}=0$ is implied by the second-order equation and the boundary condition $\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h}-\boldsymbol{j})\times\boldsymbol{n}_{|\partial\Omega}=0$. One now checks that the equivalent variational formulation in $\boldsymbol{H}(\mathbf{curl};\Omega)$ writes

$$\left\{\begin{array}{l} \textit{Find } \boldsymbol{h}\in\boldsymbol{H}(\mathbf{curl};\Omega) \textit{ such that} \\ (\varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{h}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega}-\omega^2(\mu\boldsymbol{h}|\boldsymbol{v})_{0,\Omega}= \\ \qquad\qquad\qquad\qquad (\varepsilon^{-1}\boldsymbol{j}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega},\ \forall\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl};\Omega). \end{array}\right. \tag{11}$$

2.2. Divergence conditions

We already remarked that the divergence conditions on the fields are consequences of the second-order equations. Also, one notices that the magnetic field $\boldsymbol{h}$ is automatically $\mathrm{div}\,\mu$-free. A similar property can be exhibited for the electric field as follows. Indeed, introduce the scalar field $\varphi_\varrho\in H^1_0(\Omega)$ such that $\mathrm{div}\,\varepsilon\nabla\varphi_\varrho=\varrho$ in $H^{-1}(\Omega)$, and write $\boldsymbol{e}=\nabla\varphi_\varrho+\boldsymbol{e}_0$. Then, $\boldsymbol{e}_0$ belongs to $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, with $\mathrm{div}\,\varepsilon\boldsymbol{e}_0=0$. Plugging this splitting of $\boldsymbol{e}$ in (8), one finds that the $\mathrm{div}\,\varepsilon$-free field $\boldsymbol{e}_0$ is governed by the equation $-\omega^2\varepsilon\boldsymbol{e}_0+\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}_0)=\imath\omega\boldsymbol{j}_0$ in Ω, with $\boldsymbol{j}_0:=\boldsymbol{j}-\imath\omega\varepsilon\nabla\varphi_\varrho\in\boldsymbol{L}^2(\Omega)$ and $\mathrm{div}\,\boldsymbol{j}_0=0$, plus the boundary condition $\boldsymbol{e}_0\times\boldsymbol{n}_{|\partial\Omega}=0$, or equivalently by the variational formulation

$$\left\{\begin{array}{l} \textit{Find } \boldsymbol{e}_0\in\boldsymbol{H}_0(\mathbf{curl};\Omega) \textit{ such that} \\ (\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}_0|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega}-\omega^2(\varepsilon\boldsymbol{e}_0|\boldsymbol{v})_{0,\Omega}= \\ \qquad\qquad\qquad\qquad \imath\omega(\boldsymbol{j}_0|\boldsymbol{v})_{0,\Omega},\ \forall\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega). \end{array}\right. \tag{12}$$

Note that both splittings (of $\boldsymbol{e}$ and $\boldsymbol{j}$) are completely characterized by the scalar field φ_ϱ. By construction, one has the orthogonality relation $(\nabla\varphi_\varrho|\boldsymbol{j}_0)_{0,\Omega}=0$ so that $\|\boldsymbol{j}\|_{0,\Omega}\approx\|\nabla\varphi_\varrho\|_{0,\Omega}+\|\boldsymbol{j}_0\|_{0,\Omega}$.

2.3. Well-posedness of the time-harmonic Maxwell problems

We refer to [25, §5] or to [13] for the solution of the variational formulation (9), with the help of the Fredholm alternative. Following [13, §3.1 and §4.2], one can provide a similar construction for the variational formulation (11). In these references, an inf-sup condition is obtained, which relies on the definition of an appropriate bijective (one-to-one and onto) operator that maps $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ into itself. This is possible as soon as ω^2 is not an eigenvalue of the Maxwell eigenproblem. We recall that, expressed with the help of the electric field, the eigenproblem writes

$$\left\{\begin{array}{l} \textit{Find } (\boldsymbol{e},\nu) \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\times\mathbb{R},\ \boldsymbol{e}\neq 0 \textit{ such that} \\ (\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} = \nu(\varepsilon\boldsymbol{e}|\boldsymbol{v})_{0,\Omega},\ \forall \boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega) \\ \operatorname{div}\varepsilon\boldsymbol{e} = 0 \text{ in } \Omega. \end{array}\right. \tag{13}$$

Denoting by $(\nu_\ell)_{\ell\in\mathbb{Z}}$ the sequence of (nonnegative) eigenvalues that goes to $+\infty$ (with finite multiplicity), well-posedness holds if, and only if, $\omega^2 \notin \{\nu_\ell\ :\ \ell\in\mathbb{Z}\}$.

2.4. Discretisation of electromagnetic fields

For the ease of exposition, we assume in this subsection that Ω is a Lipschitz polyhedron. The case of a curved Lipschitz polyhedron is easily addressed, but it involves more technicalities. To define finite dimensional subspaces $(\boldsymbol{V}_h^+)_h$ of $\boldsymbol{H}(\mathbf{curl};\Omega)$, resp. $(\boldsymbol{V}_h)_h$ of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, we consider a family of simplicial meshes of Ω, and we choose the so-called Nédélec's first family of edge finite elements [29, 28]. Note that, because we are dealing with electromagnetic fields with low regularity, using the first-order finite elements will be sufficient for our purposes. However, all the analysis is valid for higher-order element methods, with marginal modifications. We consider that $\overline{\Omega}$ is triangulated by a shape regular family of meshes $(\mathcal{T}_h)_h$, made up of (closed) simplices, generically denoted by K. A mesh is indexed by $h := \max_K h_K$ (the meshsize), where h_K is the diameter of K. We use the notation $L^p(K)$, respectively $H^s(K)$, and $\int_K\,dx$ etc. in lieu of $L^p(int(K))$, resp. $H^s(int(K))$, and $\int_{int(K)}\,dx$ etc.
Let us introduce Nédélec's $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming (first family, first-order) finite element spaces

$$\boldsymbol{V}_h^+ := \{\boldsymbol{v}_h\in\boldsymbol{H}(\mathbf{curl};\Omega)\ :\ \boldsymbol{v}_{h|K}\in\mathcal{R}_1(K),\ \forall K\in\mathcal{T}_h\},\ \ \boldsymbol{V}_h := \boldsymbol{V}_h^+\cap\boldsymbol{H}_0(\mathbf{curl};\Omega),$$

where $\mathcal{R}_1(K)$ is the six-dimensional vector space of polynomials on K defined by

$$\mathcal{R}_1(K) := \{\boldsymbol{v}\in\boldsymbol{P}_1(K)\ :\ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x},\ \boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3\}.$$

It is shown in [29, Theorem 1] that any element $\boldsymbol{v}$ in $\mathcal{R}_1(K)$ is uniquely determined by the degrees of freedom in the moment set $M_E(\boldsymbol{v})$:

$$M_E(\boldsymbol{v}) := \left(\int_e \boldsymbol{v}\cdot\boldsymbol{t}\,dl\right)_{e\in A_K}. \tag{14}$$

Above, A_K is the set of edges of K, and $\boldsymbol{t}$ is a unit vector along the edge e.
One can then define the global set of moments on $\boldsymbol{V}_h^+$, resp. on $\boldsymbol{V}_h$, by taking

one degree of freedom as above per edge of $\mathcal{T}_h$, resp. per interior edge of $\mathcal{T}_h$. We recall that the basic approximability properties for the edge finite element write (cf. [28, Lemma 7.10])

$$\begin{array}{l} \lim_{h\to 0}\left(\inf_{\boldsymbol{v}_h\in \boldsymbol{V}_h^+}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\right)=0,\ \forall \boldsymbol{v}\in \boldsymbol{H}(\mathbf{curl};\Omega)\,, \\ \lim_{h\to 0}\left(\inf_{\boldsymbol{v}_h\in \boldsymbol{V}_h}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\right)=0,\ \forall \boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl};\Omega). \end{array} \tag{15}$$

Assuming for simplicity that the integrals are computed exactly, the discrete electric problem writes

$$\left\{ \begin{array}{l} \textit{Find } \boldsymbol{e}_h\in \boldsymbol{V}_h \textit{ such that} \\ (\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}_h|\,\mathbf{curl}\,\boldsymbol{v}_h)_{0,\Omega}-\omega^2(\varepsilon\boldsymbol{e}_h|\boldsymbol{v}_h)_{0,\Omega}= \\ \qquad\qquad\qquad\qquad\qquad \imath\omega(\boldsymbol{j}|\boldsymbol{v}_h)_{0,\Omega},\ \forall \boldsymbol{v}_h\in \boldsymbol{V}_h. \end{array} \right. \tag{16}$$

According(2) to [25, §5], [28, §7], or to [13], one has

$$\exists h_0>0,\ \forall h<h_0,\ \|\boldsymbol{e}-\boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\lesssim \inf_{\boldsymbol{v}_h\in \boldsymbol{V}_h}\|\boldsymbol{e}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\,. \tag{17}$$

In particular, it follows from (15) that

$$\lim_{h\to 0}\|\boldsymbol{e}-\boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}=0. \tag{18}$$

On the other hand, the discrete magnetic problem writes

$$\left\{ \begin{array}{l} \textit{Find } \boldsymbol{h}_h\in \boldsymbol{V}_h^+ \textit{ such that} \\ (\varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{h}_h|\,\mathbf{curl}\,\boldsymbol{v}_h)_{0,\Omega}-\omega^2(\mu\boldsymbol{h}_h|\boldsymbol{v}_h)_{0,\Omega}= \\ \qquad\qquad\qquad\qquad\qquad (\varepsilon^{-1}\boldsymbol{j}|\,\mathbf{curl}\,\boldsymbol{v}_h)_{0,\Omega},\ \forall \boldsymbol{v}_h\in \boldsymbol{V}_h^+. \end{array} \right. \tag{19}$$

Again, one has

$$\exists h_0>0,\ \forall h<h_0,\ \|\boldsymbol{h}-\boldsymbol{h}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\lesssim \inf_{\boldsymbol{v}_h\in \boldsymbol{V}_h^+}\|\boldsymbol{h}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\,, \tag{20}$$

and as a consequence

$$\lim_{h\to 0}\|\boldsymbol{h}-\boldsymbol{h}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}=0. \tag{21}$$

The aim of the rest of the paper is to refine the convergence estimates (18) and (21), under *minimal regularity assumptions* on the data.

3. Building splittings of electromagnetic fields

In this section, we present some abstract tools, which then yield precise regularity results for the couple of electromagnetic fields.

[2] We do not discuss the issue of the threshold value h_0, cf. for instance [19].

Let ξ be a real-valued tensor field. We introduce

$$\begin{aligned}
&\boldsymbol{X}_N(\Omega,\xi) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\ :\ \xi\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega)\},\\
&\boldsymbol{X}_T(\Omega,\xi) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega)\ :\ \xi\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega)\},\\
&\boldsymbol{Z}_B(\Omega,\xi) := \{\boldsymbol{v} \in \boldsymbol{X}_B(\Omega,\xi)\ :\ \mathbf{curl}\,\boldsymbol{v} = 0,\ \mathrm{div}\,\xi\boldsymbol{v} = 0 \text{ in } \Omega\},\ B \in \{N,T\}.
\end{aligned}$$

The function spaces $\boldsymbol{X}_N(\Omega,\xi)$ and $\boldsymbol{X}_T(\Omega,\xi)$ are endowed with the graph norm $\boldsymbol{v} \mapsto (\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\xi\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{div};\Omega)})^{1/2}$. Briefly, we recall that the Maxwell electric, resp. the Maxwell magnetic, problems are well-posed within the Fredholm alternative framework (cf. §2.3) thanks to the Weber compact embedding results [33] stated in the next Theorem.

Theorem 1. *Let ξ be a real-valued tensor field such that $\xi, \xi^{-1} \in \mathbb{L}^\infty_{sym}(\Omega)$, $\lambda_{min}(\xi) > 0$ a.e. in Ω. Then both $\boldsymbol{X}_N(\Omega,\xi)$ and $\boldsymbol{X}_T(\Omega,\xi)$ are compactly embedded into $\boldsymbol{L}^2(\Omega)$.*

If we write $\boldsymbol{e} = \nabla\varphi_\varrho + \boldsymbol{e}_0$ where $\varphi_\varrho \in H^1_0(\Omega)$ is characterized by $\varrho = \mathrm{div}\,\varepsilon\nabla\varphi_\varrho$, then $\boldsymbol{e}_0 \in \boldsymbol{X}_N(\Omega,\varepsilon)$; obviously, $\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e} = \mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}_0 \in \boldsymbol{X}_T(\Omega,\mu)$. On the other hand, $\boldsymbol{h} \in \boldsymbol{X}_T(\Omega,\mu)$; in addition, if we use now $\boldsymbol{j}_0 = \boldsymbol{j} - \imath\omega\varepsilon\nabla\varphi_\varrho$, then $\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h} - \boldsymbol{j}_0) \in \boldsymbol{X}_N(\Omega,\varepsilon)$.

3.1. Abstract geometrical setting

The domain Ω can be topologically non-trivial, or with a non-connected boundary. We recall some basic results concerning these categories.

First, the notion of trivial topology: given a vector field $\boldsymbol{v}$ defined on Ω such that $\mathbf{curl}\,\boldsymbol{v} = 0$ in Ω, does there exist a continuous, single-valued function p such that $\boldsymbol{v} = \nabla p$? This question is addressed with the help of (co)homology theory [24]:
either $\mathbf{(Top)}_{I=0}$ *'given any curl-free vector field $\boldsymbol{v} \in \boldsymbol{C}^1(\Omega)$, there exists $p \in C^0(\Omega)$ such that $\boldsymbol{v} = \nabla p$ in Ω';*
or $\mathbf{(Top)}_{I>0}$ *'there exist I non-intersecting manifolds, $\Sigma_1, \ldots, \Sigma_I$, with boundaries $\partial\Sigma_i \subset \partial\Omega$, such that, if we let $\dot\Omega = \Omega \setminus \bigcup_{i=1}^I \Sigma_i$, given any curl-free vector field $\boldsymbol{v}$, there exists $\dot p \in C^0(\dot\Omega)$ such that $\boldsymbol{v} = \nabla\dot p$ in $\dot\Omega$'.*
When $I = 0$, $\dot\Omega = \Omega$. For short, we write $\mathbf{(Top)}_I$ to cover both cases.
Regarding the practical definition of the manifolds, or cuts, $(\Sigma_i)_{i=1,\cdots,I}$, finding them to enforce $\mathbf{(Top)}_{I>0}$ is inexpensive in terms of algorithmic complexity, see [24, Chapter 6]. In particular, one can build cuts that are piecewise plane. We keep this assumption from now on. Finally, we assume, that $\dot\Omega$ is a connected set.
The domain Ω is said to be topologically trivial when $I = 0$. When $I > 0$, the set $\dot\Omega$ has pseudo-Lipschitz boundary in the sense of [3], the continuation operator from $L^2(\dot\Omega)$ to $L^2(\Omega)$ is denoted by $\widetilde{\ }$, whereas the jump across Σ_i is denoted by $[\cdot]_{\Sigma_i}$, for $i = 1, \cdots, I$. The definition of the jump depends on the (fixed) orientation of the normal vector field to Σ_i. For all i, we let $\langle\cdot,\cdot\rangle_{\Sigma_i}$ denote the

duality pairing between $H^{1/2}(\Sigma_i)$ and $(H^{1/2}(\Sigma_i))'$.
One has the integration by parts formula [3, Lemma 3.10]:

$$(\boldsymbol{v}|\nabla\dot{q})_{0,\dot{\Omega}}+(\operatorname{div}\boldsymbol{v}|\dot{q})_{0,\dot{\Omega}} = \sum_{1\le i'\le I} \langle \boldsymbol{v}\cdot\boldsymbol{n}, [\dot{q}]_{\Sigma_{i'}}\rangle_{\Sigma_{i'}},\ \forall \boldsymbol{v}\in \boldsymbol{H}_0(\operatorname{div};\Omega),\ \forall \dot{q}\in H^1(\dot{\Omega}).$$

In this configuration, we also introduce the subspace $P(\dot{\Omega})$ of $H^1(\dot{\Omega})$:

$$P(\dot{\Omega}) := \{\dot{q}\in H^1(\dot{\Omega})\ :\ [\dot{q}]_{\Sigma_i} = cst_i,\ 1\le i\le I\}.$$

Above, for $i\neq i'$, cst_i and $cst_{i'}$ may be different. If $\dot{q}\in P(\dot{\Omega})$ with vanishing jumps $([\dot{q}]_{\Sigma_i})_{i=1,\cdots,I}$, then $\widetilde{\dot{q}}\in H^1(\Omega)$.

Second, when the boundary $\partial\Omega$ is not connected, let $(\Gamma_k)_{k=0,\cdots,K}$ be its (maximal) connected components. Otherwise, $\Gamma_0=\partial\Omega$. For all k, we let $\langle\cdot,\cdot\rangle_{\Gamma_k}$ denote the duality pairing between $H^{1/2}(\Gamma_k)$ and $H^{-1/2}(\Gamma_k) = (H^{1/2}(\Gamma_k))'$. We introduce the subspace $H^1_{\partial\Omega}(\Omega)$ of $H^1(\Omega)$:

$$H^1_{\partial\Omega}(\Omega) := \{q\in H^1(\Omega)\ :\ q_{|\Gamma_0}=0,\ q_{|\Gamma_k}=cst_k,\ 1\le k\le K\}.$$

Above, for $k\neq k'$, cst_k and $cst_{k'}$ may be different.

For domains that fit into the above categories, one can build scalar potentials for curl-free elements, and also vector potentials for divergence-free elements, provided some compatibility conditions are fulfilled. Those results are recalled in the next subsection (see [22, 3] for details) since they are a crucial ingredient to derive the splittings of electromagnetic fields.

3.2. Scalar and vector potentials

Let us begin by the extraction of scalar potentials.

Theorem 2. *Let Ω be a domain such that* $(\textbf{Top})_I$ *is fulfilled. Then, given $\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)$, there holds*

$$\mathbf{curl}\,\boldsymbol{v}=0\ \text{in}\ \Omega \iff \exists\dot{p}\in P(\dot{\Omega}),\ \boldsymbol{v}=\widetilde{\nabla\dot{p}}\ \text{in}\ \Omega.$$

The scalar potential $\dot{p}$ is unique up to a constant, and $|\dot{p}|_{1,\dot{\Omega}} = \|\boldsymbol{v}\|_{0,\Omega}$.

Remark 1. When $I>0$, if $\dot{p}\in H^1(\dot{\Omega})\setminus P(\dot{\Omega})$, then $\mathbf{curl}\,\widetilde{\nabla\dot{p}}\neq 0$ in Ω.

Theorem 3. *Let Ω be a domain. Then, given $\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)$, there holds*

$$\left.\begin{array}{l}\mathbf{curl}\,\boldsymbol{v}=0\ \text{in}\ \Omega,\\ \boldsymbol{v}\times\boldsymbol{n}_{|\partial\Omega}=0\end{array}\right\}\iff \exists p\in H^1_{\partial\Omega}(\Omega),\ \boldsymbol{v}=\nabla p\ \text{in}\ \Omega.$$

The scalar potential p is unique, and $|p|_{1,\Omega}=\|\boldsymbol{v}\|_{0,\Omega}$.

Let us continue by the extraction of vector potentials.

Theorem 4. *Let Ω be a domain. Then, given $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, there holds*

$$\left.\begin{array}{l}\operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega,\\ \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_k} = 0,\ \forall k\end{array}\right\} \iff \left\{\begin{array}{l}\exists \boldsymbol{w} \in \boldsymbol{H}^1_{zmv}(\Omega),\\ \operatorname{div}\boldsymbol{w} = 0 \text{ in } \Omega,\end{array}\right. \quad \boldsymbol{v} = \mathbf{curl}\,\boldsymbol{w} \text{ in } \Omega.$$

Furthermore, one may choose the vector potential $\boldsymbol{w}$ such that $\|\boldsymbol{w}\|_{1,\Omega} \lesssim \|\boldsymbol{v}\|_{0,\Omega}$.

Theorem 5. *Let Ω be a domain such that* **(Top)**$_I$ *is fulfilled. Then, given $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, there holds*

$$\left.\begin{array}{l}\operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega,\\ \boldsymbol{v}\cdot\boldsymbol{n}_{|\partial\Omega} = 0,\\ \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_i} = 0,\ \forall i\end{array}\right\} \iff \left\{\begin{array}{l}\exists \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),\\ \operatorname{div}\boldsymbol{w} = 0 \text{ in } \Omega,\\ \langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_k} = 0,\ \forall k\end{array}\right. \quad \boldsymbol{v} = \mathbf{curl}\,\boldsymbol{w} \text{ in } \Omega.$$

The vector potential $\boldsymbol{w}$ is unique, and $\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{v}\|_{0,\Omega}$.

Remark 2. As indicated in the statement of Theorems 2 to 5, the assumptions on the field $\boldsymbol{v}$ are necessary and sufficient to guarantee the existence of the potential. Regarding the bounds, the constants hidden in $\lesssim$ depend only on the geometry of the domain Ω.

As a consequence of these results, one may derive auxiliary results on the measure of the couple of electromagnetic fields.

Corollary 1. *Let Ω be a domain. Let ξ be a real-valued tensor field such that $\xi, \xi^{-1} \in \mathbb{L}^\infty_{sym}(\Omega)$, $\lambda_{min}(\xi) > 0$ a.e. in Ω. There holds:*

$$\|\boldsymbol{v}\|_{0,\Omega} \lesssim \|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega} + \|\operatorname{div}\xi\boldsymbol{v}\|_{-1,\Omega} + \|\mathtt{P}_N\boldsymbol{v}\|_{\boldsymbol{Z}_N(\Omega,\xi)},\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\,;$$
$$\|\boldsymbol{v}\|_{0,\Omega} \lesssim \|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega} + \|\operatorname{div}\xi\boldsymbol{v}\|_{0,\Omega} + \|\mathtt{P}_N\boldsymbol{v}\|_{\boldsymbol{Z}_N(\Omega,\xi)},\ \forall \boldsymbol{v} \in \boldsymbol{X}_N(\Omega,\xi).$$

Above, $\mathtt{P}_N$ is an idempotent operator acting from $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ onto $\boldsymbol{Z}_N(\Omega,\xi)$.

Corollary 2. *Let Ω be a domain such that* **(Top)**$_I$ *is fulfilled. Let ξ be a real-valued tensor field such that $\xi, \xi^{-1} \in \mathbb{L}^\infty_{sym}(\Omega)$, $\lambda_{min}(\xi) > 0$ a.e. in Ω. There holds:*

$$\|\boldsymbol{v}\|_{0,\Omega} \lesssim \|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega} + \|\operatorname{div}\xi\boldsymbol{v}\|_{0,\Omega} + \|\mathtt{P}_T\boldsymbol{v}\|_{\boldsymbol{Z}_T(\Omega,\xi)},\ \forall \boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\xi).$$

Above, $\mathtt{P}_T$ is an idempotent operator acting from $\boldsymbol{H}(\mathbf{curl};\Omega)$ onto $\boldsymbol{Z}_T(\Omega,\xi)$.

3.3. Assumption on the coefficients

Let us proceed with the proper assumptions on the coefficients ε and μ that are needed later on.

Definition 1. $\mathcal{P} := \{\Omega_j\}_{j=1,\cdots,J}$ is a *partition* of Ω if $(\Omega_j)_{j=1,\cdots,J}$ are disjoint domains, and $\overline{\Omega} = \cup_{j=1}^J \overline{\Omega_j}$.

Given a partition $\mathcal{P}$, define the interfaces $F_{jj'} := \partial\Omega_j \cap \partial\Omega_{j'}$ and $\mathcal{F}_{int} := \{F_{jj'},\ 1 \le j \ne j' \le J\}$; $F_j = \partial\Omega_j \cap \partial\Omega$ and $\mathcal{F}_{bdry} := \{F_j,\ 1 \le j \le J\}$; $\mathcal{F} := \mathcal{F}_{int} \cup \mathcal{F}_{bdry}$. By convention, if the Hausdorf dimension of $F_{jj'}$ (resp. F_j) is lower than 2, then $F_{jj'} = \emptyset$ (resp. $F_j = \emptyset$). For a field v defined on Ω, we denote by v_j its restriction to Ω_j, for all j. Define further:

$$
\begin{aligned}
&PH^t(\Omega) := \{v \in L^2(\Omega)\ :\ v_j \in H^t(\Omega_j),\ 1 \le j \le J\},\ t > 0\,;\\
&PW^{1,\infty}(\Omega) := \{\zeta \in L^\infty(\Omega)\ :\ \zeta_j \in W^{1,\infty}(\Omega_j),\ 1 \le j \le J\}\,;\\
&(g|g')_{0,\mathcal{F}'} := \sum_{F\in\mathcal{F}'} (g|g')_{0,F},\ \forall g, g' \in L^2(\mathcal{F}'),\ \mathcal{F}' \in \{\mathcal{F}_{int}, \mathcal{F}_{bdry}, \mathcal{F}\}\,;\\
&PH^{1/2}(\mathcal{F}_{int}) := \{g \in L^2(\mathcal{F}_{int})\ :\ g_{|F} \in H^{1/2}(F),\ F \in \mathcal{F}_{int}\}.
\end{aligned}
$$

Above, the reference to $\mathcal{P}$ is omitted to simplify the notations.
Classically, in a domain Ω, one has $PH^t(\Omega) = H^t(\Omega)$ for all partitions $\mathcal{P}$ and for all $t \in]0, 1/2[$. On the other hand, when the partition is trivial, that is $\mathcal{P} = \{\Omega\}$, one has $PH^t(\Omega) = H^t(\Omega)$ for all $t > 0$, etc.

Definition 2. Let ξ be a real-valued tensor field such that $\xi, \xi^{-1} \in \mathbb{L}^\infty_{sym}(\Omega)$, $\lambda_{min}(\xi) > 0$ a.e. in Ω. ξ fulfills the *coefficient assumption* if there exists a partition $\mathcal{P}$ of Ω such that $\xi \in \mathbb{PW}^{1,\infty}(\Omega)$.

Remark 3. If ξ fulfills the coefficient assumption on a partition, then ξ^{-1} fulfills the coefficient assumption on the same partition.

Let ξ fulfill the coefficient assumption, and define

$$\boldsymbol{H}_B(\Omega, \xi) := \boldsymbol{X}_B(\Omega, \xi) \cap \boldsymbol{PH}^1(\Omega),\ B \in \{N, T\}.$$

If ξ is smooth on Ω, then one may choose $\mathcal{P} = \{\Omega\}$. In the particular case where ξ is equal to the identity, one writes $\boldsymbol{X}_N(\Omega)$ instead of $\boldsymbol{X}_N(\Omega, 1)$, etc., and one has obviously

$$\boldsymbol{H}_B(\Omega) \subset \boldsymbol{H}^1(\Omega),\ B \in \{N, T\},$$

where $\subset$ refers to an algebraical and topological embedding.

3.4. Case of constant coefficients

To begin with, we recall the Birman-Solomyak splitting of elements of $\boldsymbol{X}_N(\Omega)$, see [6, Theorem 4.1]. This fundamental result complements those on the extraction of scalar and vector potentials.

Theorem 6. *Let Ω be a domain. Then, there exists a continuous splitting operator acting from $\boldsymbol{X}_N(\Omega)$ to $\boldsymbol{H}_N(\Omega) \times H^1_0(\Omega)$.*
More precisely, given $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega)$,

$$\exists(\boldsymbol{v}_{reg}, q) \in \boldsymbol{H}_N(\Omega) \times H^1_0(\Omega),\ \boldsymbol{v} = \boldsymbol{v}_{reg} + \nabla q \text{ in } \Omega, \tag{22}$$

and one has

$$\|\boldsymbol{v}_{reg}\|_{1,\Omega} + \|q\|_{1,\Omega} + \|\Delta q\|_{0,\Omega} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{X}_N(\Omega)}. \tag{23}$$

On the other hand, having at hand additional results for the function spaces $\boldsymbol{Z}_B(\Omega)$, $B \in \{N, T\}$ is very useful to tackle the general case. Indeed since we consider fairly general geometrical settings (topologically non-trivial domains or non-connected boundary), we must take into account the null spaces. Let us recall the characterizations of $\boldsymbol{Z}_N(\Omega)$ and $\boldsymbol{Z}_T(\Omega)$ provided by [3].

First, we remark that, for all $1 \leq k' \leq K$, there exists one, and only one $q_{k'} \in H^1_{\partial\Omega}(\Omega)$ such that $\Delta q_{k'} = 0$ in Ω and $q_{k'|\Gamma_k} = \delta_{k'k}$ for $1 \leq k \leq K$. Defining $\boldsymbol{v}_{k'} := \nabla q_{k'} \in \boldsymbol{L}^2(\Omega)$, one checks that

$$\mathbf{curl}\, \boldsymbol{v}_{k'} = 0,\ \mathrm{div}\, \boldsymbol{v}_{k'} = 0 \text{ in } \Omega,\ \boldsymbol{v}_{k'} \times \boldsymbol{n}_{|\partial\Omega} = 0,\ 1 \leq k' \leq K.$$

Let $Q_N(\Omega) := \mathrm{Span}_{1 \leq k' \leq K}(q_{k'})$ be the vector space of potentials, of dimension K. Notice that an element q of $Q_N(\Omega)$ is obviously characterized by its boundary values $(q_{|\Gamma_{k'}})_{k'=1,\cdots,K}$. One has the characterization below for the null space $\boldsymbol{Z}_N(\Omega)$.

Proposition 1. *Let Ω be a domain. One has $\boldsymbol{Z}_N(\Omega) = Span_{1 \leq k' \leq K}(\nabla q_{k'})$. In addition, $\boldsymbol{v} \in \boldsymbol{Z}_N(\Omega)$ can be characterized by the fluxes $(\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_k})_{k=1,\cdots,K}$.*

As far as the regularity of elements of the null space $\boldsymbol{Z}_N(\Omega)$ is concerned, the previous characterization allows one to derive it easily.

Corollary 3. *Let Ω be a domain. There holds $\boldsymbol{Z}_N(\Omega) \subset \boldsymbol{H}^{1/2}(\Omega)$.*

Proof. Let $\boldsymbol{v} \in \boldsymbol{Z}_N(\Omega)$: according to Proposition 1, there exists $p_z \in Q_N(\Omega)$ such that $\boldsymbol{v} = \nabla p_z$. By construction, $\Delta p_z = 0$ in Ω, and moreover $p_{z|\partial\Omega} \in H^1(\partial\Omega)$. Thanks to [26], one has $p_z \in H^{3/2}(\Omega)$, hence $\nabla p_z \in \boldsymbol{H}^{1/2}(\Omega)$. This proves the claim. □

In particular, $\|\cdot\|_{1/2,\Omega}$ and all ℓ_p-norms measuring the fluxes $(\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_k})_{k=1,\cdots,K}$ in $\mathbb{C}^K$ are equivalent norms over the finite-dimensional vector space $\boldsymbol{Z}_N(\Omega)$.

Regarding the null space $\boldsymbol{Z}_T(\Omega)$, one considers, for all $1 \leq i' \leq I$, the scalar field $\dot{p}_{i'}$ defined on $\dot{\Omega}$ as the solution to:

$$\left\{ \begin{array}{l} \text{Find } \dot{p}_{i'} \in P_{zmv}(\dot{\Omega}) \text{ such that} \\ (\nabla \dot{p}_{i'} | \nabla \dot{q})_{0,\dot{\Omega}} = [\,\bar{\dot{q}}\,]_{\Sigma_{i'}},\ \forall \dot{q} \in P_{zmv}(\dot{\Omega}) \end{array} \right. .$$

Then, $\boldsymbol{v}_{i'} = \widetilde{\nabla \dot{p}_{i'}} \in \boldsymbol{L}^2(\Omega)$ is such that

$$\mathbf{curl}\, \boldsymbol{v}_{i'} = 0,\ \mathrm{div}\, \boldsymbol{v}_{i'} = 0 \text{ in } \Omega,\ \boldsymbol{v}_{i'} \cdot \boldsymbol{n}_{|\partial\Omega} = 0, \text{ and } \langle \boldsymbol{v}_{i'} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_i} = \delta_{ii'},\ 1 \leq i' \leq I.$$

Let us define a second vector space $Q_T(\dot{\Omega}) := \mathrm{Span}_{1 \leq i' \leq I}(\dot{p}_{i'})$ of scalar potentials, of dimension I. Its elements $\dot{p}$ may be characterized by their jumps $([\dot{p}]_{\Sigma_{i'}})_{1 \leq i' \leq I}$. Next, if $\dot{p} \in Q_T(\dot{\Omega})$ fulfills $[\dot{p}]_{\Sigma_i} = 0$ for all i, then $p = \widetilde{\dot{p}}$ belongs to $H^1(\Omega)$. On the other hand, using $\dot{q} = \dot{p}$ in the variational formulation that defines $\dot{p}$ yields $\nabla \dot{p} = 0$ in $\dot{\Omega}$: it follows that $\dot{p} = 0$ and $p = 0$. Hence, (continuations to Ω of) non-zero elements of $Q_T(\dot{\Omega})$ do not belong to $H^1(\Omega)$.

Proposition 2. *Let* Ω *be a domain such that* **(Top)**$_I$ *is fulfilled. One has* $\boldsymbol{Z}_T(\Omega) = Span_{1\le i'\le I}(\widetilde{\nabla \dot{p}_{i'}})$. *In addition, an element* $\boldsymbol{v}$ *of* $\boldsymbol{Z}_T(\Omega)$ *can be characterized by the fluxes* $(\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_i})_{i=1,\cdots,I}$.

Regarding the regularity of elements of the null space $\boldsymbol{Z}_T(\Omega)$, one may prove the result below.

Corollary 4. *Let* Ω *be a domain such that* **(Top)**$_I$ *is fulfilled. There holds* $\boldsymbol{Z}_T(\Omega) \subset \boldsymbol{H}^{1/2}(\Omega)$.

Proof. Let $\boldsymbol{v} \in \boldsymbol{Z}_T(\Omega)$, $\boldsymbol{v} \neq 0$: according to Proposition 2, there exists $\dot{p} \in P_{zmv}(\dot{\Omega})$ such that $\boldsymbol{v} = \widetilde{\nabla \dot{p}}$. As noted above, $\widetilde{\dot{p}} \notin H^1(\Omega)$.
However, one may address this difficulty by using a partition of unity. Let $(\chi_i)_{i=1,\cdots,I}$ be such that for all i: $\chi_i \in C^\infty(\overline{\Omega},[0,1])$ with connected support, $\chi_i = 1$ in a neighborhood of Σ_i, and $\mathrm{supp}(\chi_{i'}) \cap \Sigma_i = \emptyset$ for $i' \neq i$. One may further define connected, open subsets $(\mathcal{O}_i)_{i=1,\cdots,I}$ of Ω such that $\mathrm{supp}(\chi_i) \cap \Omega \subset \mathcal{O}_i$ and $\mathcal{O}_{i'} \cap \Sigma_i = \emptyset$, for $i \neq i'$. Each subset is split into two parts, $\mathcal{O}_i^-$ and $\mathcal{O}_i^+$, according to the orientation of the normal vector to Σ_i, so that $[z]_{\Sigma_i} = z_{|\partial\mathcal{O}_i^+} - z_{|\partial\mathcal{O}_i^-}$. By defining $\chi_0 = 1 - \sum_{1\le i\le I}\chi_i$, one gets a partition of unity $(\chi_\iota)_{\iota=0,\cdots,I}$ on $\overline{\Omega}$.
Next, let $\dot{p}_\iota = \chi_\iota \dot{p}$ for all ι: by construction, $\widetilde{\dot{p}_0} \in H^1(\Omega)$, whereas $\dot{p}_i \in P(\dot{\Omega})$ for $1 \le i \le I$. Introduce, for $1 \le i \le I$, $p_i \in L^2(\mathcal{O}_i)$ defined as $p_i = \dot{p}_i$ in $\mathcal{O}_i^-$ and $p_i = \dot{p}_i - [\dot{p}_i]_{\Sigma_i}$ in $\mathcal{O}_i^+$. As $[p_i]_{\Sigma_i} = 0$, it holds $p_i \in H^1(\mathcal{O}_i)$, and in addition $\Delta p_i \in L^2(\mathcal{O}_i)$ and $\partial_n p_{i|\partial\mathcal{O}_i} \in L^2(\partial\mathcal{O}_i)$. So we obtain that $p_i \in H^{3/2}(\mathcal{O}_i)$, cf. [26, 15], which implies $\widetilde{\nabla \dot{p}_i} = \nabla p_i \in \boldsymbol{H}^{1/2}(\mathcal{O}_i)$. It follows that $\widetilde{\nabla \dot{p}_i}$ belongs to $\boldsymbol{H}^{1/2}(\Omega)$ because $\widetilde{\nabla \dot{p}_i}$ vanishes in a neighborhood of $\partial\mathcal{O}_i \cap \Omega$ (and $\widetilde{\nabla \dot{p}_i} = 0$ in $\Omega \setminus \overline{\mathcal{O}_i}$). Likewise, $\widetilde{\nabla \dot{p}_0} = \nabla \widetilde{\dot{p}_0}$ belongs to $\boldsymbol{H}^{1/2}(\Omega)$. Using the definition of the partition of unity, one concludes that $\boldsymbol{v} = \widetilde{\nabla \dot{p}} \in \boldsymbol{H}^{1/2}(\Omega)$. □

If we let
$$P^{3/2}(\dot{\Omega}) := \{\dot{q} \in P(\dot{\Omega}) \ : \ \widetilde{\nabla \dot{q}} \in \boldsymbol{H}^{1/2}(\Omega)\},$$
we have proven in passing that $Q_T(\dot{\Omega}) \subset P^{3/2}_{zmv}(\dot{\Omega})$. Moreover, $\|\cdot\|_{1/2,\Omega}$ and all ℓ_p-norms measuring the fluxes $(\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_i})_{i=1,\cdots,I}$ in $\mathbb{C}^I$ are equivalent norms over $\boldsymbol{Z}_T(\Omega)$.

In the next subsection, we proceed with the splittings of elements of $\boldsymbol{X}_N(\Omega,\xi)$, resp. of elements of $\boldsymbol{X}_T(\Omega,\xi)$.

3.5. Splittings of fields

We provide now splittings into a regular part, and a gradient part, of elements of $\boldsymbol{X}_N(\Omega,\varepsilon)$ ("electric case"), resp. of elements of $\boldsymbol{X}_T(\Omega,\mu)$ ("magnetic case"), called *regular/gradient splittings*. Since we are dealing with general geometrical settings, an additional part is present, which belongs to the null space with the same boundary condition.
Let us begin with the "electric case".

Theorem 7. *Let Ω be a domain such that* $\mathbf{(Top)}_I$ *is fulfilled, and assume that ξ fulfills the coefficient assumption. Then, there exists a continuous splitting operator acting from $\boldsymbol{X}_N(\Omega,\xi)$ to $\boldsymbol{H}_N(\Omega)\times\boldsymbol{Z}_N(\Omega)\times H^1_0(\Omega)$.*
More precisely, given $\boldsymbol{v}\in\boldsymbol{X}_N(\Omega,\xi)$,

$$\exists(\boldsymbol{v}_{reg},\boldsymbol{z},p_0)\in\boldsymbol{H}_N(\Omega)\times\boldsymbol{Z}_N(\Omega)\times H^1_0(\Omega),\ \boldsymbol{v}=\boldsymbol{v}_{reg}+\boldsymbol{z}+\nabla p_0\ \text{in}\ \Omega\,; \quad (24)$$

the scalar field p_0 is governed by the variational formulation below, for some $f\in L^2(\Omega)$ and $g_{\mathcal{F}}\in PH^{1/2}(\mathcal{F}_{int})$:

$$\left\{\begin{array}{l}\text{Find } p_0\in H^1_0(\Omega) \text{ such that}\\ (\xi\nabla p_0|\nabla\psi)_{0,\Omega}=-(\xi\boldsymbol{z}|\nabla\psi)_{0,\Omega}+(f|\psi)_{0,\Omega}\\ \qquad\qquad\qquad\qquad +(g_{\mathcal{F}}|\psi)_{0,\mathcal{F}_{int}},\ \forall\psi\in H^1_0(\Omega)\,;\end{array}\right. \quad (25)$$

one has

$$\left\{\begin{array}{l}\|\boldsymbol{v}_{reg}\|_{1,\Omega}+\|\boldsymbol{v}_{reg}\|_{\boldsymbol{X}_N(\Omega)}+\|\boldsymbol{z}\|_{1/2,\Omega}\lesssim\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)},\\ \|\xi\boldsymbol{z}\|_{\boldsymbol{PH}^{1/2}(\Omega)}+\|f\|_{0,\Omega}+\|g_{\mathcal{F}}\|_{1/2,\mathcal{F}_{int}}\lesssim\|\boldsymbol{v}\|_{\boldsymbol{X}_N(\Omega,\xi)}.\end{array}\right. \quad (26)$$

In addition, one may choose $\mathtt{p}_N\ :\ \boldsymbol{v}\mapsto\boldsymbol{z}$, acting from $\boldsymbol{X}_N(\Omega,\xi)$ to $\boldsymbol{Z}_N(\Omega)$ to be an idempotent operator.

Remark 4. In the splitting (24) of $\boldsymbol{v}\in\boldsymbol{X}_N(\Omega,\xi)$, all three terms $\boldsymbol{v}_{reg},\boldsymbol{z},\nabla p_0$ have vanishing tangential components on the boundary $\partial\Omega$. Regarding regularity in (24), one has $\boldsymbol{v}_{reg}\in\boldsymbol{H}^1(\Omega)$, resp. $\boldsymbol{z}=\nabla p_z\in\boldsymbol{H}^{1/2}(\Omega)$ with $p_z\in H^1(\Omega)$, resp. $\nabla p_0\in\boldsymbol{L}^2(\Omega)$. Since ξ fulfills the coefficient assumption, the variational formulation (25) is well-posed. In the first bound in (26), the constant hidden in $\lesssim$ depends only on the geometry, whereas, in the last bound in (26), the constant hidden in $\lesssim$ also depends on $\|\xi\|_{\mathbb{PW}^{1,\infty}(\Omega)}$. The idea of the proof follows closely [16, §3].

Proof. Let $\boldsymbol{y}=\mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{div};\Omega)$. By construction $\mathrm{div}\,\boldsymbol{y}=0$ in Ω, and one checks that $\langle\boldsymbol{y}\cdot\boldsymbol{n},1\rangle_{\Sigma_i}=0$ for all i. According to Theorem 5 on vector potentials, there exists $\boldsymbol{w}\in\boldsymbol{X}_N(\Omega)$ with $\mathrm{div}\,\boldsymbol{w}=0$ in Ω, $\langle\boldsymbol{w}\cdot\boldsymbol{n},1\rangle_{\Gamma_k}=0$ for all k, such that $\boldsymbol{y}=\mathbf{curl}\,\boldsymbol{w}$ in Ω and $\|\boldsymbol{w}\|_{\boldsymbol{X}_N(\Omega)}\lesssim\|\boldsymbol{y}\|_{0,\Omega}$. Next, we know that there exists a Birman-Solomyak splitting of $\boldsymbol{w}$, see Theorem 6:

$$\exists\boldsymbol{v}_{reg}\in\boldsymbol{H}_N(\Omega),\ \exists q\in H^1_0(\Omega),\ \boldsymbol{w}=\boldsymbol{v}_{reg}+\nabla q\ \text{in}\ \Omega,$$

with continuous dependence (23). By construction, $\mathbf{curl}(\boldsymbol{v}-\boldsymbol{v}_{reg})=0$ in Ω, with $(\boldsymbol{v}-\boldsymbol{v}_{reg})\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$. According to Theorem 3 on scalar potentials, there exists $p\in H^1_{\partial\Omega}(\Omega)$ such that $\boldsymbol{v}=\boldsymbol{v}_{reg}+\nabla p$ in Ω. Using the definition of the space of scalar potentials $Q_N(\Omega)$, one may further split p as $p=p_0+p_z$ in Ω, with $p_0\in H^1_0(\Omega)$, and $p_z\in Q_N(\Omega)$. Introducing finally $\boldsymbol{z}=\nabla p_z\in\boldsymbol{Z}_N(\Omega)$, we have proved that there holds

$$\boldsymbol{v}=\boldsymbol{v}_{reg}+\boldsymbol{z}+\nabla p_0\ \text{in}\ \Omega,$$

with $\boldsymbol{v}_{reg} \in \boldsymbol{H}_N(\Omega)$, $\boldsymbol{z} \in \boldsymbol{Z}_N(\Omega)$, $p_0 \in H^1_0(\Omega)$, which is precisely (24).
Let us proceed with the definition of p_0 as the solution to (25). Let $\psi \in H^1_0(\Omega)$, then

$$(\xi\nabla p_0|\nabla\psi)_{0,\Omega} = -(\xi\boldsymbol{z}|\nabla\psi)_{0,\Omega} + (\xi\boldsymbol{v}|\nabla\psi)_{0,\Omega} - (\xi\boldsymbol{v}_{reg}|\nabla\psi)_{0,\Omega}.$$

Below, we study the last two terms separately.
Consider first $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega,\xi)$. One has in particular $\xi\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$, so by integration by parts on Ω one gets

$$(\xi\boldsymbol{v}|\nabla\psi)_{0,\Omega} = -(\mathrm{div}\,\xi\boldsymbol{v}|\psi)_{0,\Omega}.$$

Consider next $\boldsymbol{v}_{reg} \in \boldsymbol{H}_N(\Omega)$. If ξ is only piecewise smooth([3]) on Ω, $\xi\boldsymbol{v}_{reg}\cdot\boldsymbol{n}$ has jumps across faces of $\mathcal{F}_{int}$. On the other hand, one has $\xi_j\boldsymbol{v}_{reg,j} \in \boldsymbol{H}^1(\Omega_j)$ for all j. Therefore, one can integrate by parts over each subdomain to find

$$\begin{aligned}
-(\xi\boldsymbol{v}_{reg}|\nabla\psi)_{0,\Omega} &= -\sum_j(\xi_j\boldsymbol{v}_{reg,j}|\nabla\psi_j)_{0,\Omega_j} \\
&= \sum_j(\mathrm{div}\,\xi_j\boldsymbol{v}_{reg,j}|\psi_j)_{0,\Omega_j} - \sum_{F\in\mathcal{F}_{int}}([\xi\boldsymbol{v}_{reg}\cdot\boldsymbol{n}]|\psi)_{0,F} \\
&= (\widetilde{\mathrm{div}\,\xi\boldsymbol{v}_{reg}}|\psi)_{0,\Omega} - \sum_{F\in\mathcal{F}_{int}}([\xi\boldsymbol{v}_{reg}\cdot\boldsymbol{n}]|\psi)_{0,F}.
\end{aligned}$$

If we introduce

$$f = -\,\mathrm{div}\,\xi\boldsymbol{v} + \widetilde{\mathrm{div}\,\xi\boldsymbol{v}_{reg}} \in L^2(\Omega), \quad g_{\mathcal{F}} = -[\xi\boldsymbol{v}_{reg}\cdot\boldsymbol{n}] \in PH^{1/2}(\mathcal{F}_{int}),$$

we obtain that p_0 is governed by (25).
We derive next the (uniform) estimates (26) to prove that the splitting operator is continuous. By construction

$$\left\{\begin{array}{l}
\|\boldsymbol{v}_{reg}\|_{1,\Omega} \lesssim \|\boldsymbol{w}\|_{\boldsymbol{X}_N(\Omega)} \lesssim \|\boldsymbol{y}\|_{0,\Omega} \leq \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\,; \\
\|\boldsymbol{v}_{reg}\|_{\boldsymbol{X}_N(\Omega)} \leq \|\boldsymbol{w}\|_{\boldsymbol{X}_N(\Omega)} + \|\nabla q\|_{\boldsymbol{X}_N(\Omega)} \lesssim \|\boldsymbol{w}\|_{\boldsymbol{X}_N(\Omega)} \leq \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.
\end{array}\right.$$

For instance, $\boldsymbol{z} \in \boldsymbol{Z}_N(\Omega)$ can be measured by the ℓ_1-norm of $(\langle\boldsymbol{z}\cdot\boldsymbol{n},1\rangle_{\Gamma_k})_{k=1,\cdots,K}$:

$$\begin{aligned}
|\langle\boldsymbol{z}\cdot\boldsymbol{n},1\rangle_{\Gamma_k}| &= |\langle\boldsymbol{z}\cdot\boldsymbol{n},q_k\rangle_{H^{1/2}(\partial\Omega)}| = |(\boldsymbol{z}|\nabla q_k)_{0,\Omega}| \\
&= |(\boldsymbol{z}+\nabla p_0|\nabla q_k)_{0,\Omega}| = |(\boldsymbol{v}-\boldsymbol{v}_{reg}|\nabla q_k)_{0,\Omega}| \\
&\leq (\|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{v}_{reg}\|_{0,\Omega})\,\|\nabla q_k\|_{0,\Omega} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.
\end{aligned}$$

Above, we used first the definition of $(q_k)_k$ given in §3.4, and then the fact that ∇p_0 and ∇q_k are orthogonal with respect to $(\cdot|\cdot)_{0,\Omega}$ (integrate by parts).
For a given j,

$$\begin{aligned}
&\|\xi\boldsymbol{z}\|_{1/2,\Omega_j} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\,, \\
&\|f\|_{0,\Omega_j} \leq \|\,\mathrm{div}\,\xi\boldsymbol{v}\|_{0,\Omega_j} + \|\,\mathrm{div}\,\xi\boldsymbol{v}_{reg}\|_{0,\Omega_j} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{X}_N(\Omega,\xi)}.
\end{aligned}$$

[3]If ξ is globally smooth on Ω, $\mathcal{P} = \{\Omega\}$ and $\mathcal{F}_{int} = \emptyset$.

And for a given $F = \partial\Omega_j \cap \partial\Omega_{j'} \in \mathcal{F}_{int}$, we find, thanks to the continuity of the trace mapping,

$$\|g_{\mathcal{F}}\|_{1/2,F} = \|[\xi \boldsymbol{v}_{reg}\cdot\boldsymbol{n}]\|_{1/2,F} \leq \|[\xi \boldsymbol{v}_{reg}]\|_{1/2,F} \lesssim \sum_{\beta=j,j'} \|\xi \boldsymbol{v}_{reg}\|_{1,\Omega_\beta} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{X}_N(\Omega,\xi)}.$$

In the last three bounds, the constant hidden in $\lesssim$ depends on $\|\xi\|_{\mathbb{PW}^{1,\infty}(\Omega)}$. Finally, to prove that the operator $\mathtt{p}_N$ is idempotent, let us build the splitting (24) for $\boldsymbol{v} \in \boldsymbol{Z}_N(\Omega)$: as $\boldsymbol{y} = \mathbf{curl}\, \boldsymbol{v} = 0$, it follows that $\boldsymbol{w} = 0$, so that $\boldsymbol{v}_{reg} = 0$ and $q = 0$ in its Birman-Solomyak splitting. Next, let $p \in H^1_{\partial\Omega}(\Omega)$ be such that $\boldsymbol{v} = \nabla p$: according to Proposition 1, p actually belongs to $Q_N(\Omega)$. By uniqueness, one obtains that $p_0 = 0$, hence the splitting (24) writes $\boldsymbol{v} = 0 + \boldsymbol{v} + 0$ in Ω, so that $\mathtt{p}_N\boldsymbol{v} = \boldsymbol{v}$. To conclude that $\mathtt{p}_N$ is an idempotent operator, one simply remarks that for all $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega)$, $\mathtt{p}_N\boldsymbol{v}$ belongs to $\boldsymbol{Z}_N(\Omega)$ by definition, so it holds $\mathtt{p}_N(\mathtt{p}_N\boldsymbol{v}) = \mathtt{p}_N\boldsymbol{v}$, or $\mathtt{p}_N^2 = \mathtt{p}_N$ in operator form. □

To carry on, one needs regularity results regarding ∇p_0, where p_0 is governed by the variational formulation (25). For that, we use an abstract shift theorem, proven in [8, Theorem 3.1], that deals with second order elliptic PDEs complemented with Dirichlet boundary conditions. This result provides a *lower bound* on the *a priori* regularity of ∇p_0 in all the configurations that we consider in this paper([4]).

Theorem 8. *Let Ω be a domain, and assume that ξ fulfills the coefficient assumption. There exists $\tau_{Dir} := \tau_{Dir}(\xi) \in]0, 1/2[$ depending only on the geometry and the coefficient ξ such that: for all $s \in [0, \tau_{Dir}[$, for all $\ell \in H^{s-1}(\Omega)$, the solution to*

$$\left\{ \begin{array}{l} \text{Find } u \in H^1_0(\Omega) \text{ such that} \\ (\xi\nabla u|\nabla\psi)_{0,\Omega} = \langle \ell, \psi\rangle_{H^1_0(\Omega)},\ \forall \psi \in H^1_0(\Omega), \end{array} \right.$$

belongs to $H^{s+1}(\Omega)$, and moreover([5]) $\|u\|_{s+1,\Omega} \lesssim_s \|\ell\|_{s-1,\Omega}$.

Combining the two theorems yields the result regarding the regular/gradient splitting of elements of $\boldsymbol{X}_N(\Omega, \xi)$. Below, $\nabla[H^{s+1}(\Omega) \cap H^1_{\partial\Omega}(\Omega)]$ denotes the range of the gradient operator from $H^{s+1}(\Omega) \cap H^1_{\partial\Omega}(\Omega)$ to $\boldsymbol{H}^s(\Omega)$.

Corollary 5. *Let Ω be a domain such that* $(\mathbf{Top})_I$ *is fulfilled, and assume that ξ fulfills the coefficient assumption. There holds*

$$\boldsymbol{X}_N(\Omega, \xi) \subset \boldsymbol{H}_N(\Omega) + \nabla[H^{s+1}(\Omega) \cap H^1_{\partial\Omega}(\Omega)],\ \forall s \in [0, \tau_{Dir}[. \qquad (27)$$

[4]In some configurations, it can happen that that the limit exponent τ_{Dir} is larger than 1/2. However, we are interested here only in the existence of such an exponent. More precise results may be derived for fairly general subclasses of the configurations, we refer to §5.2.

[5]The symbol $\lesssim_s$ means that the value of the given constant appearing in the inequality depends on s: $\forall s,\ \exists C_s,\ \forall \ell \in H^{s-1}(\Omega),\ \|u\|_{s+1,\Omega} \leq C_s \|\ell\|_{s-1,\Omega}$.

Proof. Let $s \in [0,\tau_{Dir}[$. Given $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega,\xi)$, we use the splitting (24), namely

$$\exists \boldsymbol{v}_{reg} \in \boldsymbol{H}_N(\Omega),\ \exists \boldsymbol{z} \in \boldsymbol{Z}_N(\Omega),\ \exists p_0 \in H^1_0(\Omega),\ \boldsymbol{v} = \boldsymbol{v}_{reg} + \boldsymbol{z} + \nabla p_0 \text{ in } \Omega,$$

where p_0 is governed by (25), with the uniform bounds (26). Hence, $\|\boldsymbol{v}_{reg}\|_{1,\Omega} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{X}_N(\Omega,\xi)}$. Furthermore, thanks to Corollary 3, one can write $\boldsymbol{z} = \nabla p_z$, with $p_z \in H^{3/2}(\Omega) \cap H^1_{\partial\Omega}(\Omega) \subset H^{s+1}(\Omega) \cap H^1_{\partial\Omega}(\Omega)$, so it holds $\|p_z\|_{1+s,\Omega} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{X}_N(\Omega,\xi)}$. Then, p_0 is characterized by (25), with a right-hand side

$$\ell\ :\ \psi \mapsto -(\xi\boldsymbol{z}|\nabla\psi)_{0,\Omega} + (f|\psi)_{0,\Omega} + (g_{\mathcal{F}}|\psi)_{0,\mathcal{F}_{int}}$$

that belongs to $(H^{1-s}_0(\Omega))' = H^{s-1}(\Omega)$. Indeed, if $\psi \in H^{1-s}_0(\Omega)$, then:

- $\nabla\psi \in \boldsymbol{H}^{-s}(\Omega) = (\boldsymbol{H}^s(\Omega))'$ (recall that $s \in [0,1/2[$), and $\xi\boldsymbol{z} \in \boldsymbol{PH}^{1/2}(\Omega) \subset \boldsymbol{H}^s(\Omega)$, so one may write the first term as $\langle \nabla\psi, \xi\boldsymbol{z}\rangle_{\boldsymbol{H}^s(\Omega)}$;
- for all $F \in \mathcal{F}_{int}$, $\psi_{|F} \in L^2(F)$.

Hence, according to the shift Theorem 8, it follows that $p_0 \in H^{s+1}(\Omega)$, with continuous dependence. So we get:

$$\|p_0\|_{1+s,\Omega} \lesssim \|\ell\|_{s-1,\Omega} \lesssim \|\xi\boldsymbol{z}\|_{\boldsymbol{PH}^{1/2}(\Omega)} + \|f\|_{0,\Omega} + \|g_{\mathcal{F}}\|_{0,\mathcal{F}_{int}} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{X}_N(\Omega,\xi)}.$$

This proves the claim. □

Let us continue with the "magnetic case".

Theorem 9. *Let Ω be a domain such that* **(Top)**$_I$ *is fulfilled, and assume that ξ fulfills the coefficient assumption. Then, there exists a continuous splitting operator acting from $\boldsymbol{X}_T(\Omega,\xi)$ to $\boldsymbol{H}^1_{zmv}(\Omega) \times \boldsymbol{Z}_T(\Omega) \times H^1_{zmv}(\Omega)$.*
More precisely, given $\boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\xi)$,

$$\exists(\boldsymbol{w}_{reg},\boldsymbol{z},q_0) \in \boldsymbol{H}^1_{zmv}(\Omega)\times\boldsymbol{Z}_T(\Omega)\times H^1_{zmv}(\Omega),\ \boldsymbol{v} = \boldsymbol{w}_{reg}+\boldsymbol{z}+\nabla q_0 \text{ in } \Omega\,; \quad (28)$$

the scalar field q_0 is governed by the variational formulation below, for some $f \in L^2(\Omega)$ and $g_{\mathcal{F}} \in PH^{1/2}(\mathcal{F})$:

$$\left\{\begin{array}{l} \text{Find } q_0 \in H^1_{zmv}(\Omega) \text{ such that} \\ (\xi\nabla q_0|\nabla\psi)_{0,\Omega} = -(\xi\boldsymbol{z}|\nabla\psi)_{0,\Omega} + (f|\psi)_{0,\Omega} \\ \qquad\qquad\qquad\qquad +(g_{\mathcal{F}}|\psi)_{0,\mathcal{F}},\ \forall\psi \in H^1_{zmv}(\Omega)\,; \end{array}\right. \quad (29)$$

one has

$$\left\{\begin{array}{l} \|\boldsymbol{w}_{reg}\|_{1,\Omega} + \|\boldsymbol{z}\|_{1/2,\Omega} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\,, \\ \|\xi\boldsymbol{z}\|_{\boldsymbol{PH}^{1/2}(\Omega)} + \|f\|_{0,\Omega} + \|g_{\mathcal{F}}\|_{1/2,\mathcal{F}} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{X}_T(\Omega,\xi)}. \end{array}\right. \quad (30)$$

Remark 5. In the splitting (28) of $\boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\xi)$, $\boldsymbol{w}_{reg}$ does not fulfill any boundary condition in general. Regarding regularity, one has $\boldsymbol{w}_{reg} \in \boldsymbol{H}^1(\Omega)$, resp. $\boldsymbol{z} = \widetilde{\nabla\dot{p}_z} \in \boldsymbol{H}^{1/2}(\Omega)$ with $\dot{p}_z \in P_{zmv}(\Omega)$, resp. $\nabla q_0 \in \boldsymbol{L}^2(\Omega)$. Since ξ fulfills the coefficient assumption, the variational formulation (29) is well-posed. In the first bound in (30), the constant hidden in $\lesssim$ depends only on the geometry, whereas in the last bound the constant hidden in $\lesssim$ also depends on $\|\xi\|_{\mathbb{PW}^{1,\infty}(\Omega)}$. Again, the idea of the proof follows closely [16, §3].

Proof. Let $\boldsymbol{y} = \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega)$. One has $\mathrm{div}\,\boldsymbol{y} = 0$ in Ω, and $\langle \boldsymbol{y}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_k} = 0$ for all k. Thanks to Theorem 4 on vector potentials, there exists $\boldsymbol{w}_{reg} \in \boldsymbol{H}^1_{zmv}(\Omega)$ with $\mathrm{div}\,\boldsymbol{w}_{reg} = 0$ in Ω such that $\boldsymbol{y} = \mathbf{curl}\,\boldsymbol{w}_{reg}$ in Ω and

$$\|\boldsymbol{w}_{reg}\|_{1,\Omega} \lesssim \|\boldsymbol{y}\|_{0,\Omega} \leq \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

By construction, $\mathbf{curl}(\boldsymbol{v} - \boldsymbol{w}_{reg}) = 0$ in Ω, with $(\boldsymbol{v} - \boldsymbol{w}_{reg}) \in \boldsymbol{H}(\mathbf{curl};\Omega)$. According to Theorem 2 on scalar potentials, there exists $\dot{q} \in P_{zmv}(\dot{\Omega})$ such that $\boldsymbol{v} = \boldsymbol{w}_{reg} + \widetilde{\nabla}\dot{q}$ in Ω. And $|\dot{q}|_{1,\dot{\Omega}} \leq \|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{w}_{reg}\|_{0,\Omega} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$.
Since elements of $Q_T(\dot{\Omega})$ can be characterized by their jumps, we introduce next $\dot{p} \in Q_T(\dot{\Omega})$ such that $[\dot{p}]_{\Sigma_i} = [\dot{q}]_{\Sigma_i}$ for all i, and then $\boldsymbol{z} = \widetilde{\nabla}\dot{p} \in \boldsymbol{Z}_T(\Omega)$. The norm $\|\dot{p}\|_{Q_T(\dot{\Omega})}$ is bounded by $\|([\dot{q}]_{\Sigma_i})_i\|_{\ell_1(\mathbb{C}^I)}$, which is itself bounded by $|\dot{q}|_{1,\dot{\Omega}}$, so one gets $\|\boldsymbol{z}\|_{1/2,\Omega} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$.
If one lets $q_0 = \widetilde{\dot{q} - \dot{p}}$, one has $q_0 \in H^1_{zmv}(\Omega)$, and in addition there holds

$$\boldsymbol{v} = \boldsymbol{w}_{reg} + \boldsymbol{z} + \nabla q_0 \text{ in } \Omega,$$

with $\boldsymbol{w}_{reg} \in \boldsymbol{H}^1_{zmv}(\Omega)$, $\boldsymbol{z} \in \boldsymbol{Z}_T(\Omega)$, $q_0 \in H^1_{zmv}(\Omega)$, ie. (28).
About the definition of q_0 as the solution to (29), let $\psi \in H^1_{zmv}(\Omega)$:

$$(\xi\nabla q_0|\nabla\psi)_{0,\Omega} = -(\xi\boldsymbol{z}|\nabla\psi)_{0,\Omega} + (\xi\boldsymbol{v}|\nabla\psi)_{0,\Omega} - (\xi\boldsymbol{w}_{reg}|\nabla\psi)_{0,\Omega}.$$

As $\xi\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega)$ one finds by integration by parts $(\xi\boldsymbol{v}|\nabla\psi)_{0,\Omega} = -(\mathrm{div}\,\xi\boldsymbol{v}|\psi)_{0,\Omega}$. For the third term, one proceeds as in the proof of Theorem 7, the only difference being that there are additional boundary terms:

$$\begin{aligned} -(\xi\boldsymbol{w}_{reg}|\nabla\psi)_{0,\Omega} \;=\; & (\mathrm{div}\,\widetilde{\xi\boldsymbol{w}_{reg}}|\psi)_{0,\Omega} \\ & - \sum_{F\in\mathcal{F}_{int}} ([\xi\boldsymbol{w}_{reg}\cdot\boldsymbol{n}]|\psi)_{0,F} - (\xi\boldsymbol{w}_{reg}\cdot\boldsymbol{n}|\psi)_{0,\partial\Omega}. \end{aligned}$$

Define next,

$$f = -\,\mathrm{div}\,\xi\boldsymbol{v} + \mathrm{div}\,\widetilde{\xi\boldsymbol{w}_{reg}} \in L^2(\Omega), \quad g_{\mathcal{F}} = -[\xi\boldsymbol{w}_{reg}\cdot\boldsymbol{n}] \in PH^{1/2}(\mathcal{F}),$$

where, for all $F \in \mathcal{F}_{bdry}$ and $z \in L^2(F)$, the "jump" $[z]$ is simply equal to z. It follows that q_0 is characterized by (29).
Finally, the first bound in (30) has already been derived, and the second one is obtained exactly as in the proof of Theorem 7, hence continuity of the splitting operator is obtained. □

In the "magnetic case", one needs regularity results regarding ∇q_0, where q_0 is now governed by (29). We use the abstract shift theorem [8, Theorem 3.1] for PDEs with Neumann boundary conditions (see footnote [4], page 15, for some comments on the optimality of the limit exponent, here τ_{Neu}).

Theorem 10. *Let Ω be a domain, and assume that ξ fulfills the coefficient assumption. There exists $\tau_{Neu} := \tau_{Neu}(\xi) \in]0, 1/2[$ depending only on the geometry and the coefficient ξ such that: for all $s \in [0, \tau_{Neu}[$, for all $\ell \in (H^{1-s}_{zmv}(\Omega))'$, the solution to*

$$\begin{cases} \text{Find } u \in H^1_{zmv}(\Omega) \text{ such that} \\ (\xi \nabla u | \nabla \psi)_{0,\Omega} = \langle \ell, \psi \rangle_{H^1_{zmv}(\Omega)}, \ \forall \psi \in H^1_{zmv}(\Omega), \end{cases}$$

belongs to $H^{s+1}(\Omega)$, and moreover $\|u\|_{s+1,\Omega} \lesssim_s \|\ell\|_{(H^{1-s}_{zmv}(\Omega))'}$.

More precise results may be derived for subclasses of the configurations, cf. §5.2. Combining the two Theorems 9 and 10 yields the result for the regular/gradient splitting of elements of $\boldsymbol{X}_T(\Omega, \xi)$. The proof is omitted, as it is very close to the one of Corollary 5.

Corollary 6. *Let Ω be a domain such that* $\textbf{(Top)}_I$ *is fulfilled, and assume that ξ fulfills the coefficient assumption. There holds*

$$\boldsymbol{X}_T(\Omega, \xi) \subset \boldsymbol{H}^1(\Omega) + \widetilde{\nabla}[P^{3/2}_{zmv}(\dot{\Omega})] + \nabla[H^{s+1}_{zmv}(\Omega)], \ \forall s \in [0, \tau_{Neu}[. \qquad (31)$$

For the sake of completeness and because this result will be used later on, we mention that it is also possible to derive a splitting of $\boldsymbol{X}_T(\Omega, \xi)$ which preserves the homogeneous boundary condition on the normal trace, under some moderate restrictions on the domain Ω.

Theorem 11. *Let Ω be a (curved) Lipschitz polyhedron such that* $\textbf{(Top)}_I$ *is fulfilled, and assume that ξ fulfills the coefficient assumption. Then, there exists a continuous splitting operator acting from $\boldsymbol{X}_T(\Omega, \xi)$ to $\boldsymbol{H}_T(\Omega) \times \boldsymbol{Z}_T(\Omega) \times H^1_{zmv}(\Omega)$. Given $\boldsymbol{v} \in \boldsymbol{X}_T(\Omega, \xi)$,*

$$\exists (\boldsymbol{v}_{reg}, \boldsymbol{z}, p_0) \in \boldsymbol{H}_T(\Omega) \times \boldsymbol{Z}_T(\Omega) \times H^1_{zmv}(\Omega), \ \boldsymbol{v} = \boldsymbol{v}_{reg} + \boldsymbol{z} + \nabla p_0 \ \text{in } \Omega, \quad (32)$$

and one may choose the operator $\mathtt{p}_T : \boldsymbol{v} \mapsto \boldsymbol{z}$, acting from $\boldsymbol{X}_T(\Omega, \xi)$ to $\boldsymbol{Z}_T(\Omega)$, to be idempotent.

Proof. (outlined) Let us begin as for Theorem 9 to derive $\boldsymbol{w}_{reg} \in \boldsymbol{H}^1_{zmv}(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{w}_{reg} = \mathbf{curl}\, \boldsymbol{v}$ in Ω and $\|\boldsymbol{w}_{reg}\|_{1,\Omega} \lesssim \|\mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega} \leq \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. A priori, $\boldsymbol{w}_{reg} \cdot \boldsymbol{n}_{|\partial\Omega} \neq 0$. But, in the (curved) Lipschitz polyhedron one has [7],

$$\exists q_{reg} \in H^2(\Omega), \ \frac{\partial q_{reg}}{\partial n}_{|\partial\Omega} = \boldsymbol{w}_{reg} \cdot \boldsymbol{n}_{|\partial\Omega}\,; \ \|q_{reg}\|_{2,\Omega} \lesssim \|\boldsymbol{w}_{reg}\|_{1,\Omega}.$$

It follows that $\boldsymbol{v}_{reg} = \boldsymbol{w}_{reg} - \nabla q_{reg} \in \boldsymbol{X}_T(\Omega) \cap \boldsymbol{H}^1(\Omega) = \boldsymbol{H}_T(\Omega)$, $\mathbf{curl}\, \boldsymbol{v}_{reg} = \mathbf{curl}\, \boldsymbol{v}$ in Ω and $\|\boldsymbol{v}_{reg}\|_{1,\Omega} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. Because $\mathbf{curl}(\boldsymbol{v} - \boldsymbol{v}_{reg}) = 0$ in Ω, with $(\boldsymbol{v} - \boldsymbol{v}_{reg}) \in \boldsymbol{H}(\mathbf{curl};\Omega)$, there exists $\dot{p} \in P_{zmv}(\dot{\Omega})$ such that $\boldsymbol{v} = \boldsymbol{v}_{reg} + \widetilde{\nabla \dot{p}}$ in Ω and $|\dot{p}|_{1,\dot{\Omega}} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$ (cf. Theorem 2 on scalar potentials). One then writes $\widetilde{\nabla \dot{p}}$ as $\widetilde{\nabla \dot{p}} = \boldsymbol{z} + \nabla p_0$, with $\boldsymbol{z} \in \boldsymbol{Z}_T(\Omega)$ and $p_0 \in H^1_{zmv}(\Omega)$.
One then follows the proof of Theorem 9 to prove that the splitting operator $\boldsymbol{v} \mapsto (\boldsymbol{v}_{reg}, \boldsymbol{z}, p_0)$ is continuous from $\boldsymbol{X}_T(\Omega, \xi)$ to $\boldsymbol{H}_T(\Omega) \times \boldsymbol{Z}_T(\Omega) \times H^1_{zmv}(\Omega)$. Finally, one checks step-by-step that the construction of the splitting (32) for $\boldsymbol{v} \in \boldsymbol{Z}_T(\Omega)$ yields the decomposition $\boldsymbol{v} = 0 + \boldsymbol{v} + 0$ in Ω, so that $\mathtt{p}_T \boldsymbol{v} = \boldsymbol{v}$. Hence $\mathtt{p}_T^2 = \mathtt{p}_T$, ie. the operator $\mathtt{p}_T$ is idempotent. □

Remark 6. In the particular case where ξ is equal to the identity, (32) may be viewed as a second Birman-Solomyak equality.

4. Interpolation and quasi-interpolation

We assume that ε, μ fulfill the coefficient assumption on the same partition $\mathcal{P} := \{\Omega_j\}_{j=1,\cdots,J}$ of Ω and, for the ease of exposition, we also assume that the domain Ω and the subdomains $\{\Omega_j\}_{j=1,\cdots,J}$ are *Lipschitz polyhedra*. A triangulation $\mathcal{T}_h$ is *compatible with the partition* $\mathcal{P}$ if, for all $K \in \mathcal{T}_h$, there exists $j \in \{1, \cdots, J\}$ such that $K \subset \overline{\Omega_j}$. On the other hand, if the domain is such that $\mathbf{(Top)}_{I>0}$ is fulfilled, we have at hand some piecewise plane cuts $(\Sigma_i)_{i=1,\cdots,I}$, cf. §3.1. A triangulation $\mathcal{T}_h$ is *compatible with the cuts* $(\Sigma_i)_{i=1,\cdots,I}$ if, for all $1 \le i \le I$ and for all $K \in \mathcal{T}_h$, $int(K) \cap \Sigma_i = \emptyset$.

Definition 3. A triangulation is *compatible* if it is both compatible with the partition and with the cuts.

From now on, we assume that $(\mathcal{T}_h)_h$ is a shape regular family of compatible meshes. We are interested in the interpolation or quasi-interpolation of $\boldsymbol{v}$ such that $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega, \varepsilon)$ and $\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{X}_T(\Omega, \mu)$ ("electric case") or, vice versa, such that $\boldsymbol{v} \in \boldsymbol{X}_T(\Omega, \mu)$ and $\varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{X}_N(\Omega, \varepsilon)$ ("magnetic case").

4.1. Classical Nédélec interpolation

Introduce the local interpolation operator

$$\Pi_K \; : \; \boldsymbol{X}(K) \to \mathcal{R}_1(K),$$

where $\boldsymbol{X}(K)$ is some function space defined on K and given $\boldsymbol{v} \in \boldsymbol{X}(K)$, $\Pi_K \boldsymbol{v}$ is by definition the only element of $\mathcal{R}_1(K)$ with moments equal to $M_E(\boldsymbol{v})$, cf. (14). Then, one defines the global interpolation operator Π_h^+ with values in $\boldsymbol{V}_h^+$ for all elements $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega)$, resp. Π_h with values in $\boldsymbol{V}_h$ for all elements $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$, such that $\boldsymbol{v}_{|K} \in \boldsymbol{X}(K)$ for all $K \in \mathcal{T}_h$, by

$$(\Pi_h^+ \boldsymbol{v})_{|K} := \Pi_K \boldsymbol{v}, \text{ resp. } (\Pi_h \boldsymbol{v})_{|K} := \Pi_K \boldsymbol{v}, \; \forall K \in \mathcal{T}_h.$$

One uses $\boldsymbol{v}_h := \Pi_h \boldsymbol{v}$ in the "electric case" (resp. $\boldsymbol{v}_h := \Pi_h^+ \boldsymbol{v}$ in the "magnetic case"), *provided that* the action of the operator Π_h (resp. of the operator Π_h^+) on $\boldsymbol{v}$ is well-defined. This yields local, simplex-by-simplex estimates with respect to h_K, and then global estimates with respect to h.

Several choices of $\boldsymbol{X}(K)$ have been proposed over the years. We list some of them below. Let $\boldsymbol{H}^\delta(\mathbf{curl}; K) := \{\boldsymbol{v} \in \boldsymbol{H}^\delta(K) \; : \; \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^\delta(K)\}$ for $\delta > 0$.

<u>Case 1</u> $\boldsymbol{X}_2(K) := \boldsymbol{H}^2(K)$ [29, Theorem 2]:

$$\|\boldsymbol{v} - \Pi_K \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};K)} \lesssim h_K \, |\boldsymbol{v}|_{2,K}.$$

Case 2 $\boldsymbol{X}_{1,1}(K) := \boldsymbol{H}^1(\mathbf{curl}; K)$ [27, Lemma 2.3]:

$$\|\boldsymbol{v} - \Pi_K \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};K)} \lesssim h_K(|\boldsymbol{v}|_{1,K} + |\,\mathbf{curl}\,\boldsymbol{v}|_{1,K}).$$

Case 3 $\boldsymbol{X}_{\frac{1}{2}+,\frac{1}{2}+}(K) := \boldsymbol{H}^\delta(\mathbf{curl}; K)$ for some $\delta \in]1/2, 1[$ [2, §5] or [14, §3]:

$$\|\boldsymbol{v} - \Pi_K \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};K)} \lesssim (h_K)^\delta(|\boldsymbol{v}|_{\delta,K} + |\,\mathbf{curl}\,\boldsymbol{v}|_{\delta,K}).$$

The fact that the action of Π_K is well-defined for elements of $\boldsymbol{H}^\delta(\mathbf{curl}; K)$ when $\delta \in]1/2, 1[$ stems from the result [3, Lemma 4.7].

Proposition 3. *Let K be a simplex and $p > 2$. Then the operator Π_K is well-defined on $\boldsymbol{X}(K) := \{\boldsymbol{v} \in \boldsymbol{L}^p(K) \; : \; \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(K),\, \boldsymbol{v} \times \boldsymbol{n}_{|\partial K} \in \boldsymbol{L}^p(\partial K)\}$.*

For $\delta \in]1/2, 1[$, $\boldsymbol{v} \in \boldsymbol{H}^\delta(K)$ implies that $\boldsymbol{v}_{|\partial K} \in \boldsymbol{H}^{\delta-1/2}(\partial K)$. Then, due to classical embedding theorems (cf. [1, Theorem 7.57]), there exists $p := p(\delta) > 2$ such that $\boldsymbol{v} \in \boldsymbol{L}^p(K)$ and $\boldsymbol{v}_{|\partial K} \in \boldsymbol{L}^p(\partial K)$.

Case 4 $\boldsymbol{X}_{\frac{1}{2}+,0+}(K) := \{\boldsymbol{v} \in \boldsymbol{H}^\delta(K) \; : \; \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^{\delta'}(K)\}$, for some $\delta \in]1/2, 1]$, $\delta' \in]0, 1]$ [5, Lemma 5.1]:

$$\|\boldsymbol{v} - \Pi_K \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};K)} \lesssim (h_K)^{\min(\delta,\delta')}(|\boldsymbol{v}|_{\delta,K} + |\,\mathbf{curl}\,\boldsymbol{v}|_{\delta',K}).$$

Indeed, the authors of [5] first note that one may still use the Proposition 3 for elements of $\boldsymbol{X}_{\frac{1}{2}+,0+}(K)$ thanks to the same embedding theorems (ie. [1, Theorem 7.57]), so that the action of Π_K is actually well-defined for elements of $\boldsymbol{X}_{\frac{1}{2}+,0+}(K)$. Then, they conclude by applying the same theory as the one developed for Case 3 (cf. [2, 14]) to find the desired local estimate.

Global estimates can be derived easily, starting from the local ones of Cases 1-4. For instance, for $\delta \in]1/2, 1]$ and $\delta' \in]0, 1]$, given $\boldsymbol{v} \in \boldsymbol{PH}^\delta(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{PH}^{\delta'}(\Omega)$, one finds

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h^+} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\delta,\delta')}\left(|\boldsymbol{v}|_{\boldsymbol{PH}^\delta(\Omega)} + |\,\mathbf{curl}\,\boldsymbol{v}|_{\boldsymbol{PH}^{\delta'}(\Omega)}\right);$$
$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\delta,\delta')}\left(|\boldsymbol{v}|_{\boldsymbol{PH}^\delta(\Omega)} + |\,\mathbf{curl}\,\boldsymbol{v}|_{\boldsymbol{PH}^{\delta'}(\Omega)}\right) \text{ if } \boldsymbol{v} \times \boldsymbol{n}_{|\partial\Omega} = 0.$$

4.2. Combined interpolation in the general case

We consider now the case where the simplex-by-simplex regularity of the fields is *minimal but provable*, cf. §3, namely $\boldsymbol{v} \in \boldsymbol{H}^\delta(K)$, $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^{\delta'}(K)$, for orders $0 < \delta, \delta' < 1/2$ that can be arbitrarily small. Precisely, we use the regularity and the splitting results of Corollaries 5 and 6, in the sense that we study the local, simplex-by-simplex interpolation of

$$\boldsymbol{v} \in \{\boldsymbol{v} \in \boldsymbol{H}^1(K) + \nabla[H^{1+\delta}(K)] \; : \; \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^{\delta'}(K)\}\,. \tag{33}$$

Remark 7. This condition is the one that holds for *conforming* triangulations, in the sense that the jumps of the scalar potential, if any, can only occur at the boundary of its simplices.

The dissymmetry appearing in (33) (fields are split, but not their curl) is used as follows. Noting that $\mathbf{curl}(\nabla\cdot) = 0$, we actually have to study the approximability of the fields in two cases:

Case a $\boldsymbol{v} \in \boldsymbol{X}_{1,0+}(K) := \{\boldsymbol{v} \in \boldsymbol{H}^1(K) \ : \ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^{\delta'}(K)\}$, for $\delta' > 0$;

Case b $\boldsymbol{v} \in \nabla[H^{1+\delta}(K)]$, for $\delta > 0$.

On the one hand, one addresses the Case a for $\boldsymbol{v} \in \boldsymbol{X}_{1,0+}(K)$ using the operator Π_K and the last interpolation estimate of §4.1 (see Case 4).

On the other hand, handling the Case b is very classical: let $\boldsymbol{v} = \nabla\varphi$ for some $\varphi \in H^{1+\delta}(K)$. Globally we introduce Lagrange's $H^1(\Omega)$-conforming (P_1 family) finite element space

$$V_h^+ := \{\varphi_h \in H^1(\Omega) \ : \ \varphi_{h|K} \in P_1(K),\ \forall K \in \mathcal{T}_h\}, \quad V_h := V_h^+ \cap H^1_{\partial\Omega}(\Omega).$$

By construction, $\nabla V_h^+ \subset \boldsymbol{V}_h^+$ and $\nabla V_h \subset \boldsymbol{V}_h$.
One can use the (modified) Clément, or the Scott-Zhang, interpolation operators from $H^1(\Omega)$ to V_h^+, resp. from $H^1_{\partial\Omega}(\Omega)$ to V_h (cf. [21, 11]). Denoting by π_K the local operator, we know that

$$|\varphi - \pi_K\varphi|_{1,K} \lesssim h_K^\delta\, |\varphi|_{1+\delta,S_K}$$

where S_K is the neighborhood of the simplex K that is defined by $S_K := int(\cup_{K_i,\, K_i\cap K\neq\emptyset} K_i)$. In other words, the local estimate on K depends on the regularity on the whole neighborhood S_K. Let π_h, π_h^+ denote the associated global operators.

In the "electric case", the field defined on Ω writes $\boldsymbol{v} = \nabla\varphi$ where $\varphi \in H^{1+\delta}(\Omega)\cap H^1_{\partial\Omega}(\Omega)$ (see Corollary 5). Introducing $\boldsymbol{v}_h \in \boldsymbol{V}_h$ equal to $\boldsymbol{v}_h = \nabla(\pi_h\varphi)$ on Ω, one aggregates the local estimates to obtain $\|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^\delta\, |\boldsymbol{v}|_{\delta,\Omega}$.
In the "magnetic case", there is no global regularity result because of the constant, non-zero jumps across the cuts: $\boldsymbol{v} = \widetilde{\nabla\varphi}$ where $\varphi \in H^{1+\delta}(\dot\Omega) \cap P_{zmv}(\dot\Omega)$ (see Corollary 6). However, one may still aggregate the local estimates as follows. If the simplex K does not intersect any cut, then $\varphi \in H^{1+\delta}(S_K)$ and there is no difficulty. On the other hand, if there exists([6]) $i \in \{1, \cdots, I\}$ such that $\partial K \cap \Sigma_i \neq \emptyset$, then one chooses an H^1-conforming continuation of $\varphi_{|K}$ to S_K, called φ_K^{cont}, replacing φ by $\varphi \pm [\varphi]_{\Sigma_i}$ for the simplices of S_K that lie opposite to

[6] The cuts do not intersect one another so, for sufficiently small h, a simplex intersects at most with one cut.

K with respect to the cut. One has $\varphi_K^{cont} \in H^{1+\delta}(S_K)$ (see again Corollary 6), with $\widetilde{\nabla\varphi} = \nabla\varphi_K^{cont}$ on S_K. Applying π_K yields a local estimate on K, namely

$$|\varphi_K^{cont} - \pi_K\varphi_K^{cont}|_{1,K} \lesssim h_K^\delta\, |\varphi_K^{cont}|_{1+\delta,S_K} = h_K^\delta\, |\nabla\varphi_K^{cont}|_{\delta,S_K} = h_K^\delta\, |\widetilde{\nabla\varphi}|_{\delta,S_K}.$$

If one defines now $\boldsymbol{v}_h \in \boldsymbol{V}_h^+$ by $\boldsymbol{v}_h = \nabla(\pi_K\varphi)$ on K, resp. $\boldsymbol{v}_h = \nabla(\pi_K\varphi_K^{cont})$ on K, this yields $\|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^\delta\, |\boldsymbol{v}|_{\delta,\Omega}$.
As a consequence, one derives

$$\begin{aligned}
&\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^\delta\, |\boldsymbol{v}|_{\delta,\Omega} \text{ for } \boldsymbol{v} \in \nabla[H^{1+\delta}(\Omega) \cap H^1_{\partial\Omega}(\Omega)]\,,\\
&\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h^+} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^\delta\, |\boldsymbol{v}|_{\delta,\Omega} \text{ for } \boldsymbol{v} \in \widetilde{\nabla}[P^{3/2}_{zmv}(\dot{\Omega})] + \nabla[H^{1+\delta}_{zmv}(\Omega)].
\end{aligned}$$

A combination of the two Cases a-b then leads to the desired result.

Proposition 4. *Let Ω be a domain such that $(\mathbf{Top})_I$ is fulfilled, and assume that ε, μ fulfill the coefficient assumption on the same partition. Let $\tau := \min(\tau_{Dir}(\varepsilon), \tau_{Neu}(\mu)) \in]0, 1/2[$. Let $(\mathcal{T}_h)_h$ be a shape regular family of compatible meshes.*
In the "electric case", let $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega,\varepsilon)$ such that $\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\mu)$. There holds, for all $s \in [0,\tau[$,

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim_s h^s\, \{\|\boldsymbol{v}\|_{\boldsymbol{X}_N(\Omega,\varepsilon)} + \|\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{X}_T(\Omega,\mu)}\}. \tag{34}$$

In the "magnetic case", let $\boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\mu)$ such that $\varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{X}_N(\Omega,\varepsilon)$. There holds, for all $s \in [0,\tau[$,

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h^+} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim_s h^s\, \{\|\boldsymbol{v}\|_{\boldsymbol{X}_T(\Omega,\mu)} + \|\varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{X}_N(\Omega,\varepsilon)}\}. \tag{35}$$

PROOF. Let us outline the proof in the "electric case". Let $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega,\varepsilon)$ such that $\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\mu)$. With the help of the embedding result of Corollaries 5 and 6, $\boldsymbol{v}$ and $\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}$ may be split continuously into a regular part and a gradient, or the continuation of a gradient. Because ε, μ fulfill the coefficient assumption, it follows that for all simplices $K \in \mathcal{T}_h$, the local assumption (33) holds, for all $\delta \in]0, \tau_{Dir}(\varepsilon)[$ and for all $\delta' \in]0, \tau_{Neu}(\mu)[$. Then, one may apply the previous construction (Cases a-b) to find the result.
The proof in the "magnetic case" is similar. □

To conclude this subsection on the interpolation of electromagnetic fields with minimal regularity, note that we have built *combined interpolation operators* in the process, that rest on both the classical operators Π_h^+, Π_h, and also on $\nabla(\pi_h^+)$, $\nabla(\pi_h)$. Under the assumptions of Proposition 4, these combined operators $\Pi^+_{comb,h}$, $\Pi_{comb,h}$ are well-defined according respectively to the maps

$$\begin{aligned}
\Pi_{comb,h} &\;:\; \{\boldsymbol{v} \in \boldsymbol{X}_N(\Omega,\varepsilon) \;:\; \mu^{-1}\,\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\mu)\} \to \boldsymbol{V}_h\,;\\
\Pi^+_{comb,h} &\;:\; \{\boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\mu) \;:\; \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{X}_N(\Omega,\varepsilon)\} \to \boldsymbol{V}_h^+.
\end{aligned}$$

4.3. Quasi-interpolation and commuting diagrams

Let us recall some classical results. Denote by $\pi_h^{g,+}$, resp. π_h^g, the standard Lagrange interpolation operator for (sufficiently smooth) elements of $H^1(\Omega)$, resp. $H^1_{\partial\Omega}(\Omega)$, and $\Pi_h^{d,+}$, resp. Π_h^d, the Raviart-Thomas interpolation operator for (sufficiently smooth) elements of $\boldsymbol{H}(\mathrm{div};\Omega)$, resp. $\boldsymbol{H}_0(\mathrm{div};\Omega)$, cf. [22, 21, 11]. Defining the finite dimensional subspaces $(\boldsymbol{V}_h^{d,+})_h$ of $\boldsymbol{H}(\mathrm{div};\Omega)$, resp. $(\boldsymbol{V}_h^d)_h$ of $\boldsymbol{H}_0(\mathrm{div};\Omega)$, based on the so-called first order Raviart-Thomas finite elements [22, 21, 11], we have that $\mathbf{curl}[\boldsymbol{V}_h^+] \subset \boldsymbol{V}_h^{d,+}$, resp. $\mathbf{curl}[\boldsymbol{V}_h] \subset \boldsymbol{V}_h^d$. In addition, the classical commuting diagram properties write, provided the degrees of freedom exist:

$$\left\{ \begin{array}{ll} \Pi_h^{d,+}(\mathbf{curl}\,\boldsymbol{v}) = \mathbf{curl}(\Pi_h^+\boldsymbol{v}), & \text{and } \Pi_h^+(\nabla q) = \nabla(\pi_h^{g,+}q)\,; \\ \Pi_h^d(\mathbf{curl}\,\boldsymbol{v}) = \mathbf{curl}(\Pi_h\boldsymbol{v}), & \text{and } \Pi_h(\nabla q) = \nabla(\pi_h^g q). \end{array} \right. \tag{36}$$

On the other hand, when one has only $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega)$, the moments (14) may not exist on some simplices. To address this difficulty, the idea developed in [31, 32] is to apply, on the simplex K, a (local) smoothing operator s_K to $\boldsymbol{v}$, and then the interpolation operator Π_K itself: this results in the so-called *quasi-interpolation operator* $\Pi_K \circ s_K$. With this approach, it is possible to obtain estimates as soon as $\boldsymbol{v} \in \boldsymbol{H}^\delta(\Omega)$ for some $\delta > 0$. Briefly, if one introduces $\widetilde{S_K}$ a local neighborhood of K (different from S_K, regardless still local in the sense of connectivity and neighboring simplices in $\mathcal{T}_h$), then it holds [31, Theorem 5]:

$$\|\boldsymbol{v} - (\Pi_K \circ s_K)\boldsymbol{v}\|_{0,K} \lesssim (h_K)^\delta\, |\boldsymbol{v}|_{\delta,\widetilde{S_K}}.$$

The minor drawback of this approach is that, due to its generality, the commuting diagram properties now require the introduction of other local smoothing operators. For short, we consider only the case with boundary conditions next. In the general situation considered in [31, 32], one introduces a smoothing operator s for elements of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ (see above), together with smoothing operators s^g for elements of $H^1_{\partial\Omega}(\Omega)$, resp. s^d for elements of $\boldsymbol{H}_0(\mathrm{div};\Omega)$. Given $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \cap \boldsymbol{H}^\delta(\Omega)$ and $q \in H^1_{\partial\Omega}(\Omega) \cap H^\delta(\Omega)$ for some $\delta > 0$, it holds now [32, §3]:

$$(\Pi_h^d \circ s^d)(\mathbf{curl}\,\boldsymbol{v}) = \mathbf{curl}((\Pi_h \circ s)\boldsymbol{v}) \text{ and } (\Pi_h \circ s)(\nabla q) = \nabla((\pi_h^g \circ s^g)q).$$

Whereas, for the combined interpolation operator $\Pi_{comb,h}$ defined on $\{\boldsymbol{v} \in \boldsymbol{X}_N(\Omega,\varepsilon)\ :\ \mu^{-1}\,\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\mu)\}$, by using [4, §3], one checks easily that

$$\Pi_h^d(\mathbf{curl}\,\boldsymbol{v}) = \mathbf{curl}(\Pi_{comb,h}\boldsymbol{v}).$$

Remark 8. We refer to [12] for the abstract theory on quasi-interpolation operators within the framework of exterior calculus.

5. Error estimates

We apply to the electromagnetic fields the results of §4.1-4.2 with the classical and combined interpolation operators.

5.1. Minimal regularity assumptions

To obtain error estimates that are better than (18) or (21), one must impose extra regularity on the source terms $\boldsymbol{j} \in \boldsymbol{L}^2(\Omega)$, $\varrho \in H^{-1}(\Omega)$ even in the low-regularity case. Let $\tau = \min(\tau_{Dir}(\varepsilon), \tau_{Neu}(\mu)) \in]0, 1/2[$ as in Proposition 4. Recall that we introduced in §2.2 the scalar charge potential $\varphi_\varrho \in H_0^1(\Omega)$ such that $\operatorname{div} \varepsilon \nabla \varphi_\varrho = \varrho$ in $H^{-1}(\Omega)$, and that we split the current density as $\boldsymbol{j} := \boldsymbol{j}_0 + \boldsymbol{j}_\varrho$, with $\boldsymbol{j}_0 \in \boldsymbol{L}^2(\Omega)$, $\operatorname{div} \boldsymbol{j}_0 = 0$ and $\boldsymbol{j}_\varrho = \imath\omega\varepsilon\nabla\varphi_\varrho \in \boldsymbol{L}^2(\Omega)$. Below, we may impose that $\boldsymbol{j}_\varrho \in \boldsymbol{H}^\tau(\Omega)$ or that $\boldsymbol{j}_0 \in \boldsymbol{H}^\tau(\Omega)$([7]).

Theorem 12. *Let Ω be a domain such that* $\mathbf{(Top)}_I$ *is fulfilled, and assume that ε, μ fulfill the coefficient assumption on the same partition. Let $\tau := \min(\tau_{Dir}(\varepsilon), \tau_{Neu}(\mu)) \in]0, 1/2[$. Let $(\mathcal{T}_h)_h$ be a shape regular family of compatible meshes.*
In the "electric case", assume that $\boldsymbol{j}_\varrho \in \boldsymbol{H}^\tau(\Omega)$. There holds, for all $s \in [0, \tau[$,

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{e} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim_s h^s \{\|\boldsymbol{j}\|_{0,\Omega} + |\boldsymbol{j}_\varrho|_{s,\Omega}\}. \tag{37}$$

In the "magnetic case", assume that $\boldsymbol{j}_0 \in \boldsymbol{H}^\tau(\Omega)$. There holds, for all $s \in [0, \tau[$,

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h^+} \|\boldsymbol{h} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim_s h^s \{\|\boldsymbol{j}\|_{0,\Omega} + |\boldsymbol{j}_0|_{s,\Omega}\}. \tag{38}$$

Remark 9. Note that because $\tau < 1/2$, we only give a lower optimal bound on the rate of convergence. In the case where the source terms are more regular then, depending on the smoothness of the coefficients ε and μ, the convergence rate may be as fast as $\lesssim h$ for order one finite elements, cf. Case 2 of §4.1. We highlight these situations with some examples in §5.2.

Proof. Regarding the "electric case", we saw in §2.2 that one may split the electric field as $\boldsymbol{e} = \nabla\varphi_\varrho + \boldsymbol{e}_0$ with $\boldsymbol{e}_0 \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$, $\operatorname{div} \varepsilon\boldsymbol{e}_0 = 0$ and the same scalar potential φ_ϱ as in the splitting of $\boldsymbol{j}$. Because of the extra regularity on $\boldsymbol{j}_\varrho$, one has $\varphi_\varrho \in H^{1+\tau}(\Omega)$ and so using the (modified) Clément, or the Scott-Zhang, interpolation operators (cf. §4.2) one obtains

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\nabla\varphi_\varrho - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^s\, |\nabla\varphi_\varrho|_{s,\Omega}.$$

On the other hand, one has $\boldsymbol{e}_0 \in \boldsymbol{X}_N(\Omega, \varepsilon)$ and $\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}_0 \in \boldsymbol{X}_T(\Omega, \mu)$. So we derive from Proposition 4 ("electric case") that, for all $s \in [0, \tau[$,

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{e}_0 - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim_s h^s \{\|\boldsymbol{e}_0\|_{\boldsymbol{X}_N(\Omega,\varepsilon)} + \|\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}_0\|_{\boldsymbol{X}_T(\Omega,\mu)}\}.$$

[7] Other conditions can be considered when ε fulfills the coefficient assumption. On the one hand, one may impose the condition $\varrho \in H^{\tau-1}(\Omega)$, which implies that $\varphi_\varrho \in H^{1+t}(\Omega)$, or that $\varepsilon^{-1}\boldsymbol{j}_\varrho \in \boldsymbol{H}^t(\Omega)$, for all $t \in]0, \tau[$ (Theorem 8), hence $\boldsymbol{j}_\varrho \in \boldsymbol{H}^t(\Omega)$, for all $t \in]0, \tau[$. So, with a slight abuse, one can consider that $\varrho \in H^{\tau-1}(\Omega)$ implies the condition $\boldsymbol{j}_\varrho \in \boldsymbol{H}^\tau(\Omega)$. On the other hand, one may impose the global condition $\boldsymbol{j} \in \boldsymbol{H}^\tau(\Omega)$. In this case, $\operatorname{div} \boldsymbol{j} \in H^{\tau-1}(\Omega)$, hence $\varrho \in H^{\tau-1}(\Omega)$ thanks to the charge conservation equation (5). Then one has $\boldsymbol{j}_\varrho \in \boldsymbol{H}^t(\Omega)$, for all $t \in]0, \tau[$, and it follows that $\boldsymbol{j}_0 = \boldsymbol{j} - \boldsymbol{j}_\varrho \in \boldsymbol{H}^t(\Omega)$, also for $t \in]0, \tau[$.

Adding up the two estimates and noting finally that, due to the well-posedness of the variational formulation (12) in $\boldsymbol{e}_0$ (§2), one has

$$\|\boldsymbol{e}_0\|_{\boldsymbol{X}_N(\Omega,\varepsilon)} + \|\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}_0\|_{\boldsymbol{X}_T(\Omega,\mu)} \lesssim \|\boldsymbol{j}\|_{0,\Omega},$$

we find (37).
Regarding the "magnetic case", one has $\boldsymbol{h} \in \boldsymbol{X}_T(\Omega,\mu)$ and $\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h} - \boldsymbol{j}_0) \in \boldsymbol{X}_N(\Omega,\varepsilon)$. Thanks to the extra regularity on $\boldsymbol{j}_0$, one may follow the proof of the Proposition 4 ("magnetic case"). First, as an element of $\boldsymbol{X}_T(\Omega,\mu)$, the magnetic field is decomposed as usual (31). Second, its curl exhibits the same regularity as before. Indeed, introduce the auxiliary field $\boldsymbol{x} := \varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h} - \boldsymbol{j}_0)$. Then given $s \in [0,\tau[$, $\boldsymbol{x} \in \boldsymbol{X}_N(\Omega,\varepsilon) \subset \boldsymbol{H}^s(\Omega)$, so that $\mathbf{curl}\,\boldsymbol{h} - \boldsymbol{j}_0 = \varepsilon\boldsymbol{x} \in \boldsymbol{H}^s(\Omega)$ and $\mathbf{curl}\,\boldsymbol{h} \in \boldsymbol{H}^s(\Omega)$ because $\boldsymbol{j}_0 \in \boldsymbol{H}^s(\Omega)$, with the bound

$$\|\,\mathbf{curl}\,\boldsymbol{h}\|_{s,\Omega} \lesssim_s \|\boldsymbol{x}\|_{\boldsymbol{X}_N(\Omega,\varepsilon)} + \|\boldsymbol{j}_0\|_{s,\Omega}.$$

It follows that

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h^+} \|\boldsymbol{h} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim_s h^s\,\{\|\boldsymbol{h}\|_{\boldsymbol{X}_T(\Omega,\mu)} + \|\boldsymbol{x}\|_{\boldsymbol{X}_N(\Omega,\varepsilon)} + \|\boldsymbol{j}_0\|_{s,\Omega}\}.$$

Due to the well-posedness of the variational formulation (11) in $\boldsymbol{h}$ (§2), one gets

$$\|\boldsymbol{h}\|_{\boldsymbol{X}_T(\Omega,\mu)} + \|\boldsymbol{x}\|_{\boldsymbol{X}_N(\Omega,\varepsilon)} \lesssim \|\boldsymbol{j}\|_{0,\Omega}.$$

Finally, we recover (38) with the help of $\|\boldsymbol{j}_0\|_{0,\Omega} \lesssim \|\boldsymbol{j}\|_{0,\Omega}$. □

Corollary 7. *Let the assumptions of Theorem 12 on the geometry and the coefficients hold.*
Assume that $\boldsymbol{j}_\varrho \in \boldsymbol{H}^\tau(\Omega)$ and let $\boldsymbol{b}_{\boldsymbol{e}} := \mathbf{curl}\,\boldsymbol{e} \in \boldsymbol{H}_0(\mathrm{div};\Omega)$. Then $\boldsymbol{b}_{\boldsymbol{e}}$ is a divergence-free field and there holds, for all $s \in [0,\tau[$,

$$\inf_{\boldsymbol{w}_h \in \boldsymbol{V}_h^d} \|\boldsymbol{b}_{\boldsymbol{e}} - \boldsymbol{w}_h\|_{\boldsymbol{H}(\mathrm{div};\Omega)} \lesssim_s h^s\,\{\|\boldsymbol{j}\|_{0,\Omega} + |\boldsymbol{j}_\varrho|_{s,\Omega}\}. \tag{39}$$

Assume that $\boldsymbol{j}_0 \in \boldsymbol{H}^\tau(\Omega)$ and let $\boldsymbol{d}_{0,\boldsymbol{h}} := \mathbf{curl}\,\boldsymbol{h} - \boldsymbol{j}_0 \in \boldsymbol{H}(\mathrm{div};\Omega)$. Then $\boldsymbol{d}_{0,\boldsymbol{h}}$ is a divergence-free field and there holds, for all $s \in [0,\tau[$,

$$\inf_{\boldsymbol{w}_h \in \boldsymbol{V}_h^{d,+}} \|\boldsymbol{d}_{0,\boldsymbol{h}} - \boldsymbol{w}_h\|_{\boldsymbol{H}(\mathrm{div};\Omega)} \lesssim_s h^s\,\{\|\boldsymbol{j}\|_{0,\Omega} + |\boldsymbol{j}_0|_{s,\Omega}\}. \tag{40}$$

Proof. Thanks to (36), one notices that

$$\inf_{\boldsymbol{w}_h \in \boldsymbol{V}_h^d} \|\boldsymbol{b}_{\boldsymbol{e}} - \boldsymbol{w}_h\|_{\boldsymbol{H}(\mathrm{div};\Omega)} \le \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\,\mathbf{curl}(\boldsymbol{e} - \boldsymbol{v}_h)\|_{\boldsymbol{H}(\mathbf{curl};\Omega)},$$

hence (39) is a straightforward consequence of (37).
The second result is derived in a similar fashion if one recalls that, since $\boldsymbol{j}_0$ is a divergence-free field that belongs to $\boldsymbol{H}^\tau(\Omega)$, one has $\|\boldsymbol{j}_0 - \Pi_h^d \boldsymbol{j}_0\|_{\boldsymbol{H}(\mathrm{div};\Omega)} \lesssim h^s\,|\boldsymbol{j}_0|_{s,\Omega}$ (cf. [4, Lemma 3.3]). □

5.2. More on regularity in a polyhedral domain

We assume here that the domain Ω and the subdomains of its partition are *Lipschitz polyhedra* (for short, a subdomain is called subpolyhedron). In this setting, we would like to guarantee when the regularity of the fields is *a priori* sufficient to apply the classical estimates of §4.1 or if one has to apply instead those of §4.2 and §5.1. When the electric permittivity or the magnetic permeability are *scalar-valued* coefficients, we write ξ instead of $\xi\mathbb{I}_3$ for $\xi \in \{\varepsilon, \mu\}$ by abuse of notation.

To start with, one has an improved regularity result for elements of $\boldsymbol{X}_B(\Omega)$, for $B \in \{N, T\}$. It is based on the regularity results for scalar fields written next (cf. [17, Corollary 23.5]), and on Birman-Solomyak splittings of elements of $\boldsymbol{X}_B(\Omega)$ from Theorems 6 and 11 (with $\xi = 1$).

Theorem 13. *Let Ω be a Lipschitz polyhedron.*
If Ω is convex, then

$$\{z \in H^1_0(\Omega) \ : \ \nabla z \in \boldsymbol{X}_N(\Omega)\} \subset H^2(\Omega)\,;$$
$$\{z \in H^1(\Omega) \ : \ \nabla z \in \boldsymbol{X}_T(\Omega)\} \subset H^2(\Omega).$$

If Ω is non-convex, then there exist $\delta_{Dir}, \delta_{Neu} \in]1/2, 1[$ that can be explicitly characterized such that,

$$\{z \in H^1_0(\Omega) \ : \ \nabla z \in \boldsymbol{X}_N(\Omega)\} \subset H^{1+s}(\Omega), \ \forall s \in]1/2, \delta_{Dir}[\,;$$
$$\{z \in H^1(\Omega) \ : \ \nabla z \in \boldsymbol{X}_T(\Omega)\} \subset H^{1+s}(\Omega), \ \forall s \in]1/2, \delta_{Neu}[.$$

In the convex case, we use $\delta_{Dir} = \delta_{Neu} = 1$.

Remark 10. The characterizations of the regularity exponents δ_{Dir}, δ_{Neu} allow one to compute them numerically in principle.

Using a partition of unity as in the proof of Corollary 4 with functions $(\chi_i)_{i=1,\cdots,I}$ that fulfill in addition the homogeneous boundary condition $\partial_n\chi_i \mid_{\partial\Omega} = 0$, the embeddings $\boldsymbol{Z}_T(\Omega) \subset \boldsymbol{H}^1(\Omega)$ (Ω convex) and $\boldsymbol{Z}_T(\Omega) \subset \boldsymbol{H}^s(\Omega)$, $\forall s \in]1/2, \delta_{Neu}[$ (Ω non-convex) follow as direct consequences of Theorem 13. Likewise, using a partition of unity $(\chi_k)_{1\le k\le K}$ such that $\chi_k = 1$ in a neighborhood of Γ_k for all $1 \le k \le K$, etc., $\boldsymbol{Z}_N(\Omega) \subset \boldsymbol{H}^1(\Omega)$ (Ω convex) and $\boldsymbol{Z}_N(\Omega) \subset \boldsymbol{H}^s(\Omega)$, $\forall s \in]1/2, \delta_{Dir}[$ (Ω non-convex) are again direct consequences of Theorem 13. Finally, with the help of the Birman-Solomyak splittings, one gets the by-product below. Note that it includes elements of $\boldsymbol{Z}_N(\Omega)$ and $\boldsymbol{Z}_T(\Omega)$ as particular cases.

Corollary 8. *Let Ω be a Lipschitz polyhedron.*
If Ω is convex, then for $B \in \{N, T\}$, $\boldsymbol{X}_B(\Omega) \subset \boldsymbol{H}^1(\Omega)$.
If Ω is non-convex, then there exist $\delta_{Dir}, \delta_{Neu} \in]1/2, 1[$ such that,

$$\boldsymbol{X}_N(\Omega) \subset \boldsymbol{H}^s(\Omega), \ \forall s \in]1/2, \delta_{Dir}[\,;$$
$$\boldsymbol{X}_T(\Omega) \subset \boldsymbol{H}^s(\Omega), \ \forall s \in]1/2, \delta_{Neu}[.$$

Next, one remarks that if the scalar-valued coefficient ξ is *smooth*, that is $\xi \in W^{1,\infty}(\Omega)$ and $\xi^{-1} \in L^\infty(\Omega)$, then given $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ such that $\operatorname{div} \xi \boldsymbol{v} \in L^2(\Omega)$, one may write $\operatorname{div} \xi \boldsymbol{v} = \nabla\xi \cdot \boldsymbol{v} + \xi \operatorname{div} \boldsymbol{v}$ in Ω, so $\operatorname{div} \boldsymbol{v} = \xi^{-1}(\operatorname{div} \xi\boldsymbol{v} - \nabla\xi \cdot \boldsymbol{v})$ belongs to $L^2(\Omega)$. Hence, one has $\boldsymbol{X}_B(\Omega, \xi) \subset \boldsymbol{X}_B(\Omega)$, for $B \in \{N, T\}$, and one can use the Corollary 8 to derive the regularity results that are needed for obtaining the convergence estimates: Case 2 (convex domain) or Case 3 (non-convex domain) of §4.1 now apply.

Theorem 14. *Let Ω be a Lipschitz polyhedron such that* **(Top)**$_I$ *is fulfilled, and assume that the* scalar *coefficients ε, μ fulfill the coefficient assumption on the trivial partition $\mathcal{P} = \{\Omega\}$. Let $\delta := \min(\delta_{Dir}, \delta_{Neu}) \in]1/2, 1]$. Let $(\mathcal{T}_h)_h$ be a shape regular family of compatible meshes.*
In the "electric case", assume that $\boldsymbol{j}_\varrho \in \boldsymbol{H}^\delta(\Omega)$. There holds, for $s = \delta$ (convex domain), or for all $s \in [0, \delta[$ (non-convex domain),

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{e} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim_s h^s \{\|\boldsymbol{j}\|_{0,\Omega} + |\boldsymbol{j}_\varrho|_{s,\Omega}\}. \tag{41}$$

In the "magnetic case", assume that $\boldsymbol{j}_0 \in \boldsymbol{H}^\delta(\Omega)$. There holds, for $s = \delta$ (convex domain), or for all $s \in [0, \delta[$ (non-convex domain),

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h^+} \|\boldsymbol{h} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim_s h^s \{\|\boldsymbol{j}\|_{0,\Omega} + |\boldsymbol{j}_0|_{s,\Omega}\}. \tag{42}$$

Remark 11. According to Theorem 13, a sufficient condition for $\varepsilon^{-1}\boldsymbol{j}_\varrho \in \boldsymbol{H}^\delta(\Omega)$ to hold is that $\varrho \in L^2(\Omega)$. Indeed, $\boldsymbol{j}_\varrho = \imath\omega\varepsilon\nabla\varphi_\varrho$, with $\varphi_\varrho \in H^1_0(\Omega)$ and $\operatorname{div} \varepsilon\nabla\varphi_\varrho = \varrho$. In particular, $\varphi_\varrho \in \{z \in H^1(\Omega) \ : \ \nabla z \in \boldsymbol{X}_N(\Omega)\}$. This observation is similar in spirit to the one made in footnote [7], page 24.

The next case we consider is when the scalar-valued coefficient ξ is *piecewise constant*. The regularity results for elements of $\boldsymbol{X}_B(\Omega, \xi)$ for $B \in \{N, T\}$ rely on the regular/singular splitting of scalar fields below (cf. [16, 30]).

Theorem 15. *Let Ω be a Lipschitz polyhedron, and assume that the scalar, piecewise constant coefficient ξ fulfills the coefficient assumption.*
For all $f \in L^2(\Omega)$ and $g_\mathcal{F} \in PH^{1/2}(\mathcal{F}_{int})$, let p be the solution to

$$\text{BC=Dir} \quad \left\{ \begin{array}{l} \text{Find } p \in H^1_0(\Omega) \text{ such that} \\ (\xi\nabla p|\nabla\psi)_{0,\Omega} = (f|\psi)_{0,\Omega} + (g_\mathcal{F}|\psi)_{0,\mathcal{F}_{int}},\ \forall\psi \in H^1_0(\Omega)\,; \end{array} \right.$$

$$\text{BC=Neu} \quad \left\{ \begin{array}{l} \text{Find } p \in H^1_{zmv}(\Omega) \text{ such that} \\ (\xi\nabla p|\nabla\psi)_{0,\Omega} = (f|\psi)_{0,\Omega} + (g_\mathcal{F}|\psi)_{0,\mathcal{F}_{int}},\ \forall\psi \in H^1_{zmv}(\Omega). \end{array} \right.$$

Then for $BC \in \{Dir, Neu\}$, p admits a continuous splitting $p = p_{reg} + p_{sing}$, with a regular part $p_{reg} \in PH^2(\Omega)$ and a singular part $p_{sing} \in P_{BC}(\Omega)$; there exists $\delta_{BC}(\xi) \in]0, 1]$ that can be explicitly characterized such that $P_{BC}(\Omega) \subset PH^{1+s}(\Omega)$ for all $s \in]0, \delta_{BC}(\xi)[$, and $P_{BC}(\Omega) \subset PH^2(\Omega)$ if $\delta_{BC}(\xi) = 1$.

Remark 12. In principle, the characterization of the regularity exponents $\delta_{Dir}(\xi)$, $\delta_{Neu}(\xi)$ allows one to compute them numerically.

For $B \in \{N, T\}$, one can then derive embedding results for $\boldsymbol{X}_B(\Omega, \xi)$. To that aim, one uses Theorem 15, which requires to reformulate the right-hand-sides defining([8]) p_0 and q_0 resp. in Theorems 7 and 11, *without* the part of the decomposition that belongs to the null spaces $\boldsymbol{Z}_B(\Omega)$. This is achieved with the help of the idempotent operators $\mathtt{p}_B$ introduced in those Theorems.
For example, in the "electric case", let $\boldsymbol{v} \in \ker(\mathtt{p}_N) \subset \boldsymbol{X}_N(\Omega)$. The splitting (24) writes $\boldsymbol{v} = \boldsymbol{v}_{reg} + 0 + \nabla p_0$ in Ω, with p_0 governed by (25) *without* the first term in the right-hand side. Hence Theorem 15 may be applied to p_0 and one concludes that

$$\ker(\mathtt{p}_N) \subset \boldsymbol{PH}^s(\Omega),\ \forall s \in]0, \delta_{Dir}(\xi)[\,;\ \ker(\mathtt{p}_N) \subset \boldsymbol{PH}^1(\Omega) \text{ if } \delta_{Dir}(\xi) = 1.$$

Next, let $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega)$: it is split continuously as $\boldsymbol{v} = \mathtt{p}_N\boldsymbol{v} + (\boldsymbol{v} - \mathtt{p}_N\boldsymbol{v})$. The first part, $\mathtt{p}_N\boldsymbol{v}$, belongs to $\boldsymbol{Z}_N(\Omega)$ and as such its regularity is governed by Corollary 8, with regularity exponent δ_{Dir}. On the other hand, the operator $\mathtt{p}_N$ is idempotent so $(\boldsymbol{v} - \mathtt{p}_N\boldsymbol{v}) \in \ker(\mathtt{p}_N)$ is governed by the regularity exponent $\delta_{Dir}(\xi)$.
Similarly for the "magnetic case".

Corollary 9. *Let the assumptions of Theorem 15 hold. Then*

$$\begin{array}{l} \boldsymbol{X}_N(\Omega, \xi) \subset \boldsymbol{PH}^s(\Omega),\ \forall s \in]0, \min(\delta_{Dir}(\xi), \delta_{Dir})[\,; \\ \boldsymbol{X}_T(\Omega, \xi) \subset \boldsymbol{PH}^s(\Omega),\ \forall s \in]0, \min(\delta_{Neu}(\xi), \delta_{Neu})[. \end{array}$$

Moreover if $\min(\delta_{Dir}(\xi), \delta_{Dir})$ *or* $\min(\delta_{Neu}(\xi), \delta_{Neu})$ *is equal to* 1*, the corresponding inclusion holds for* $s = 1$*.*

Remark 13. It can happen that $\delta_{Dir}(\xi) > \delta_{Dir}$, or $\delta_{Neu}(\xi) > \delta_{Neu}$.

Let us now highlight four practical situations, denoted by [c1], [c2], [c3], [c4] below. Two subpolyhedra are adjacent if their boundaries intersect.

Theorem 16. *Let* Ω *be a Lipschitz polyhedron, and assume that the* scalar *coefficient* ξ *is piecewise constant on the partition* $\mathcal{P} := \{\Omega_j\}_{j=1,\cdots,J}$*. In addition, assume that:*

either: [c1] Ω *is convex and the maximal number of adjacent subpolyhedra is equal to two ;*

or: [c2] *there exists some* j *such that* $\partial\Omega \subset \partial\Omega_j$ *and the maximal number of adjacent subpolyhedra is equal to two.*

Then $\delta_{Dir}(\xi), \delta_{Neu}(\xi) \in]1/2, 1]$.

[8]Note that $g_{\mathcal{F}} \in PH^{1/2}(\mathcal{F}_{int})$ in the statement of Theorem 15. So, in the "magnetic case", one can use the splitting with boundary condition of Theorem 11, but not the one of Theorem 9.

Remark 14. The situation where $\delta_{Dir}(\xi)$ or $\delta_{Neu}(\xi)$ is equal to 1 can occur only in a convex domain with special geometry, such as a partition with flat interfaces intersecting the boundary at right angles. The second situation [c2] covers the case of isolated inclusions of media in an otherwise homogeneous material.

Regarding the convergence rates for the situations [c1] and [c2], one gets results similar to those of Theorem 14 with $\delta := \min(\delta_{Dir}(\xi), \delta_{Neu}(\xi), \delta_{Dir}, \delta_{Neu}) \in]1/2, 1]$. In particular, one may again apply the classical results of §4.1 when the assumptions of Theorem 16 hold.

On the other hand, as soon as there are three adjacent subpolyhedra or more, the regularity exponents $\delta_{Dir}(\xi)$ or $\delta_{Neu}(\xi)$ can become arbitrarily close to 0.

[c3] Let us give an illustration: let Ω be the unit cube, $\mathcal{P} := \{\Omega_j\}_{j=1,2,3,4}$ where

$$\Omega_1 :=]0, \tfrac{1}{2}[\times]0, \tfrac{1}{2}[\times]0, 1[\,; \quad \Omega_2 :=]\tfrac{1}{2}, 1[\times]0, \tfrac{1}{2}[\times]0, 1[\,;$$
$$\Omega_3 :=]\tfrac{1}{2}, 1[\times]\tfrac{1}{2}, 1[\times]0, 1[\,; \quad \Omega_4 :=]0, \tfrac{1}{2}[\times]\tfrac{1}{2}, 1[\times]0, 1[.$$

Define ξ such that $\xi_1 = \xi_3 = 1$, and $\xi_2 = \xi_4 = \underline{\xi}$, for some parameter $\underline{\xi} \geq 1$. We call this configuration the *checkerboard case*. In this case, one may compute directly the regularity exponents by studying singular solutions of the problem "Find $z \in H^1(\Omega)$ such that $\operatorname{div} \xi \nabla z = 0$ in some neighborhood of the line $x_1 = x_2 = \frac{1}{2}$ plus homogeneous boundary condition". One checks first that $\partial_3 z \in PH^1(\Omega)$ and then that $\delta_{Dir}(\xi)$ and $\delta_{Neu}(\xi)$ are equal and, in addition, that their common value δ is related to the parameter $\underline{\xi}$ by the relation (see for instance [16, §8]):

$$\underline{\xi}^{-1/2} = \tan\left(\frac{\delta\pi}{4}\right).$$

Hence δ is a decreasing function of $\underline{\xi}$, with $\delta = 1$ if and only if $\underline{\xi} = 1$, $\lim_{\underline{\xi}\to+\infty} \delta = 0$, and in particular $\delta < 1/2$ as soon as $\underline{\xi} > 5.8284$. So in the situation [c3] when $\underline{\xi} > 5.8284$, one must use the results of §4.2 and §5.1 to derive convergence rates, cf. Theorem 12.

[c4] The last case we consider is when the tensor- or scalar-valued coefficient ξ is *piecewise smooth*, ie. when it fulfills the coefficient assumption on a non-trivial partition. One can apply the *frozen coefficients* technique, developed in [23, §5.2]. Briefly, it is proven there that one can derive regular/singular splittings, where the singular part is governed by equations with constant coefficients. The results are derived rigorously in 2D configurations ; they can be extended for instance to the checkerboard case. What is more, the constant coefficients are simply the limit of the value of the coefficients at the corners of the interface. In principle, one may still compute the regularity exponents in these configurations. We refer to [18, §5] for similar results.

6. Conclusion

We have presented some results on the numerical approximation of low-regularity electromagnetic fields by edge finite elements. In particular, we addressed the case of general geometrical settings, including topologically non-trivial domains or domains with a non-connected boundary, and tensor-valued, piecewise smooth electric permittivity and magnetic permeability. In all cases, a convergence rate in h^δ is recovered, where h is the meshsize, for some exponent $\delta \in]0,1]$. It relies either on classical estimates, cf. [2, 14, 28, 5] when $\delta > 1/2$, or on the combined interpolation operator when $\delta < 1/2$. The optimality of the value of δ has first been discussed with respect to abstract shift theorems. In some simple configurations, typically for scalar-valued permittivity and permeability, the value of δ has been further characterized.

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