# Compactness of linearized kinetic operators

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**Abstract** This article reviews various results on the compactness of the linearized Boltzmann operator and of its generalization to mixtures of non-reactive monatomic gases.

## 1 Introduction

Investigating the compactness properties of linearized operators arising in kinetic theory is a crucial step to establish fluid dynamical approximations to solutions of the corresponding kinetic equations.

The starting point of this research line was given by Hilbert, in the same paper in which he introduced what we now call the Hilbert expansion [23]. Since then, a significant number of results allowed to clarify the main aspects of these compactness properties, both for the archetypal of all kinetic models, the classical Boltzmann equation, and, more recently, for the variant of the system of Boltzmann equations describing the behaviour of non-reactive mixtures constituted with monatomic gases.

In the mixture case, the interaction between the different species induces some peculiarities in the structure of the linearized kinetic operators, which can reflect

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some specific physical phenomena (such as uphill diffusion in the purely diffusive case, see [26, 29, 16, 3, 5, 24]). Consequently, it is not surprising that compactness properties in the mixture case cannot be deduced through a straightforward adaptation of the standard methods of proof from the mono-species case. In [4], the authors indeed observed that, when there are different involved molecular masses, the standard approach, which is mostly due to Grad [20], degenerates. A new method of proof is needed to recover the linearized operator compactness. Let us point out that this new argument does not hold either when the molecular masses become equal. Hence, both aforementioned strategies must be seen as complementary when dealing with mixtures.

The study of compactness properties for mixtures is only at its beginning, and there are still many unexplored situations. We quote for example the study of linearized kinetic operators for mixtures of polyatomic gases: the non-reactive case, for instance as defined in [7, 14], and the chemical reacting one as in [15]. Those models require a supplementary internal variables, such as the internal energy of the molecules. The presence of such a variable induces significant difficulties for the analysis of the compactness properties.

The article is divided into two parts. The next section is dedicated to the study of the compactness properties of the linearized standard Boltzmann operator for a monatomic perfect gas, including discussions related to Grad's angular cut-off assumption. Then, in Section 3, we consider the extension to a non-reactive mixture of ideal monatomic gases.

## 2 The classical Boltzmann equation case

This section deals with the compactness properties of the classical linearized Boltzmann kernel. The subject has a long history, since the first study comes back to Hilbert [23], who applied his new theory of integral operators to this specific problem.

### *2.1 Boltzmann's equation*

As a starting point, we briefly introduce the classical Boltzmann equation. This equation is the first and most studied kinetic model since the nineteenth century, after the pioneering works of Boltzmann himself [1, 2] and Maxwell [26]. Since this equation has been widely studied, we only introduce its basic aspects and refer to the many reference texts on the the Boltzmann equation, for instance [11, 12, 30], where the arguments below are more accurately discussed.

The Boltzmann equation describes the time evolution of a system composed by a large number of particles, described by a distribution function $f$ defined on the phase space of the system. The particles are supposed to be identical and monatomic. They

follow the classical mechanics laws, with only translational degrees of freedom. If the particles are contained in a domain $\Omega_x \subseteq \mathbb{R}^3$, the quantity $f(t,x,v)$ can be defined for any $(t,x,v) \in \mathbb{R}^+ \times \Omega_x \times \mathbb{R}^3$, and, for all $t$, the integral

$$\iint_{X\times V} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$$

can be interpreted as the number of particles in the space volume $X \subseteq \Omega_x$ with velocity in $V \subseteq \mathbb{R}^3$. A reasonable assumption on $f$ is

$$f(t,\cdot,\cdot) \in L^1_{\mathrm{loc}}(\Omega_x; L^1(\mathbb{R}^3)), \qquad \forall\, t \in \mathbb{R}^+,$$

which ensures that there is always a finite number of particles in a bounded domain of the space. For the sake of simplicity, we also assume that the system is isolated, so that there is no external effect on the particles.

If, moreover, the particles do not interact with each other, the time evolution of $f$ is driven by the so-called free transport equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0. \tag{1}$$

When the interaction between particles cannot be neglected, Equation (1) does not hold any more, and one has to add in (1) a right-hand-side term. When only binary and local collisions are allowed, the effect of the interactions is described by a quadratic (with respect to $f$) collision operator $Q(f,f)$.

If the pairwise interactions between particles of the system are elastic, then momentum and kinetic energy are conserved during the interaction process. Hence, if we denote by $v'$ and $v'_*$ the pre-collisional velocities, and by $v$ and $v_*$ the post-collisional ones, the following microscopic conservation laws hold:

$$v + v_* = v' + v'_*, \qquad {v}^2 + {v_*}^2 = {v'}^2 + {v'_*}^2.$$

These conservation laws allow to fix four of the six degrees of freedom of the interaction. The remaining degrees of freedom of the binary interaction can be described in several ways. For our purposes, we only consider two possible descriptions. The first one, the so-called $\sigma$-representation, is the representation of the pre-post velocities in the centre of mass of two particles: we introduce $\sigma \in S^2$, and write

$$v' = \frac{1}{2}(v + v_* + |v - v_*|\sigma), \qquad v'_* = \frac{1}{2}(v + v_* - |v - v_*|\sigma). \tag{2}$$

The second one is the $\omega$-representation, defined as

$$v' = v - (\omega \cdot (v - v_*))\,\omega, \qquad v'_* = v_* + (\omega \cdot (v - v_*))\,\omega, \tag{3}$$

where $\omega \in S^2$.

Hence, the time evolution of $f$ is governed by the Boltzmann equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \tag{4}$$

where the collision operator $Q$ can be defined either in the $\sigma$-representation (2) by

$$Q(f,g) = \int_{\mathbb{R}^3} \int_{S^2} B(\sigma, v, v_*) \left[ f(t,x,v') g(t,x,v'_*) - f(t,x,v) g(t,x,v_*) \right] \, \mathrm{d}\sigma \, \mathrm{d}v_* \tag{5}$$

or in the $\omega$-representation

$$Q(f,g) = \int_{\mathbb{R}^3} \int_{S^2} B(\omega, v, v_*) \left[ f(t,x,v') g(t,x,v'_*) - f(t,x,v) g(t,x,v_*) \right] \, \mathrm{d}\omega \, \mathrm{d}v_*. \tag{6}$$

Either way, in the study of the Boltzmann equation, particular care has to be given to the properties of the collision cross-section $B : S^2 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^+$ of the system, which describes the details of the interactions between the particles.

In general, by symmetry arguments and thanks to the Galilean invariance, it is possible to prove that $B$, which is nonnegative, in fact depends on $|v - v_*|$ and $\cos\theta := \sigma \cdot (v - v_*)/|v - v_*|$, where $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^3$. For the sake of simplicity from now, on, we write $B$ as a function of $\sigma$ and $V := v - v_*$. If we assume the collisions to be microreversible, we can state that

$$B(\sigma, v - v_*) = B(\sigma, v' - v'_*), \quad \forall \sigma \in S^2, \ \forall v, v_* \in \mathbb{R}^3. \tag{7}$$

The choice of collision kernel $B$ has a deep influence on the properties of the Boltzmann equation. By limiting ourselves to the classical elastic case, it is possible to prove that

$$B(\sigma, v - v_*) = K |v - v_*|, \qquad K > 0, \tag{8}$$

for a gas of three-dimensional hard spheres. In the case of inverse $s$-power forces, $B$ can be factorized as

$$B(\sigma, v - v_*) = \Phi(|v - v_*|) b(\cos\theta), \tag{9}$$

where, in three space dimensions,

$$\Phi(|V|) = |V|^\gamma, \quad \gamma = (s-5)/(s-1),$$

and $b$ is a locally smooth function with a non integrable singularity when $\theta$ tends to 0, i.e.

$$b(\cos\theta) \sin\theta \sim K \theta^{-(1+\nu)}, \quad \nu = 2/(s-1).$$

Factorized collision kernels like (9) are very popular in the study of the classical Boltzmann equation. By convention, $\Phi$ is named the *kinetic* collision kernel, and $b$ the *angular* one. The class of kinetic collision kernels of the form $\Phi(|V|) = |V|^\gamma$ is usually split in three sub-classes, depending on the value of $\gamma$. When $\gamma > 0$, the kernel is said to derive from hard potentials, when $\gamma < 0$, the kernel is said to derive from soft potentials and, when $\gamma = 0$, the kinetic collision kernel does not play any role. In this latter situation, the corresponding Boltzmann equation describes the

behaviour a gas of Maxwell molecules. Even if it is only a mathematical model, it is very popular in the literature, since it considerably simplifies the study of the Boltzmann equation. Let us point out that Maxwell and Boltzmann themselves used this model, because it allows to carry out many explicit computations.

In order to handle more easily the angular cross section, Grad [19] (see also [11]) suggested a working hypothesis, nowadays known as the *Grad angular cut-off assumption*. It consists in postulating that the collision kernel is integrable with respect to the angular variable. Note that the great majority of mathematical works about the Boltzmann equation is based on this Grad cut-off assumption, which could be considered, from the physical point of view, as a short-range assumption [30].

To conclude these considerations about the cross-section, let us emphasize that, whenever we use $B$ in this article, the precise dependence of $B$ on the independent variables will be clear in the local context: general dependence with respect to $\omega$ or $\sigma$ along with $V$, or the more precise form involving $|V|$ and $\cos\theta$.

Now define the normalized centred Maxwellian

$$M(v) = \left(\frac{1}{2\pi}\right)^{3/2} e^{-v^2/2}, \quad v \in \mathbb{R}^3,$$

and a perturbation $g$ to $M$ as

$$f = M + M^{1/2} g.$$

The linearized collision operator $\mathscr{L}$ is studied for instance in [17] and can be defined by

$$\mathscr{L} g = \frac{1}{\sqrt{M}} \left[ Q(\sqrt{M} g, M) + Q(M, \sqrt{M} g) \right]. \tag{10}$$

More precisely, $\mathscr{L}$ can be written as

$$\mathscr{L} = \mathscr{K} - \nu \,\mathrm{Id},$$

where $\mathscr{K}$ is given by

$$\mathscr{K} g(v) = \left(\frac{1}{2\pi}\right)^{3/2} \iint_{\mathbb{R}^3 \times S^2} B(\sigma, v - v_*)\, e^{-v^2/4}\, e^{-v_*^2/2} \left[ e^{v_*'^2/4} g(v_*') - e^{v_*^2/4} g(v_*) + e^{v'^2/4} g(v') \right] \mathrm{d}\sigma \, \mathrm{d}v_*, \tag{11}$$

and the collision frequency $\nu$ by

$$\nu(v) = \left(\frac{1}{2\pi}\right)^{3/2} \iint_{\mathbb{R}^3 \times S^2} B(\sigma, v - v_*) e^{-v_*^2/2} \, \mathrm{d}\sigma \, \mathrm{d}v_*. \tag{12}$$

## 2.2 Earlier compactness results

The first result of compactness for $\mathscr{K}$ is given by Hilbert in [23] for the three-dimensional hard sphere case: Hilbert uses the now so-called Hilbert's theory of integral operators to write $\mathscr{K}$ as a kernel operator, and then obtains a compactness property.

Then, in [22], and in the same cross-section setting, Hecke presents a variant of the previous result: he proves that the linearized Boltzmann kernel is roughly of Hilbert-Schmidt type.

Carleman [10] provides an improvement to the latter result and significantly simplifies the proof. We must emphasize that those various compactness results are established in different $L^2$ settings, which may involve $v$ in the weights.

## 2.3 Grad's procedure

In [20], Grad presents an extension of Hilbert's result in both Maxwell and hard potential cases, by supposing $\gamma \in [0,1]$, and by using his angular cut-off assumption [19]. He requires that the form of the cross-section $B$ is either (8) or (9), with a uniformly bounded angular cross-section $b$. More precisely, Grad imposes the following general condition on $B$:

$$B(\omega, V) \leq a\,|\sin\theta|\,|\cos\theta| \left(|V| + \frac{1}{|V|^{1-\delta}}\right), \quad \forall\, \omega \in S^2,\ \forall V \in \mathbb{R}^3, \qquad (13)$$

where $a > 0$, $0 < \delta < 1$. This allows to adapt Hecke's argument and prove that the kernel of $\mathscr{K}$ is Hilbert-Schmidt in $L^2(M\mathrm{d}v)$.

Let us provide more details about Grad's procedure. In order to prove its compactness in $L^2$, the operator $\mathscr{K}$ is written as the sum of two operators, $\mathscr{K}_1$ and $\mathscr{K}_2$, where, for any $v \in \mathbb{R}^3$,

$$\begin{aligned}
\mathscr{K}_1 g(v) &= -\left(\frac{1}{2\pi}\right)^{3/2} \iint_{\mathbb{R}^3 \times S^2} B(\omega, v - v_*)\, e^{-\frac{1}{4}v^2}\, e^{-\frac{1}{4}{v_*}^2}\, g(v_*)\, \mathrm{d}\omega\, \mathrm{d}v_*, \\
\mathscr{K}_2 g(v) &= \left(\frac{1}{2\pi}\right)^{3/2} \iint_{\mathbb{R}^3 \times S^2} B(\omega, v - v_*)\, e^{-\frac{1}{4}v^2}\, e^{-\frac{1}{2}{v_*}^2} \\
&\qquad \times \left[e^{\frac{1}{4}{v'_*}^2}\, g(v'_*) + e^{\frac{1}{4}v'^2}\, g(v')\right] \mathrm{d}\omega\, \mathrm{d}v_*.
\end{aligned}$$

Both operators $\mathscr{K}_1$ and $\mathscr{K}_2$ have a kernel form. This is straightforward for $\mathscr{K}_1$: indeed, if we set

$$k_1(v, v_*) = -\left(\frac{1}{2\pi}\right)^{3/2} \int_{S^2} B(\omega, v - v_*)\, e^{-\frac{1}{4}v^2}\, e^{-\frac{1}{4}v_*^2}\, \mathrm{d}\omega \qquad \forall\, v, v_* \in \mathbb{R}^3,$$

we clearly have

$$\mathscr{K}_1 g(v) = \int_{\mathbb{R}^3} k_1(v, v_*)\, g(v_*)\, \mathrm{d}v_*, \qquad \forall v \in \mathbb{R}^3.$$

The analogous result for $\mathscr{K}_2$ is more intricate and is detailed in the next lines.

We begin by using the microscopic collision rules (3) to write $\mathscr{K}_2 g$ in terms of $v$, $v_*$ and $v'$ only (hence, any dependence on $v'_*$ disappears). The following lemma holds:

**Lemma 1.** *There exists a nonnegative function $\tilde{B}$ satisfying (13), such that, for all $v \in \mathbb{R}^3$,*

$$\mathscr{K}_2 g(v) = \iint_{\mathbb{R}^3 \times S^2} \tilde{B}(\omega, v - v_*)\, e^{-\frac{1}{4}v^2 - \frac{1}{2}v_*^2 + \frac{1}{4}v'^2}\, g(v')\, \mathrm{d}\omega\, \mathrm{d}v_*. \tag{14}$$

*Proof.* The key point of the proof lies in some geometrical properties of symmetry in the microscopic collision process. By involving the relative velocity $V = v - v_*$, it is possible to choose a unit vector $\omega^\perp \in \mathrm{Span}(V, \omega)$, orthogonal to $\omega$. Then we clearly have

$$V = \omega(\omega \cdot V) + \omega^\perp(\omega^\perp \cdot V).$$

It allows to write that

$$v - (\omega \cdot V)\omega = v_* + (\omega^\perp \cdot V)\omega^\perp, \qquad v_* + (\omega \cdot V)\omega = v - (\omega^\perp \cdot V)\omega^\perp. \tag{15}$$

Note that the post-collision relative velocity for the same pre-collisional $V$, but with respect to $\omega^\perp$, are obtained by a simple exchange between $v'$ and $v'_*$. This means that the transformation $\omega \mapsto \omega^\perp$ induces $v' \mapsto v'_*$ and $v'_* \mapsto v'$.

Hence, by replacing $\omega$ by $\omega^\perp$, we get

$$\iint_{\mathbb{R}^3 \times S^2} B(\omega, V) e^{-\frac{1}{2}v_*^2}\, e^{\frac{1}{4}(v_* + (\omega \cdot V)\omega)^2}\, g(v_* + (\omega \cdot V)\omega)\, \mathrm{d}\omega\, \mathrm{d}v_* =$$

$$\iint_{\mathbb{R}^3 \times S^2} B(\omega^\perp, V) e^{-\frac{1}{2}v_*^2}\, e^{\frac{1}{4}(v_* + (\omega^\perp \cdot V)\omega^\perp)^2}\, g(v_* + (\omega^\perp \cdot V)\omega^\perp)\, \mathrm{d}\omega^\perp\, \mathrm{d}v_*.$$

The change of variables from $\omega$ to $\omega^\perp$ is a rotation, so that its Jacobian equals 1. By (15), the previous integral becomes

$$\iint_{\mathbb{R}^3 \times S^2} B(\omega^\perp, V) e^{-\frac{1}{2}v_*^2}\, e^{\frac{1}{4}v'^2}\, g(v')\, \mathrm{d}\omega^\perp\, \mathrm{d}v_* \\ = \iint_{\mathbb{R}^3 \times S^2} B(\omega^\perp, V) e^{-\frac{1}{2}v_*^2}\, e^{\frac{1}{4}v'^2}\, g(v')\, \mathrm{d}\omega\, \mathrm{d}v_*.$$

Let us set

$$\tilde{B}(\omega, V) = \left(\frac{1}{2\pi}\right)^{3/2} \Big[ B(\omega, V) + B(\omega^\perp, V) \Big].$$

The estimate (13) on $B$ guarantees that

$$\tilde{B}(\omega,V) \leq 2a \left(\frac{1}{2\pi}\right)^{3/2} |\sin\theta||\cos\theta| \left(|V| + |V|^{\delta-1}\right). \tag{16}$$

The thesis of the lemma is hence proved.

The previous lemma allows to prove the following result.

**Proposition 1.** *There exists $C > 0$ such that*

$$\mathscr{K}_2 g(v) \leq C \int_{\mathbb{R}^3} g(\eta)\, k_2(\eta,v)\, \mathrm{d}\eta, \quad \forall v \in \mathbb{R}^3,$$

*where*

$$k_2(\eta,v) = e^{-\frac{1}{8}(\eta-v)^2 - \frac{1}{8}\frac{(\eta^2-v^2)^2}{(\eta-v)^2}} |\eta - v|^{-1}, \quad \forall \eta, v \in \mathbb{R}^3.$$

*Proof.* Using the change of variables $v_* \mapsto V_* = v_* - v$, of Jacobian equal to 1, in (14), we can write

$$\mathscr{K}_2 g(v) = \iint_{\mathbb{R}^3 \times S^2} e^{-\frac{1}{4}v^2} e^{-\frac{1}{2}(V_*+v)^2} e^{\frac{1}{4}v'^2} g(v')\, \tilde{B}(\omega,V_*)\, \mathrm{d}\omega\, \mathrm{d}V_*. \tag{17}$$

Then, let us denote $V_* = p + q$, with $p = \omega(\omega \cdot V_*)$ and $q = V_* - \omega(\omega \cdot V_*)$. Note that the component $q$ belongs to the plane $\Pi = \{\omega\}^\perp = \{p\}^\perp$.

Consider now the change of variables

$$(V_*, \omega) \mapsto (p,q), \qquad \mathbb{R}^3 \times S^2 \to \mathbb{R}^3 \times \Pi. \tag{18}$$

We have to be very careful with the integration order in the change of variables, because $q$ strongly depends on $p$. More precisely, we first integrate with respect to $q$ since $\Pi = \{p\}^\perp$, then we combine the one-dimensional integration in the direction $\omega$ with the integral on $\omega \in S^2$ to obtain a three-dimensional integration over all rectangular components of $|p|\omega$. Moreover, the Jacobian of (18) is given by

$$\mathrm{d}V_*\, \mathrm{d}\omega = \frac{2}{p^2 \sin(p, p+q)}\, \mathrm{d}p\, \mathrm{d}q.$$

Since it is clear that $v' = v + p$, (17) becomes, by the abuse of notation $\tilde{B}(p, p+q) = \tilde{B}(\omega, V_*)$,

$$\mathscr{K}_2 g(v) = 2 \int_{\mathbb{R}^3} \int_{\Pi} \tilde{B}(p, p+q) e^{-\frac{1}{4}v^2 - \frac{1}{2}(p+q+v)^2 + \frac{1}{4}(v+p)^2} \\ g(v+p)\, |p|^{-2}\, |\sin(p, p+q)|^{-1}\, \mathrm{d}q\, \mathrm{d}p. \tag{19}$$

Using the fact that $p \cdot q = 0$, we deduce

$$-\frac{1}{4}v^2 + \frac{1}{4}(v+p)^2 - \frac{1}{2}(p+q+v)^2 = -\frac{1}{8}p^2 - \frac{1}{2}\left[q + \frac{1}{2}(2v+p)\right]^2,$$

which allows to write

$$\mathscr{K}_2 g(v) = 2\int_{\mathbb{R}^3}\int_{\Pi} \tilde{B}(p,p+q) e^{-\frac{1}{8}p^2-\frac{1}{2}\left[q+\frac{1}{2}(2v+p)\right]^2} \\ g(v+p)\,|p|^{-2}\,|\sin(p,p+q)|^{-1}\,\mathrm{d}q\,\mathrm{d}p.$$

Let $z = v + p/2$, and consider $z_1$ its component parallel to $\omega$, and denote $z_2 = z - z_1 \in \Pi$. Then, using the straightforward equality $(q+v+p/2)^2 = {z_1}^2 + (q+z_2)^2$, $\mathscr{K}_2 g$ becomes

$$\mathscr{K}_2 g(v) = 2\int_{\mathbb{R}^3} e^{-\frac{1}{8}p^2-\frac{1}{2}{z_1}^2}\, g(v+p)\,|p|^{-2} \\ \int_{\Pi} \tilde{B}(p,p+q) e^{-\frac{1}{2}(q+z_2)^2}\,|\sin(p,p+q)|^{-1}\,\mathrm{d}q\,\mathrm{d}p. \quad (20)$$

We are led to prove that the integral

$$\Delta := \frac{1}{|p|}\int_{\Pi} \tilde{B}(p,p+q) e^{-\frac{1}{2}(q+z_2)^2}\,|\sin(p,p+q)|^{-1}\,\mathrm{d}q$$

is upper-bounded, uniformly with respect to $p \in \mathbb{R}^3$ and $z_2 \in \Pi$. From (16), we obtain, for some constant $C > 0$,

$$\frac{\tilde{B}(p,p+q)}{|\sin(p,p+q)|} \leq C\,|\cos(p,p+q)|\left(|p+q| + |p+q|^{\delta-1}\right).$$

Using $|\tan(p,p+q)| = |q|/|p|$, we can write

$$\frac{\tilde{B}(p,p+q)}{|\sin(p,p+q)|} \leq C\left(1+\frac{q^2}{p^2}\right)^{-\frac{1}{2}}\left[(p^2+q^2)^{\frac{1}{2}} + (p^2+q^2)^{\frac{\delta-1}{2}}\right].$$

This implies that

$$\frac{\tilde{B}(p,p+q)}{|p|\,|\sin(p,p+q)|} \leq C\left[1+\left(p^2+q^2\right)^{\frac{\delta}{2}-1}\right] \leq C\left[1+|q|^{\delta-2}\right],$$

using the fact that $\delta < 1$. It is now convenient to split the range of integration, in the expression of $\Delta$, into $|q| \leq 1$ and $|q| \geq 1$, and get

$$\Delta \leq C\left(\int_{|q|\leq 1}(1+|q|^{\delta-2})\mathrm{d}q + \int_{|q|\geq 1} e^{-\frac{1}{2}(q+z_2)^2}\mathrm{d}q\right).$$

The right-hand-side of the estimate is clearly upper-bounded by a universal constant.

To conclude the proof, we perform the change of variable $p \mapsto \eta = p + v$ in (20) after using the uniform upper bound of $\Delta$. Then the thesis of the proposition is a consequence of the following equalities:

$$z_1{}^2 = \left(z \cdot \frac{\eta - v}{|\eta - v|}\right)^2 = \left(\frac{1}{2}(\eta + v) \cdot \frac{(\eta - v)}{|\eta - v|}\right)^2 = \frac{1}{4}\frac{(\eta^2 - v^2)^2}{|\eta - v|^2}.$$

The compactness of $\mathscr{K}$ then appears as a consequence of the following properties:

- uniform decay at infinity:

$$\|\mathscr{K} g\|_{L^2(B(0,R)^c)} \leq \zeta(R)\, \|g\|_{L^2(\mathbb{R}^3)}, \qquad \forall R > 0,$$

  where $B(0,R)$ is the open ball of $\mathbb{R}^3_v$ centred at 0 and of radius $R$, and $\zeta(R)$ goes to 0 when $R$ goes to $+\infty$;
- equicontinuity: for any $\varepsilon > 0$, there exists $\rho > 0$ such that, for all $w \in B(0,\rho)$,

$$\|(\tau_w - \mathrm{Id})\mathscr{K} g\|_{L^2(\mathbb{R}^3)} \leq \varepsilon \|g\|_{L^2(\mathbb{R}^3)},$$

  where Id is the identity and $\tau_w$ the translation operator

$$\tau_w \mathscr{K} g\,(v) = \mathscr{K} g\,(v + w), \qquad \forall v, w \in \mathbb{R}^3.$$

In [20], Grad provided the required estimates on the kernels $k_1$ and $k_2$, which allows to prove the compactness of $\mathscr{K}$.

## *2.4 Extensions in the cut-off case*

Caflisch [8, 9] extended Grad's result to the soft potential case by treating Grad cutoff kernels with $\gamma \in (-1, 1]$ in three space dimensions.

In [18], Golse and Poupaud are interested in studying the stationary solutions of the three-dimensional linearized Boltzmann equation in a half-space. An important step in their proof strategy consists in obtaining the compactness of the linearized Boltzmann operator in $L^2(M\mathrm{d}v)$ for Grad cutoff kernels with $\gamma \in (-2, 1]$. By defining

$$f^* := M^{1/2} f \text{ and } L^*(f*) = M^{-1/2}\mathscr{L}(f),$$

they introduce the operator

$$K^*(f^*) := f^* - L^*(f^*).$$

Subsequently, to prove the compactness of $K^*$, they use some growth estimates for the operator and an iteration technique, which allows to deduce that $(K^*)^4$ is a Hilbert-Schmidt operator on $L^2(\mathrm{d}v)$. Since $K^*$ is self-adjoint on $L^2(\mathrm{d}v)$, the linearized Boltzmann operator itself is compact in $L^2(M\mathrm{d}v)$.

More recently, Guo [21] extended Caflisch's result for Grad cutoff kernels to the range $\gamma \in (-3, 1]$ in three space dimensions.

The last result that we quote in this subsection is due to Levermore and Sun [25]. They prove a $L^p$ compactness result for the gain parts of the linearized Boltzmann

collision operator (in any dimension $D$) associated with weakly cutoff collision kernels that derive from a power-law intermolecular potential. In their proof, they assume that the cross-section has the form

$$B(|v-v_*|,\cos\theta) = |v-v_*|^\gamma b(\cos\theta), \quad \gamma \in (-D,+\infty)$$

where $b \in L^1(S^{D-1})$ is an even function. We really need these assumptions on $b$ in order for $B$ to be locally integrable in all its variables, which allows to give sense to both the gain and loss parts of the collision operator. In fact, the linearized Boltzmann operator $\mathscr{L}$ is split in the following way:

$$\mathscr{L} = \nu \times (\mathrm{Id} + \mathscr{K}_1 - \mathscr{K}_2 - \mathscr{K}_3) f,$$

where the loss operator $\mathscr{K}_1$ and the gain operators $\mathscr{K}_2$ and $\mathscr{K}_3$ are respectively given by

$$\begin{aligned}
\mathscr{K}_1 f(v) &= \frac{1}{\nu(v)} \iint_{\mathbb{R}^D \times S^{D-1}} B(\sigma, v-v_*) f(v_*) M(v_*) \mathrm{d}\sigma \, \mathrm{d}v_*, \\
\mathscr{K}_2 f(v) &= \frac{1}{\nu(v)} \iint_{\mathbb{R}^D \times S^{D-1}} B(\sigma, v-v_*) f(v') M(v_*) \mathrm{d}\sigma \, \mathrm{d}v_*, \\
\mathscr{K}_3 f(v) &= \frac{1}{\nu(v)} \iint_{\mathbb{R}^D \times S^{D-1}} B(\sigma, v-v_*) f(v'_*) M(v_*) \mathrm{d}\sigma \, \mathrm{d}v_*,
\end{aligned}$$

and the collision frequency $\nu$ is the $D$-dimensional analogous of (12).

They first prove that, under the assumptions written above, the operators $\mathscr{K}_j : L^p(\nu M \mathrm{d}v) \to L^p(\nu M \mathrm{d}v)$ are compact, $1 \le j \le 3$. Once proved the compactness result for $L^p$ with $p = 2$, the result for every $p \in (1,\infty)$ is deduced thanks to a straightforward interpolation argument and the following compactness criterion, which generalizes the classical Hilbert-Schmidt property.

**Lemma 2.** *Let $K$ be an integral operator given by*

$$Kf(v) = \int_{\mathbb{R}^D} k(v,v') f(v') \, \mathrm{d}\mu(v'),$$

*where $\mathrm{d}\mu$ is a $\sigma$-finite, positive measure over $\mathbb{R}^D$. Let the kernel $k(v,v')$ be symmetric in $v$ and $v'$ and, for some $r \in [1,2]$, satisfy the bound*

$$\|k\|_{L^s(L^r)} := \left( \int_{\mathbb{R}^D} \left( \int_{\mathbb{R}^D} |k(v,v')|^r \mathrm{d}\mu(v') \right)^{s/r} \mathrm{d}\mu(v) \right)^{1/s} < +\infty,$$

*where $s \in [2,+\infty]$ is defined by $1/r + 1/s = 1$. Let $p$, $q \in [r,s]$ such that $1/p + 1/q = 1$. Then, for any $f \in L^p(\mathrm{d}\mu)$ and $g \in L^q(\mathrm{d}\mu)$, the following estimate holds:*

$$\int_{\mathbb{R}^D} |g(v)Kf(v)|\mathrm{d}\mu(v) \leq \iint_{\mathbb{R}^D\times\mathbb{R}^D} |k(v,v')f(v')g(v)|\mathrm{d}\mu(v)\mathrm{d}\mu(v')$$
$$\leq \|k\|_{L^s(L^r)}\|f\|_{L^p}\|g\|_{L^q}.$$

*Consequently,* $K : L^p(\mathrm{d}\mu) \to L^p(\mathrm{d}\mu)$ *is bounded and satisfies* $|||K|||_{L^p} \leq \|k\|_{L^s(L^r)}$. *Moreover, if* $r \in (1,2]$ *then* $K : L^p(\mathrm{d}\mu) \to L^p(\mathrm{d}\mu)$ *is compact.*

## *2.5 Extensions to kernels without cut-off*

In [27], among other topics, Mouhot and Strain investigate compactness properties of both linearized Boltzmann and Landau operators to obtain explicit spectral gap and coercivity estimates. In this section, we only discuss the Boltzmann case, since the study of the Landau operator is not the purpose of this review article. They improve an earlier result of Pao [28], by using a completely different approach, and establish the Fredholm alternative for a broad class of collision kernels without any small deflection cut-off assumption. First note that it is well-known that $\mathscr{L}$ is an unbounded symmetric operator on $L^2$, see, for example, [11].

The cross-sections considered in their article have the form

$$B(|v-v_*|,\cos\theta) = |v-v_*|^\gamma b(\cos\theta), \qquad \gamma \in (-3,+\infty), \tag{21}$$

where $b$ behaves as follows:

$$b(\cos\theta) \underset{\theta\to 0}{\sim} b^*(\theta)\,(\sin\theta/2)^{-2-\alpha}, \qquad \alpha \in [0,2), \tag{22}$$

where $b^*$ is a nonnegative function, bounded and non-zero near $\theta$ vanishes. When $\alpha \geq 0$, the angular singularity is not integrable: hence, we indeed deal with the non cut-off case.

By using the change of variable $\sigma \mapsto -\sigma$, the angular cross-section $b$ can be replaced by its symmetric form

$$\tilde{b}(\cos\theta) = \frac{\mathbf{1}_{[0,\pi/2]}(\theta)}{2}\left[b(\cos\theta) + b(\cos(\pi-\theta))\right].$$

In what follows, we are mostly interested in establishing the compactness of the collisional operator, and we do not take into account the (of course interesting) consequences on the spectral gap estimate.

The first part of the proof consists in a technical estimate on the linearized collision operator by assuming that the cross-section $B$ is of variable hard spheres type, i.e. it does not depend on the angular variable:

$$B_q(|v-v_*|) = |v-v_*|^q, \qquad q \in (-3,+\infty).$$

The linearized collision operator $\mathscr{L}$ corresponding to $B_q$, is then written in the following form:

$$\mathscr{L} g = \nu g - Kg,$$

where the multiplicative local part $\nu$ can be seen as convolution

$$\nu(v) := \iint_{\mathbb{R}^3 \times S^2} M(v) B_q(|v - v_*|) \, \mathrm{d}v_* \, \mathrm{d}\sigma = \left|S^2\right| \left( |\cdot|^q * M \right)(v),$$

and the non-local part writes

$$Kg(v) := \iint_{\mathbb{R}^3 \times S^2} B_q(|v - v_*|) \Big[ g(v') M^{1/2}(v'_*) + g(v'_*) M^{1/2}(v') - g(v_*) M^{1/2}(v) \Big] M^{1/2}(v_*) \, \mathrm{d}v_* \, \mathrm{d}\sigma.$$

In fact, $K$ itself is composed by a pure convolution part

$$K^c g(v) := \left|S^2\right| \left[ |\cdot|^q * (M^{1/2} g) \right](v) M^{1/2}(v)$$

and by the remainder

$$K^+ g(v) := \iint_{\mathbb{R}^3 \times S^2} B_q(|v - v_*|) \Big[ g(v') M^{1/2}(v'_*) + g(v'_*) M^{1/2}(v') \Big] M^{1/2}(v_*) \, \mathrm{d}v_* \, \mathrm{d}\sigma.$$

Using the change of variable $\sigma \mapsto -\sigma$ in a part of the integral, the previous expression becomes

$$K^+ g(v) := 2 \iint_{\mathbb{R}^3 \times S^2} B_q(|v - v_*|) g(v') M^{1/2}(v'_*) M^{1/2}(v_*) \, \mathrm{d}v_* \, \mathrm{d}\sigma.$$

At the formal level, and rigorously only when $B_q$ is locally integrable with respect to the angular variable, we can associate to $B_q$ a kernel $k_q := k_q(v, v')$ such that

$$K^+ g(v) = \int_{\mathbb{R}^3} g(v') k_B(v, v') \, \mathrm{d}v', \qquad v \in \mathbb{R}^3.$$

Hence, the authors can apply Grad's strategy, as in Subsection 2.3, by first studying again the locally integrable case. In that situation, they prove the following preliminary results. The first one provides an explicit expression to the kernel.

**Lemma 1** *For $q > -1$, the explicit formula holds:*

$$k_q(v, v') = \frac{8}{|v' - v| \, (2\pi)^{3/2}} \exp\left[ -\frac{|v' - v|^2}{8} - \frac{|v' - v + 2(v \cdot \omega)\omega|^2}{8} \right] \times \left( \int_{\{\omega\}^\perp} |v' - v + z|^{q - (3-2)} \exp\left[ -\frac{|z + (v - (v \cdot \omega)\omega)|^2}{2} \right] \mathrm{d}z \right).$$

The second one gives an a priori estimate on the kernel.

**Proposition 2** *The kernel $k_q$ is symmetric with respect to $v$ and $v'$, and, for any $q > -1$ and $s \in \mathbb{R}$, satisfies the estimate*

$$\int_{\mathbb{R}^3} k_q(v,v')\,(1+|v'|)^s\,\mathrm{d}v' \leq C_{q,s}\,(1+|v|)^{q+s-2}, \qquad v \in \mathbb{R}^3,$$

*where $C_{q,s}$ is a constant which only depends on $q$ and $s$.*

The previous results are then extended to the non locally integrable case. More precisely, consider a cross-section $B$ satisfying a condition of the type

$$B(|v-v_*|,\cos\theta) \geq K\,B_{\gamma,\alpha}(|v-v_*|,\cos\theta)\,\mathbf{1}_{[0,\theta_0]}(\theta), \qquad v,\,v_* \in \mathbb{R}^3, \quad \theta \in [0,\pi],$$

where $K > 0$ and $\theta_0 \in (0,\pi]$ are constants, and $B_{\gamma,\alpha}$ is given, for any $\gamma \in (-3,+\infty)$, $\alpha \in [0,2)$, by

$$B_{\gamma,\alpha}(|v-v_*|,\cos\theta) = |v-v_*|^\gamma \sin^{-2-\alpha}(\theta/2), \qquad v,\,v_* \in \mathbb{R}^3, \quad \theta \in [0,\pi].$$

In order to use their preliminary results, the authors focus on a fictitious self-adjoint operator on $L^2$ defined by

$$\hat{L}g(v) = \iint_{\mathbb{R}^3\times S^2} |v-v_*|^{\gamma+\alpha+2}\,\mathbf{1}_{[0,1]}(|v-v'|)\,\mathbf{1}_{[0,\theta_0]}(\theta)\,M^{1/2}(v)\,M(v_*) \left[-\frac{g(v')}{M(v')^{1/2}} - \frac{g(v'_*)}{M(v'_*)^{1/2}} + \frac{g(v)}{M(v)^{1/2}} + \frac{g(v_*)}{M(v_*)^{1/2}}\right] \mathrm{d}v_*\,\mathrm{d}\sigma,$$

where both $v'$ and $\theta$ are considered as functions of $v$, $v_*$ and $\sigma$.

This operator is then written as the sum of several operators

$$\hat{L} = \hat{\nu}\,\mathrm{Id} - \hat{K}^+ + \hat{K}^c,$$

and the authors prove that the right-hand side is the sum of Hilbert-Schmidt operators, which implies the operator compactness in $L^2$.

In fact, this Hilbert-Schmidt property is straightforward for the multiplicative operator $\hat{\nu}$ Id. Indeed, it is clear that there exists a constant $C > 0$ such that

$$\hat{\nu}(v) = \iint_{\mathbb{R}^3\times S^2} |v-v_*|^{\gamma+\alpha+2}\,\mathbf{1}_{[0,1]}(|v-v'|)\,\mathbf{1}_{[0,\theta_0]}(\theta)\,M(v_*)\,\mathrm{d}v_*\,\mathrm{d}\sigma \geq C\,(1+|v|)^{\gamma+\alpha}.$$

Note that an analogous result is immediate for $\hat{K}^c$.

Unfortunately, the situation is more intricate with the operator $\hat{K}^+$. In the case when $\gamma+\alpha = 0$, $\hat{K}^+$ can be written as a limit of Hilbert-Schmidt operators. Indeed, the kernel of $\hat{K}^+$ is, by simple inspection,

$$\hat{k} := k_2(v,v')\,\mathbf{1}_{[0,1]}(|v-v'|)\,\mathbf{1}_{[0,\theta_0]}(\theta)$$
$$= \frac{8}{|v'-v|\,(2\pi)^{3/2}}\exp\left\{-\frac{|v'-v|^2}{8}-\frac{|v'-v+2(v\cdot\omega)\omega|^2}{8}\right\}$$
$$\times\left(\int_{\omega^\perp}|v'-v+z|\exp\left\{-\frac{|z+(v-(v\cdot\omega)\omega)|^2}{2}\right\}\mathrm{d}z\right)\mathbf{1}_{[0,1]}(|v-v'|)\,\mathbf{1}_{[0,\theta_0]}(\theta).$$

This kernel is then approximated as follows: it is split into

$$\hat{k} = \hat{k}^c_\varepsilon + \hat{k}^r_\varepsilon,$$

with

$$\hat{k}^c_\varepsilon = \left[\mathbf{1}_{[\varepsilon,1]}(|v-v'|)\times\mathbf{1}_{[\varepsilon,1]}\left(\left|\frac{v}{|v|}\cdot\frac{(v-v')}{|v-v'|}\right|\right)\right]\hat{k},$$

and, obviously,

$$\hat{k}^r_\varepsilon = \hat{k}-\hat{k}^c_\varepsilon.$$

The authors prove that $\hat{k}^r_\varepsilon$ is symmetric in $v,v'$ and that

$$\lim_{\varepsilon\to 0}\sup_{v\in\mathbb{R}^3}\int_{\mathbb{R}^3}|\hat{k}^r_\varepsilon|\,\mathrm{d}v' = 0.$$

Therefore, the sequence of operators $\hat{K}^{+,c}_\varepsilon$ associated to kernels $\hat{k}^c_\varepsilon$ converges to $\hat{K}^+$ in $L^2$ when $\varepsilon$ goes to 0. Hence, we only have to prove that each $\hat{K}^{+,c}_\varepsilon$ is compact. Note, then, that the kernel $\hat{k}^c_\varepsilon$ satisfies

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}\left(\hat{k}^c_\varepsilon\right)^2\mathrm{d}v\,\mathrm{d}v'$$
$$\leq C\int_{\mathbb{R}^3}\int_\varepsilon^1(1+r)^2\,(1+|v|\,\sin\theta)^2\,e^{-\frac{r^2}{4}}$$
$$\int_0^\pi e^{-\frac{(r+2|v|\cos\theta)^2}{4}}\,\sin\theta\,\mathbf{1}_{[\varepsilon,1]}(|\cos\theta|)\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}v$$
$$\leq C\int_{\mathbb{R}^3}(1+|v|)^2\,e^{-\varepsilon^2|v|^2}\,\mathrm{d}v,$$

which clearly is a finite quantity. Consequently, $\hat{K}^{+,c}_\varepsilon$ is a Hilbert-Schmidt operator.

When $\gamma+\alpha\neq 0$, one considers the following symmetric weighted modification of $\hat{L}$:

$$\tilde{L} = (1+|\cdot|)^{-(\gamma+\alpha)/2}\,\hat{L}\left((1+|\cdot|)^{-(\gamma+\alpha)/2}\,\cdot\right)$$

and the corresponding decomposition $\tilde{L} = \tilde{\nu} - \tilde{K}^+ + \tilde{K}^c$. Then $\tilde{\nu}$ is uniformly strictly positive and upper-bounded.

The authors conclude their argument by proving that $\tilde{K}^c$ is a Hilbert-Schmidt operator. They first focus on the term $\tilde{K}^+$. Its kernel is

$$(1+|v|)^{-(\gamma+\alpha)/2}\,k_{\gamma+\alpha+2}(v,v')\,(1+|v'|)^{-(\gamma+\alpha)/2}\,\mathbf{1}_{[0,1]}(|v-v'|)\,\mathbf{1}_{[0,\theta_0]}(\theta)$$

and similar computations as above allow to prove again that $\tilde{K}^+$ can be written as a limit of Hilbert-Schmidt operators.

# 3 The compactness properties for the linearized kinetic operators for mixtures

In this section, mainly following [13] for the definition of the linearized operator, and [4] for the compactness result, we investigate the case of an ideal gas mixture, with monatomic species.

## *3.1 Building the linearized collision operator for mixtures*

Each of the $I \geq 2$ species are described through a distribution function $f_i$, $1 \leq i \leq I$. As in the mono-species case, this function depends on time $t \in \mathbb{R}_+$, space position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. In the following, we also use the macroscopic density of species $i$, defined by

$$n_i(t,x) = \int_{\mathbb{R}^3} f_i(t,x,v)\,\mathrm{d}v.$$

The interactions between molecules are assumed to remain elastic, so that two colliding molecules of species $i$ and $j$, $1 \leq i,j \leq I$, with respective molecular (or molar) masses $m_i$ and $m_j$, see their velocities modified through the collision rules

$$v' = \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j} T_\omega (v - v_*), \tag{23}$$

$$v'_* = \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j} T_\omega (v - v_*), \tag{24}$$

where $\omega \in S^2$ and $T_\omega$ denotes the symmetry with respect to the plane $\{\omega\}^\perp$, i.e.

$$T_\omega z = z - 2(\omega \cdot z)\omega, \qquad \forall z \in \mathbb{R}^3.$$

Then the collision operator related to species $i$ and $j$ is given by

$$Q_{ij}(f,g)(v) = \iint_{\mathbb{R}^3 \times S^2} \left[ f(v')g(v'_*) - f(v)g(v_*) \right] B_{ij}(\omega, v - v_*)\,\mathrm{d}\omega\,\mathrm{d}v_*. \tag{25}$$

The cross-sections $B_{ij}$, $1 \leq i,j \leq I$, are chosen as in the mono-species case, i.e. they satisfy (7) and the same condition as in the classical case (13), namely

$$B_{ij}(\omega, V) \leq a\,|\sin\theta|\,|\cos\theta| \left( |V| + \frac{1}{|V|^{1-\delta}} \right), \quad \forall\, \omega \in S^2,\ \forall V \in \mathbb{R}^3, \tag{26}$$

where $a > 0$ and $0 < \delta < 1$ are again given constants which do not depend on $i$, $j$, and $\theta$ denotes the oriented angle between $\omega$ and $V$.

The time evolution of each distribution function $f_i$, $1 \leq i \leq I$, is then given by

$$\frac{\partial f_i}{\partial t} + v \cdot \nabla_x f_i = \sum_{j=1}^{I} Q_{ij}(f_i, f_j). \tag{27}$$

One can write weak forms of the collision operators using the changes of variables $(v, v_*) \mapsto (v_*, v)$ and $(v, v_*) \mapsto (v', v'_*)$ with a fixed $\omega \in S^2$. It is worth noticing that cases $i = j$ and $i \neq j$ are intrinsically different, see [15, 6] for more details. Moreover, we can formally write, for any $i$ and $j$, and any functions (of $v$) $f$ and $g$

$$\int_{\mathbb{R}^3} Q_{ij}(f,g)(v)\,\mathrm{d}v = 0, \tag{28}$$

$$\int_{\mathbb{R}^3} Q_{ij}(f,g)(v) \begin{pmatrix} m_i v \\ m_i v^2/2 \end{pmatrix} \mathrm{d}v + \int_{\mathbb{R}^3} Q_{ji}(g,f)(v) \begin{pmatrix} m_j v \\ m_j v^2/2 \end{pmatrix} \mathrm{d}v = 0. \tag{29}$$

One can also write an H-theorem [15, 6], which allows to obtain Maxwell functions as equilibrium. From now on, let us denote $M_i$ the normalized, centred Maxwell function related to species $i$

$$M_i(v) = \left(\frac{m_i}{2\pi}\right)^{3/2} e^{-\frac{m_i}{2}v^2}, \quad \forall v \in \mathbb{R}^3. \tag{30}$$

Now we are ready to write the linearized collision operator $\mathscr{L}$ for mixtures. We shall work in a $L^2$ setting again. More precisely, for any function $g \in L^2(\mathbb{R}^3)^I$ of $v$, we shall write the $L^2$ norm of $g$ as

$$\|g\|_{L^2}^2 = \sum_{j=1}^{I} \|g_j\|_{L^2}^2 = \sum_{j=1}^{I} \int_{\mathbb{R}^3} g_j(v)^2\,\mathrm{d}v.$$

Consider now macroscopic densities $(n_1, \ldots, n_I)$ as given and define the standard perturbation $g = (g_1, \ldots, g_I)$ to $M = (M_1, \ldots, M_I)$ by

$$f_i = n_i M_i + n_i M_i^{1/2} g_i, \qquad 1 \leq i \leq I.$$

By defining the $i$-th component of $\mathscr{L}g$ as

$$[\mathscr{L}g]_i = {M_i}^{-1/2} \sum_{j=1}^{I} n_i n_j \left( Q_{ij}(M_i, M_j^{1/2} g_j) + Q_{ij}(M_i^{1/2} g_i, M_j) \right), \tag{31}$$

for any function $g = (g_1, \ldots, g_I)$ and $1 \leq i \leq I$, as well as the $i$-th component of $\mathscr{Q}(g,g)$ by

$$[\mathscr{Q}(g,g)]_i = M_i^{-1/2} \sum_{j=1}^{I} n_i n_j Q_{ij}(M_i^{1/2} g_i, M_j^{1/2} g_j), \qquad 1 \leq i \leq I, \tag{32}$$

the Boltzmann equations on the components of the perturbation $g$ write

$$\partial_t g_i + v \cdot \nabla_x g_i + [\mathscr{L} g]_i = [\mathscr{Q}(g,g)]_i, \qquad 1 \leq i \leq I. \tag{33}$$

If we introduce the operator $\mathscr{K}$, whose $i$-th component of $\mathscr{K} g$ is given by

$$[\mathscr{K} g]_i(v) = \sum_{j=1}^{I} \left(\frac{m_i m_j}{4\pi^2}\right)^{3/4} n_i n_j \iint_{\mathbb{R}^3 \times S^2} B_{ij}(\omega, v - v_*)\, e^{-\frac{1}{4} m_i v^2}\, e^{-\frac{1}{2} m_j v_*^2} \tag{34}$$

$$\left[\left(\frac{m_i}{2\pi}\right)^{3/4} \left(e^{\frac{1}{4} m_j v_*'^2} g_j(v_*') - e^{\frac{1}{4} m_j v_*^2} g_j(v_*)\right) + \left(\frac{m_j}{2\pi}\right)^{3/4} e^{\frac{1}{4} m_i v'^2} g_i(v')\right] \mathrm{d}\omega\, \mathrm{d}v_* \tag{35}$$

and the positive function $\nu = \nu(v)$, whose $i$-th component writes

$$\nu_i(v) = \sum_{j=1}^{I} n_i n_j \left(\frac{m_i m_j}{4\pi^2}\right)^{3/2} \iint_{\mathbb{R}^3 \times S^2} e^{-\frac{1}{2} m_j v_*^2} B_{ij}(\omega, v - v_*)\, \mathrm{d}\omega\, \mathrm{d}v_*, \tag{36}$$

we can immediately state that

$$\mathscr{L} = \mathscr{K} - \nu\, \mathrm{Id}\,. \tag{37}$$

The following result holds.

**Theorem 1.** *The operator $\mathscr{K}$, defined by (34), is compact from $L^2(\mathbb{R}^3)^I$ to $L^2(\mathbb{R})^I$.*

The detailed proof can be found in [4]. We discuss below its main features, emphasizing on the major strategy differences with respect to Grad's proof in the mono-species case.

## *3.2 Elements of proof for the compactness*

First, we write

$$\mathscr{K} = \mathscr{K}_1 + \mathscr{K}_2 + \mathscr{K}_3 + \mathscr{K}_4,$$

where the $i$-th component of each $\mathscr{K}_\ell g$, $1 \leq \ell \leq 4$, is given by

$$[\mathscr{K}_1 g]_i(v) = -\sum_{j=1}^{I} n_i n_j \left(\frac{m_i m_j}{4\pi^2}\right)^{3/4} \times \iint_{\mathbb{R}^3 \times S^2} e^{-\frac{1}{4} m_i v^2}\, e^{-\frac{1}{4} m_j v_*^2}\, g_j(v_*) B_{ij}(\omega, v - v_*)\, \mathrm{d}\omega\, \mathrm{d}v_*,$$

$$[\mathscr{K}_2 g]_i(v) = \sum_{j\neq i} n_i n_j \left(\frac{m_i m_j}{4\pi^2}\right)^{3/4} \times \iint_{\mathbb{R}^3\times S^2} e^{-\frac{1}{4}m_i v^2} e^{-\frac{1}{2}m_j v_*^2} e^{\frac{1}{4}m_j v_*'^2} g_j(v_*') B_{ij}(\omega, v-v_*)\,\mathrm{d}\omega\,\mathrm{d}v_*,$$

$$[\mathscr{K}_3 g]_i(v) = {n_i}^2 \left(\frac{m_i}{2\pi}\right)^{3/2} \times \iint_{\mathbb{R}^3\times S^2} e^{-\frac{1}{4}m_i v^2} e^{-\frac{1}{2}m_i v_*^2} \left[e^{\frac{1}{4}m_i v_*'^2} g_i(v_*') + e^{\frac{1}{4}m_i v'^2} g_i(v')\right] B_{ii}(\omega, v-v_*)\,\mathrm{d}\omega\,\mathrm{d}v_*,$$

$$[\mathscr{K}_4 g]_i(v) = \sum_{j\neq i} n_i n_j \left(\frac{m_j}{2\pi}\right)^{3/2} \times \iint_{\mathbb{R}^3\times S^2} e^{-\frac{1}{4}m_i v^2} e^{-\frac{1}{2}m_j v_*^2} e^{\frac{1}{4}m_i v'^2} g_i(v') B_{ij}(\omega, v-v_*)\,\mathrm{d}\omega\,\mathrm{d}v_*.$$

The compactness for $\mathscr{K}$ is obtained by successively proving the compactness property for each $\mathscr{K}_\ell$. It is crucial to dissociate the cases when $i = j$ or not, because the proofs are quite different.

### 3.2.1 Compactness of $\mathscr{K}_1$

Denote, for any $i$, $j$,

$$k_1^{ij}(v, v_*) = \int_{S^2} e^{-\frac{1}{4}m_i v^2} e^{-\frac{1}{4}m_j v_*^2} B_{ij}(\omega, v-v_*)\,\mathrm{d}\omega, \quad \forall v, v_* \in \mathbb{R}^3.$$

We immediately have, for any $i$,

$$[\mathscr{K}_1 g]_i(v) = -\sum_{j=1}^{I} n_i n_j \left(\frac{m_i m_j}{4\pi^2}\right)^{3/4} \int_{\mathbb{R}^3} g_j(v_*) k_1^{ij}(v, v_*)\,\mathrm{d}v_*, \quad \forall v \in \mathbb{R}^3.$$

The compactness property can then be deduced, because $\mathscr{K}_1$ is somehow a Hilbert-Schmidt operators with associated kernels $(k_1^{ij})$.

### 3.2.2 Compactness of $\mathscr{K}_2$

The proof strategy here is very different from Grad's [20]. Once again, we aim to recover the Hilbert-Schmidt structure for $\mathscr{K}_2$. We first write $\mathscr{K}_2$ in another form. Thanks to the microscopic conservation of kinetic energy during a collision, we have

$$-\frac{1}{4}m_i v^2 - \frac{1}{2}m_j v_*^2 + \frac{1}{4}m_j v_*'^2 = -\frac{1}{4}m_j v_*^2 - \frac{1}{4}m_i v'^2.$$

Consequently, $[\mathscr{K}_2]_i$ can be rewritten pas

$$[\mathscr{K}_2 g]_i(v) = \sum_{j\neq i} n_i n_j \left(\frac{m_i m_j}{4\pi^2}\right)^{3/4} \times \iint_{\mathbb{R}^3\times S^2} e^{-\frac{1}{4}m_j v_*^2} e^{-\frac{1}{4}m_i v'^2} g_j(v_*') B_{ij}(\omega, v-v_*)\,\mathrm{d}\omega\,\mathrm{d}v_*. \quad (38)$$

To recover a kernel form in (38), it would be very convenient to replace $v_*$ and $v'$ by $v$ and $v_*'$ in the exponential terms, and then perform a change of variables $v_* \mapsto v_*'$, $\omega$ remaining unchanged.

It is indeed possible thanks to the following result, which only holds when $m_i \neq m_j$ (in the monatomic case, that is equivalent to $i \neq j$).

**Proposition 2.** *There exists $\rho > 0$ such that, for any $i$, $j$ with $i \neq j$,*

$$m_i v'^2 + m_j v_*^2 \geq \rho\left(m_i v^2 + m_j v_*'^2\right) \quad (39)$$

*for any $v$, $v_* \in \mathbb{R}^3$ and $v'$, $v_*'$ given by (23).*

*Proof.* The proof of Proposition 2 is quite simple. Let us choose $i$ and $j \neq i$. Collision rules (23) can be rewritten as

$$v' = \left(\mathrm{I}_3 - 2\frac{m_j}{m_i+m_j}\omega\,\omega^\intercal\right)v + 2\frac{m_j}{m_i+m_j}\omega\,\omega^\intercal v_*, \quad (40)$$

$$v_*' = \left(\mathrm{I}_3 - 2\frac{m_i}{m_i+m_j}\omega\,\omega^\intercal\right)v_* + 2\frac{m_i}{m_i+m_j}\omega\,\omega^\intercal v, \quad (41)$$

where $\mathrm{I}_3$ is the identity matrix of $\mathbb{R}^3$. Now we set

$$A(\omega) := \mathrm{I}_3 - 2\frac{m_i}{m_i+m_j}\omega\,\omega^\intercal \in \mathbb{R}^{3\times 3}.$$

From (41), we easily get

$$A(\omega)v_* = v_*' - 2\frac{m_i}{m_i+m_j}\omega\,\omega^\intercal v.$$

Fortunately, $A(\omega)$ is an invertible matrix, since

$$\det A(\omega) = \frac{m_j - m_i}{m_i+m_j}$$

and $j \neq i$. Note that, in the mono-species case, the proof already fails at this stage, since the corresponding matrix $A(\omega)$ is not invertible.

Consequently, we can write $v_*$ in terms of both $v$ and $v_*'$:

$$v_* = \left(\mathrm{I}_3 - A(\omega)^{-1}\right) v + A(\omega)^{-1} v'_*, \tag{42}$$

where we used the equality

$$-2\frac{m_i}{m_i+m_j} A(\omega)^{-1}\omega\,\omega^{\intercal} = \mathrm{I}_3 - A(\omega)^{-1}.$$

Then we obtain an expression of $v'$ with respect to $v$ and $v'_*$ by putting (42) in (40):

$$v' = \left(\frac{m_i+m_j}{m_i}\,\mathrm{I}_3 - \frac{m_j}{m_i} A(\omega)^{-1}\right) v - \frac{m_j}{m_i}\left(\mathrm{I}_3 - A(\omega)^{-1}\right) v'_*.$$

Consider now the following block matrix in $\mathbb{R}^{6\times 6}$

$$\mathbb{A}(\omega) = \begin{bmatrix} \dfrac{m_i+m_j}{m_i}\,\mathrm{I}_3 - \dfrac{m_j}{m_i} A(\omega)^{-1} & -\sqrt{\dfrac{m_j}{m_i}}\left(\mathrm{I}_3 - A(\omega)^{-1}\right) \\ \sqrt{\dfrac{m_j}{m_i}}\left(\mathrm{I}_3 - A(\omega)^{-1}\right) & A(\omega)^{-1} \end{bmatrix}. \tag{43}$$

The previous matrix is invertible (check that $\det \mathbb{A}(\omega) = -1$) and we have $\mathbb{A}(\omega)^{-1} = \mathbb{A}(\omega)$. Moreover, it is clear that

$$\begin{bmatrix} \sqrt{m_i}\, v' \\ \sqrt{m_j}\, v_* \end{bmatrix} = \mathbb{A}(\omega) \begin{bmatrix} \sqrt{m_i}\, v \\ \sqrt{m_j}\, v'_* \end{bmatrix}.$$

The best constant $\rho$ satisfying (39) is obtained by computing

$$\inf_{v, v'_* \in \mathbb{R}^3} \frac{\left|\mathbb{A}(\omega)\begin{bmatrix}\sqrt{m_i}\, v & \sqrt{m_j}\, v'_*\end{bmatrix}^{\intercal}\right|^2}{\left|\begin{bmatrix}\sqrt{m_i}\, v & \sqrt{m_j}\, v'_*\end{bmatrix}^{\intercal}\right|^2} = \|\mathbb{A}(\omega)^{-1}\|_2{}^{-2} = \|\mathbb{A}(\omega)\|_2{}^{-2}.$$

Since $\omega \mapsto \|\mathbb{A}(\omega)\|_2{}^{-2}$ is clearly a continuous positive function of $\omega$ on the compact set $S^2$, it reaches its minimum. Hence, we are led to set

$$\rho = \min_{j\neq i} \min_{\omega \in S^2} \|\mathbb{A}(\omega)\|_2{}^{-2} > 0$$

to satisfy (39).

Using Proposition 2 and (26) for each $B_{ij}$, we obtain the existence of a constant $C > 0$, only depending on all the molecular masses, such that, for any $i$,

$$\begin{aligned}[\mathscr{K}_2 g]_i(v) \leq C \sum_{j\neq i} n_i n_j\, e^{-\frac{\rho}{4} m_i v^2} \\ \times \iint_{\mathbb{R}^3\times S^2} e^{-\frac{\rho}{4} m_j v'^2_*}\, g_j(v'_*)\left(|v - v_*| + |v - v_*|^{\delta - 1}\right) \mathrm{d}v_*\, \mathrm{d}\omega.\end{aligned}$$

We then perform the change of variable $v_* \mapsto v'_*$, whose Jacobian is $1/\det A(\omega)$. Noticing that

$$v - v_* = A(\omega)^{-1}\left(v - v'_*\right) \quad \text{and} \quad \|A(\omega)\|_2{}^{-1} \leq \frac{\left|A(\omega)^{-1}\left(v - v'_*\right)\right|}{|(v - v'_*)|} \leq \|A(\omega)^{-1}\|_2,$$

we can state that

$$|v - v_*| + |v - v_*|^{\delta-1} \leq \|A(\omega)^{-1}\|_2 \left|v - v'_*\right| + \|A(\omega)\|_2{}^{1-\delta} \left|v - v'_*\right|^{\delta-1}.$$

Eventually, we write

$$[\mathscr{K}_2 g]_i(v) \leq C \sum_{j\neq i} n_i n_j \iint_{\mathbb{R}^3\times S^2} e^{-\frac{\rho}{4} m_i v^2} e^{-\frac{\rho}{4} m_j v'^2_*} g_j(v'_*) \times \left(|v - v'_*| + |v - v'_*|^{\delta-1}\right) \mathrm{d}\omega \, \mathrm{d}v'_*.$$

We thus recover a kernel form to study, which allows us to conclude on the compactness of $\mathscr{K}_2$.

### 3.2.3 Compactness of $\mathscr{K}_3$ and $\mathscr{K}_4$

Since $\mathscr{K}_3$ appears as a mono-species operator, it can be treated following Grad's strategy. The proof of the compactness of $\mathscr{K}_4$ is more intricate and involves both kinds of strategies, Grad's and the one from Subsection 3.2.2.

## References


1. L. Boltzmann. Weitere studien über das wärmegleichgewicht unter gasmolekülen. *Sitzungsberichte Akad. Wiss.*, 66:275–370, 1873.
2. L. Boltzmann. *Lectures on gas theory*. Translated by Stephen G. Brush. University of California Press, Berkeley, 1964.
3. D. Bothe. On the Maxwell-Stefan approach to multicomponent diffusion. In *Parabolic problems*, volume 80 of *Progr. Nonlinear Differential Equations Appl.*, pages 81–93. Birkhäuser/Springer Basel AG, Basel, 2011.
4. L. Boudin, B. Grec, M. Pavić, and F. Salvarani. Diffusion asymptotics of a kinetic model for gaseous mixtures. *Kinet. Relat. Models*, 6(1):137–157, 2013.
5. L. Boudin, B. Grec, and F. Salvarani. A mathematical and numerical analysis of the Maxwell-Stefan diffusion equations. *Discrete Contin. Dyn. Syst. Ser. B*, 17(5):1427–1440, 2012.
6. L. Boudin, B. Grec, and F. Salvarani. The Maxwell-Stefan diffusion limit for a kinetic model of mixtures. *Acta Appl. Math.*, 136:79–90, 2015.
7. J.-F. Bourgat, L. Desvillettes, P. Le Tallec, and B. Perthame. Microreversible collisions for polyatomic gases and Boltzmann's theorem. *European J. Mech. B Fluids*, 13(2):237–254, 1994.
8. R. E. Caflisch. The Boltzmann equation with a soft potential. I. Linear, spatially-homogeneous. *Comm. Math. Phys.*, 74(1):71–95, 1980.

9. R. E. Caflisch. The Boltzmann equation with a soft potential. II. Nonlinear, spatially-periodic. *Comm. Math. Phys.*, 74(2):97–109, 1980.
10. T. Carleman. *Problèmes mathématiques dans la théorie cinétique des gaz*. Publ. Sci. Inst. Mittag-Leffler. 2. Almqvist & Wiksells Boktryckeri Ab, Uppsala, 1957.
11. C. Cercignani. *The Boltzmann equation and its applications*, volume 67 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1988.
12. C. Cercignani, R. Illner, and M. Pulvirenti. *The mathematical theory of dilute gases*, volume 106 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.
13. E. S. Daus, A. Jüngel, C. Mouhot, and N. Zamponi. Hypocoercivity for a linearized multispecies Boltzmann system. *ArXiv e-prints*, Apr. 2015.
14. L. Desvillettes. Sur un modèle de type Borgnakke-Larsen conduisant à des lois d'énergie non linéaires en température pour les gaz parfaits polyatomiques. *Ann. Fac. Sci. Toulouse Math. (6)*, 6(2):257–262, 1997.
15. L. Desvillettes, R. Monaco, and F. Salvarani. A kinetic model allowing to obtain the energy law of polytropic gases in the presence of chemical reactions. *Eur. J. Mech. B Fluids*, 24(2):219–236, 2005.
16. J. B. Duncan and H. L. Toor. An experimental study of three component gas diffusion. *AIChE Journal*, 8(1):38–41, 1962.
17. R. T. Glassey. *The Cauchy problem in kinetic theory*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.
18. F. Golse and F. Poupaud. Stationary solutions of the linearized Boltzmann equation in a half-space. *Math. Methods Appl. Sci.*, 11(4):483–502, 1989.
19. H. Grad. Principles of the kinetic theory of gases. In *Handbuch der Physik (herausgegeben von S. Flügge), Bd. 12, Thermodynamik der Gase*, pages 205–294. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958.
20. H. Grad. Asymptotic theory of the Boltzmann equation. II. In *Rarefied Gas Dynamics (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962), Vol. I*, pages 26–59. Academic Press, New York, 1963.
21. Y. Guo. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal.*, 169(4):305–353, 2003.
22. E. Hecke. Über die Integralgleichung der kinetischen Gastheorie. *Math. Z.*, 12(1):274–286, 1922.
23. D. Hilbert. Begründung der kinetischen Gastheorie. *Math. Ann.*, 72(4):562–577, 1912.
24. H. Hutridurga and F. Salvarani. On the Maxwell-Stefan diffusion limit for a mixture of monatomic gases. *ArXiv e-prints*, 2015.
25. C. D. Levermore and W. Sun. Compactness of the gain parts of the linearized Boltzmann operator with weakly cutoff kernels. *Kinet. Relat. Models*, 3(2):335–351, 2010.
26. J. C. Maxwell. On the dynamical theory of gases. *Phil. Trans. R. Soc.*, 157:49–88, 1866.
27. C. Mouhot and R. M. Strain. Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff. *J. Math. Pures Appl. (9)*, 87(5):515–535, 2007.
28. Y. P. Pao. Boltzmann collision operator with inverse-power intermolecular potentials. I, II. *Comm. Pure Appl. Math.*, 27:407–428; ibid. 27 (1974), 559–581, 1974.
29. J. Stefan. Ueber das Gleichgewicht und die Bewegung insbesondere die Diffusion von Gasgemengen. *Akad. Wiss. Wien*, 63:63–124, 1871.
30. C. Villani. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics, Vol. I*, pages 71–305. North-Holland, Amsterdam, 2002.