Inverse Reduced-Order Modeling

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We propose a general probabilistic formulation of reduced-order modeling in the case the system state is hidden and characterized by some uncertainty. The objective is to integrate noisy and incomplete observations in the process of building a reduced-order model. We call this problematic *inverse reduced-order modeling*. This problematic arises in many scientific domains where there exists a need of accurate low-order descriptions of highly-complex phenomena, which can not be directly and/or deterministically observed. Among others, it concerns geophysical studies dealing with image data, which are important for the characterization of global warming or the prediction of natural disasters.

The description of the inverse reduced-order modeling problematic is as follows. Let $\{X_t\}_t$ be a sequence of random vectors, where each vector X_t takes its value in $\mathbb{R}^n$. This sequence is assumed to satisfy the general probabilistic model

$$\begin{cases} X_t = f_t(\Theta), \\ \Theta \sim \eta(d\theta), \end{cases}$$

with functions $f_t : \Omega \to \mathbb{R}^n$ and where Θ denotes some random parameters defined over some probability space $(\Omega, \mathcal{F}, \eta)$ with the posterior probability measure $\eta : \mathcal{F} \to \mathbb{R}$. The f_t's do not need to be necessarily linear. The random parameter is not necessarily Gaussian. We are interested in lowering the computational complexity of this high-dimensional model. Assuming that elements of the approximated trajectory $\{\tilde{X}_t\}_t$ belong to a k-dimensional subspace of $\mathbb{R}^n$, we seek for a reduced-order model admitting a representation

$$\begin{cases} \tilde{X}_t = \tilde{f}_t(\Theta, \mathbf{u}), \\ \Theta \sim \eta(d\theta), \end{cases}$$

where approximated functions $\tilde{f}_t : \Omega \times \mathcal{M} \to \mathbb{R}^n$ of the sequence $\{\tilde{f}_t\}_t$ are computable with a low complexity and parametrized by $\mathbf{u}$ in some finite-dimensional vector space $\mathcal{M}$. Moreover, the approximated functions should minimize some expectation criterion

$$\arg\min_{\mathbf{u} \in \mathcal{M}} \langle \eta, \phi(\cdot, \mathbf{u}) \rangle,$$

where the cost $\phi : \Omega \times \mathcal{M} \to \mathbb{R}^+$ measures some distance between the true and approximated sequence: $\phi(\theta, \mathbf{u}) = d(\{f_t(\theta)\}_t, \{\tilde{f}_t(\theta, \mathbf{u})\}_t)$.

Standard low-rank approximations such as principal orthogonal decomposition (POD), principal oscillating patterns (POP) and principal interacting patterns (PIP) can be generalized to this probabilistic context. We show that the POD and POP generalized problems admit closed-form solutions, and besides, we provide some sub-optimal solvers in the PIP case.

Nevertheless, all these solutions depend on the computation of some expectation with respect to the posterior probability measure η. In order to evaluate the expectations of interest, we invoke Bayesian inference and assume that the posterior follows a Gibbs-Boltzmann distribution. This family characterizes a large variety of random processes and in particular state-space models. We study the two following situations.

- The posterior measure corresponds to a linear Gaussian state-space model. We first examine this particular setting and show how *Kalman filtering* can be used to derive exact solutions to the inverse reduced-order modeling problems.
- The general case where the posterior measure does not admit an explicit form. Expectations can nevertheless be efficiently and accurately approached by sequential Monte-Carlo simulation, and in particular with *sampling with importance resampling* algorithms. We provide in this general case some approximated solution to inverse reduced-order modeling problems, taking the explicit form of a function of the simulated samples.

Finally, we illustrate the potential of the proposed methodology by showing through numerical experiments the gain in accuracy obtained by exploiting the probabilistic reformulation of POD in the linear Gaussian case.