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Identification of Preferences from Market Data

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Andrés Carvajal and Alvaro Riascos

Abstract

We offer a new proof that the equilibrium manifold (under complete markets) identifies individual demands globally. Moreover, under observation of only a subset of the equilibrium manifold, we find domains on which aggregate and individual demands are identifiable. Our argument avoids the assumption of Balasko (2004) requiring the observation of the complete manifold.

KEYWORDS: General equilibrium, consumer demand, identification

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1 Introduction

We offer a new proof that the equilibrium manifold (under complete markets) identifies individual demands.[1] This same result has previously been obtained by Balasko (2004), Chiappori et al. (2004) and Matzkin (2005). Balasko's result has been criticized for its assumption that equilibrium prices are observed for situations in which the incomes of all individuals but one are zero. Under regularity assumptions, Chiappori et al. obtain local identification of individual demands using local knowledge of the manifold. Balasko, however, has claimed that the argument given by Chiappori et al. implicitly requires that preferences be analytic. Matzkin determines the largest class of fundamentals for which identification is possible.

We use Balasko's idea on how to recover the aggregate demand function from the equilibrium manifold, and hence avoid Chiappori et al.'s usage of the implicit function and the Cartan-Kaehler theorems. We then use a slightly different argument than Chiappori et al. to identify individual demands from the aggregate demand function, so we also avoid Balasko's observational assumption.

2 The Model

Consider a profile of utility functions $\left(u_i : \mathbb{R}^L_{++} \longrightarrow \mathbb{R}\right)_{i\in\mathcal{I}}$, where $\mathcal{I} = \{1, ..., I\}$ is a finite set of agents (a society) and $L \geq 2$ is the number of commodities.

Condition 1 *For each $i \in \mathcal{I}$, u^i represents locally nonsatiated and strictly convex preferences, is continuous and its upper-contour sets have interior closures.*

Let $S^{L-1}_{++} = \{p \in \mathbb{R}^L_{++} : p_1 = 1\}$ be the set of normalized prices and denote endowments by $\left(w^i\right)_{i\in\mathcal{I}} \in \mathbb{R}^{LI}_{++}$. Individual demand functions are $\left(f^i : S^{L-1}_{++} \times \mathbb{R}^L_{++} \to \mathbb{R}^L_{++}\right)_{i\in\mathcal{I}}$, the aggregate demand function is $F : S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \to \mathbb{R}^L_{++}$ and the equilibrium manifold is

$$M = \left\{(p, w) \in S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} : F(p, w) = \sum_{i\in\mathcal{I}} w^i\right\}$$

Notice that we maintain strictly positive endowments and that the equilibrium manifold allows for variations in the aggregate endowment. In Balasko's argument, behavior at null endowments is used, unlike here. In Chiappori et al. the assumption of the observation of a relatively open subset of the equilibrium manifold implies that the aggregate endowment is allowed to vary as well: their argument allows for perturbations to each agent's endowments, keeping prices and the endowments of other consumers fixed (see Carvajal et al., 2004).

Henceforth, we maintain the assumption that there is a profile $\left(u^i\right)_{i\in\mathcal{I}}$ that satisfies condition 1, and assume that some subset of its equilibrium manifold, M, is observed. We

[1] When the economy has generically complete real assets structures, we conjecture that our results hold generically on prices and endowments.

study whether unobserved fundamentals can be uniquely determined from that subset. We do not test the existence of profile $\left(u^i\right)_{i\in\mathcal{I}}$. Our results generalize Balasko's theorem showing that M uniquely and globally determines aggregate demand F. We also consider the case in which only a subset of M is observed and show that identification of F over a subset of its domain is possible. Then, we show that F uniquely determines individual demands $\left(f^i\right)_{i\in\mathcal{I}}$. That is, we prove identification of individual demands from the equilibrium manifold. It then follows from MasColell (1977) that individual preferences can also be identified.[2]

3 From the Equilibrium Manifold to Aggregate Demand

Under our assumptions, it is well known that F is continuous and satisfies that

$$\left(p\cdot \widehat{w}^i = p\cdot w^i\right)_{i\in\mathcal{I}} \Longrightarrow F(p,w) = F(p,\widehat{w})$$

and that each f^i satisfies Walras's law.

For any $\Phi : S^{L-1}_{++}\times\mathbb{R}^{LI}_{++} \longrightarrow \mathbb{R}^L_{++}$ and $D\subseteq S^{L-1}_{++}\times\mathbb{R}^{LI}_{++}$, denote by $\Phi_{|D}$ the restriction of Φ to D.

We say that $E\subseteq M$ **identifies** F **over** $D\subseteq S^{L-1}_{++}\times\mathbb{R}^{LI}_{++}$ if for every continuous function $\Phi : S^{L-1}_{++}\times\mathbb{R}^{LI}_{++} \longrightarrow \mathbb{R}^L_{++}$ that satisfies Walras's law and is such that

$$\left(p\cdot \widehat{w}^i = p\cdot w^i\right)_{i\in\mathcal{I}} \Longrightarrow \Phi(p,w) = \Phi(p,\widehat{w})$$

and

$$\left\{ (p,w)\in S^{L-1}_{++}\times\mathbb{R}^{LI}_{++} \,\middle|\, \Phi(p,w) = \sum_{i\in\mathcal{I}} w^i \right\} \supseteq E$$

it is true that $\Phi_{|D} = F_{|D}$.

If in the previous definition we drop the requirement that Φ be continuous, then we say that E **strongly identifies** F **over** D. Clearly, strong identification implies identification.

Intuitively, we say that E identifies F over D if, for any function that cannot be ruled out as aggregate demand function, we have that, on the restricted domain D, that function is identical to the true aggregate demand function. A function cannot be ruled out as aggregate demand function when (i) it satisfies the properties that are necessary for it to be an aggregate demand, and (ii) it is consistent with the observed data (because all the observed equilibria are equilibrium according to it).

We say that **the equilibrium manifold identifies (strongly identifies) aggregate demand globally** if M identifies (strongly identifies) F over $S^{L-1}_{++}\times\mathbb{R}^{LI}_{++}$. It is straightforward that the equilibrium manifold identifies aggregate demand globally if and only if F

[2] The literature on integrability studies conditions under which it is possible to construct preferences that rationalize a given demand function. It does not tell us when the preferences that rationalize a demand function are unique, which MasColell (1977) does.

is the only continuous function $\Phi : S_{++}^{L-1} \times \mathbb{R}_{++}^{LI} \longrightarrow \mathbb{R}_{++}^{L}$, satisfying Walras's law, such that $\left(p \cdot \widehat{w}^i = p \cdot w^i\right)_{i\in\mathcal{I}} \Longrightarrow \Phi(p,w) = \Phi(p,\widehat{w})$ and

$$\left\{ (p,w) \in S_{++}^{L-1} \times \mathbb{R}_{++}^{LI} \,\middle|\, \Phi(p,w) = \sum_{i\in\mathcal{I}} w^i \right\} = M$$

In the case of global identification, every function that satisfies the properties of an aggregate demand can be ruled out, with the exception of the true aggregate demand.

Some properties of the concept of identification (strong identification) are straightforward.

Theorem 1 *Let* $\underline{E} \subseteq E \subseteq M$, $\widetilde{E} \subseteq M$, $D \subseteq \widehat{D} \subseteq S_{++}^{L-1} \times \mathbb{R}_{++}^{LI}$ *and* $\widetilde{D} \subseteq S_{++}^{L-1} \times \mathbb{R}_{++}^{LI}$. *Then,*

1. *If* $D \subseteq E$, *then* E *strongly identifies* F *over* D.
2. *If* E *identifies (strongly)* F *over* $\widehat{D}$ *, then it identifies (strongly)* F *over* D.
3. *If* $\underline{E}$ *identifies (strongly)* F *over* D, *then* E *identifies (strongly)* F *over* D.
4. *If* E *identifies (strongly)* F *over* D *and* $\widetilde{E}$ *identifies (strongly)* F *over* $\widetilde{D}$, *then* $E \cup \widetilde{E}$ *identifies (strongly)* F *over* $D \cup \widetilde{D}$.
5. *If* E *identifies* F *over* D, *then it identifies* F *over the closure of* D *in* $S_{++}^{L-1} \times \mathbb{R}_{++}^{LI}$, *which we denote by* $\overline{D}$.

Proof. Fix $\Phi : S_{++}^{L-1} \times \mathbb{R}_{++}^{LI} \longrightarrow \mathbb{R}_{++}^{L}$ such that $\left(p \cdot \widehat{w}^i = p \cdot w^i\right)_{i\in\mathcal{I}} \Longrightarrow \Phi(p,w) = \Phi(p,\widehat{w})$ and $\left\{ (p,w) \in S_{++}^{L-1} \times \mathbb{R}_{++}^{LI} \,\middle|\, \Phi(p,w) = \sum_{i\in\mathcal{I}} w^i \right\} \supseteq E$ and let $(p,w) \in D$.

If $D \subseteq E$, it follows that $\Phi(p,w) = \sum_{i\in\mathcal{I}} w^i$, whereas, since $E \subseteq M$, $F(p,w) = \sum_{i\in\mathcal{I}} w^i$, which proves 1.

If E identifies F over $\widehat{D}$, then $\Phi_{|\widehat{D}} = F_{|\widehat{D}}$ and since $D \subseteq \widehat{D}$, obviously $\Phi_{|D} = F_{|D}$, proving 2.

If $\underline{E}$ identifies F over D, since $\underline{E} \subseteq E \subseteq \left\{ (\widetilde{p},\widetilde{w}) \in S_{++}^{L-1} \times \mathbb{R}_{++}^{LI} \,\middle|\, \Phi(\widetilde{p},\widetilde{w}) = \sum_{i\in\mathcal{I}} \widetilde{w}^i \right\}$, it follows that $\Phi_{|D} = F_{|D}$, proving 3.

Suppose that E and $\widetilde{E}$ identify F over D and $\widetilde{D}$, respectively, and suppose that $E \cup \widetilde{E} \subseteq \left\{ (\widetilde{p},\widetilde{w}) \in S_{++}^{L-1} \times \mathbb{R}_{++}^{LI} \,\middle|\, \Phi(\widetilde{p},\widetilde{w}) = \sum_{i\in\mathcal{I}} \widetilde{w}^i \right\}$; then $\Phi_{|D} = F_{|D}$ and $\Phi_{|\widetilde{D}} = F_{|\widetilde{D}}$, which implies that $\Phi_{|D\cup\widetilde{D}} = F_{|D\cup\widetilde{D}}$, proving 4.

Finally, suppose further that Φ is continuous and $(p,w) \in \overline{D}$. Let $(p_n,w_n)_{n=1}^{\infty}$ be a sequence in D_E converging to (p,w). If E identifies F over D, it follows that for every $n \in \mathbb{N}$, $\Phi(p_n,w_n) = F(p_n,w_n)$, and then, by continuity, $\Phi(p,w) = F(p,w)$. ■

The key result is the following theorem, which generalizes the idea of Balasko.

Theorem 2 *Let $E \subseteq M$, and define*

$$D_E = \left\{ (p,w) \in S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \,\middle|\, \left(\exists \widehat{w} \in \mathbb{R}^{LI}_{++}\right) : (p,\widehat{w}) \in E \text{ and } \left(p \cdot \widehat{w}^i\right)_{i\in\mathcal{I}} = \left(p \cdot w^i\right)_{i\in\mathcal{I}} \right\}$$

E identifies F over any $D \subseteq \overline{D_E}$, strongly over any $D \subseteq D_E$.

Proof. By theorem 1, parts 2 and 5, it suffices to show that E strongly identifies F over D_E.

Let $\Phi : S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \longrightarrow \mathbb{R}^{L}_{++}$ be such that $\left(p \cdot \widehat{w}^i = p \cdot w^i\right)_{i\in\mathcal{I}} \Longrightarrow \Phi(p,w) = \Phi(p,\widehat{w})$ and $\left\{ (p,w) \in S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \,\middle|\, \Phi(p,w) = \sum_{i\in\mathcal{I}} w^i \right\} \supseteq E$. Fix $(p,w) \in D_E$. By definition, we can fix $\widehat{w} \in \mathbb{R}^{LI}_{++}$ such that $(p,\widehat{w}) \in E$ and $p \cdot \widehat{w}^i = p \cdot w^i$ for all i. Then, by construction, $\Phi(p,w) = \Phi(p,\widehat{w})$ and $F(p,\widehat{w}) = F(p,w)$. Since $E \subseteq \left\{ (\widetilde{p},\widetilde{w}) \in S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \,\middle|\, \Phi(\widetilde{p},\widetilde{w}) = \sum_{i\in\mathcal{I}} \widetilde{w}^i \right\}$, it follows that $\Phi(p,\widehat{w}) = \sum_{i\in\mathcal{I}} \widehat{w}^i$, whereas since $E \subseteq M$, $F(p,\widehat{w}) = \sum_{i\in\mathcal{I}} \widehat{w}^i$. It follows that $\Phi(p,w) = F(p,w)$. ■

Corollary 1 (Balasko) *The equilibrium manifold strongly identifies aggregate demand globally.*

Proof. It suffices to show that for each $(p,w) \in S^{L-1}_{++} \times \mathbb{R}^{LI}_{++}$, there exists $\left(\widehat{w}^i\right)_{i\in\mathcal{I}} \in \mathbb{R}^{LI}_{++}$ such that $\left(p, \left(\widehat{w}^i\right)_{i\in\mathcal{I}}\right) \in M$ and $p \cdot \widehat{w}^i = p \cdot w^i$ for all i. Let $\widehat{w}^i = f^i(p,w^i) \in \mathbb{R}^{L}_{++}$. The result follows by Walras's Law and the fact that, then, $\widehat{w}^i = f^i(p,\widehat{w}^i)$. ■

Knowledge of the manifold for individual incomes arbitrarily close to zero may, however, be unrealistic. In particular, suppose that $K > 0$ and define

$$M^K = \left\{ (p,w) \in M \,\middle|\, (\forall i \in \mathcal{I}) : p \cdot w^i > K \right\}$$

Corollary 2 *M^K identifies F over $D^K = \left\{ (p,w) \in S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \,\middle|\, (\forall i \in \mathcal{I}) : p \cdot w^i > K \right\}$.*

Proof. It suffices to show that for each $(p,w) \in D^K$, there exists $\left(\widehat{w}^i\right)_{i\in\mathcal{I}} \in \mathbb{R}^{LI}_{++}$ such that $\left(p, \left(\widehat{w}^i\right)_{i\in\mathcal{I}}\right) \in M^K$ and $p \cdot \widehat{w}^i = p \cdot w^i$ for all i. Again, let $\widehat{w}^i = f^i(p,w^i) \in \mathbb{R}^{L}_{++}$. The result follows by Walras's Law, $p \cdot \widehat{w}^i = p \cdot w^i > K$. ■

The largest domain on which, given $E \subseteq M$, strong identification is possible, is determined next.

Theorem 3 *If $E \subseteq M$ strongly identifies F over D, then $D \subseteq D_E$.*

Proof. Denote by $F_2(p,w)$ the second component of $F(p,w)$ and define $\Phi : S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \longrightarrow \mathbb{R}^{L}_{++}$ by

$$\Phi(p,w) = \begin{cases} F(p,w), \text{ if } (p,w) \in D_E \\ F(p,w) + \frac{F_2(p,w)}{2} \begin{bmatrix} p_2 & -1 & 0 & \cdots & 0 \end{bmatrix}^{\top}, \text{ if } (p,w) \notin D_E \end{cases}$$

Φ is well defined: it maps into $\mathbb{R}^L_{++}$, satisfies Walras's law, $\left(p \cdot \widehat{w}^i = p \cdot w^i\right)_{i \in \mathcal{I}} \Longrightarrow \Phi(p,w) = \Phi(p,\widehat{w})$ and $\left\{ (p,w) \in S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \middle| \Phi(p,w) = \sum_{i \in \mathcal{I}} w^i \right\} \supseteq E$, because $D_E \cap M \supseteq E$.

Suppose that there is $(p,w) \in D \backslash D_E$. Then, by definition, $\Phi_2(p,w) = \frac{F_2(p,w)}{2}$, and, since $(p,w) \in D$ and E identifies F over D then $\Phi(p,w) = F(p,w)$, implying that $F_2(p,w) = 0$, which is impossible. ■

Identification, however, requires a more involved argument, which assumes compactness of E. We present that argument in Appendix A1. This theorem determines the largest restriction of the domain over which identification of aggregate demand is possible. The "dual" question of how small a subset of the manifold can be if it is to allow for identification of the aggregate demand over a given subset of its domain, which is of lesser empirical relevance, is dealt with in Appendix A2.

Theorem 4 *If $E \subseteq M$ is compact and identifies F over D, then $D \subseteq \overline{D_E}$.*

Proof. See Appendix A1. ■

4 From Aggregate Demand to Individual Demands

If, contrary to our assumption, equilibrium prices are observable for situations in which the incomes of all individuals but one are zero, the argument above still holds and then it is straightforward that aggregate demand identifies individual demands: for all i, $f^i(p,w^i) = F(p,(\mathbf{0},\mathbf{0},...,w^i,...,\mathbf{0}))$. That argument, however, has been criticized by Chiappori et al. (2004): if observation of situations in which the endowments of all consumers but one are pegged at zero is possible, then the section of the manifold at that boundary is that one consumer's inverse demand. In our case, an immediate argument for identification of individual demands can obviously still be made by taking sequences of endowments that converge to zero for all individuals but one, for whom the endowment is kept constant,[3] but that argument would fail when, for example, only M^K is observed.

We now show that, under an additional assumption, one can identify individual demands, without resorting to boundary analysis. Our proof is somewhat similar to the one presented by Chiappori et al., but simpler in the sense that it does not need to invoke the Cartan-Kaehler theorem. As a consequence, for us it suffices that only the first few derivatives of the demand exist, so analyticity is not required.

For the sake of simplicity, we initially study the global setting introduced in the previous section. The case in which the aggregate demand is not globally known is presented afterwards.

[3] A referee expressed concern for this fact, which motivated us to develop corollary 2. We thank her or him for the observation.

4.1 The global case:

In this case, we can weaken Chiappori et al.'s regularity assumption as follows:

Condition 2 (Regularity) *For every individual i, $u^i \in \mathbf{C}^4\left(\mathbb{R}^L_{++}\right)$ and is differentiably strictly concave, and for every $p \in S^{L-1}_{++}$, there exist $w \in \mathbb{R}^L_{++}$, and $l, l' \in \{1, ..., L\} \setminus \{1\}$, such that $\frac{\partial^2 f^i_l}{\partial\left(w^i_1\right)^2} \neq 0$ and*

$$\begin{vmatrix} \frac{\partial^2 f^i_l}{\partial\left(w^i_1\right)^2}(p,w) & \frac{\partial^2 f^i_{l'}}{\partial\left(w^i_1\right)^2}(p,w) \\ \frac{\partial^3 f^i_l}{\partial\left(w^i_1\right)^3}(p,w) & \frac{\partial^3 f^i_{l'}}{\partial\left(w^i_1\right)^3}(p,w) \end{vmatrix} \neq 0$$

The previous condition is weaker not only because of the lower degree of differentiability required, but also because w only needs an existential, and not a universal, quantifier. The condition is indeed restrictive as it requires, for example, that income effects do not vanish. It also requires that there be at least three commodities. Intuitively, the condition requires that preferences be "complex enough" so as to generate the independence of income effects. As Chiappori et al. have pointed out, it suffices that individual demands have rank at least two for the condition to be met everywhere. Appendix A3, at the end of the paper, illustrates the point.

Under condition 2, each $f^i \in \mathbf{C}^3\left(S^{L-1}_{++} \times \mathbb{R}^L_{++}\right)$ and further satisfies Slutsky symmetry. By definition $\left(f^i\right)_{i\in\mathcal{I}}$ is such that $\sum_{i\in\mathcal{I}} f^i = F$.

We say that **aggregate demand globally identifies individual demands** if, given F, $\left(f^i\right)_{i\in\mathcal{I}}$ is the only profile of functions $\left(\varphi^i : S^{L-1}_{++} \times \mathbb{R}^L_{++} \to \mathbb{R}^L_{++} \in \mathbf{C}^3\left(S^{L-1}_{++} \times \mathbb{R}^L_{++}\right)\right)_{i\in\mathcal{I}}$, satisfying Walras's law and Slutsky symmetry, and such that $\sum_{i\in\mathcal{I}} \varphi^i = F$.

Theorem 5 *Aggregate demand identifies individual demands.*

Proof. Let $\left(\varphi^i : S^{L-1}_{++} \times \mathbb{R}^L_{++} \to \mathbb{R}^L_{++} \in \mathbf{C}^3\left(S^{L-1}_{++} \times \mathbb{R}^L_{++}\right)\right)_{i\in\mathcal{I}}$ satisfy Walras's law and Slutsky symmetry and be such that $F = \sum_{i\in\mathcal{I}} \varphi^i$. Fix $i \in \mathcal{I}$ and define $\theta^i : S^{L-1}_{++} \times \mathbb{R}^L_{++} \longrightarrow \mathbb{R}^L$ and $\gamma^i : S^{L-1}_{++} \longrightarrow \mathbb{R}^L$ by $\theta^i(p,w) = F(p, (\mathbf{1}, \mathbf{1}, ..., w, ..., \mathbf{1}))$, where w occupies the i^{th} position, and $\gamma^i(p) = -\sum_{j\in\mathcal{I}\setminus\{i\}} \varphi^j(p, \mathbf{1})$.[4]

By Slutsky symmetry, for every $l, l' \in \{1, ..., L\} \setminus \{1\}$, everywhere in $S^{L-1}_{++} \times \mathbb{R}^L_{++}$,

$$\frac{\partial \varphi^i_l}{\partial p_{l'}} + \left(\varphi^i_{l'} - w^i_{l'}\right)\frac{\partial \varphi^i_l}{\partial w^i_1} = \frac{\partial \varphi^i_{l'}}{\partial p_l} + \left(\varphi^i_l - w^i_l\right)\frac{\partial \varphi^i_{l'}}{\partial w^i_1}$$

Since $\varphi^i\left(p, w^i\right) = \theta^i\left(p, w^i\right) + \gamma^i(p)$, substituting,

$$\frac{\partial \theta^i_l}{\partial p_{l'}} + \frac{\partial \gamma^i_l}{\partial p_{l'}} + \left(\theta^i_{l'} + \gamma^i_{l'} - w^i_{l'}\right)\frac{\partial \theta^i_l}{\partial w^i_1} = \frac{\partial \theta^i_{l'}}{\partial p_l} + \frac{\partial \gamma^i_{l'}}{\partial p_l} + \left(\theta^i_l + \gamma^i_l - w^i_l\right)\frac{\partial \theta^i_{l'}}{\partial w^i_1}$$

[4] This is the step that simplifies the proof of Chiappori et al.

Fix $p \in S_{++}^{L-1}$. Taking that $l, l' \neq 1$ and deriving once and twice with respect to w_1^i gives

$$\frac{\partial^2 \theta_l^i}{\partial w_1^i \partial p_{l'}} + \left(\theta_{l'}^i + \gamma_{l'}^i - w_{l'}^i\right) \frac{\partial^2 \theta_l^i}{\partial \left(w_1^i\right)^2} = \frac{\partial^2 \theta_{l'}^i}{\partial w_1^i \partial p_l} + \left(\theta_l^i + \gamma_l^i - w_l^i\right) \frac{\partial^2 \theta_{l'}^i}{\partial \left(w_1^i\right)^2}$$

and

$$\frac{\partial^3 \theta_l^i}{\partial \left(w_1^i\right)^2 \partial p_{l'}} + \frac{\partial \theta_{l'}^i}{\partial w_1^i} \frac{\partial^2 \theta_l^i}{\partial \left(w_1^i\right)^2} + \left(\theta_{l'}^i + \gamma_{l'}^i - w_{l'}^i\right) \frac{\partial^3 \theta_l^i}{\partial \left(w_1^i\right)^3} =$$
$$\frac{\partial^3 \theta_{l'}^i}{\partial \left(w_1^i\right)^2 \partial p_l} + \frac{\partial \theta_l^i}{\partial w_1^i} \frac{\partial^2 \theta_{l'}^i}{\partial \left(w_1^i\right)^2} + \left(\theta_l^i + \gamma_l^i - w_l^i\right) \frac{\partial^3 \theta_{l'}^i}{\partial \left(w_1^i\right)^3}$$

which rewrites as (recall that p is fixed)

$$\Delta_{l,l'}\left(w^i\right) \begin{bmatrix} \gamma_{l'}^i(p) \\ \gamma_l^i(p) \end{bmatrix} = \Gamma_{l,l'}\left(w^i\right)$$

where

$$\Delta_{l,l'}\left(w^i\right) = \begin{bmatrix} \frac{\partial^2 \theta_l^i}{\partial \left(w_1^i\right)^2}\left(p, w^i\right) & -\frac{\partial^2 \theta_{l'}^i}{\partial \left(w_1^i\right)^2}\left(p, w^i\right) \\ \frac{\partial^3 \theta_l^i}{\partial \left(w_1^i\right)^3}\left(p, w^i\right) & -\frac{\partial^3 \theta_{l'}^i}{\partial \left(w_1^i\right)^3}\left(p, w^i\right) \end{bmatrix}$$

and $\Gamma_{l,l'}\left(w^i\right)$ is

$$\begin{bmatrix} \frac{\partial^2 \theta_{l'}^i}{\partial w_1^i \partial p_l} - \frac{\partial^2 \theta_l^i}{\partial w_1^i \partial p_{l'}} + \left(\theta_l^i - w_l^i\right) \frac{\partial^2 \theta_{l'}^i}{\partial \left(w_1^i\right)^2} - \left(\theta_{l'}^i - w_{l'}^i\right) \frac{\partial^2 \theta_l^i}{\partial \left(w_1^i\right)^2} \\ \frac{\partial^3 \theta_{l'}^i}{\partial \left(w_1^i\right)^2 \partial p_l} - \frac{\partial^3 \theta_l^i}{\partial \left(w_1^i\right)^2 \partial p_{l'}} + \frac{\partial \theta_l^i}{\partial w_1^i} \frac{\partial^2 \theta_{l'}^i}{\partial \left(w_1^i\right)^2} - \frac{\partial \theta_{l'}^i}{\partial w_1^i} \frac{\partial^2 \theta_l^i}{\partial \left(w_1^i\right)^2} + \left(\theta_l^i - w_l^i\right) \frac{\partial^3 \theta_{l'}^i}{\partial \left(w_1^i\right)^3} - \left(\theta_{l'}^i - w_{l'}^i\right) \frac{\partial^3 \theta_l^i}{\partial \left(w_1^i\right)^3} \end{bmatrix}$$

Notice that the resulting $\Delta_{l,l'}\left(w^i\right)$ and $\Gamma_{l,l'}\left(w^i\right)$ do not contain $\gamma_{l'}^i$ or $\gamma_{l'}^i$, but only θ^i, which is determined by F.

By regularity, for some $w^i \in \mathbb{R}_+^L$ and $l, l' \in \{1, ..., L\}$, $\Delta_{l,l'}\left(w^i\right)$ is invertible, so

$$\begin{bmatrix} \gamma_{l'}^i(p) \\ \gamma_l^i(p) \end{bmatrix} = \left(\Delta_{l,l'}\left(w^i\right)\right)^{-1} \Gamma_{l,l'}\left(w^i\right) \tag{*}$$

whereas, for every other $l'' \in \{1, ..., L\} \setminus \{1\}$, by Slutsky symmetry,

$$\gamma_{l''}^i(p) = \frac{\frac{\partial^2 \theta_{l''}^i}{\partial w_1^i \partial p_l} - \frac{\partial^2 \theta_l^i}{\partial w_1^i \partial p_{l''}} + \left(\theta_l^i\left(p, w^i\right) + \gamma_l^i(p) - w_l^i\right) \frac{\partial^2 \theta_{l''}^i}{\partial \left(w_1^i\right)^2} - \left(\theta_{l''}^i\left(p, w^i\right) - w_{l''}^i\right) \frac{\partial^2 \theta_l^i}{\partial \left(w_1^i\right)^2}}{\frac{\partial^2 \theta_l^i}{\partial \left(w_1^i\right)^2}}$$

and, by Walras's law, $\gamma_1^i(p) = -\sum_{l''=2}^L p_{l''} \gamma_{l''}^i(p)$.

Since $\varphi^i\left(p, w^i\right) = \theta^i\left(p, w^i\right) + \gamma^i(p)$ and the expression on the right hand side of equation (*) depends only on F, it follows that $\varphi_l^i = f_l^i$, which implies that $\varphi^i = f^i$.

It follows that $\left(\varphi^i\right)_{i=1}^I = \left(f^i\right)_{i=1}^I$. ■

4.2 Restricted observation

It must be noticed that the choice of $\left(w^j\right)_{j\in\mathcal{I}\setminus\{i\}} = (\mathbf{1})_{j\in\mathcal{I}\setminus\{i\}}$ in the proof of theorem 5 is arbitrary and that in the local case, or when only $F_{|D}$ is available, such choice can be modified as needed.

In the case in which only M^K has been observed, and hence F has been identified only over D^K, condition 2 needs to be strengthened to require that $p \cdot w > K$. If only local information of F is available, one must strengthen the second part of the assumption so that the profile of endowments at which the condition is met lies in the observed domain. In any case, $\mathbf{C}^4$ still suffices. For the sake of simplicity, assume that we strengthen condition 2 by substituting the existential quantifier of w by the universal quantifier.

Let $D \subseteq S^{L-1}_{++} \times \mathbb{R}^{LI}_{++}$ and, for every i, let $D^i \subseteq S^{L-1}_{++} \times \mathbb{R}^{L}_{++}$. We say that $F_{|D}$ **identifies** $\left(f^i\right)_{i\in\mathcal{I}}$ **over** $\left(D^i\right)_{i\in\mathcal{I}}$ if for every $\left(\varphi^i : S^{L-1}_{++} \times \mathbb{R}^{L}_{++} \longrightarrow \mathbb{R}^{L}_{++}\right)_{i\in\mathcal{I}}$, satisfying Walras's law, $\left(p \cdot \widehat{w}^i = p \cdot w^i\right)_{i\in\mathcal{I}} \Longrightarrow \left(\varphi^i(p, w^i)\right)_{i\in\mathcal{I}} = \left(\varphi^i(p, \widehat{w}^i)\right)_{i\in\mathcal{I}}$ and Slutsky symmetry, and such that

$$\left\{ (p, w) \in S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \middle| \sum_{i\in\mathcal{I}} \varphi^i\left(p, w^i\right) = F\left(p, w\right) \right\} \supseteq D$$

it is true that $\left(\varphi^i_{|D^i}\right)_{i\in\mathcal{I}} = \left(f^i_{|D^i}\right)_{i\in\mathcal{I}}$.

Intuitively, we say that $F_{|D}$ identifies $\left(f^i\right)_{i\in\mathcal{I}}$ over $\left(D^i\right)_{i\in\mathcal{I}}$ if, for any profile of functions that cannot be ruled out as individual demands, we have that, on the restricted domains $\left(D^i\right)_{i\in\mathcal{I}}$, those functions are identical to the true demand functions. A profile of functions cannot be ruled out as individual demands when (i) it satisfies the properties that are necessary for a profile of individual demands, and (ii) it is consistent with the observed data (because at all observed prices and endowments, the aggregate of the functions equals the observed aggregate demand).

Theorem 6 *Let $D \subseteq S^{L-1}_{++} \times \mathbb{R}^{LI}_{++}$ and, for each $i \in \mathcal{I}$, denote*

$$D^i = \overline{\left\{ (p, w^i) \in S^{L-1}_{++} \times \mathbb{R}^{L}_{++} \left| \left(\exists \left(\widehat{p}, \widehat{w}\right) \in D^0\right) : \widehat{p} = p \text{ and } \widehat{p} \cdot \widehat{w}^i = \widehat{p} \cdot w^i \right.\right\}}$$

where D^0 is the interior of D. $F_{|D}$ identifies $\left(f^i\right)_{i\in\mathcal{I}}$ over $\left(D^i\right)_{i\in\mathcal{I}}$.

Proof. Fix $\left(\varphi^i : S^{L-1}_{++} \times \mathbb{R}^{L}_{++} \longrightarrow \mathbb{R}^{L}_{++}\right)_{i\in\mathcal{I}}$, satisfying Walras's law and Slutsky symmetry, and such that for every $(p, w) \in D$, $\sum_{i\in\mathcal{I}} \varphi^i\left(p, w^i\right) = F\left(p, w\right)$.

Fix $i \in \mathcal{I}$ and $\left(p, w^i\right) \in D^i_0$, where D^i_0 is the projection of D^0 into the space of $\left(p, w^i\right)$. By definition, there exists $\left(w^j\right)_{j\in\mathcal{I}\setminus\{i\}}$ such that $\left(p, \left(w^j\right)_{j=1}^{I}\right) \in D^0$. Since D^0 is open, there exists $\epsilon > 0$ such that $\left\{ (\widetilde{p}, \widetilde{w}) \in S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \middle| \, \|(\widetilde{p}, \widetilde{w}) - (p, w)\| < \epsilon \right\} \subseteq D^0$. Denote $O = \left\{ \widetilde{p} \in S^{L-1}_{++} \middle| \, \|\widetilde{p} - p\| < \epsilon \right\}$ and $O^i = \left\{ \left(\widetilde{p}, \widetilde{w}^i\right) \in S^{L-1}_{++} \times \mathbb{R}^{L}_{++} \middle| \, \left\|\left(\widetilde{p}, \widetilde{w}^i\right) - \left(p, w^i\right)\right\| < \epsilon \right\}$, and define $\theta^i : O^i \longrightarrow \mathbb{R}^L$ and $\gamma^i : O \longrightarrow \mathbb{R}^L$, as $\theta^i\left(\widetilde{p}, \widetilde{w}^i\right) = F\left(\widetilde{p}, \left(w^1, w^2, ..., \widetilde{w}^i, ..., w^I\right)\right)$, where

$\widetilde{w}^i$ occupies the i^{th} position, and $\gamma^i(\widetilde{p}) = -\sum_{j\in\mathcal{I}\setminus\{i\}}\varphi^j\left(\widetilde{p}, w^j\right)$. By Slutsky symmetry and regularity, as in the proof of theorem 5, $\varphi^i\left(p, w^i\right) = f^i\left(p, w^i\right)$.

Now, let $\left(\overline{p}, \overline{w}^i\right) \in D^i$. By definition, there exists $\left(p_n, w_n^i\right)_{n=1}^{\infty}$ such that $\left(p_n, w_n^i\right) \to \left(\overline{p}, \overline{w}^i\right)$ and, for each $n \in \mathbb{N}$, there exists $\widetilde{w}_n \in \mathbb{R}_{++}^{LI}$ such that $(p_n, \widetilde{w}_n) \in D^0$ and $p_n \cdot \widetilde{w}_n^i = p_n \cdot w_n^i$. Therefore, $\varphi^i\left(p_n, \widetilde{w}_n^i\right) = \varphi^i\left(p_n, w_n^i\right)$ and $f^i\left(p_n, \widetilde{w}_n^i\right) = f^i\left(p_n, w_n^i\right)$ and, by continuity, $\varphi^i\left(p_n, \widetilde{w}_n^i\right) = \varphi^i\left(p_n, w_n^i\right) \to \varphi^i\left(\overline{p}, \overline{w}^i\right)$ and $f^i\left(p_n, \widetilde{w}_n^i\right) = f^i\left(p_n, w_n^i\right) \to f^i\left(\overline{p}, \overline{w}^i\right)$. Since $\left(p_n, \widetilde{w}_n^i\right) \in D_0^i$, it follows from our previous argument that $\varphi^i\left(p_n, \widetilde{w}_n^i\right) = f^i\left(p_n, \widetilde{w}_n^i\right)$, and hence that $\varphi^i\left(\overline{p}, \overline{w}^i\right) = f^i\left(\overline{p}, \overline{w}^i\right)$. ■

5 Concluding remarks

We offer a new proof that the equilibrium manifold identifies individual demands. We do so by combining the ideas of Balasko (2004) and Chiappori et al. (2004) in such a way that we avoid Balasko's argument at the boundary of the space of endowments and Chiappori et al.'s analyticity assumption. We also believe that our proof retains much of the simplicity that distinguishes Balasko's argument from Chiappori et al.'s more involved idea.

This result is surprising as it implies that the belief derived from the Sonnenschein-Mantel-Debreu result suggesting that all the structure imposed by individual rationality disappears upon aggregation lacks foundation. The fundamental questions that remain open in this literature, namely to what extent the result extends to more realistic economies (e.g. with production or with externalities) or more realistic data sets (e.g. a path of endowments and equilibrium prices in a dynamic economy) are not addressed by the paper.

Appendix A1: proof of theorem 4

Lemma 1 *If E is compact, then $D_E = \overline{D_E}$.*

Proof. Let $(p_n, w_n)_{n=1}^{\infty}$ be a sequence defined in D_E such that $(p_n, w_n) \longrightarrow (p, w)$. By definition, there exists $(\widehat{w}_n)_{n=1}^{\infty}$ such that $(p_n, \widehat{w}_n)_{n=1}^{\infty}$ is a sequence in E such that $\left(p_n \cdot \widehat{w}_n^i\right)_{n=1}^{\infty} = \left(p_n \cdot w_n^i\right)_{n=1}^{\infty}$ for all i. Since E is compact, we can take a convergent subsequence $\left(p_{k(n)}, \widehat{w}_{k(n)}\right)_{n=1}^{\infty}$ such that $\left(p_{k(n)}, \widehat{w}_{k(n)}\right) \longrightarrow (p, \widehat{w}) \in E$ for some $\widehat{w} \in \mathbb{R}_{++}^{LI}$. By construction, $p_{k(n)} \cdot w_{k(n)} \longrightarrow p \cdot w$, while $p_{k(n)} \cdot \widehat{w}_{k(n)} \longrightarrow p \cdot \widehat{w}$. Since $\left(p_{k(n)} \cdot \widehat{w}_{k(n)}\right)_{n=1}^{\infty} = \left(p_{k(n)} \cdot w_{k(n)}\right)_{n=1}^{\infty}$, it follows that $p \cdot w = p \cdot \widehat{w}$. ■

With the previous lemma, we can simply prove that $D \subseteq D_E$, as follows:

Proof of Theorem 4. Suppose that $D \nsubseteq D_E$. Let $(\overline{p}, \overline{w}) \in D \backslash D_E$. Since, by the lemma, D_E is closed, we know that for some $\delta > 0$, $B_\delta(\overline{p}) \times \prod_{i\in\mathcal{I}} B_\delta\left(\overline{w}^i\right) \subseteq (D_E)^c$, where $(D_E)^c$ denotes the complement of D_E in $S_{++}^{L-1} \times \mathbb{R}_{++}^{LI}$.

Define the following sets:

$$\begin{aligned}
K &= \left\{ (p,k) \in S_{++}^{L-1} \times \mathbb{R}_{++}^{I} \middle| (\exists (\widehat{p}, \widehat{w}) \in E) : \widehat{p} = p \ \&\ \left(\widehat{p} \cdot \widehat{w}^i\right)_{i\in\mathcal{I}} = k \right\} \\
\mathbb{K} &= \left\{ (p,k) \in S_{++}^{L-1} \times \mathbb{R}_{++}^{I} \middle| \left(\exists (\widehat{p}, \widehat{w}) \in \overline{B_{\frac{\delta}{2}}(\overline{p})} \times \prod_{i\in\mathcal{I}} \overline{B_{\frac{\delta}{2}}\left(\overline{w}^i\right)}\right) : \widehat{p} = p \ \&\ \left(\widehat{p} \cdot \widehat{w}^i\right)_{i\in\mathcal{I}} = k \right\} \\
\mathcal{K} &= K \cup \left(S_{++}^{L-1} \times \left(\prod_{i\in\mathcal{I}} \left(\underline{k}^i, \overline{k}^i\right)\right)^c \right)
\end{aligned}$$

where for each i,

$$\begin{aligned}
\underline{k}^i &= \min_{(p,w)\in\overline{B_\delta(\overline{p})}\times\overline{B_\delta(\overline{w}^i)}} p \cdot w \\
\overline{k}^i &= \max_{(p,w)\in\overline{B_\delta(\overline{p})}\times\overline{B_\delta(\overline{w}^i)}} p \cdot w
\end{aligned}$$

Since E is compact, K and, hence, $\mathcal{K}$ are closed. Similarly, $\mathbb{K}$ is closed.

We divide the proof in four consecutive steps.

Step 1: We first prove that $\mathcal{K} \cap \mathbb{K} = \phi$.

Suppose that $(p,k) \in \mathcal{K} \cap \mathbb{K}$. Since $\mathcal{K} \cap \mathbb{K} = K \cap \mathbb{K} \cup \left(S_{++}^{L-1} \times \left(\prod_{i\in\mathcal{I}} \left(\underline{k}^i, \overline{k}^i\right)\right)^c \right) \cap \mathbb{K}$, then $(p,k) \in K \cap \mathbb{K}$ or $(p,k) \in \left(S_{++}^{L-1} \times \left(\prod_{i\in\mathcal{I}} \left(\underline{k}^i, \overline{k}^i\right)\right)^c \right) \cap \mathbb{K}$. If $(p,k) \in K \cap \mathbb{K}$, then, there exist $\widehat{w}, \widetilde{w} \in \mathbb{R}_{++}^{LI}$, such that $(p, \widehat{w}) \in E$, $(p, \widetilde{w}) \in \overline{B_{\frac{\delta}{2}}(\overline{p})} \times \prod_{i\in\mathcal{I}} \overline{B_{\frac{\delta}{2}}\left(\overline{w}^i\right)}$ and $\left(p \cdot \widehat{w}^i\right)_{i\in\mathcal{I}} = \left(p \cdot \widetilde{w}^i\right)_{i\in\mathcal{I}}$, which is impossible because $B_\delta(\overline{p}) \times \prod_{i\in\mathcal{I}} B_\delta\left(\overline{w}^i\right) \subseteq \overline{D}_E^c$. If $(p,k) \in \left(S_{++}^{L-1} \times \left(\prod_{i\in\mathcal{I}} \left(\underline{k}^i, \overline{k}^i\right)\right)^c \right) \cap \mathbb{K}$ then $k \in \left(\prod_{i\in\mathcal{I}} \left(\underline{k}^i, \overline{k}^i\right)\right)^c$ and there exists $(p, \widetilde{w}) \in$

$\overline{B_{\frac{\delta}{2}}(\overline{p})} \times \prod_{i\in\mathcal{I}} \overline{B_{\frac{\delta}{2}}(\overline{w}^i)}$ such that $\left(p\cdot\widetilde{w}^i\right)_{i\in\mathcal{I}} = \left(k^i\right)_{i\in\mathcal{I}}$. By the second implication, since $(p,\widetilde{w}) \in \overline{B_{\frac{\delta}{2}}(\overline{p})} \times \prod_{i\in\mathcal{I}} \overline{B_{\frac{\delta}{2}}(\overline{w}^i)}$ then $(p,\widetilde{w}) \in \overline{B_\delta(\overline{p})} \times \prod_{i\in\mathcal{I}} \overline{B_\delta(\overline{w}^i)}$ therefore $\left(p\cdot\widetilde{w}^i\right)_{i\in\mathcal{I}} \in \left(\prod_{i\in\mathcal{I}}\left(\underline{k}^i, \overline{k}^i\right)\right)$ a contradiction.

Step 2: We now construct a function Φ.
Define $d : S^{L-1}_{++} \times \mathbb{R}^I_{++} \longrightarrow \mathbb{R}_{++}$ by

$$d(p,k) = F_2\left(p, \left(\frac{k^i}{p\cdot\mathbf{1}}\mathbf{1}\right)_{i\in\mathcal{I}}\right)$$

which is continuous, and fix

$$0 < r < \min_{(p,k)\in\overline{B_\delta(\overline{p})}\times\prod\limits_{i\in\mathcal{I}}\left[\underline{k}^i,\overline{k}^i\right]} d(p,k)$$

Since $\mathcal{K}$ and $\mathbb{K}$ are closed disjoint sets (step 1) one can construct a continuous function, $\Delta_1 : S^{L-1}_{++} \times \mathbb{R}^I_{++} \longrightarrow [0,r]$, such that:

$$\begin{array}{rcl}
(\forall\,(p,k)\in\mathcal{K}) & : & \Delta_1(p,k) = 0 \\
(\forall\,(p,k)\in\mathcal{K}^c) & : & \Delta_1(p,k) \neq 0 \\
(\forall\,(p,k)\in\mathbb{K}) & : & \Delta_1(p,k) = r
\end{array}$$

For the same reason we can define a continuous function $\Delta_0 : S^{L-1}_{++} \longrightarrow [0,1]$ such that

$$\begin{array}{rcl}
\left(\forall p\in B_\delta(\overline{p})^c\right) & : & \Delta_0(p) = 0 \\
\left(\forall p\in B_\delta(\overline{p})\right) & : & \Delta_0(p) \neq 0 \\
\left(\forall p\in \overline{B_{\frac{\delta}{2}}(\overline{p})}\right) & : & \Delta_0(p) = 1
\end{array}$$

Finally, define $\Phi : S^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \longrightarrow \mathbb{R}^L$ as

$$\Phi(p,w) = F(p,w) + \Delta_0(p)\,\Delta_1\left(p, \left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\begin{bmatrix} p_2 & -1 & 0 & \cdots & 0\end{bmatrix}^\top$$

Step 3: We now prove that Φ is well defined: it maps into $\mathbb{R}^L_{++}$, is continuous, satisfies Walras's law, $\left(p\cdot w^i = p\cdot\widehat{w}^i\right)_{i\in\mathcal{I}} \Longrightarrow \Phi(p,w) = \Phi(p,\widehat{w})$ and

$$E \subseteq \left\{(p,w)\in S^{L-1}_{++}\times\mathbb{R}^{LI}_{++}\,\middle|\,\Phi(p,w) = \sum_{i\in\mathcal{I}} w^i\right\}$$

By construction, Φ is obviously continuous and satisfies Walras's law. To show that it is well defined suppose that (p,w) is such that $\Phi(p,w) \neq F(p,w)$. By construction

$\Delta_0(p)\,\Delta_1\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\neq 0$ therefore, $p\in B_\delta(\overline{p})$ and $\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\notin\mathcal{K}$. By definition of $\mathcal{K}$ this implies $\left(p\cdot w^i\right)_{i\in\mathcal{I}}\prod_{i\in\mathcal{I}}\left[\underline{k}^i,\overline{k}^i\right]$, therefore, $\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\in\overline{B_\delta(\overline{p})}\times\prod_{i\in\mathcal{I}}\left[\underline{k}^i,\overline{k}^i\right]$ and, hence, $d\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)>r$. Now, since

$$\left(p\cdot\left(\frac{p\cdot w^i}{p\cdot\mathbf{1}}\mathbf{1}\right)\right)_{i\in\mathcal{I}}=\left(p\cdot w^i\right)_{i\in\mathcal{I}}$$

it follows that

$$\begin{aligned}\Phi_2(p,w) &= F_2(p,w)-\Delta_0(p)\,\Delta_1\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\\ &= F_2\left(p,\left(\frac{p\cdot w^i}{p\cdot\mathbf{1}}\mathbf{1}\right)_{i\in\mathcal{I}}\right)-\Delta_0(p)\,\Delta_1\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\\ &= d\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)-\Delta_0(p)\,\Delta_1\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\\ &> r-\Delta_0(p)\,\Delta_1\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\\ &\geq 0\end{aligned}$$

since, by construction, $\Delta_0(p)\,\Delta_1\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\leq r$. This proves that Φ *maps into* $\mathbb{R}^L_{++}$.

Now, let $(p,w)\in E$. By construction, $\left(p,\left(p\cdot w^i\right)_{i\in\mathcal{I}}\right)\in\mathcal{K}$ and, hence, $\Phi(p,w)=F(p,w)$ whereas $F(p,w)=\sum_{i\in\mathcal{I}}w^i$. Then, $E\subseteq\left\{(p,w)\in S^{L-1}_{++}\times\mathbb{R}^{LI}_{++}\middle|\,\Phi(p,w)=\sum_{i\in\mathcal{I}}w^i\right\}$.

Step 4: Finally, using that $\Phi_{|D}=F_{|D}$, we derive a contradiction.

Since $(\overline{p},\overline{w})\in D$, $\overline{p}\in\overline{B_{\frac{\delta}{2}}(\overline{p})}$ and $\left(\overline{p},\left(\overline{p}\cdot\overline{w}^i\right)_{i\in\mathcal{I}}\right)\in\mathbb{K}$, therefore $F_2(\overline{p},\overline{w})=\Phi_2(\overline{p},\overline{w})$, $\Delta_0(\overline{p})=1$ and $\Delta_1\left(\overline{p},\left(\overline{p}\cdot\overline{w}^i\right)_{i\in\mathcal{I}}\right)=r$. Hence,

$$\begin{aligned}F_2(\overline{p},\overline{w}) &= \Phi_2(\overline{p},\overline{w})\\ &= F_2(\overline{p},\overline{w})-\Delta_0(\overline{p})\,\Delta_1\left(\overline{p},\left(\overline{p}\cdot\overline{w}^i\right)_{i\in\mathcal{I}}\right)\\ &= F_2(\overline{p},\overline{w})-r\\ &< F_2(\overline{p},\overline{w})\end{aligned}$$

■

Appendix A2: duality

Given $D\subseteq S^{L-1}_{++}\times\mathbb{R}^{LI}_{++}$, define

$$E_D=\left\{(p,w)\in M\middle|\left(\exists\widetilde{w}\in\mathbb{R}^{LI}_{++}\right):(p,\widetilde{w})\in D\text{ and }\left(f^i\left(p,\widetilde{w}^i\right)\right)_{i\in\mathcal{I}}=w\right\}$$

Theorem 7 *For any* $D \subseteq S_{++}^{L-1} \times \mathbb{R}_{++}^{LI}$*,* E_D *identifies* F *over* D*.*

Proof. It suffices to show that $D \subseteq D_{E_D}$. Let $(p,w) \in D$, $\widehat{w} = \left(f^i\left(p,w^i\right)\right)_{i\in\mathcal{I}}$ and $\widetilde{w} = w$. It is straightforward that $(p,\widehat{w}) \in M$ and that $(p,\widetilde{w}) \in D$ and $\left(f^i\left(p,\widetilde{w}^i\right)\right)_{i\in\mathcal{I}} = \widehat{w}$, so $(p,\widehat{w}) \in E_D$. Moreover, since $\left(p \cdot \widehat{w}^i\right)_{i\in\mathcal{I}} = \left(p \cdot w^i\right)_{i\in\mathcal{I}}$, it follows that $(p,w) \in D_{E_D}$. ■

Even if D is compact, it is not true that if E identifies F over D, then $E \supseteq E_D$ However,

Theorem 8 *For any* $D \subseteq S_{++}^{L-1} \times \mathbb{R}_{++}^{LI}$*, if* E*, compact, identifies* F *over* D*, then for every* $(p,w) \in E_D$ *there exists* $\left(\delta^i\right)_{i\in\mathcal{I}} \in \left([p]^{\perp}\right)^I$ *such that* $\sum_{i\in\mathcal{I}} \delta^i = 0$ *and* $(p, w+\delta) \in E$*.*

Proof. Since E is compact, it follows from theorem 4 that $D \subseteq D_E$. Fix $(p,w) \in \underline{E}_D \subseteq M$. By definition, there exists $\widetilde{w} \in \mathbb{R}_{++}^{LI}$ such that$(p,\widetilde{w}) \in D$ and $\left(f^i\left(p,\widetilde{w}^i\right)\right)_{i\in\mathcal{I}} = w$. Since $D \subseteq \overline{D}_E$, it follows that for some $\widehat{w} \in \mathbb{R}_{++}^{LI}$, $(p,\widehat{w}) \in E$ and $\left(p \cdot \widehat{w}^i\right)_{i\in\mathcal{I}} = \left(p \cdot \widetilde{w}^i\right)_{i\in\mathcal{I}}$. Define $\delta^i = \widehat{w}^i - w^i$, for each i. That $(p, w+\delta) \in E$ is immediate. Also, since $p \cdot \delta^i = p \cdot \left(\widehat{w}^i - w^i\right) = p \cdot \widehat{w}^i - p \cdot f^i\left(p,\widetilde{w}^i\right) = 0$, because $p \cdot \widehat{w}^i = p \cdot \widetilde{w}^i$, it follows that $\left(\delta^i\right)_{i\in\mathcal{I}} \in \left([p]^{\perp}\right)^I$. Then, since $(p,w), (p,\widehat{w}) \in M$, it is immediate that $\sum_{i\in\mathcal{I}} \delta^i = \sum_{i\in\mathcal{I}} \left(\widehat{w}^i - w^i\right) = \sum_{i\in\mathcal{I}} \widehat{w}^i - \sum_{i\in\mathcal{I}} w^i = F(p,\widehat{w}) - F(p,\widetilde{w}) = 0$, because $\left(p \cdot \widehat{w}^i\right)_{i\in\mathcal{I}} = \left(p \cdot \widetilde{w}^i\right)_{i\in\mathcal{I}}$. ■

Appendix A3: an example of regularity

Consider a particular case of a deflated income demand system (see Lewbel, 2003, and Banks et al., 1997) for three commodities, $l = 1,2,3$:

$$f_l(p,m) = \frac{m}{p_l}\left(A_l(p) + B_l(p)\log\left(\frac{m}{a(p)}\right) + C_l(p)\log\left(\frac{m}{a(p)}\right)^2\right)$$

where m denotes income, which suffices for the purposes of the example, A_l, B_l and C_l are homogeneous of degree zero in p and a is homogeneous of degree 1 in $p \in \mathbb{R}_{++}^3$ (these two conditions guarantee that f_l is homogeneous of degree zero in p and w) and

$$\sum_{l=1}^{3} A_l(p) + \sum_{l=1}^{3} B_l(p)\log\left(\frac{m}{a(p)}\right) + \sum_{l=1}^{3} C_l(p)\log\left(\frac{m}{a(p)}\right)^2 = 1$$

for all p and m.

Now, the rank of system $f(p,m)$ is, by definition, the rank of:

$$\begin{bmatrix} A_1(p) & B_1(p) & C_1(p) \\ A_2(p) & B_2(p) & C_2(p) \\ A_3(p) & B_3(p) & C_3(p) \end{bmatrix}$$

If B_l and C_l are zero then the system is of rank 1, the utility function is homothetic and clearly the regularity condition does not hold. If $B_2(p)\,C_3(p) - B_2(p)\,C_3(p) \neq 0$ the system has rank at least 2. Below, we prove that, for this case, the regularity condition holds.

Restricting p to S^2_{++},

$$\begin{aligned}\frac{\partial f_l}{\partial m}(p,m) &= \frac{1}{p_l}\left(A_l(p) + B_l(p)\log\left(\frac{m}{a(p)}\right) + C_l(p)\log\left(\frac{m}{a(p)}\right)^2\right)\\ &\quad + \frac{1}{p_l}\left(B_l(p) + 2C_l(p)\log\left(\frac{m}{a(p)}\right)\right)\end{aligned}$$

$$\frac{\partial^2 f_l}{\partial m^2}(p,m) = \frac{1}{p_l m}\left(B_l(p) + 2C_l(p)\left(\log\left(\frac{m}{a(p)}\right)+1\right)\right)$$

$$\frac{\partial^3 f_1}{\partial m^3}(p,m) = -\frac{1}{p_l\,(m)^2}\left(B_l(p) + 2C_l(p)\log\left(\frac{m}{a(p)}\right)\right)$$

It follows that the regularity condition is satisfied if

$$\begin{vmatrix} \frac{1}{p_2 m}\left(B_2(p) + 2C_2(p)\left(\log\left(\frac{m}{a(p)}\right)+1\right)\right) & \frac{1}{p_3 m}\left(B_3(p) + 2C_3(p)\left(\log\left(\frac{m}{a(p)}\right)+1\right)\right) \\ -\frac{1}{p_2 m^2}\left(B_2(p) + 2C_2(p)\log\left(\frac{m}{a(p)}\right)\right) & -\frac{1}{p_3 m^2}\left(B_3(p) + 2C_3(p)\log\left(\frac{m}{a(p)}\right)\right) \end{vmatrix} \neq 0$$

$$\Leftrightarrow \begin{vmatrix} \left(B_2(p) + 2C_2(p)\left(\log\left(\frac{m}{a(p)}\right)+1\right)\right) & \left(B_3(p) + 2C_3(p)\left(\log\left(\frac{m}{a(p)}\right)+1\right)\right) \\ \left(B_2(p) + 2C_2(p)\log\left(\frac{m}{a(p)}\right)\right) & \left(B_3(p) + 2C_3(p)\log\left(\frac{m}{a(p)}\right)\right) \end{vmatrix} \neq 0$$

$$\Leftrightarrow \begin{vmatrix} C_2(p) & C_3(p) \\ \left(B_2(p) + 2C_2(p)\log\left(\frac{m}{a(p)}\right)\right) & \left(B_3(p) + 2C_3(p)\log\left(\frac{m}{a(p)}\right)\right) \end{vmatrix} \neq 0$$

$$\Leftrightarrow C_2(p)B_3(p) - C_3(p)B_2(p) \neq 0$$

A rank 3 system clearly satisfies this condition and, if the condition is satisfied, then the rank is at least 2.

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