# On the Stability of a Periodic Solution of Distributed Parameters Biochemical System

Abdou Khadry Dramé, Frédéric Mazenc, Peter Wolenski

# On the Stability of a Periodic Solution of Distributed Parameters Biochemical System

Abdou K. Drame[1], Frédéric Mazenc[2] and Peter R. Wolenski[3]

[1] Dept. of Math., LaGuardia C. College, City Univ. of New York, USA (adrame@lagcc.cuny.edu)

[2] EPI INRIA DISCO, CNRS-L2S-Supelec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France (frederic.mazenc@lss.supelec.fr)

[3] Dept. of Math., Louisiana State University, Baton Rouge, LA, USA (wolenski@math.lsu.edu)

**Résumé**

This paper studies the stability of periodic solutions of distributed parameters biochemical system with periodic input $S_{in}(t)$. We prove that if $S_{in}(t)$ is periodic then the system has a periodic solution that is input to state stable when small perturbations are acting on the input concentration $S_{in}(t)$.

**Keywords.** Stability, Lyapunov functional, biochemical system, partial differential equations.

## 1 Introduction

The aim of this work is to prove the existence and stability of periodic solutions of a model describing a biochemical reactor with periodic input $S_{in}(t)$. Periodic solutions arise in many bioengineering systems because of the often periodically time varying environments. In the last decades, the existence of such periodic solutions has been extensively investigated by many authors to understand oscillations observed in many chemostat experiments (see eg. [25], [16], [20], and references therein). The chemostat is an experimental device used to understand the dynamics of biological, biochemical or ecological systems and in which the components of the systems are only time varying. Parallel to chemostat systems, the dynamical analysis and control of tubular (bio)chemical reactors have also motivated many research activities over the last decades (see eg. [1], [2], [5], [6], [11], [24], etc. and references therein). These studies are mostly focused on existence and asymptotic behavior of state trajectories, control and observability of the systems, in which a linearization of the system is the underlying tool. Following the ideas in the theoretical and experimental results in chemostat studies, recently Drame et al. ([7], [8]) studied the existence of periodic and almost periodic solutions of distributed parameters biochemical systems. It was shown that periodic solutions of a time delay system exist with a constant input $S_{in}$ in [7], and with time varying input $S_{in}(\cdot)$ in [8], but both studies lack a stability analysis. Pilyugin and Waltman [22] studied a reaction-diffusion system describing an unstirred chemostat and prove the existence of periodic solutions based on a system reduction technique. However, the system in [22] is monotone and the method cannot be applied to the systems considered here or in [7], [8].

It is natural to assume in a tubular biochemical reactor's model that the input nutrient concentration $S_{in}(t)$ is time dependent and periodic in the time $t$. The dynamical system under consideration in the current paper uses this assumption in a model inolving a diffusion-transport partial differential equation coupled with a nonlinear ordinary differential equation. The coupling term involves both the biomass and substrate. The justification of the model is derived from work performed on anaerobic digestion in the pilot fixed bed reactor of the LBE-INRA in Narbonne (France) and is validated on the process (see [1], [23]). Our main result involves a Lyapunov functional technique that analyzes and proves that if we replace $S_{in}(t)$ by $S_{in}(t) + a(t)$, where $a(t)$ is a small perturbation, then this will have small effect on the periodic solution. The paper is organized as follows. The model, background, and preliminary results are given in Section 2. Section 3 introduces an auxiliary system and gives an existence result for a solution. Section 4 is devoted to the existence problem of a periodic solution of the main system under study. The main new results are contained in Section 5 where a stability analysis is presented.

# 2 Notation, the model and preliminary results

## 2.1 Notation and Schauder's Fixed Point Theorem

The notation is standard and will be simplified whenever no confusion can arise from the context. The Euclidean norm of vectors of any dimension is denoted $|\cdot|$. For a funtion $\varphi \in L^2(0,1)$, the $L^2$ norm is $\|\varphi\|_{L^2} = \sqrt{\int_0^1 |\varphi(m)|^2 dm}$. We let $\mathcal{Z} = C[0,\,1] \times C[0,\,1]$.

The set of modulus functions is denoted by $\mathcal{K}_\infty$ and consists of all continuous functions $\gamma : [0,\infty) \to [0,\infty)$ satisfying (i) $\gamma(0) = 0$, (ii) $\gamma(.)$ is strictly increasing, and (iii) $\gamma(r) \nearrow +\infty$ as $r \to \infty$.

We recall the Schauder's Fixed Point Theorem [17, p. 126], which will be later invoked to provide existence result. Let $\mathcal{X}$ be Banach space and $\mathcal{D} \subseteq \mathcal{X}$. Recall that a completely continuous function $A(.) : \mathcal{D} \to \mathcal{X}$ is a continuous function that maps bounded sets into relatively compact ones.

**Theorem 2.1 (Schauder's Theorem)** *Suppose that $\mathcal{D}$ is a closed bounded convex subset of a Banach space $\mathcal{X}$ and $A(.) : \mathcal{D} \to \mathcal{X}$ is a completely continuous function with $A(\mathcal{D}) \subseteq \mathcal{D}$. Then there is a point $z \in \mathcal{D}$ such that $Az = z$.*

## 2.2 The Model

Applying the mass balance principles to the limiting substrate concentration $S(t,z)$ and the living biomass concentration $X(t,z)$ leads to the following dynamical system :

$$\begin{aligned} \frac{\partial S}{\partial t}(t,z) &= d\frac{\partial^2 S}{\partial z^2}(t,z) - q\frac{\partial S}{\partial z}(t,z) - k\mu(S(t,z),X(t,z))X(t,z) \;, \\ \frac{\partial X}{\partial t}(t,z) &= -k_d X(t,z) + \mu(S(t,z),X(t,z))X(t,z) \;, \end{aligned} \tag{2.1}$$

with the boundary conditions

$$d\frac{\partial S}{\partial z}(t,0) - qS(t,0) + qS_{in}(t) = 0 \quad \text{and} \quad \frac{\partial S}{\partial z}(t,L) = 0 \quad \text{for all} \quad t \geq 0 \tag{2.2}$$

and initial conditions :

$$S(0,z) = S_0(z), \quad X(0,z) = X_0(z), \quad \text{for all} \quad z \in [0,\,L] \;, \tag{2.3}$$

with $S_0 \in C[0,\,L]$ and $X_0 \in C[0,\,L]$. The parameters $d, q, k, k_d, \mu, L$ are all positive and represent respectively the diffusion coefficient, the superficial fluid velocity, the yield coefficient, the death rate of the biomass, the specific growth function or growth response and the length of the reactor. The inlet limiting substrate concentration is the function $S_{in}(.)$ defined on $[0,\infty)$. We assume without loss of generality that the length $L$ is 1.

The novelty of this paper is fashioned on the following assumption :

**Assumption A1.** *The input $S_{in}(t)$ is a* periodic *continuous positive function of $t$ of period $p > 0$.*

We also need some technical assumptions. The following one guarantees the existence and (some) regularity of the solutions to (2.1)-(2.3).

**Assumption A2.** *The function $\mu$ is of class $C^2$, nonnegative and bounded on $\mathbb{R}^2$ by a constant $\bar{\mu} > 0$. Moreover, there is a constant $\bar{\kappa}$ such that*

$$\left|\frac{\partial \mu}{\partial S}(m_1, m_2)\right| \leq \bar{\kappa} \quad , \quad \left|\frac{\partial \mu}{\partial X}(m_1, m_2)\right| \leq \bar{\kappa} \tag{2.4}$$

*for all $(m_1, m_2) \in \mathbb{R}^2$.*

The next assumption prevents the so-called crowding effect phenomenon (see for instance [12]). It also ensures that the biomass component $X$ of a solution will remain bounded.

**Assumption A3.** *There is a constant $\bar{X} > 0$ such that, for all $X \geq \bar{X}$, and all $S \in \mathbb{R}$,*

$$\mu(S, X) < k_d \ . \tag{2.5}$$

# 3 Auxiliary system and existence of solutions

An auxiliary system is introduced first to ease the analysis of the system (2.1)-(2.3), and secondly, we will demonstrate the existence of solutions for the system (2.1)-(2.3).

## 3.1 Auxiliary system

We introduce the following auxiliary system :

$$\begin{aligned} \frac{\partial w}{\partial t}(t, z) = d\frac{\partial^2 w}{\partial z^2}(t, z) - q\frac{\partial w}{\partial z}(t, z) \quad & \text{for all } (t, z) \in (0, \infty) \times (0, 1), \\ d\frac{\partial w}{\partial z}(t, 0) - qw(t, 0) + qS_{in}(t) = 0 \quad \text{and} \quad \frac{\partial w}{\partial z}(t, 1) = 0 \quad & \text{for all } t \geq 0. \end{aligned} \tag{3.1}$$

The techniques used in [8] can show that the auxiliary system (3.1) has a stable periodic solution (under Assumption **A1**). The following result can be proved in same way as [8, Lemma 3.2].

**Lemma 3.1** *Assume **A1**. The equation* (3.1) *admits a solution $w_p(t, z)$ that is periodic with respect to $t$ of period $p$.*

We next prove an error estimate for a perturbation of $w_p(t, z)$ that will be used in a robustness result in Section 5. Consider a small perturbation $a(t)$ of the input concentration. That is, replace $S_{in}(t)$ by $S_{in}(t) + a(t)$ in equation (3.1) for all $t \geq 0$. We shall prove the following perturbation result for $w_p(t, z)$.

**Lemma 3.2** *Let $a(.) \in C[0,\infty) \cap L^2(0,\infty)$, and suppose $\psi(t,z)$ is a solution to the corresponding perturbed auxiliary problem :*

$$\begin{aligned} \frac{\partial \psi}{\partial t}(t,z) &= d\frac{\partial^2 \psi}{\partial z^2}(t,z) - q\frac{\partial \psi}{\partial z}(t,z) \quad \text{for all } (t,z) \in (0,\infty)\times(0,1), \\ d\frac{\partial \psi}{\partial z}(t,0) &= q(\psi(t,0) - S_{in}(t) - a(t)) \quad \text{and} \quad \frac{\partial \psi}{\partial z}(t,1) = 0 \text{ for all } t \geq 0. \end{aligned} \tag{3.2}$$

*Now let $\bar{w}(t,z) = e^{-\frac{q}{2d}z}(\psi(t,z) - w_p(t,z))$. We have*

$$\int_0^1 |\bar{w}(t,z)|^2 dz \leq e^{-\lambda_1 t}\int_0^1 |\bar{w}(0,z)|^2 dz + q\int_0^t e^{-\lambda_1(t-s)}a^2(s)ds \quad \text{for all } t > 0, \tag{3.3}$$

*where $\lambda_1 = \frac{q^2}{2d}$.*

**Proof of Lemma 3.2 :** We first claim that $\bar{w}(.,.)$ satisfies

$$\begin{aligned} \frac{\partial \bar{w}(t)}{\partial t} &= d\frac{\partial^2 \bar{w}(t)}{\partial z^2} - \frac{q^2}{4d}\bar{w}(t,z) \qquad \text{for all } (t,z) \in (0,\infty)\times(0,1), \\ d\frac{\partial \bar{w}}{\partial z}(t,0) &= \frac{q}{2}\bar{w}(t,0) - qa(t) \quad \text{and } d\frac{\partial \bar{w}}{\partial z}(t,1) = -\frac{q}{2}\bar{w}(t,1) \text{ for all } t > 0. \end{aligned} \tag{3.4}$$

Indeed, for $(t,z) \in (0,\infty)\times(0,1)$, we calculate

$$\begin{aligned} \frac{\partial^2}{\partial z^2}\bar{w}(t,z) &= \frac{\partial}{\partial z}\left[-\frac{q}{2d}e^{-\frac{q}{2d}z}\big(\psi(t,z) - w_p(t,z)\big) + e^{-\frac{q}{2d}z}\left(\frac{\partial \psi}{\partial z}(t,z) - \frac{\partial w_p}{\partial z}(t,z)\right)\right] \\ &= \frac{q^2}{4d^2}e^{-\frac{q}{2d}z}\big(\psi(t,z) - w_p(t,z)\big) - \frac{q}{2d}e^{-\frac{q}{2d}z}\left(\frac{\partial \psi}{\partial z}(t,z) - \frac{\partial w_p}{\partial z}(t,z)\right) \\ &\qquad -\frac{q}{2d}e^{-\frac{q}{2d}z}\left(\frac{\partial \psi}{\partial z}(t,z) - \frac{\partial w_p}{\partial z}(t,z)\right) + e^{-\frac{q}{2d}z}\left(\frac{\partial^2 \psi}{\partial z^2}(t,z) - \frac{\partial^2 w_p}{\partial z^2}(t,z)\right) \\ &= \frac{q^2}{4d^2}\bar{w}(t,z) - \frac{q}{d}e^{-\frac{q}{2d}z}\left(\frac{\partial \psi}{\partial z}(t,z) - \frac{\partial w_p}{\partial z}(t,z)\right) + e^{-\frac{q}{2d}z}\left(\frac{\partial^2 \psi}{\partial z^2}(t,z) - \frac{\partial^2 w_p}{\partial z^2}(t,z)\right) \\ &= \frac{q^2}{4d^2}\bar{w}(t,z) + e^{-\frac{q}{2d}z}\left[-\frac{q}{d}\frac{\partial \psi}{\partial z}(t,z) + \frac{\partial^2 \psi}{\partial z^2}(t,z)\right] - e^{-\frac{q}{2d}z}\left[-\frac{q}{d}\frac{\partial w_p}{\partial z}(t,z) + \frac{\partial^2 w_p}{\partial z^2}(t,z)\right]. \end{aligned}$$

Multiplying through by $d$, rearranging terms, and using the first equations in (3.1) and (3.2) yields

$$\begin{aligned} d\frac{\partial^2}{\partial z^2}\bar{w}(t,z) - \frac{q^2}{4d}\bar{w}(t,z) &= e^{-\frac{q}{2d}z}\left[-q\frac{\partial \psi}{\partial z}(t,z) + d\frac{\partial^2 \psi}{\partial z^2}(t,z) + q\frac{\partial w_p}{\partial z}(t,z) - d\frac{\partial^2 w_p}{\partial z^2}(t,z)\right] \\ &= e^{-\frac{q}{2d}z}\left(\frac{\partial \psi}{\partial t}(t,z) - \frac{\partial w_p}{\partial t}(t,z)\right) = \frac{\partial \bar{w}}{\partial t}(t,z), \end{aligned}$$

which proves the first equation in (3.4). To verify the boundary conditions in (3.4), note that

$$d\frac{\partial}{\partial z}\bar{w}(t,z) = d\left[-\frac{q}{2d}e^{-\frac{q}{2d}z}\big(\psi(t,z) - w_p(t,z)\big) + e^{-\frac{q}{2d}z}\left(\frac{\partial \psi}{\partial z}(t,z) - \frac{\partial w_p}{\partial z}(t,z)\right)\right]. \tag{3.5}$$

Evaluating (3.5) at the beginning boundary value $z = 0$ gives

$$\begin{aligned} d\frac{\partial}{\partial z}\bar{w}(t,0) &= -\frac{q}{2}\big(\psi(t,0) - w_p(t,0)\big) + \left(d\frac{\partial \psi}{\partial z}(t,0) - d\frac{\partial w_p}{\partial z}(t,0)\right) \\ &= \frac{q}{2}\big(\psi(t,0) - w_p(t,0)\big) - qa(t) \\ &= \frac{q}{2}\bar{w}(t,0) - qa(t), \end{aligned} \tag{3.6}$$

where we inserted the boundary conditions in (3.1) and (3.2) to deduce (3.6). Evaluating (3.5) at the ending boundary value $z = 1$ and using the ending boundary conditions in (3.1) and (3.2) yields the latter part of (3.4). This finishes the proof of a our claim.

Define $V : L^2[0,1] \to \mathbb{R}$ by $V(\phi) = \int_0^1 \frac{1}{2}(\phi(z))^2 dz$. Using (3.4), we differentiate $t \to V(\bar{w}(t,.))$ with respect to $t$ to get

$$\begin{aligned}
\frac{dV}{dt}(\bar{w}(t,.)) &= \int_0^1 \frac{\partial \bar{w}(t,z)}{\partial t}\bar{w}(t,z)dz \\
&= \int_0^1 \left( d\frac{\partial^2 \bar{w}(t,z)}{\partial z^2} - \frac{q^2}{4d}\bar{w}(t,z)\right)\bar{w}(t,z)dz \\
&= -d\int_0^1 \left(\frac{\partial \bar{w}(t,z)}{\partial z}\right)^2 dz - \frac{q}{2}(\bar{w}^2(t,0) + \bar{w}^2(t,1)) + qa(t)\bar{w}(t,0) - \frac{q^2}{2d}V(\bar{w}(t,.)),
\end{aligned}$$

where the last equality is justified by an integration by parts and the boundary conditions in (3.4). From the elementary inequality $qa(t)\bar{w}(t,0) \le \frac{q}{2}a^2(t) + \frac{q}{2}\bar{w}^2(t,0)$, one has

$$\frac{dV}{dt}(\bar{w}(t,.)) \le -d\int_0^1 \left(\frac{\partial \bar{w}(t,z)}{\partial z}\right)^2 dz - \frac{q}{2}(\bar{w}^2(t,0) + \bar{w}^2(t,1)) + \frac{q}{2}a^2(t) + \frac{q}{2}\bar{w}^2(t,0) - \frac{q^2}{2d}V(\bar{w}(t,.)),$$

and dropping the nonpositive terms leads to

$$\frac{dV}{dt}(\bar{w}(t,.)) \le -\lambda_1 V(\bar{w}(t,.)) + \frac{q}{2}a^2(t).$$

That is, using the definition of $V$,

$$\frac{d}{dt}\left(\int_0^1 \frac{1}{2}|\bar{w}(t,z)|^2 dz\right) \le -\lambda_1 \left(\int_0^1 \frac{1}{2}|\bar{w}(t,z)|^2 dz\right) + \frac{q}{2}a^2(t).$$

Integrating this differential inequality with respect to $t$, we get

$$\int_0^1 |\bar{w}(t,z)|^2 dz \le e^{-\lambda_1 t}\int_0^1 |\bar{w}(0,z)|^2 dz + q\int_0^t e^{-\lambda_1(t-s)}a^2(s)ds, \qquad \text{for all } t > 0$$

which completes the proof of Lemma 3.2. □

Now let us recall that the proof of the existence of periodic solution of the distributed parameters system (2.1)-(2.3) will rely on tools of functional analysis such as semigroup theory and Schauder's fixed point Theorem. To that end, we introduce the following change of variables.

$$u_1(t,z) = S(t,z) - w_p(t,z) \quad \text{and} \quad u_2(t,z) = X(t,z) \text{ for all } t \ge 0 \text{ and } 0 \le z \le 1 \ , \tag{3.7}$$

where $S$ and $X$ are as in (2.1)-(2.3). Since $L = 1$, $u = (u_1, u_2)$ satisfies the equations

$$\begin{aligned}
\frac{\partial u_1}{\partial t}(t,z) &= d\frac{\partial^2 u_1}{\partial z^2}(t,z) - q\frac{\partial u_1}{\partial z}(t,z) + f_{1p}(t, u_1(t,z), u_2(t,z)) \ , \\
\frac{\partial u_2}{\partial t}(t,z) &= -k_d u_2(t,z) + f_{2p}(t, u_1(t,z), u_2(t,z)) \ , \\
d\frac{\partial u_1}{\partial z}(t,0) &= qu_1(t,0) \quad \text{and} \quad \frac{\partial u_1}{\partial z}(t,1) = 0 \ ,
\end{aligned} \tag{3.8}$$

with, for all $t \geq 0$, $m_1 \in \mathbb{R}$, $m_2 \in \mathbb{R}$,

$$\begin{array}{rcl} f_{1p}(t, m_1, m_2) & = & -k\mu(w_p(t) + m_1, m_2)m_2 \ , \\ f_{2p}(t, m_1, m_2) & = & \mu(w_p(t) + m_1, m_2)m_2 \ , \end{array} \tag{3.9}$$

and $f_p = (f_{1p}, f_{2p})$.
In sections 4 and 5, the existence and stability of periodic solution of the system (3.8) will be discussed. Given that we are interested in how a small perturbation $a(t)$ in $S_{in}(t)$ will affect the stability of such periodic solution, we also introduce the following variables.

$$v_1(t, z) = \tilde{S}(t, z) - \psi(t, z) \quad \text{and} \quad v_2(t, z) = \tilde{X}(t, z) \text{ for all } t \geq 0 \text{ and } 0 \leq z \leq 1 \ , \tag{3.10}$$

where $(\tilde{S}, \tilde{X})$ is solution of a perturbation of the system (2.1)-(2.3), where $S_{in}(t)$ is replaced by $S_{in}(t) + a(t)$ for all $t \geq 0$. Then, $v = (v_1, v_2)$ satisfies the equations

$$\begin{array}{rcl} \dfrac{\partial v_1}{\partial t}(t, z) & = & d\dfrac{\partial^2 v_1}{\partial z^2}(t, z) - q\dfrac{\partial v_1}{\partial z}(t, z) + f_1(t, v_1(t, z), v_2(t, z)) \ , \\ \dfrac{\partial v_2}{\partial t}(t, z) & = & -k_d v_2(t, z) + f_2(t, v_1(t, z), v_2(t, z)) \ , \\ d\dfrac{\partial v_1}{\partial z}(t, 0) & = & q v_1(t, 0) \quad \text{and} \quad \dfrac{\partial v_1}{\partial z}(t, 1) = 0 \ , \end{array} \tag{3.11}$$

with, for all $t \geq 0$, $m_1 \in \mathbb{R}$, $m_2 \in \mathbb{R}$,

$$\begin{array}{rcl} f_1(t, m_1, m_2) & = & -k\mu(\psi(t) + m_1, m_2)m_2 \ , \\ f_2(t, m_1, m_2) & = & \mu(\psi(t) + m_1, m_2)m_2 \ , \end{array} \tag{3.12}$$

and $f = (f_1, f_2)$.
Now, let us define the operators

$$D(A_1) = \left\{ r \in C^2[0,\, 1] : d\frac{\partial r}{\partial z}(0) = qr(0);\ \frac{\partial r}{\partial z}(1) = 0 \right\} \ ,$$

$$A_1 r = d\frac{\partial^2 r}{\partial z^2} - q\frac{\partial r}{\partial z} \ ,$$

$$D(A_2) = C[0,\, 1] \quad \text{and} \quad A_2 = -k_d I \ ,$$

where $I$ denoted the identity and finally

$$D(A) = D(A_1) \otimes D(A_2) \quad \text{and} \quad A = diag(A_1,\, A_2) \ .$$

The arguments used in [6] can show that the operator $A$ is the infinitesimal generator of a $C_0$-semigroup of bounded linear operators $T(t)$ on $\mathcal{Z}$, given by $T(t) = diag(T_1(t),\, T_2(t))$, where $T_1(t)$ and $T_2(t)$ are the $C_0$-semigroups generated by $A_1$ and $A_2$, respectively. Moreover, the semigroup $T_1(t)$ is compact in $C^1[0,\, 1]$ and $T(t)$ is analytic, and we have

$$|T_1(t)| \leq e^{-\frac{q^2}{4d}t} \quad \text{and} \quad |T_2(t)| \leq e^{-k_d t} \quad \text{for all} \quad t \geq 0 \ . \tag{3.13}$$

The systems (3.8) and (3.11) can be written as the following abstract Cauchy problems

$$\begin{array}{rcl} \dfrac{du(t)}{dt} & = & Au(t) + f_p(t, u(t)) \ , \\ u(0) & = & u_0 \in \mathcal{Z} \ . \end{array} \tag{3.14}$$

and

$$\frac{dv(t)}{dt} = Av(t) + f(t, v(t)) \; , \qquad v(0) = v_0 \in \mathcal{Z} \; . \tag{3.15}$$

Observe that $\psi(t,z) = w_p(t,z) + e^{\frac{q}{2d}z}\bar{w}(t,z)$ for all $t \geq 0$ and $z \in [0,\,1]$, where $\bar{w}(t,z)$ satisfies the inequality (3.3) in Lemma 3.2. Hence, we can see that the system (3.15) is a perturbation of the system (3.14).
Throughout the sequel, we consider mild and classical solutions of (3.14) (similarly mild and classical solutions of (3.15)), defined as follows :

**Definition 3.1**

- *A mild solution of* (3.14) *is a continuous function* $u : [0, t_u) \to \mathcal{Z}$*, with* $t_u > 0$*, satisfying*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f_p(s, u(s))ds, \; 0 \leq t < t_u \; .$$

- *A function* $u \in C([0, t_u), \mathcal{Z}) \bigcap C^1((0, t_u), \mathcal{Z})$ *satisfying* $u(t) \in D(A)$*, for* $0 < t < t_u$*, and satisfying* (3.14) *is called a classical solution.*

### 3.2 Result of existence of solutions

We recall that from Assumption **A2**, the functions $f_p$, $f : [0,\,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ defined in (3.9) and (3.12) are continuously differentiable. Also, the semigroup $T(t)$ is a $C_0$-semigroup on $\mathcal{Z}$. Then, by the usual existence and regularity theorem (see [7, Theorem 3.1], [17, Theorem 1], [21, Theorem 1.5, p. 187]), we have the following theorem :

**Theorem 3.3** *Let the system* (3.14) *be such that Assumptions* ***A1*** *and* ***A2*** *hold. Then for any* $u_0 \in \mathcal{Z}$*, the system* (3.14) *(and equivalently* (2.1)*-*(2.3)*) has a unique mild solution* $u(t)$ *with initial condition* $u_0$*. Moreover,* $u(t)$ *is a classical solution of* (3.14) *for all* $t > 0$*. Finally, if we denote by* $\mathbb{T}(t)\theta = u(t, u_0)$ *the solution of* (3.14)*, then* $\mathbb{T}(t)$ *is a nonlinear* $C_0$*-semigroup on* $\mathcal{Z}$*.*

Observe that similar result holds for the system (3.15).

## 4 Existence of periodic solutions of the system (3.14)

In this section, we prove the existence of a periodic solution of the system (3.14), which implies that the distributed parameters biochemical system (2.1)-(2.3) admits a periodic solution too.

### 4.1 Technical result

We introduce the set $\mathcal{Y} \subset \mathcal{Z}$ defined by

$$\mathcal{Y} = \left\{ r \in \mathcal{Z} \; : \; |r_1(z)| \leq \frac{4d\bar{\mu}\bar{X}}{q^2} \; , \; |r_2(z)| \leq \bar{X} \; , \forall z \in [0,\,1] \right\} \; , \tag{4.1}$$

where $\bar{\mu}$ is the constant in Assumption **A2** and $\bar{X}$ is the constant in Assumption **A3** .

We establish the following invariance result for the system (3.14).

**Lemma 4.1** *Assume that Assumptions* ***A1-A3*** *hold. Then, for any* $u_0 \in \mathcal{Y}$*, the solution,* $u(t,.)$*, of the system* (3.14)*, with initial condition* $u(0,.) = u_0$*, satisfies* $u(t,.) \in \mathcal{Y}$ *for all* $t \geq 0$*.*

**Proof of Lemma 4.1.** Let $u(t,z) = (u_1(t,z), u_2(t,z))$ be a solution of the system (3.14). Let us start with the analysis of the second component $u_2(t,z)$. We have

$$\frac{\partial u_2}{\partial t}(t,z) = [-k_d + \mu(w_p(t) + u_1(t,z), u_2(t,z))]\, u_2(t,z)$$

and $|u_2(0,z)| \leq \bar{X}$. From Assumption **A3**, it follows that, for all $t \geq 0$,

$$|u_2(t,z)| \leq \bar{X} \ . \tag{4.2}$$

Now, let us study the component $u_1(t,z)$. Using the integral form, for all $t \geq 0$, we have

$$u_1(t,z) = T_1(t)u_{01}(z) + \int_0^t T_1(t-s) f_{1p}(s, u(s,z))ds \ .$$

Then, using $|u_1(0,z)| \leq \frac{4d\bar{\mu}\bar{X}}{q^2}$, the first inequality in (3.13), the definition of $f_{1p}$, Assumption **A2** and the inequality (4.2), we obtain, for all $t \geq 0$,

$$|u_1(t,z)| \quad \leq \quad e^{-\frac{q^2}{4d}t}\frac{4d\bar{\mu}\bar{X}}{q^2} + \bar{\mu}\bar{X}\int_0^t e^{-\frac{q^2}{4d}(t-s)}ds = \frac{4d\bar{\mu}\bar{X}}{q^2} \ .$$

This completes the proof of Lemma 4.1 □

## 4.2 Existence of a periodic solution

Now, we can establish the existence result of periodic solution of the system (3.14).

**Theorem 4.2** *Assume that Assumptions **A1** to **A3** hold. Then, the system* (3.14) *admits a periodic solution of period* $p$*. We denote it* $u_p$.

**Proof of Theorem 4.2.** The proof of Theorem 4.2 relies on Poincare map and Schauder's fixed point theorem, which is recalled in Section 2.1.

Observe that the function $f_p$ in the system (3.8) is periodic in $t$ of period $p$, uniformly with respect to $u$, since the function $w_p(t,z)$ in the expression of $f_p$ is a periodic solution of the equation (3.1) of period $p$.

Let us define the following Poincare map

$$\begin{array}{rcl} P : \mathcal{Y} & \rightarrow & \mathcal{Y} \\ u_0 & \rightarrow & P(u_0) = u(p) \ , \end{array}$$

where $u(p)$ is the value of the solution, $u(t)$, of (3.14), with initial condition $u_0$, at time $t = p$.
**(i)** First of all, we can see that the set $\mathcal{Y}$ is bounded, closed and convex.
**(ii)** From Lemma 4.1, it follows that $P(\mathcal{Y}) \subseteq \mathcal{Y}$.
**(iii)** Now, we have to prove that $P$ is completely continuous.

- **Continuity.** Since the solutions of (3.14) depend continuously on initial conditions and parameters, then $P$ is continuous.
- **Compactness.** Let $(u_0^n)_{n\geq 0}$ be a sequence in $\mathcal{Y}$ and let $x_n = (x_{1n}, x_{2n}) = u^n(p) = P(u_0^n)$ for any integer $n \geq 0$. First, notice that Lemma 4.1 implies that $x_n \in \mathcal{Y}$ for any integer $n \geq 0$ and that the sequences $u^n(t,z)$ and $x_n$ are bounded in $\mathcal{Z}$ : there is $\bar{B}$ such that for all $z \in [0,1]$, $|x_n(z)| \leq \bar{B}$ and for all $t \geq 0$ and all $z \in [0,1]$, $|u^n(t,z)| \leq \bar{B}$.

**(a)** Since the semigroup $T_1$ is compact and the function $f_{1p}$ is bounded, then the first component, $(x_{1n})_{n\geq 0}$ is precompact. So, all we need to prove is that the second component $(x_{2n})_{n\geq 0}$ is precompact.

**(b)** Since the sequence $(x_{2n})_{n\geq 0}$ is bounded in $C[0,\,1]$ and $x_{2n}(z)\in\mathbb{R}$ for all $z\in[0,\,1]$, then it is enough to prove that $(x_{2n})_{n\geq 0}$ is equicontinuous. Let $z_0\in[0,\,1]$, observe that for all , $z\in[0,\,1]$, $t\in[0,p]$, we have

$$u_2^n(t,z)-u_2^n(t,z_0)=T_2(t)(u_{02}^n(z)-u_{02}^n(z_0))+\int_0^t T_2(t-s)\left(f_{2p}(s,u^n(s,z))-f_{2p}(s,u^n(s,z_0))\right)ds\,.$$

Denoting $l_2$ the Lipschitz constant of $f_{2p}$ with respect to $u$ over the set $\mathcal{S}=\{u\in\mathbb{R}^2:|u|\leq\bar{B}\}$, we get for all $t\in[0,p]$,

$$\begin{aligned}|u_2^n(t,z)-u_2^n(t,z_0)| &\leq e^{-k_dt}|u_{02}^n(z)-u_{02}^n(z_0)|\\ &\quad+l_2\int_0^t e^{-k_d(t-s)}\left(|u_1^n(s,z)-u_1^n(s,z_0)|+|u_2^n(s,z)-u_2^n(s,z_0)|\right)ds\\ &\leq e^{-k_dt}|u_{02}^n(z)-u_{02}^n(z_0)|+l_2\int_0^t e^{-k_d(t-s)}|u_1^n(s,z)-u_1^n(s,z_0)|ds\\ &\quad+l_2\int_0^t e^{-k_d(t-s)}|u_2^n(s,z)-u_2^n(s,z_0)|ds\,.\end{aligned}$$

From **(a)**, for any $\xi>0$, there exists a constant $\eta_0>0$ such that if $|z-z_0|\leq\eta_0$ then $|u_1^n(s,z)-u_1^n(s,z_0)|<\xi$ for all $s>0$.
Also, by continuity we have : for any $\xi>0$, there exists a constant $\eta_1>0$ such that if $|z-z_0|\leq\eta_1$ then $|u_{02}^n(z)-u_{02}^n(z_0)|<\xi$.
Then, for any $\xi>0$, there exists $\eta_2<\min\{\eta_0,\eta_1\}$ such that if $|z-z_0|\leq\eta_2$ then

$$\begin{aligned}|u_2^n(t,z)-u_2^n(t,z_0)| &\leq \xi+l_2\xi\int_0^t e^{-k_d(t-s)}ds+l_2\int_0^t e^{-k_d(t-s)}|u_2^n(s,z)-u_2^n(s,z_0)|ds\\ &\leq \xi+\tfrac{l_2\xi}{k_d}+l_2\int_0^t e^{k_d(s-t)}|u_2^n(s,z)-u_2^n(s,z_0)|ds\\ &\leq \xi+\tfrac{l_2\xi}{k_d}+l_2\int_0^t |u_2^n(s,z)-u_2^n(s,z_0)|ds\,.\end{aligned}$$

By applying the Gronwall's inequality, it follows that, for all $t\in[0,p]$,

$$|u_2^n(t,z)-u_2^n(t,z_0)|\leq\left(\xi+\frac{l_2\xi}{k_d}\right)\left(e^{l_2p}-1\right)\,.$$

Then, for any $\epsilon>0$, taking $\xi=\dfrac{\epsilon}{\left(1+\frac{l_2}{k_d}\right)(e^{l_2p}-1)}$ in the inequality above, there exists $\eta>0$ such that if $|z-z_0|\leq\eta$ then $|x_{2n}(z)-x_{2n}(z_0)|<\epsilon$.
Therefore, the sequence $(x_{2n})_{n\geq 0}$ is equicontinuous in $C[0,1]$ and by the Ascoli-Arzela's theorem, $(x_{2n})_{n\geq 0}$ is precompact.

So, the Poincare map $P$ is completely continuous.

Combining **(i)**, **(ii)**, **(iii)** and the Schauder's theorem (see Section 2.1), we deduce that the Poincare map $P$ has a fixed point in $\mathcal{Y}$. Since $f_p$ is periodic in $t$ of period $p$, the system (3.14) has a periodic solution of period $p$. This completes the proof of Theorem 4.2. □

## 5 The Stability Problem

In this section, we prove the following stability problem :

(SP) Let $v(t,z)$ be a solution of (3.15) and let $x_1(t,z) = e^{-\frac{q}{2d}z}\left(v_1(t,z) - u_{1,p}(t,z)\right)$ and $x_2(t,z) = e^{-\frac{q}{2d}z}\left(v_2(t,z) - u_{2,p}(t,z)\right)$, where $u_p = (u_{1p}, u_{2p})$ is the periodic solution of (3.14) given in Theorem 4.2. Then, the deviation $x = (x_1, x_2)$ satisfies the following ISS inequality : There exist $\lambda_1 > 0$ and $\lambda_1 > 0$ such that

$$|x(t,.)|_{L^2} \leq e^{-\frac{\lambda_1}{2}t}|x(0,.)|_{L^2} + \lambda_2 \sup_{m\in[0,t]} \sqrt{\int_0^1 |\bar{w}(m,z)|^2 dz}, \quad \text{for all } t \geq 0. \tag{5.1}$$

We also recall, from (5.1) and (3.3), that

$$\text{If } |a(t)| \to 0 \text{ as } t \to \infty, \text{ then } \|\bar{w}(t)\|_{L^2} \to 0 \text{ as } t \to \infty.$$

where $a(t)$ is the small perturbation introduced in (3.2).

### 5.1 Change of coordinates and extra assumption

Let $v(t,z)$ be a solution of (3.15). To ease the forthcoming analysis, we introduce the error variables :

$$x_1(t,z) = e^{-\frac{q}{2d}z}\left(v_1(t,z) - u_{1,p}(t,z)\right) \quad , \quad x_2(t,z) = e^{-\frac{q}{2d}z}\left(v_2(t,z) - u_{2,p}(t,z)\right) \ , \tag{5.2}$$

where $u_p = (u_{1,p}, u_{2,p})$ is the periodic solution of the system (3.14) corresponding to $w_p(t,z)$ and such that, for all $t \geq 0$, $z \in [0,1]$,

$$|u_{1,p}(t,z)| \leq \frac{4d\bar{\mu}\bar{X}}{q^2} \qquad |u_{2,p}(t,z)| \leq \bar{X} \ . \tag{5.3}$$

Let us recall that the existence of such a solution is guaranteed by Theorem 4.2.
Since

$$\frac{\partial x_1(t,z)}{\partial z} = e^{-\frac{q}{2d}z}\left[-\frac{q}{2d}\left(v_1(t,z) - u_{1,p}(t,z)\right) + \left(\frac{\partial v_1(t,z)}{\partial z} - \frac{\partial u_{1,p}(t,z)}{\partial z}\right)\right]$$

and

$$d\frac{\partial^2 x_1(t,z)}{\partial z^2} = \frac{q^2}{4d}x_1(t,z) + e^{\frac{-q}{2d}z}\left[-q\left(\frac{\partial v_1(t,z)}{\partial z} - \frac{\partial u_{1,p}(t,z)}{\partial z}\right) + d\left(\frac{\partial^2 v_1(t,z)}{\partial z^2} - \frac{\partial^2 u_{1,p}(t,z)}{\partial z^2}\right)\right]$$

we deduce that

$$\begin{aligned}
\frac{\partial x_1(t,z)}{\partial t} &= d\frac{\partial^2 x_1(t,z)}{\partial z^2} - \frac{q^2}{4d}x_1(t,z) - ke^{-\frac{q}{2d}z}\mu(\psi(t,z) + v_1(t,z), v_2(t,z))v_2(t,z) \\
&+ ke^{-\frac{q}{2d}z}\mu(w_p(t,z) + u_{1,p}(t,z), u_{2,p}(t,z))u_{2,p}(t,z).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{\partial x_2(t,z)}{\partial t} &= -k_d x_2(t,z) + e^{-\frac{q}{2d}z}\mu(\psi(t,z) + v_1(t,z), v_2(t,z))v_2(t,z) \\
&- e^{-\frac{q}{2d}z}\mu(w_p(t,z) + u_{1,p}(t,z), u_{2,p}(t,z))u_{2,p}(t,z).
\end{aligned}$$

This system writes as

$$
\begin{array}{rcl}
\dfrac{\partial x_1(t,z)}{\partial t} &=& d\dfrac{\partial^2 x_1(t,z)}{\partial z^2} - \dfrac{q^2}{4d}x_1(t,z) - ke^{-\frac{q}{2d}z}\zeta_1(t,z,u_p(t,z))\,[\psi(t,z)-w_p(t,z)] \\
&-& ke^{-\frac{q}{2d}z}\,[\zeta_1(t,z,u_p(t,z))\,[v_1(t,z)-u_{1,p}(t,z)] + \zeta_2(t,z,u_p(t,z))\,[v_2(t,z)-u_{2,p}(t,z)]] \\
\dfrac{\partial x_2(t,z)}{\partial t} &=& -k_d x_2(t,z) + e^{-\frac{q}{2d}z}\zeta_1(t,z,u_p(t,z))\,[\psi(t,z)-w_p(t,z)] \\
&+& e^{-\frac{q}{2d}z}\,[\zeta_1(t,z,u_p(t,z))\,[v_1(t,z)-u_{1,p}(t,z)] + \zeta_2(t,z,u_p(t,z))\,[v_2(t,z)-u_{2,p}(t,z)]] \\
d\dfrac{\partial x_1}{\partial z}(t,0) &=& \dfrac{q}{2}x_1(t,0) \quad \text{and} \quad d\dfrac{\partial x_1}{\partial z}(t,1) = -\dfrac{q}{2}x_1(t,1,
\end{array}
$$

where

$$\zeta_1(t,z,u_p(t,z)) = \left(\frac{\partial \mu}{\partial S}\,(w_p(t,z)+u_{1,p}(t,z),u_{2,p}(t,z))\right)u_{2,p}(t,z),$$

and

$$\zeta_2(t,z,u_p(t,z)) = \left(\frac{\partial \mu}{\partial X}\,(w_p(t,z)+u_{1,p}(t,z),u_{2,p}(t,z))\right)u_{2,p}(t,z)+\mu(w_p(t,z)+u_{1,p}(t,z),u_{2,p}(t,z)).$$

Then, we have

$$
\begin{array}{rcl}
\dfrac{\partial x_1(t,z)}{\partial t} &=& d\dfrac{\partial^2 x_1(t,z)}{\partial z^2} - \dfrac{q^2}{4d}x_1(t,z) - k\zeta_1(t,z,u_p(t,z))\,[\bar{w}(t,z)+x_1(t,z)] \\
&-& k\zeta_2(t,z,u_p(t,z))x_2(t,z) \\
\dfrac{\partial x_2(t,z)}{\partial t} &=& -k_d x_2(t,z) + \zeta_1(t,z,u_p(t,z))\,[\bar{w}(t,z)+x_1(t,z)] + \zeta_2(t,z,u_p(t,z))x_2(t,z) \\
d\dfrac{\partial x_1}{\partial z}(t,0) &=& \dfrac{q}{2}x_1(t,0) \quad \text{and} \quad d\dfrac{\partial x_1}{\partial z}(t,1) = -\dfrac{q}{2}x_1(t,1).
\end{array}
\tag{5.4}
$$

where $\bar{w}(t,z) = e^{-\frac{q}{2d}z}\,(\psi(t,z)-w_p(t,z))$.

Notice for later use that Assumption **A2** ensures that, for all $t \geq 0$, $z \in [0,1]$,

$$|\zeta_1(t,z,u_p(t,z))| \leq \bar{\zeta}_1 = \bar{\kappa}\bar{X}\ ,\ |\zeta_2(t,z,u_p(t,z))| \leq \bar{\zeta}_2 = \bar{\kappa}\bar{X} + \bar{\mu}\ . \tag{5.5}$$

To carry out the stability analysis, we observe that the system (5.4) is in the following (more general) form (with $\alpha(t,z)$ in place of $\bar{w}(t,z)$).

$$
\begin{array}{rcl}
\dfrac{\partial y_1(t,z)}{\partial t} &=& d\dfrac{\partial^2 y_1(t,z)}{\partial z^2} - \dfrac{q^2}{4d}y_1(t,z) - k\zeta_1(t,z,u_p(t,z))y_1(t,z) \\
&-& k\zeta_2(t,z,u_p(t,z))y_2(t,z) - k\zeta_1(t,z,u_p(t,z))\alpha(t,z) \\
\dfrac{\partial y_2(t,z)}{\partial t} &=& -k_d y_2(t,z) + \zeta_1(t,z,u_p(t,z))y_1(t,z) + \zeta_2(t,z,u_p(t,z))y_2(t,z) \\
&+& \zeta_1(t,z,u_p(t,z))\alpha(t,z) \\
d\dfrac{\partial y_1}{\partial z}(t,0) &=& \dfrac{q}{2}y_1(t,0) \quad \text{and} \quad d\dfrac{\partial y_1}{\partial z}(t,1) = -\dfrac{q}{2}y_1(t,1).
\end{array}
\tag{5.6}
$$

where $\alpha \in C\,([0,\,\infty);\ C[0,\,1])$. For any $y_0$ and control $\alpha$, we denote by $y(t,z;y_0,\alpha)$ the solution of (5.6) satisfying $y(0) = y_0$. For example, $x(t,z,\alpha) = (x_1(t,z,\alpha), x_2(t,z,\alpha))$ is the solution of (5.4) for the perturbation $\bar{w}(t,z) = \alpha(t,z)$.

Now we recall some definition on Lyapunov functionals (see e.g. [14, Definition 3.62] and [19, Definition 2.1]).

**Definition 5.1** *Let $V : \mathcal{Z} \to \mathbb{R}$ be a continuously differentiable function. The functional $V$ is called a weak Lyapunov functional for* (5.6) *if there are two functions $K_S$ and $K_M$ of class $\mathcal{K}_\infty$ such that, for all functions $\phi \in \mathcal{Z}$,*

$$K_S(|\phi|_{L^2(0,1)}) \leq V(\phi) \leq \int_0^1 K_M(|\phi(z)|)dz \tag{5.7}$$

*where $|\phi|^2_{L^2(0,1)} = |\phi_1|^2_{L^2(0,1)} + |\phi_2|^2_{L^2(0,1)}$, and, in the absence of $\alpha$, for all solutions of* (5.4) *and for all $t \geq 0$*

$$\frac{dV(y(t,.))}{dt} \leq 0.$$

*The function is said to be ISS Lyapunov functional for* (5.6) *if, in addition, there exists $\Lambda_1 > 0$ and a function $\Lambda_2$ of class $\mathcal{K}_\infty$ such that, for all continuous functions $\alpha$, for all solutions of* (5.6)*, and for all $t \geq 0$,*

$$\frac{dV(y(t,.))}{dt} \leq -\Lambda_1 V(y(t,.)) + \int_0^1 \Lambda_2(|\alpha(t,z)|)dz. \tag{5.8}$$

## 5.2 Main result

Since Lemma 4.1 ensures that $u_p$ is bounded, there is a constant $\bar{P} > 0$ such that $|u_p(t,z)| \leq \bar{P}$ for all $t \geq 0$, $z \in [0,1]$. Observe that from (5.3), we deduce that a possible choice for $\bar{P}$ is $\bar{P} = \sqrt{1 + \frac{16d^2\bar{\mu}^2}{q^4}}\bar{X}$.

We introduce the following technical assumption :

**Assumption A4.** *The inequalities*

$$\frac{q^2}{4d} - k\bar{\kappa}\bar{P} > 0 \ , \ k_d - \bar{\mu}\bar{P} > 0 \quad \textit{and} \quad 4\left(\frac{q^2}{4d} - k\bar{\kappa}\bar{P}\right)\left(k_d - \bar{\mu}\bar{P}\right) > \left(\bar{\kappa} + k\bar{\mu}\right)^2 \bar{P}^2 \tag{5.9}$$

*are satisfied.*

**Remark.** Selecting $\bar{P} = \sqrt{1 + \frac{16d^2\bar{\mu}^2}{q^4}}\bar{X}$, the inequality (5.9) can be checked. Observe that if $\bar{X}$ is sufficiently small, then Assumption **A4** is satisfied.

Define the functional $V : \mathcal{Z} \to \mathbb{R}$ by :

$$V(\phi) = \int_0^1 \frac{1}{2}\left((\phi_1(z))^2 + (\phi_2(z))^2\right)dz, \text{ for every } \phi \in \mathcal{Z}.$$

We are ready to state and prove the following result :

**Theorem 5.1** *Assume that Assumptions **A1-A4** hold, there exists $\Lambda_1 > 0$ and a function $\Lambda_2$ of class $\mathcal{K}_\infty$ such that, for all continuous functions $\alpha$, for all solutions of* (5.6)*, and for all $t \geq 0$,*

$$\frac{dV(y(t,.))}{dt} \leq -\Lambda_1 V(y(t,.)) + \int_0^1 \Lambda_2(|\alpha(t,z)|)dz.$$

*and therefore, the functional $V$ is an ISS Lyapunov functional for the system* (5.6).

**Proof of Theorem 5.1 :** First of all, let's prove inequality (5.7) for $V$. Let

$$\begin{array}{rccc} K_S : & [0,\infty) & \to & [0,\infty) \\ & x & \to & \frac{1}{4}x^2. \end{array}$$

We have $K_S \in \mathcal{K}_\infty$ and for every $\phi \in \mathcal{Z}$, we get

$$K_S(|\phi|_{L^2(0,1)}) = \frac{1}{4}|\phi|^2_{L^2(0,1)} = \frac{1}{4}\int_0^1 \left[|\phi_1(z)|^2 + |\phi_2(z)|^2\right] dz \le V(\phi). \qquad (\star)$$

Let

$$\begin{array}{rccc} K_M: & [0,\infty) & \to & [0,\infty) \\ & x & \to & x^2. \end{array}$$

We have $K_S \in \mathcal{K}_\infty$ and for every $\phi \in \mathcal{Z}$, we get

$$\int_0^1 K_M(|\phi(z)|)dz = \int_0^1 \left[|\phi_1(z)|^2 + |\phi_2(z)|^2\right] dz \ge V(\phi). \qquad (\star\star)$$

Then, combining $(\star)$ and $(\star\star)$, we

$$K_S(|\phi|_{L^2(0,1)}) \le V(\phi) \le \int_0^1 K_M(|\phi(z)|)dz.$$

Next, let's consider the time derivative of $V$ along the trajectory $y(t,z)$ of (5.6). We get

$$\frac{dV(y(t,.))}{dt} = \int_0^1 \frac{\partial y_1(t,z)}{\partial t} y_1(t,z)dz + \int_0^1 \frac{\partial y_2(t,z)}{\partial t} y_2(t,z)dz.$$

We will study these two integrals above separately. Let us prove the following lemmas :

**Lemma 5.2** *Assume that Assumptions **A1-A4** hold. Then,*

$$\begin{array}{rcl} \int_0^1 \frac{\partial y_2(t,z)}{\partial t} y_2(t,z))dz & = & -k_d \int_0^1 |y_2(t,z)|^2 dz + \int_0^1 y_2(t,z)\zeta_1(t,z,u_p)\alpha(t,z)dz \\ & + & \int_0^1 y_2(t,z)\zeta_1(t,z,u_p)y_1(t,z)dz + \int_0^1 y_2(t,z)\zeta_2(t,z,u_p)y_2(t,z)dz. \end{array}$$

**Proof of Lemma 5.2 :** It is straightforward to see that

$$\begin{array}{rcl} \int_0^1 \frac{\partial y_2(t,z)}{\partial t} y_2(t,z)dz & = & -k_d \int_0^1 |y_2(t,z)|^2 dz + \int_0^1 y_2(t,z)\zeta_1(t,z,u_p)\alpha(t,z)dz \\ & + & \int_0^1 y_2(t,z)\zeta_1(t,z,u_p)y_1(t,z)dz + \int_0^1 y_2(t,z)\zeta_2(t,z,u_p)y_2(t,z)dz \end{array}$$

This completes proof of Lemma 5.2. □

**Lemma 5.3** *Assume that Assumptions **A1-A4** hold. Then,*

$$\begin{array}{rcl} \int_0^1 \frac{\partial y_1(t,z)}{\partial t} y_1(t,z)dz & = & -\frac{q^2}{4d} \int_0^1 |y_1(t,z)|^2 dz - k\int_0^1 \zeta_1(t,z,u_p)y_1(t,z)\alpha(t,z)dz \\ & - & k\int_0^1 y_1(t,z)\zeta_1(t,z,u_p)y_1(t,z)dz - k\int_0^1 y_2(t,z)\zeta_2(t,z,u_p)y_1(t,z)dz. \end{array}$$

**Proof of Lemma 5.3 :** We have

$$\begin{array}{rcl} \int_0^1 \frac{\partial y_1(t,z)}{\partial t} y_1(t,z)dz & = & \int_0^1 \left(d\frac{\partial^2 y_1(t,z)}{\partial z^2} - \frac{q^2}{4d} y_1(t,z) - k\zeta_1(t,z,u_p)\alpha(t,z)\right) y_1(t,z)dz \\ & - & k\int_0^1 \left(\zeta_1(t,z,u_p)y_1(t,z) + \zeta_2(t,z,u_p)y_2(t,z)\right) y_1(t,z)dz \\ & = & -\frac{q^2}{4d}\int_0^1 |y_1(t,z)|^2 dz + d\int_0^1 y_1(t,z)\frac{\partial^2 y_1(t,z)}{\partial z^2}dz \\ & - & k\int_0^1 \zeta_1(t,z,u_p)y_1(t,z)\alpha(t,z)dz - k\int_0^1 y_1(t,z)\zeta_1(t,z,u_p)y_1(t,z)dz \\ & - & k\int_0^1 y_2(t,z)\zeta_2(t,z,u_p)y_1(t,z)dz \end{array}$$

Using the boundary conditions in (5.6), we get

$$\begin{aligned}
\int_0^1 \frac{\partial y_1(t,z)}{\partial t} y_1(t,z)dz &= -\frac{q^2}{4d}\int_0^1 |y_1(t,z)|^2 dz - d\int_0^1 \left(\frac{\partial y_1(t,z)}{\partial z}\right)^2 dz - \frac{q}{2}\left(y_1^2(t,0)+y_1^2(t,1)\right) \\
&- k\int_0^1 \zeta_1(t,z,u_p)y_1(t,z)\alpha(t,z)dz - k\int_0^1 y_1(t,z)\zeta_1(t,z,u_p)y_1(t,z)dz \\
&- k\int_0^1 y_2(t,z)\zeta_2(t,z,u_p)y_1(t,z)dz
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^1 \frac{\partial y_1(t,z)}{\partial t} y_1(t,z)dz &\le -\frac{q^2}{4d}\int_0^1 |y_1(t,z)|^2 dz - k\int_0^1 \zeta_1(t,z,u_p)y_1(t,z)\alpha(t,z)dz \\
&- k\int_0^1 y_1(t,z)\zeta_1(t,z,u_p)y_1(t,z)dz - k\int_0^1 y_2(t,z)\zeta_2(t,z,u_p)y_1(t,z)dz.
\end{aligned}$$

This completes proof of Lemma 5.3. □

Combining the inequalities in Lemmas 5.2 and 5.3, we get

$$\begin{aligned}
\frac{dV(y(t,.))}{dt} &\le -\frac{q^2}{4d}\int_0^1 |y_1(t,z)|^2 dz - k\int_0^1 \zeta_1(t,z,u_p)y_1(t,z)\alpha(t,z)dz \\
&- k\int_0^1 \zeta_1(t,z,u_p)|y_1(t,z)|^2 dz - k\int_0^1 \zeta_2(t,z,u_p)y_2(t,z)y_1(t,z)dz. \\
&- k_d\int_0^1 |y_2(t,z)|^2 dz + \int_0^1 \zeta_1(t,z,u_p)y_2(t,z)\alpha(t,z)dz \\
&+ \int_0^1 \zeta_1(t,z,u_p)y_2(t,z)y_1(t,z)dz + \int_0^1 \zeta_2(t,z,u_p)|y_2(t,z)|^2 dz.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{dV(y(t,.))}{dt} &\le -\int_0^1 \left(\frac{q^2}{4d} + k\zeta_1(t,z,u_p)\right)|y_1(t,z)|^2 dz - \int_0^1 \left(k_d + \zeta_2(t,z,u_p)\right)|y_2(t,z)|^2 dz \\
&+ \int_0^1 \left(\zeta_1(t,z,u_p) - k\zeta_2(t,z,u_p)\right)y_2(t,z)y_1(t,z)dz \\
&+ \int_0^1 \zeta_1(t,z,u_p)\left(y_2(t,z) - ky_1(t,z)\right)\alpha(t,z)dz.
\end{aligned}$$

Observe that $|\zeta_2(.)| \le \bar{\mu}\bar{P}$ and $|\zeta_1(.)| \le \bar{\kappa}\bar{P}$, we deduce that

$$\begin{aligned}
\frac{dV(y(t,.))}{dt} &\le -\left(\frac{q^2}{4d} - k\bar{\kappa}\bar{P}\right)\int_0^1 |y_1(t,z)|^2 dz - \left(k_d - \bar{\mu}\bar{P}\right)\int_0^1 |y_2(t,z)|^2 dz \\
&+ \left(\bar{\kappa}\bar{P} + k\bar{\mu}\bar{P}\right)\int_0^1 |y_2(t,z)y_1(t,z)|dz + \bar{\kappa}\bar{P}\int_0^1 |y_2(t,z)\alpha(t,z)|dz \\
&+ k\bar{\kappa}\bar{P}\int_0^1 |y_1(t,z)\alpha(t,z)|dz.
\end{aligned}$$

Using Young inequality, we get

$$\begin{aligned}
\bar{\kappa}\bar{P}\int_0^1 |y_2(t,z)\alpha(t,z)|dz + k\bar{\kappa}\bar{P}\int_0^1 |y_1(t,z)\alpha(t,z)|dz &\le \frac{\bar{\kappa}\bar{P}\eta}{2}\int_0^1 |y_2(t,z)|^2 dz \\
&+ \frac{k\bar{\kappa}\bar{P}\eta}{2}\int_0^1 |y_1(t,z)|^2 dz \\
&+ \frac{\bar{\kappa}\bar{P}}{2\eta}(k+1)\int_0^1 |\alpha(t,z)|^2 dz.
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dV(y(t,.))}{dt} &\leq -\left(\frac{q^2}{4d} - k\bar{\kappa}\bar{P} - \frac{k\bar{\kappa}\bar{P}\eta}{2}\right)\int_0^1 |y_1(t,z)|^2 dz - \left(k_d - \bar{\mu}\bar{P} - \frac{\bar{\kappa}\bar{P}\eta}{2}\right)\int_0^1 |y_2(t,z)|^2 dz \\ &+ \left(\bar{\kappa}\bar{P} + k\bar{\mu}\bar{P})\right)\int_0^1 |y_2(t,z)y_1(t,z)|dz + \frac{\bar{\kappa}\bar{P}}{2\eta}(k+1)\int_0^1 |\alpha(t,z)|^2 dz\end{aligned}$$

with

$$0 < \eta < \min\left(\frac{\frac{q^2}{4d} - k\bar{\kappa}\bar{P}}{k\bar{\kappa}\bar{P}}; \quad \frac{k_d - \bar{\mu}\bar{P}}{\bar{\kappa}\bar{P}}\right). \tag{5.10}$$

From Assumption **A4**, we get

$$\begin{aligned}\frac{dV(y(t,.))}{dt} &\leq -\left(\frac{q^2}{4d} - k\bar{\kappa}\bar{P} - k\bar{\kappa}\bar{P}\eta\right)\int_0^1 \frac{1}{2}|y_1(t,z)|^2 dz - \left(k_d - \bar{\mu}\bar{P} - \bar{\kappa}\bar{P}\eta\right)\int_0^1 \frac{1}{2}|y_2(t,z)|^2 dz \\ &+ \frac{\bar{\kappa}\bar{P}}{2\eta}(k+1)\int_0^1 |\alpha(t,z)|^2 dz.\end{aligned}$$

Observe that, with the choice of $\eta$ in (5.10) and Assumption **A4**, we have

$$\left(\frac{q^2}{4d} - k\bar{\kappa}\bar{P} - k\bar{\kappa}\bar{P}\eta\right) > 0 \text{ and } \left(k_d - \bar{\mu}\bar{P} - \bar{\kappa}\bar{P}\eta\right) > 0.$$

Now, we take

$$0 < \Lambda_1 = \min\left(\left(\frac{q^2}{4d} - k\bar{\kappa}\bar{P} - k\bar{\kappa}\bar{P}\eta\right); \quad \left(k_d - \bar{\mu}\bar{P} - \bar{\kappa}\bar{P}\eta\right)\right)$$

and we define the function, $\Lambda_2$, of class $\mathcal{K}_\infty$ by $\Lambda_2(y) = \dfrac{\bar{\kappa}\bar{P}}{2\eta}(k+1)y^2$. Hence,

$$\frac{dV(y(t,.))}{dt} \leq -\Lambda_1 V(y(t,.)) + \int_0^1 \Lambda_2(|\alpha(t,z)|)dz. \tag{5.11}$$

This completes the proof of Theorem 5.1. □

Finally, observe that the inequality (5.11), established by Theorem 5.1, implies the ISS property (5.1) if one take $\alpha(t,z) = \bar{w}(t,z)$.

## Références


[1] O. Bernard, Z. Hadj-Sadok, D. Dochain, A. Genovesi and J.-P. Steyer, Dynamical model development and parameter identification for an anaerobic wastewater treatment process, *Biotech. Bioeng.*, 2001, Vol. 75, pp. 424-438.

[2] P.D. Christofides and P. Daoutidis, Nonlinear feedback control of parabolic PDE systems, In *Nonlinear Model Based Process Control*; R. Berber and C. Kravaris (eds), Kluwer, Dordrecht, 1998.

[3] M. Damak, K. Ezzinbi, L. Souden, Weighted pseudo-almost periodic solutions for some neutral partial functional differential equations, *Electronic Journal of differential equations*, Vol. 2012 (2012), No. 47, pp. 1-13.

[4] T. Diagana, Almost periodic solutions for some higher-order nonautonomous differential equations with operator coefficients, *Mathematical and Computer Modelling*, 2011, doi :10.1016/j.mcm.2011.06.050.

[5] D. Dochain, Contribution to the Analysis and Control of Distributed Parameters Systems with Applications to (bio)chemical Processes and Robotics, *Thèse de Doctorat d'Etat*, CESAME, Catholic University of Louvain, Belgium, 1994.

[6] A. K. Drame, D. Dochain and J. J. Winkin, Asymptotic behaviour and stability for solutions of a biochemical reactor distributed parameter model, *IEEE Transactions on Automatic Control*, Vol. 53, No 1 (2008), pp. 412-416. (Extended version : Internal Report of University of Namur FUNDP, Belgium, 06/2007).

[7] A. K. Drame, D. Dochain, J. J. Winkin and P. R. Wolenski, Periodic trajectories of distributed parameter biochemical systems with time delay, *Applied Mathematics and Computation*, 218 (2012), pp. 7395-7405.

[8] A. K. Drame, M. R. Kothari and P. R. Wolenski, Almost periodic solutions of distributed parameter biochemical systems with time delay and time varying input, *Electron. J. Diff. Equ.*, Vol. 2013 (2013), No. 161, pp. 1-11.

[9] H. R. Henriquez, B. D. Andrade, M. Rabelo, Existence of almost periodic solutions for a class of abstract impulsive differential equations, *ISRN Mathematical Analysis*, Vol. 2011 (2011), pp. 1-21.

[10] P. Hess, Periodic Parabolic Boundary Value Problems and Positivity, *John Wiley & Sons*, New York, 1991.

[11] M. Laabissi, J.J. Winkin, D. Dochain, M.E. Achhab, Dynamical analysis of a tubular biochemical reactor infinite-dimensional nonlinear model, *Proc. 44th IEEE-CDC-ECC*, 2005, Seville, Spain, pp. 5965-5970.

[12] C. Lobry, F. Mazenc, Effect of Intra-specific Competition on Persistence in Competition Models, *Electron. J. Diff. Eqns.*, No. 125, pp. 1-10, Vol. 2007 (2007).

[13] F. Long, Positive almost periodic solution for a class of Nicholson's blowflies model with a linear harvesting term, *Nonlinear Analysis : RWA*, 13, (2012), pp. 686-693.

[14] Z. H. Luo, B. Z. Guo and O. Morgul, Stability and Stabilization of infinite dimensional systems and applications, *Communications and Control Engineering*, Springer-Verlag, New York, 1999.

[15] N. MacDonald, Time delay in simple chemostat models, *Biotechnol. Bioeng.* , XVIII, PP. 805-812, 1976.

[16] N. MacDonald, Time delay in simple chemostat models, *in M.J. Bazin (Ed.), Microbial Population Dynamics, CRC, Boca Raton, FL*, 1982.

[17] R. H. Martin, JR, Nonlinear operators and differential equations in Banach spaces, *Wiley and Sons, Inc.*, New-York, 1976.

[18] R. H. Martin, JR and H. L. Smith, Abstract functional differential equations and rection-diffusion systems, *Transactions of American Mathematical Society*, Vol. 321, No. 1 (1990), pp. 1-44.

[19] F. Mazenc and C. Prieur, Strict Lyapunov functionals for nonlinear parabolic partial differential equations, *Mathematics Control and Related Fields*, Vol. 1, No. 2, pp. 231-250, 2011.

[20] T. Patarinska, D. Dochain, S.N. Agathos, L. Ganovski, Modeling of continuous microbial cultivation taking into account the memory effects, *Bioprocess Eng.*, 22, pp. 517-527, 2000.

[21] A. Pazy, Semigroups of linear operators and applications to partial differential equations, *Springer Verlag*, Berlin, 1983.

[22] S. S. Pilyugin and P. Waltman, Competition in the unstirred chemostat with periodic input and washout, *SIAM J. on Applied Mathematics*, Vol. 59, No. 4 (1999), pp. 1157-1177.

[23] O. Schoefs, D. Dochain, H. Fibrianto and J.-P. Steyer, Modelling and identification of a partial differential equation model for an anaerobic wastewater treatment process, *Proc. 10th World Congress on Anaerobic Digestion (AD01-2004), Montreal, Canada.*

[24] J. J. Winkin, D. Dochain and P. Ligarius, Dynamical analysis of distributed parameters tubular recators, *Automatica*, Vol. 36, 2000, pp. 349-361.

[25] H. Xia, G.S.K. Wolkowicz, L. Wang, Transient osciallations induced by delayed growth response in the chemostat, *J. Math. Biol.* 50 (2005) pp. 489-530.