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ASYMPTOTIC ANALYSIS OF A NEUMANN PROBLEM IN A DOMAIN WITH CUSP. APPLICATION TO THE COLLISION PROBLEM OF RIGID BODIES IN A PERFECT FLUID*

ALEXANDRE MUNNIER† AND KARIM RAMDANI‡

Abstract. We study a two dimensional collision problem for a rigid solid immersed in a cavity filled with a perfect fluid. We are led to investigate the asymptotic behavior of the Dirichlet energy associated with the solution of a Laplace–Neumann problem as the distance $\varepsilon > 0$ between the solid and the cavity's bottom tends to zero. Denoting by $\alpha > 0$ the tangency exponent at the contact point, we prove that the solid always reaches the cavity in finite time, but with a nonzero velocity for $\alpha < 2$ (real shock case), and with null velocity for $\alpha \geqslant 2$ (smooth landing case). Our proof is based on a suitable change of variables sending to infinity the cusp singularity at the contact. More precisely, for every $\varepsilon \geqslant 0$, we transform the Laplace–Neumann problem into a generalized Neumann problem set on a domain containing a horizontal strip $]0, \ell_\varepsilon[\times]0, 1[$, where $\ell_\varepsilon \to +\infty$.

1. Introduction: motivation, problem setting, and statement of the main results. In this paper, we are interested in the asymptotic analysis of a singular perturbed problem for the Laplace–Neumann equation in a domain Ω_ε depending on a small parameter ε. The typical configuration we have in mind is the one depicted on Figure 1, namely, the situation where Ω_ε is the domain located between two smooth surfaces situated at a distance ε and touching each other at one single cusp point (the origin) in the limit case $\varepsilon = 0$.

More precisely, we denote by C a smooth, open, bounded and connected set in $\mathbb{R}^2$ and we assume that C is symmetric with respect to the ordinate axis, that the origin belongs to the boundary of C, and that, near the origin, the boundary $\partial\mathsf{C}$ is locally a straight line, the domain C being locally situated above $\partial\mathsf{C}$. We denote by S_0 a compact, connected set, symmetric with respect to the ordinate axis as well and such that, for some $\varepsilon^* > 0$, the inclusion $S_\varepsilon := S_0 + \varepsilon e_2 \subset \mathsf{C}$ holds for every $0 < \varepsilon \leqslant \varepsilon^*$ (throughout the paper, $\{e_1, e_2\}$ stands for an orthonormal basis). With these settings, we have $\Omega_\varepsilon := \mathsf{C} \setminus S_\varepsilon$ for every $0 \leqslant \varepsilon \leqslant \varepsilon^*$ and we assume that there exists $\delta^* > 0$ such that Ω_ε is locally, near the origin, described by

$$\{\xi := (\xi_1, \xi_2) \in \mathbb{R}^2 \,:\, |\xi_1| < \delta^*, \, 0 < \xi_2 < H_\varepsilon(\xi_1)\},$$

where

$$H_\varepsilon(\xi_1) := \kappa|\xi_1|^{1+\alpha} + \varepsilon,$$

the constants $\kappa > 0$ and $\alpha > 0$ (called the tangency exponent) being given.

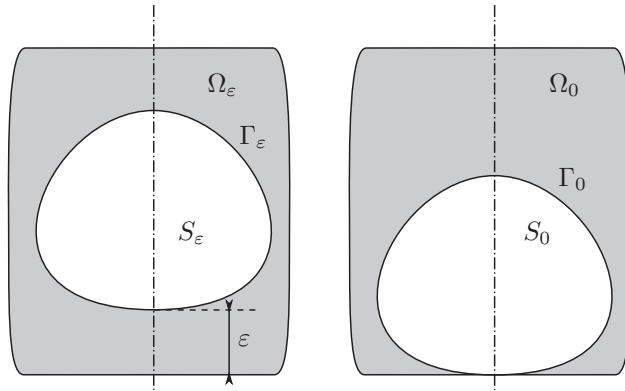FIG. 1. *The symmetric domains Ω_ε for $\varepsilon > 0$ and the singular limit domain Ω_0.*

The Neumann problem we shall consider is the following one:

(1.1a) $$-\Delta U_\varepsilon = F_\varepsilon \quad \text{in } \Omega_\varepsilon,$$

(1.1b) $$\partial_n U_\varepsilon = G_\varepsilon \quad \text{on } \Gamma_\varepsilon,$$

(1.1c) $$\partial_n U_\varepsilon = 0 \quad \text{on } \partial\mathsf{C},$$

where $\Gamma_\varepsilon := \partial S_\varepsilon$ denotes the boundary of the inclusion, n is the unit normal to $\partial\Omega_\varepsilon$ directed toward the exterior of Ω_ε, and F_ε and G_ε are given functions, respectively, defined on Ω_ε and Γ_ε and satisfying the compatibility condition

(1.2) $$\int_{\Omega_\varepsilon} F_\varepsilon \, \mathrm{d}\xi + \int_{\Gamma_\varepsilon} G_\varepsilon \, \mathrm{d}s = 0.$$

Our main objective in this paper is to study the asymptotic behavior of U_ε as $\varepsilon \to 0^+$, and more specifically, the behavior of the associated Dirichlet energy $\int_{\Omega_\varepsilon} |\nabla U_\varepsilon|^2 \, \mathrm{d}\xi$ as $\varepsilon \to 0^+$. For simplicity, only volume data F_ε and boundary data G_ε symmetric with respect to the ordinate axis will be considered.

Let us now describe the physical problem motivating such an asymptotic analysis.

1.1. Underlying fluid-structure contact problem. We are interested in investigating the possibility of a collision between a neutrally buoyant rigid solid with the bottom of the bounded cavity where it is immersed. In addition to the solid, the cavity is supposed to be filled with a perfect fluid.

Sticking to the notation of the previous section, we denote by C the cavity (C has the same properties as in the previous section), and for every time $t > 0$, by S_t the domain occupied by the solid and by Ω_t the fluid domain.

To simplify, we shall assume furthermore that at the initial time, the following hold.

1. $S_{t=0} = S_{\varepsilon^*}$ (S_{ε^*} has the same properties as above, in particular, regarding topology and symmetry).
2. The flow is irrotational, which entails, according to Helmholtz's third theorem, that it will remain irrotational for every time.
3. The flow is circulation free.
4. The velocity of the solid is vertical.

With these settings, for symmetry reasons, the motion of the solid will take place along the ordinate axis only and S_t and Ω_t will remain symmetric with respect to this axis at every moment.

In particular, the lowest point of S_t (which we assume, for the time being, to be unique) has coordinates $(0, \varepsilon(t))$ and the velocity of the solid is therefore $(0, \varepsilon'(t))$ (here and subsequently, the prime denotes the time derivative).

Classically in ideal fluid theory, according to hypothesis 2 above, we introduce at every time $t \geqslant 0$ the Kirchhoff potential $\varphi(t, \cdot)$ related to the vertical motion of the solid. This function solves a Laplace equation in Ω_t with Neumann boundary conditions, namely,

$$(1.3a) \qquad\qquad -\Delta\varphi(t, \cdot) = 0 \quad \text{in } \Omega_t,$$

$$(1.3b) \qquad\qquad \partial_n\varphi(t, \cdot) = n_2 \quad \text{on } \Gamma_t,$$

$$(1.3c) \qquad\qquad \partial_n\varphi(t, \cdot) = 0 \quad \text{on } \partial\mathsf{C},$$

where $\Gamma_t := \partial S_t$ and $n = (n_1, n_2)$ stands for the unit normal to $\partial\Omega_t$ directed towards the outside of the fluid. The Eulerian velocity of the fluid reads

$$u(t, \cdot) = \varepsilon'(t)\nabla\varphi(t, \cdot) \quad \text{in } \Omega_t \qquad (t \geqslant 0).$$

Notice at this point that the domains, and thus also the potential function, depend on t only through $\varepsilon(t)$. Consequently, from now on, we shall return to the notation of the previous section and we will denote by Ω_ε, S_ε, Γ_ε, and $\varphi(\varepsilon, \cdot)$, respectively, Ω_t, S_t, Γ_t, and $\varphi(t, \cdot)$.

The dynamics governing the motion of the solid can now be derived easily from the conservation of energy of the frictionless fluid/solid system. We denote by m_s the mass of the solid and by ϱ_f the density of the fluid. Recall that the solid is assumed to be neutrally buoyant, so the total energy of the system reduces to the kinetic energy which reads merely

$$E(\varepsilon, \varepsilon') := \frac{1}{2}(m_s + m_f(\varepsilon))|\varepsilon'|^2,$$

where

$$(1.4) \qquad\qquad m_f(\varepsilon) := \varrho_f \int_{\Omega_\varepsilon} |\nabla\varphi(\varepsilon, \xi)|^2 \, \mathrm{d}\xi$$

is the so-called added mass of the solid. Denoting by $\varepsilon'_0 < 0$ the initial value of $\varepsilon'(t)$ (the initial velocity being $(0, \varepsilon'_0)$), the identity $E(\varepsilon(t), \varepsilon'(t)) = E(\varepsilon^*, \varepsilon'_0)$ for every $t > 0$ leads to the following first order autonomous Cauchy problem for ε:

$$(1.5a) \qquad\qquad \varepsilon'(t) = \varepsilon'_0\sqrt{\frac{m_s + m_f(\varepsilon^*)}{m_s + m_f(\varepsilon(t))}}, \quad t > 0,$$

$$(1.5b) \qquad\qquad \varepsilon(t)|_{t=0} = \varepsilon^* > 0.$$

It is proved in [4] in a more general context that the function

$$\varepsilon \in \,]0, \varepsilon^*] \mapsto m_f(\varepsilon) \in \mathbb{R}^+$$

is analytic, so there is no regularity issue as long as $\varepsilon(t) > 0$. Actually, classical results for ODEs ensure that the solution exists as long as $\varepsilon(t) > 0$ (i.e., as long as the solid does not touch the boundary of the cavity). Considering the Cauchy problem (1.5), it is clear that the asymptotic behavior of the solid when getting closer to the cavity's bottom relies on the asymptotic behavior of $m_f(\varepsilon)$ as $\varepsilon \to 0^+$. The following cases can occur:

1. The added mass $m_f(\varepsilon)$ is uniformly bounded for every $\varepsilon \geqslant 0$. It entails that ε' is bounded from above by a negative constant and hence the solid will collide with the cavity's boundary in finite time with nonzero velocity (real shock case).
2. The added mass $m_f(\varepsilon)$ goes to $+\infty$ as ε goes to 0. Depending on the strength of the blowup, two subcases are to be considered:
   (a) The solid reaches the boundary of the cavity in finite time with zero velocity ("smooth landing" case, no shock);
   (b) the solution to the Cauchy problem (1.5) exists for every time $t \geqslant 0$. In this case $\varepsilon(t) \to 0$ as $t \to +\infty$ (infinite time touchdown case).

The collision problem between rigid bodies moving in a fluid has been addressed for instance in [5, 8], and more recently in [20] where the authors prove the lack of collision for a one dimensional model in which the fluid motion is governed by Burgers' equations and the solids are reduced to material points. This result has been generalized, but for a viscous fluid driven by the Navier–Stokes equations, in two and three dimensions in [9] and [10]. These studies assert that "frontal collisions" cannot occur in a viscous fluid, contrary to what happens in a perfect fluid. Indeed, in [11] it is proved that a ball immersed in a perfect fluid can hit a wall with nonzero velocity in finite time. In the present paper, we aim to extend this result to more general two dimensional configurations.

It is worth comparing our work with [6], where the authors consider a three dimensional axisymmetric solid falling on a plane. The fluid is assumed to obey the stationary Stokes equations. The purpose of the article is to study the influence of the solid roughness and of the boundary conditions on the collision problem. There is a striking similarity of the role played therein by the viscous energy and the one played by the added mass in what follows. The comparison of the results is not so easy though, since in [6] the roughness parameter is meant to tend to zero along with the distance between the solid and the wall.

1.2. Back to the model problem: A singularly perturbed boundary value problem. As already mentioned above, we will restrict our analysis to symmetric configurations (geometry, sources). For the sake of simplicity, we will use the same notation to denote the full domains C, S_ε, and Ω_ε and their intersections with the half-plane $\{\xi_1 < 0\}$. In addition to $\Gamma_\varepsilon := \partial S_\varepsilon$ and $\partial\mathsf{C}$, the boundary $\partial\Omega_\varepsilon$ is hence from now on composed of $\Gamma_\varepsilon^b := \{(0,\xi_2) : 0 < \xi_2 < \varepsilon\}$ and $\Gamma_\varepsilon^t := \partial\Omega_\varepsilon \setminus (\partial\mathsf{C} \cup \Gamma_\varepsilon \cup \Gamma_\varepsilon^b)$ (see Figure 2). The analysis of problem (1.3) (in the symmetric case considered here) leads to solving the following problem set in the half-cavity:

$$(1.6\text{a}) \qquad\qquad -\Delta U_\varepsilon = 0 \qquad \text{in } \Omega_\varepsilon,$$

$$(1.6\text{b}) \qquad\qquad \partial_n U_\varepsilon = n \cdot e_2 \quad \text{on } \Gamma_\varepsilon,$$

$$(1.6\text{c}) \qquad\qquad \partial_n U_\varepsilon = 0 \qquad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon,$$

which is nothing but a particular case of system (1.1), specifying $F_\varepsilon = 0$ and $G_\varepsilon = n \cdot e_2$.

As already mentioned concerning the general system (1.1), the main objective of this paper is to study the convergence of the U_ε solution to system (1.6) and obtain the first order term of the asymptotics of the Dirichlet energy associated with U_ε, namely, the quantity

$$(1.7) \qquad\qquad E_\varepsilon := \int_{\Omega_\varepsilon} |\nabla U_\varepsilon(\xi)|^2 \, \mathrm{d}\xi.$$
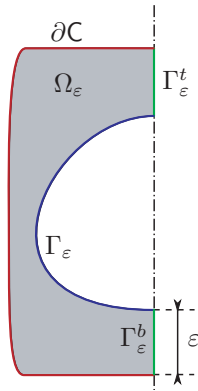
FIG. 2. *The new domain Ω_ε ($\varepsilon \geqslant 0$).*

Notice that, up to a multiplicative constant, this quantity coincides with the added mass defined in (1.4).

Deriving the asymptotics of the Dirichlet energy (1.7) requires solving two main difficulties:

1. The solution U_ε for $\varepsilon > 0$ and the solution U_0 for $\varepsilon = 0$ (if it exists) are not defined on the same domains (respectively, Ω_ε and Ω_0) and thus, they cannot be "compared" in a simple way.
2. The domain Ω_0 is strongly singular due to the presence of a cusp at the contact point.

A possible way to overcome the first difficulty is to artificially extend U_ε inside the solid (see, for instance, [19]), but this is far from being obvious due to the second one. Here we propose another approach, that we first explain formally. The key ingredient we use is a suitable change of coordinates $x = \Psi_\varepsilon(\xi)$ defined for every $\varepsilon \geqslant 0$ (i.e., including the limit case) such that, denoting $\omega_\varepsilon := \Psi_\varepsilon(\Omega_\varepsilon)$, we have

$$\omega_\varepsilon = D \cup R_\varepsilon \text{ with } D \cap R_\varepsilon = \varnothing$$

and where (see Figure 8)

- D is a fixed domain (i.e., independent of $\varepsilon \geqslant 0$);
- R_ε stands for the rectangle $]0, \ell_\varepsilon[\times]0, 1[$, where $\ell_\varepsilon \nearrow \ell_0 := +\infty$ as ε goes to 0.

Denoting by n the unit outer normal to $\partial\omega_\varepsilon$ and setting τ the tangent vector to $\partial\omega_\varepsilon$ such that $\tau^\perp = n$ and

$$u_\varepsilon := U_\varepsilon(\Psi_\varepsilon^{-1}), \qquad f_\varepsilon := F_\varepsilon(\Psi_\varepsilon^{-1})|\det D\Psi_\varepsilon^{-1}|,$$
$$g_\varepsilon := G_\varepsilon(\Psi_\varepsilon^{-1})|D\Psi_\varepsilon^{-1}\tau|, \qquad \gamma_\varepsilon := \Psi_\varepsilon(\Gamma_\varepsilon),$$

we will show that the general problem (1.1) is transformed into a new boundary value problem set in ω_ε:

$$(1.8a) \qquad -\operatorname{div}(\mathbb{A}_\varepsilon \nabla u_\varepsilon) = f_\varepsilon \quad \text{in } \omega_\varepsilon,$$
$$(1.8b) \qquad \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot n = g_\varepsilon \quad \text{on } \gamma_\varepsilon,$$
$$(1.8c) \qquad \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot n = 0 \quad \text{on } \partial\omega_\varepsilon \setminus \gamma_\varepsilon.$$

Notice that the compatibility condition (1.2) for the functions F_ε and G_ε yields

$$\int_{\omega_\varepsilon} f_\varepsilon \, dx + \int_{\gamma_\varepsilon} g_\varepsilon \, ds = 0.$$

Here, $\mathbb{A}_\varepsilon$ denotes the 2 by 2 matrix with continuous coefficients defined by

$$(1.9) \qquad \mathbb{A}_\varepsilon := \left[(D\Psi_\varepsilon) \circ \Psi_\varepsilon^{-1} \right] \left[(D\Psi_\varepsilon) \circ \Psi_\varepsilon^{-1} \right]^{\mathbf{T}} |\det(D\Psi_\varepsilon^{-1})|.$$

The Dirichlet energy E_ε defined by (1.7) takes the form

$$E_\varepsilon = \int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx.$$

Regarding the boundary value problem (1.8), we note the following:
1. In the new system of coordinates $x = (x_1, x_2)$, comparing the solution u_ε and the solution u_0 is now possible since $(\omega_\varepsilon)_{\varepsilon \geqslant 0}$ is an increasing sequence of domains, all of them included in the (unbounded) domain ω_0. In the new system of coordinates, the cusp singularity is sent to infinity.
2. The operator involved is not the Laplace operator anymore but the second order operator $-\operatorname{div}(\mathbb{A}_\varepsilon \nabla \cdot)$ (which depends on $\varepsilon \geqslant 0$). However, as we will see later, this operator is uniformly elliptic with respect to $\varepsilon \geqslant 0$ and $x \in \omega_\varepsilon$.

Motivated by the physical problem (1.3), we will mainly focus on system (1.8), obtained from system (1.6) after applying the change of variables, i.e., for a volume source term $f_\varepsilon = 0$ and boundary data $g_\varepsilon = D\Psi_\varepsilon^{-1}\tau \cdot e_1$. Recalling that the cusp is locally described by the equation $\xi_2 = H_0(\xi_1) = \kappa |\xi_1|^{1+\alpha}$ (with $\kappa, \alpha > 0$), our main result can be stated as follows.

THEOREM 1. *For every $\varepsilon > 0$, let u_ε be a solution to*

$$(1.10a) \qquad\qquad -\operatorname{div}(\mathbb{A}_\varepsilon \nabla u_\varepsilon) = 0 \qquad\qquad in\ \omega_\varepsilon,$$

$$(1.10b) \qquad\qquad \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot n = D\Psi_\varepsilon^{-1}\tau \cdot e_1 \qquad on\ \gamma_\varepsilon,$$

$$(1.10c) \qquad\qquad \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot n = 0 \qquad\qquad on\ \partial\omega_\varepsilon \setminus \gamma_\varepsilon.$$

Then the following alternative holds true:
1. *For $\alpha < 2$: System (1.10) with $\varepsilon = 0$ admits a finite energy solution u_0. Moreover,*

$$\|\nabla u_\varepsilon - \nabla u_0\|_{L^2(\omega_\varepsilon)} \to 0 \ and \ E_\varepsilon \to E_0 < \infty \ as \ \varepsilon \to 0^+.$$

2. *For $\alpha \geqslant 2$: System (1.10) with $\varepsilon = 0$ has no finite energy solution and two kinds of blowup are possible as $\varepsilon \to 0^+$ for the Dirichlet energy E_ε:*

$$(1.11) \qquad E_\varepsilon \underset{\varepsilon=0}{\sim} \begin{cases} \dfrac{1}{3}\kappa^{-1}|\ln(\varepsilon)| & if\ \alpha = 2, \\[2mm] \dfrac{1}{3}\varepsilon^{\frac{3}{1+\alpha}-1}\kappa^{-\frac{3}{1+\alpha}}\dfrac{3\pi/(1+\alpha)}{\sin(3\pi/(1+\alpha))} & if\ \alpha > 2. \end{cases}$$

Let us emphasize that the behavior of the Dirichlet energy only depends on the nature of the cusp (i.e., the constants κ and α) and not on other geometric features of the fluid domain.

This result follows immediately from the gathering of Theorems 4 and 5 below.

Regarding the collision problem, we claim the following.

COROLLARY 1. *In case* 1 *of the theorem ($\alpha < 2$), the solid will collide with the cavity's boundary in finite time with nonzero velocity (real shock case). In case* 2 *($\alpha \geqslant 2$), the solid reaches the cavity's bottom in finite time but with null velocity (smooth landing case).*

This corollary is a restatement of Corollary 2 for the case $\alpha < 2$. The case $\alpha \geqslant 2$ results from the following lemma (whose proof is postponed to Appendix B) and the estimates (1.11).

LEMMA 1. *Assume that $E_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$ and that there exists $\beta < 2$ such that $E_\varepsilon = O(\varepsilon^{-\beta})$, then the solid reaches the cavity's bottom in finite time but with null velocity (smooth landing case).*

Several references can be found in the literature regarding the asymptotics of the Dirichlet problem near a tangency point of smooth components of the boundary; see, for instance, the papers of Maz'ya, Nazarov, and Plamenevskij [13, 14, 15] and their book [16, Chapter 14]. The Neumann problem has been investigated more recently. In particular the singular behavior of the limit problem is derived in [17, 18] while the full asymptotics with respect to the small parameter ε is studied in [3, 2]. In particular, Cardone, Nazarov, and Sokolowski provide in [2] the first order asymptotics for the Neumann problem with thin ligaments in arbitrary dimension. However these papers deal with the case where the tangency exponent α of the cusp is an even integer $2m$.

In this paper, we propose a new method to obtain the first order approximation of the solution for the two dimensional Neumann Laplacian problem for arbitrary tangency exponent $\alpha > 0$. Our method relies on the use of a suitable change of variables leading to the study of Neumann problems set on the domains ω_ε and on a precise description of the asymptotic behavior at infinity of the solutions u_ε of these problems when ε tends to 0. Let us point out the main advantages of our approach :

1. Since ω_ε defines an increasing sequence, the solutions u_ε and the (potential) limit solution u_0 can be easily compared on the domain ω_ε in which they are both defined.

2. The cases $\varepsilon > 0$ and the limit case $\varepsilon = 0$ can be handled exactly in the same way in the new geometry ω_ε. Indeed, using an appropriate weighted Sobolev space (with a decaying weight as $x_1 \to \infty$), we will deal with uniformly elliptic Neumann problems on the domains ω_ε for all $\varepsilon \geqslant 0$.

3. In the new variables, the strength of the cusp (i.e., the parameter α) appears only in the operators and the boundary data of the Neumann problems and does not appear in the weight used in the functional spaces. Let us also emphasize that this makes our approach valid for arbitrary $\alpha > 0$, integer or not.

4. Finally, let us point out that our method can be, in principle, generalized to higher dimensions. The limit problem initially set on $\Omega_\varepsilon \subset \mathbb{R}^d$ is then transformed into a problem set on $\mathbb{R}^{d-1}_+ \times (0, 1)$.

1.3. Outline. The paper is organized as follows. In section 2 we collect some preliminary but elementary remarks on the asymptotic behavior of the Dirichlet energy as ε tends to 0. The change of variables near the cusp and its main properties are given in section 3. For the sake of clarity, its full construction (near and far from the cusp) is described in Appendix A. The rest of the paper deals with the analysis of the general boundary problems (1.8) set in ω_ε and obtained after applying the change of variables to system (1.1). In section 4, we describe the functional framework used to study the asymptotic behavior of the solutions u_ε of these problems as ε goes to 0.

We introduce appropriate weighted Sobolev spaces and we prove some useful lemmas (a trace theorem and a Poincaré–Wirtinger inequality involving constants which are uniform with respect to ε). In section 5, we provide a well-posedness result for the problem (1.8) with $\varepsilon = 0$, set in the unbounded domain ω_0 and a convergence result of u_ε (towards u_0) in the energy space for well prepared data (i.e., data having a suitable decay rate at infinity). These results are applied in section 6 to investigate the asymptotic behavior of the Dirichlet energy E_ε for the particular system (1.10). We show that for $\alpha < 2$ (recall that α is the coefficient describing the strength of the cusp) and $\varepsilon = 0$, the Neumann datum in (1.10b) is well prepared. This leads to the well-posedness of the limit problem and to a finite limit energy E_0. On the contrary, for $\alpha \geqslant 2$, the boundary data in (1.10b) do not have the decay rate required to apply the results of section 5. In this case, we prove the existence of a singular (nondecaying) solution for the problem (1.10) when $\varepsilon = 0$ and the blowup of the Dirichlet energy E_ε as ε tends to 0^+. In view of the collision issue, the first term of the asymptotics of E_ε is also given. Finally, in section 7, we show through some examples how the method can be adapted to deal with more general configurations.

2. Some preliminary remarks on the asymptotic behavior. In order to get a first intuition about the behavior of the solution U_ε of (1.6) as ε goes to 0, we collect here some general remarks about the problem and some comparison results obtained thanks to elementary considerations.

First of all, we recall a result proved by Nazarov, Sokolowski, and Taskinen in [18, section 5], providing a non existence result of finite energy solutions for problem 1.6, when $\varepsilon = 0$. The proof being short, it is given for the sake of completeness.

PROPOSITION 1. *The variational formulation*

$$(2.1) \qquad \int_{\Omega_0} \nabla U \cdot \nabla V \, d\xi = \int_{\Gamma_0} G_0 \, V \, d\sigma \qquad \forall V \in H^1(\Omega_0),$$

where $G_0 := n \cdot e_2$ has no solution $U \in H^1(\Omega_0)$ if $\alpha \geqslant 2$.

Proof. Using a contradiction argument, let us assume that there exists $U \in H^1(\Omega_0)$ satisfying the variational formulation (2.1). Given a neighborhood of the cusp $\mathcal{V}_0 \subset \Omega_0$ and a function $\chi \in C_0^\infty(\mathbb{R})$ with support in $]1/2, 1[$ such that $\int_\mathbb{R} \chi > 0$, define the sequence of test functions

$$V_k(\xi) = \begin{cases} 0 & \text{if } \xi \in \Omega_0 \setminus \overline{\mathcal{V}_0}, \\ 2^{\frac{k\alpha}{2}} \chi(2^k \xi_1) & \text{if } \xi \in \mathcal{V}_0. \end{cases}$$

It can be easily checked that the sequence $(V_k)_k$ is bounded in $H^1(\Omega_0)$ and using the dominated convergence theorem that

$$\lim_{k \to \infty} \int_{\Omega_0} \nabla U \cdot \nabla V_k \, d\xi = 0.$$

On the other hand, for the right-hand side of (2.1), we note that

$$\int_{\Gamma_0} G_0 \, V_k \, d\sigma = 2^{\frac{k\alpha}{2}} \int_\mathbb{R} \chi(2^k \xi_1) \, d\xi_1$$

$$= 2^{k\left(\frac{\alpha}{2} - 1\right)} \int_\mathbb{R} \chi(s) \, ds,$$
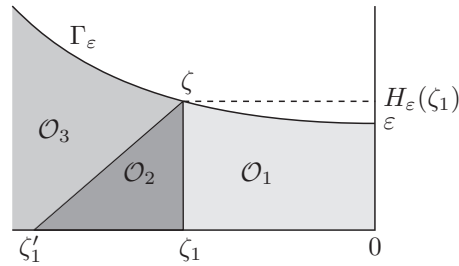
FIG. 3. *The partition of Ω_ε into $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$.*

which tends to $+\infty$ if $\alpha > 2$ and to a nonzero finite limit if $\alpha = 2$, leading to a contradiction. $\square$

The above result suggests that the limit energy E_0 is infinite for $\alpha \geqslant 2$. Using the Dirichlet principle, i.e., the identity

$$(2.2) \qquad \frac{1}{2}E_\varepsilon = \max_{U \in H^1(\Omega_\varepsilon)} \left\{ \int_{\Gamma_\varepsilon} n_2 U \, \mathrm{d}\sigma - \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla U|^2 \, \mathrm{d}\xi \right\},$$

available for every $\varepsilon > 0$, we first prove the following energy blowup for $\alpha > 2$.

PROPOSITION 2. *For every $\alpha > 2$, there exists a constant $C_\alpha > 0$ such that the Dirichlet energy* (1.7) *satisfies*

$$(2.3a) \qquad E_\varepsilon \geqslant C_\alpha \, \varepsilon^{\frac{3}{\alpha+1} - 1} \quad \forall \varepsilon > 0.$$

In the case where the solid has locally a flat bottom (i.e., contact would occur along a segment), there exists $C_\infty > 0$ such that

$$(2.3b) \qquad E_\varepsilon \geqslant C_\infty \varepsilon^{-1} \quad \forall \varepsilon > 0.$$

Remark 1. Surprisingly enough, we notice by comparison with the results of Theorem 1 and Proposition 6 that the estimates (2.3) are sharp. Estimates (2.3) prevent the rigid body from colliding with the cavity's wall with nonzero velocity but do not permit us to decide between the two remaining choices: "smooth landing" in finite time or "infinite time touchdown."

Proof. The main idea consists in building a suitable test function in the Dirichlet principle (2.2). We seek this function as a piecewise polynomial. For the sake of simplicity and unless necessary, we will drop in the notation the dependence on ε of the quantities introduced in the proof.

Let us begin by introducing the following partition of Ω_ε. The set $\mathcal{O}_1$ and $\mathcal{O}_2$ are as pictured on Figure 3 and $\mathcal{O}_3 := \Omega_\varepsilon \setminus (\overline{\mathcal{O}_1 \cup \mathcal{O}_2})$. The constants $\zeta_1 < 0$ and $\zeta_1' < 0$ will be specified later on.

Denoting by ζ the point $(\zeta_1, H_\varepsilon(\zeta_1))$, we define the following polynomial functions:

$$W_1(\xi) := -\frac{1}{2\varepsilon}(\xi_1^2 - \xi_2^2) \quad \text{and} \quad W_2(\xi) := \frac{1}{2\varepsilon}(\xi_2 + H_\varepsilon(\zeta_1))(\xi_2 - h(\xi_1)) + W_1(\zeta),$$

where

$$h(\xi_1) := \frac{H_\varepsilon(\zeta_1)}{\zeta_1 - \zeta_1'}(\xi_1 - \zeta_1').$$

Finally, the test function to be used in (2.2) reads

$$
(2.4) \qquad W(\xi) := \begin{cases} W_1(\xi) & \text{in } \mathcal{O}_1, \\ W_2(\xi) & \text{in } \mathcal{O}_2, \\ W_1(\zeta) & \text{in } \mathcal{O}_3. \end{cases}
$$

One can easily check that $W \in H^1(\Omega_\varepsilon)$ and that on the boundary Γ_ε of the solid we have

$$
W(\xi) = \begin{cases} W_1(\xi) & \text{on } \overline{\mathcal{O}_1} \cap \Gamma_\varepsilon, \\ W_1(\zeta) & \text{otherwise on } \Gamma_\varepsilon. \end{cases}
$$

Based on formula (2.2), we can obtain a lower bound for E_ε as follows:

$$
\frac{1}{2} E_\varepsilon \geqslant \int_{\Gamma_\varepsilon} n_2 W \, \mathrm{d}\sigma - \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla W|^2 \, \mathrm{d}\xi.
$$

Since $\int_{\Gamma_\varepsilon} n_2 \, \mathrm{d}\sigma = 0$ and W is a constant function on $\Gamma_\varepsilon \setminus \bar{\mathcal{O}}_1$, we can rewrite the inequality above as

$$
(2.5) \quad \frac{1}{2} E_\varepsilon \geqslant \int_{\Gamma_\varepsilon \cap \mathcal{O}_1} n_2 W_1 \, \mathrm{d}\sigma - W_1(\zeta) \int_{\Gamma_\varepsilon \cap \mathcal{O}_1} n_2 \, \mathrm{d}\sigma - \frac{1}{2} \int_{\mathcal{O}_1} |\nabla W_1|^2 \, \mathrm{d}\xi - \frac{1}{2} \int_{\mathcal{O}_2} |\nabla W_2|^2 \, \mathrm{d}\xi.
$$

We can now compute explicitly every term arising in the right-hand side of this estimate. We have

$$
\int_{\Gamma_\varepsilon \cap \mathcal{O}_1} n_2 W_1 \, \mathrm{d}\sigma = -\frac{1}{2\varepsilon} \int_0^{|\zeta_1|} \left[\xi_1^2 - (H_\varepsilon(\xi_1))^2 \right] \, \mathrm{d}\xi_1,
$$

and then, after some elementary algebra, we get

$$
(2.6) \qquad \int_{\Gamma_\varepsilon \cap \mathcal{O}_1} n_2 W_1 \, \mathrm{d}\sigma = \frac{1}{\varepsilon} \left[\frac{\kappa^2 |\zeta_1|^{3+2\alpha}}{6+4\alpha} - \frac{|\zeta_1|^3}{6} \right] + \frac{\kappa |\zeta_1|^{2+\alpha}}{2+\alpha} + \varepsilon \frac{|\zeta_1|}{2}.
$$

Addressing the second term in the right-hand side of (2.5) and observing that $\int_{\Gamma_\varepsilon \cap \mathcal{O}_1} n_2 \, \mathrm{d}\sigma = |\zeta_1|$, we get

$$
-W_1(\zeta) \int_{\Gamma_\varepsilon \cap \mathcal{O}_1} n_2 \, \mathrm{d}\sigma = \frac{|\zeta_1|}{2\varepsilon} \left[\zeta_1^2 - (H_\varepsilon(\zeta_1))^2 \right]
$$

$$
(2.7) \qquad\qquad = \frac{1}{2\varepsilon} \left[|\zeta_1|^3 - \kappa^2 |\zeta_1|^{3+2\alpha} \right] - \kappa |\zeta_1|^{2+\alpha} - \varepsilon \frac{|\zeta_1|}{2}.
$$

The third term is computed as follows:

$$
\int_{\mathcal{O}_1} |\nabla W_1|^2 \, \mathrm{d}\xi = \frac{1}{\varepsilon^2} \int_0^{|\zeta_1|} \int_0^{H_\varepsilon(\xi_1)} (\xi_1^2 + \xi_2^2) \, \mathrm{d}\xi,
$$

and this expression leads to:

$$
\begin{aligned}
(2.8) \qquad \int_{\mathcal{O}_1} |\nabla W_1|^2 \, \mathrm{d}\xi &= \frac{1}{\varepsilon^2} \left[\frac{\kappa |\zeta_1|^{4+\alpha}}{4+\alpha} + \frac{\kappa^3 |\zeta_1|^{4+3\alpha}}{12+9\alpha} \right] + \frac{1}{\varepsilon} \left[\frac{\kappa^2 |\zeta_1|^{3+2\alpha}}{3+2\alpha} + \frac{|\zeta_1|^3}{3} \right] \\
&\quad + \frac{\kappa |\zeta_1|^{2+\alpha}}{2+\alpha} + \varepsilon \left[\frac{|\zeta_1|}{3} \right].
\end{aligned}
$$

For the last term of (2.5), we have

$$\int_{\mathcal{O}_2} |\nabla W_2|^2 \, d\xi = \int_{\zeta_1'}^{\zeta_1} \int_0^{H_0(\xi_1)} |\partial_{\xi_1} W_2(\xi)|^2 + |\partial_{\xi_2} W_2(\xi)|^2 \, d\xi,$$

where

$$\partial_{\xi_1} W_2(\xi) = -\frac{1}{2\varepsilon}(H_\varepsilon(\zeta_1) + \xi_2) H_0'(\xi_1) \quad \text{and} \quad \partial_{\xi_2} W_2(\xi) = \frac{1}{2\varepsilon}(H_\varepsilon(\zeta_1) - H_0(\xi_1)) + \frac{1}{\varepsilon}\xi_2.$$

After a tedious but straightforward computation, we obtain that

(2.9)
$$\int_{\mathcal{O}_2} |\partial_{\xi_1} W_2(\xi)| \, d\xi = \frac{11}{48} \frac{1}{\varepsilon^2} \frac{(H_\varepsilon(\zeta_1))^5}{|\zeta_1' - \zeta_1|} \quad \text{and}$$
$$\int_{\mathcal{O}_2} |\partial_{\xi_2} W_2(\xi)| \, d\xi = \frac{7}{48} \frac{1}{\varepsilon^2} (H_\varepsilon(\zeta_1))^3 |\zeta_1' - \zeta_1|.$$

Now, we choose $\zeta_1 = -(\varepsilon/\kappa)^{\frac{1}{\alpha+1}}$ (so that $H_\varepsilon(\zeta_1) = 2\varepsilon$) and $\zeta_1' = \zeta_1 - \varepsilon$. Substituting (2.6), (2.7), (2.8), and (2.9) into (2.5), we obtain the following asymptotic expansion:

$$\int_{\Gamma_\varepsilon \cap \mathcal{O}_1} n_2 W_1 \, d\sigma - W_1(\zeta) \int_{\Gamma_\varepsilon \cap \mathcal{O}_1} n_2 \, d\sigma - \frac{1}{2} \int_{\mathcal{O}_1} |\nabla W_1|^2 \, d\xi - \frac{1}{2} \int_{\mathcal{O}_2} |\nabla W_2|^2 \, d\xi$$
$$= \left(\frac{\alpha+1}{6\alpha+24}\right) \kappa^{-\frac{3}{\alpha+1}} \varepsilon^{\frac{3}{\alpha+1}-1} + o\left(\varepsilon^{\frac{3}{\alpha+1}-1}\right).$$

For ε small enough, we get (2.3a). For a solid with a flat bottom, it suffices to replace H_0 by 0 in all the estimates and ζ_1 by a small constant (such that $H_0 = 0$ on $]\zeta_1, 0[$), to get the claimed result (2.3b). The proof is now complete. $\square$

The two following propositions allow comparison of the Dirichlet energy after simple changes in the geometry.

PROPOSITION 3. *Let us denote by $E_\varepsilon^{[\mathsf{C}, S_0]}$ the Dirichlet energy corresponding to a solid of shape S_0 in a cavity C. If $\mathsf{C}^1 \subset \mathsf{C}^2$ then*

(2.10)
$$E_\varepsilon^{[\mathsf{C}^1, S_0]} \geqslant E_\varepsilon^{[\mathsf{C}^2, S_0]} \quad \forall \varepsilon > 0.$$

In other words, this proposition asserts that the bigger the cavity is, the lower is the Dirichlet energy.

Proof. For every $\varepsilon > 0$, we set $\Omega_\varepsilon^k := \mathsf{C}^k \setminus S_\varepsilon$ ($k = 1, 2$). For every function $U \in H^1(\Omega_\varepsilon^2)$, its restriction to Ω_ε^1 belongs to $H^1(\Omega_\varepsilon^1)$ and we have the obvious inequality

$$\int_{\Gamma_\varepsilon} n_2 U \, d\sigma - \frac{1}{2} \int_{\Omega_\varepsilon^2} |\nabla U|^2 \, d\xi \leqslant \int_{\Gamma_\varepsilon} n_2 U \, d\sigma - \frac{1}{2} \int_{\Omega_\varepsilon^1} |\nabla U|^2 \, d\xi.$$

The conclusion follows then from the Dirichlet principle (2.2). $\square$

The next proposition tells us that the Dirichlet energy can be compared for configurations that are images one from the other by a global C^1 diffeormorphism.

PROPOSITION 4. *Let B be a large ball containing a cavity C. For every $0 \leqslant \varepsilon \leqslant \varepsilon^*$, consider the usual configuration involving a solid S_0, its domain S_ε, its boundary $\Gamma_\varepsilon := \partial S_\varepsilon$, and the fluid domain $\Omega_\varepsilon := \mathsf{C} \setminus S_\varepsilon$.*

Let $\tilde{H}_0 : \mathbb{R} \to \mathbb{R}$ be a given C^1 function. Let $\mathcal{U}$ be an open set containing Γ_ε for every $\varepsilon \geqslant 0$ small enough (see Figure 4).
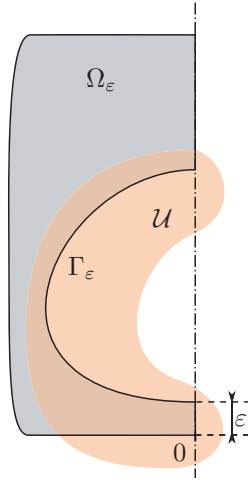
FIG. 4. *The open set $\mathcal{U}$ containing the solid's boundary for every $\varepsilon \geqslant 0$.*

For every C^1 *diffeomorphism* $\Phi : B \to B$ *such that*

$$(2.11) \qquad \Phi(\xi_1, \xi_2) = (\xi_1, \xi_2 + \tilde{H}_0(\xi_1)) \quad \forall \xi = (\xi_1, \xi_2) \in \mathcal{U},$$

the following estimate holds true (for every $\varepsilon > 0$ small enough):

$$c_1 E_\varepsilon^{[\mathsf{C}, S_0]} \leqslant E_\varepsilon^{[\tilde{\mathsf{C}}, \tilde{S}_0]} \leqslant c_2 E_\varepsilon^{[\mathsf{C}, S_0]},$$

where $\tilde{\mathsf{C}} := \Phi(\mathsf{C})$, $\tilde{S}_0 := \Phi(S_0)$, and $c_1 > 0$ and $c_2 > 0$ are two constants depending only on C, S_0, and Φ.

Notice that the assumption (2.11) entails that

$$\Phi(S_\varepsilon) = \tilde{S}_\varepsilon \quad \text{and} \quad \Phi(\Omega_\varepsilon) = \tilde{\Omega}_\varepsilon$$

for every $\varepsilon \geqslant 0$ small enough, where $\Omega_\varepsilon := \mathsf{C} \setminus S_\varepsilon$ and $\tilde{\Omega}_\varepsilon := \tilde{\mathsf{C}} \setminus \tilde{S}_\varepsilon$ and that the local parameterization of the fluid domain $\tilde{\Omega}_\varepsilon$ near the origin is now given by

$$\{\xi \in \mathbb{R}^2 : |\xi_1| < \delta^*, \ \tilde{H}_0(\xi_1) < \xi_2 < H_\varepsilon(\xi_1) + \tilde{H}_0(\xi_1)\}.$$

Proof. Let $\phi : [0,1] \to \Gamma_0$ be a parameterization of the boundary of the solid when $\varepsilon = 0$. Then $\phi_\varepsilon = \phi + \varepsilon e_2$ is a parameterization of Γ_ε and $\Phi \circ \phi_\varepsilon$ a parameterization of $\tilde{\Gamma}_\varepsilon := \Phi(\Gamma_\varepsilon) = \partial \tilde{S}_\varepsilon$ for every $\varepsilon \geqslant 0$.

For every $\varepsilon > 0$ and every $v \in H^1(\tilde{\Omega}_\varepsilon)$, we have

$$\int_{\tilde{\Gamma}_\varepsilon} n_2 \, v \, d\sigma = \int_0^1 [(D\Phi(\phi(s))\phi'(s))^\perp \cdot e_2] v(\Phi(\phi(s)) \, ds$$

$$= -\int_0^1 [\phi'(s) \cdot D\Phi(\phi(s))^{\mathbf{T}} e_1] v(\Phi(\phi(s)) \, ds.$$

From assumption (2.11), we infer that $D\Phi(\phi(s))^{\mathbf{T}} e_1 = e_1$ and therefore

$$(2.12) \qquad\qquad \int_{\tilde{\Gamma}_\varepsilon} n_2 \, v \, d\sigma = \int_{\Gamma_\varepsilon} n_2 \, \tilde{v} \, d\sigma,$$

where $\tilde{v} := v \circ \Phi$.

On the other hand, we have upon the change of variables $x = \Phi(\xi)$ the identity

$$(2.13) \qquad \int_{\tilde{\Omega}_\varepsilon} |\nabla v|^2 \, \mathrm{d}\xi = \int_{\Omega_\varepsilon} \mathbb{A}\nabla\tilde{v} \cdot \nabla\tilde{v} \, \mathrm{d}x,$$

where $\mathbb{A} := (D\Phi)^{-1}(D\Phi)^{-\mathbf{T}}|\det D\Phi|$. The matrix $\mathbb{A}$ is positive definite and hence there exist two positive constants λ_1 and λ_2 such that

$$\lambda_1|X|^2 \leqslant \mathbb{A}(\xi)X \cdot X \leqslant \lambda_2|X|^2$$

for every $X \in \mathbb{R}^2$ and every $\xi \in \overline{\mathsf{C}}$. A straightforward calculation shows that we have $\lambda_1 \leqslant 1 \leqslant \lambda_2$. Gathering (2.12) and (2.13), we get

$$\int_{\tilde{\Gamma}_\varepsilon} n_2 \, v \, \mathrm{d}\sigma - \frac{1}{2}\int_{\tilde{\Omega}_\varepsilon} |\nabla v|^2 \, \mathrm{d}\xi = \int_{\Gamma_\varepsilon} n_2 \, \tilde{v} \, \mathrm{d}\sigma - \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{A}\nabla\tilde{v} \cdot \nabla\tilde{v} \, \mathrm{d}x.$$

Then, since $\lambda_1 \leqslant 1$,

$$\int_{\Gamma_\varepsilon} n_2 \, \tilde{v} \, \mathrm{d}\sigma - \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{A}\nabla\tilde{v} \cdot \nabla\tilde{v} \, \mathrm{d}x \leqslant \lambda_1\left(\int_{\Gamma_\varepsilon} \lambda_1^{-1} n_2 \, \tilde{v} \, \mathrm{d}\sigma - \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla\tilde{v}|^2 \, \mathrm{d}x\right)$$

$$\leqslant \lambda_1 \max_{\tilde{v}\in H^1(\Omega_\varepsilon)}\left(\int_{\Gamma_\varepsilon} \lambda_1^{-1} n_2 \, \tilde{v} \, \mathrm{d}\sigma - \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla\tilde{v}|^2 \, \mathrm{d}x\right).$$

According to (2.2), we have

$$\max_{\tilde{v}\in H^1(\Omega_\varepsilon)}\left(\int_{\Gamma_\varepsilon} \lambda_1^{-1} n_2 \, \tilde{v} \, \mathrm{d}\sigma - \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla\tilde{v}|^2 \, \mathrm{d}x\right) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla\tilde{v}^*|^2 \, \mathrm{d}x,$$

where $\tilde{v}^*$ solves

$$\begin{aligned} -\Delta\tilde{v}^* &= 0 && \text{in } \Omega_\varepsilon, \\ \partial_n\tilde{v}^* &= \lambda_1^{-1}n_2 && \text{on } \Gamma_\varepsilon, \\ \partial_n\tilde{v}^* &= 0 && \text{on } \partial\mathsf{C}. \end{aligned}$$

By linearity, we obviously have

$$\int_{\Omega_\varepsilon} |\nabla\tilde{v}^*|^2 \, \mathrm{d}x = \lambda_1^{-2} E_\varepsilon^{[\mathsf{C},S_0]}.$$

Summing up, we have proved that

$$\int_{\tilde{\Gamma}_\varepsilon} n_2 \, v \, \mathrm{d}\sigma - \frac{1}{2}\int_{\tilde{\Omega}_\varepsilon} |\nabla v|^2 \, \mathrm{d}\xi \leqslant \frac{1}{2}\lambda_1^{-1} E_\varepsilon^{[\mathsf{C},S_0]},$$

and thus

$$(2.14) \qquad E_\varepsilon^{[\mathsf{C},S_0]} \geqslant \lambda_1 E_\varepsilon^{[\tilde{\mathsf{C}},\tilde{S}_0]}.$$

Remarking that Φ^{-1} enjoys the properties required for Φ to get (2.14), we deduce that we also have

$$E_\varepsilon^{[\tilde{\mathsf{C}},\tilde{S}_0]} \geqslant \lambda_2^{-1} E_\varepsilon^{[\mathsf{C},S_0]}.$$

The claim of the proposition follows. $\square$
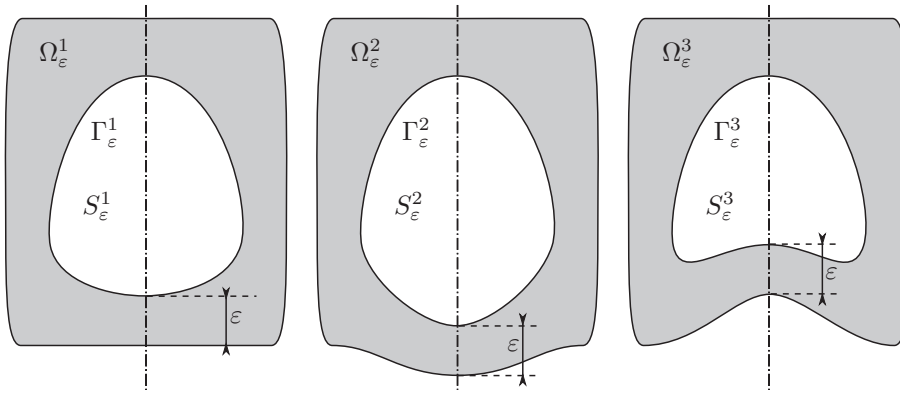
FIG. 5. *According to Proposition 4, the Dirichlet energy behaves similarly as $\varepsilon \to 0^+$ for all of these cases.*
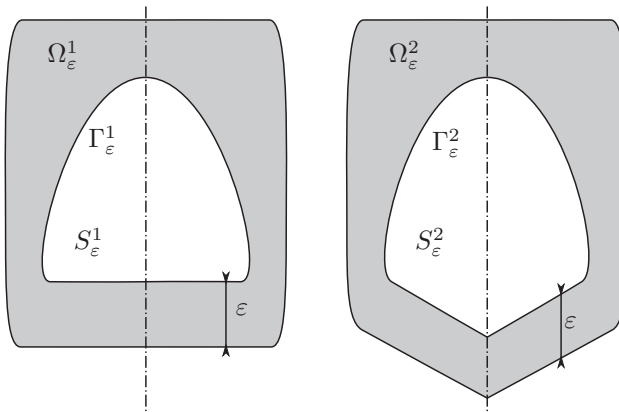
FIG. 6. *Another example of two configurations where, according to Proposition 4, the Dirichlet energy can be compared as $\varepsilon \to 0^+$ (notice on this example how we take advantage of working with a half-configuration and then recover a full configuration by symmetry).*

Typical illustrations of the above result are given in Figures 5 and 6.

Application. Combining Propositions 2, 3, and 4, we can deduce an estimate for the case where the bottom of the solid is concave (see Figure 7, on the right) and where there are two contact points for $\varepsilon = 0$. Indeed, with the notation of Figure 7, according to Proposition 3, for every $\varepsilon > 0$ we have

$$E_\varepsilon^{[\mathsf{C}^1, S_0^1]} \geqslant E_\varepsilon^{[\mathsf{C}^2, S_0^2]}$$

and according to Proposition 4, there exists a constant $c > 0$ such that

$$c\, E_\varepsilon^{[\mathsf{C}^2, S_0^2]} \geqslant E_\varepsilon^{[\mathsf{C}^3, S_0^3]}.$$

Using now Proposition 2, we infer the existence of a constant $C > 0$ such that

$$C\varepsilon^{-1} \geqslant E_\varepsilon^{[\mathsf{C}^3, S_0^3]}.$$

Consequently, the energy blowup is no greater in case 3 than in case 1.

FIG. 7. *The energy blowup in these 3 cases can be compared thanks to Propositions 3 and 4.*
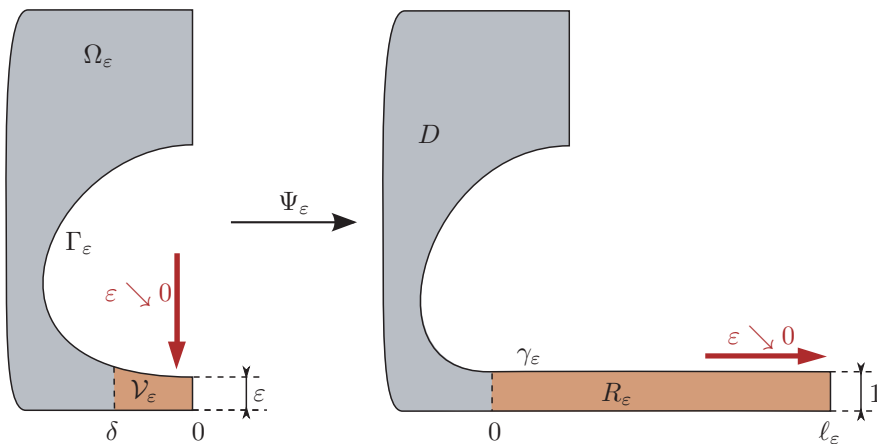
FIG. 8. *The domain Ω_ε and its image ω_ε by Ψ_ε. In particular, $D := \Psi_\varepsilon(\Omega_\varepsilon \setminus \mathcal{V}_\varepsilon)$ does not depend on $\varepsilon \geqslant 0$.*

3. From the physical domain to the semi-infinite strip. In this section, we describe the change of coordinates $x = \Psi_\varepsilon(\xi)$, $0 \leqslant \varep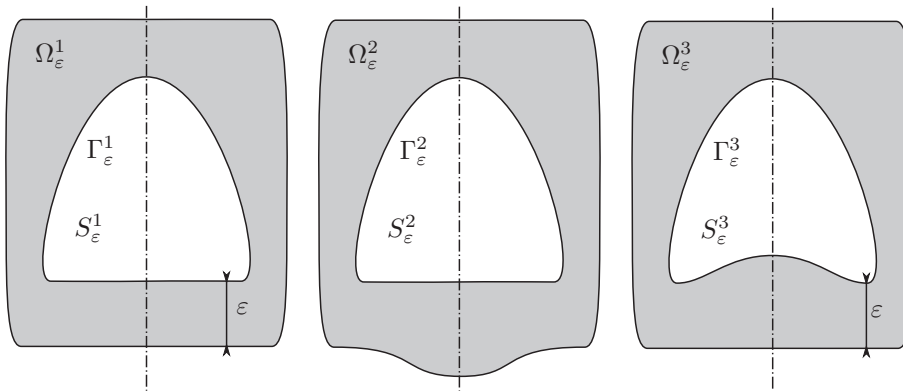silon \leqslant \varepsilon^*$, used in the sequel to transform the Laplace–Neumann problem (1.8) set on Ω_ε into an elliptic Neumann problem set on $\omega_\varepsilon := \Psi_\varepsilon(\Omega_\varepsilon) = D \cup R_\varepsilon$ (see Figure 8), where D is a fixed domain and $R_\varepsilon =]0, \ell_\varepsilon[\times]0, 1[$. Our change of variables is a generalization to the case $\varepsilon > 0$ of the one introduced by Ibuki [12] and used later by Grisvard in [7] and Acosta et al. in [1] to study the well-posedness and the regularity of Laplace problems in domains with cusps (in other words this corresponds in our problem to the limit case $\varepsilon = 0$). The full description of the diffeomorphism $\Psi_\varepsilon \in C^1(\overline{\Omega}_\varepsilon, \overline{\omega}_\varepsilon)$ is given in Appendix A. For the sake of clarity, we only give here its definition on some neighborhood of the contact region. More precisely, for every $\varepsilon \geqslant 0$ and given $\delta < 0$ small enough, let

$$(3.1) \qquad \mathcal{V}_\varepsilon := \{\xi \in \mathbb{R}^2 : \delta < \xi_1 < 0, 0 < \xi_2 < H_\varepsilon(\xi_1)\} \subset \Omega_\varepsilon.$$

Then, we set in $\mathcal{V}_\varepsilon$

$$(3.2) \qquad \Psi_\varepsilon(\xi) = \begin{pmatrix} \rho_\varepsilon(\xi_1) \\ \dfrac{\xi_2}{H_\varepsilon(\xi_1)} \end{pmatrix} \qquad \forall \xi \in \mathcal{V}_\varepsilon,$$

where the function $\rho_\varepsilon : [\delta, 0[\to \mathbb{R}^+$ is given by

$$(3.3) \qquad \rho_\varepsilon(\xi_1) := \int_\delta^{\xi_1} \frac{\mathrm{d}s}{H_\varepsilon(s)}.$$

Introducing

$$\ell_\varepsilon := \lim_{\xi_1 \to 0^+} \rho_\varepsilon(\xi_1),$$

we note that $\ell_\varepsilon < +\infty$ for $\varepsilon > 0$ and $\ell_0 := +\infty$. More precisely, based on the identity

$$\int_0^{+\infty} \frac{\mathrm{d}s}{s^{1+\alpha} + 1} = \frac{\pi/(\alpha+1)}{\sin(\pi/(\alpha+1))},$$

we can easily verify that

$$(3.4) \qquad \ell_\varepsilon \underset{\varepsilon=0}{\sim} \varepsilon^{-\frac{\alpha}{\alpha+1}} \kappa^{-\frac{1}{\alpha+1}} \left(\frac{\pi/(\alpha+1)}{\sin(\pi/(\alpha+1))} \right).$$

Setting $R_\varepsilon := \Psi_\varepsilon(\mathcal{V}_\varepsilon)$, we get

$$(3.5) \qquad R_\varepsilon =]0, \ell_\varepsilon[\times]0, 1[.$$

We define the reciprocal function to ρ_ε as being

$$(3.6) \qquad \mu_\varepsilon := \rho_\varepsilon^{-1} : [0, \ell_\varepsilon[\to [\delta, 0[.$$

When $\varepsilon = 0$, the function μ_ε can be made explicit and we have

$$(3.7) \qquad \mu_0(x_1) = -(\alpha\kappa)^{-\frac{1}{\alpha}}(x_1 + \hat{x}_1)^{-\frac{1}{\alpha}},$$

where $\hat{x}_1 = (\alpha\kappa)^{-1}|\delta|^{-\alpha}$. For all $\varepsilon \geqslant 0$, the function Ψ_ε^{-1} admits the following expression in R_ε:

$$(3.8) \qquad \Psi_\varepsilon^{-1}(x) = \begin{pmatrix} \mu_\varepsilon(x_1) \\ x_2 H_\varepsilon(\mu_\varepsilon(x_1)) \end{pmatrix} \qquad \forall x \in R_\varepsilon.$$

We collect, in the following lemma, some properties of the function μ_ε ($\varepsilon \geqslant 0$) that will be useful in what follows.

LEMMA 2.
1. *The following uniform convergence result holds true:*

$$(3.9) \qquad \|\mu_\varepsilon - \mu_0\|_{C^0([0,\ell_\varepsilon])} \to 0 \ \text{as } \varepsilon \to 0^+.$$

2. *There exist three positive constants C_1, C_2, and C_3, depending on α, κ, and δ only, such that, for every $\varepsilon \geqslant 0$ and every $x_1 \in [0, \ell_\varepsilon[$,*

$$(3.10a) \qquad |\mu_\varepsilon(x_1)| \leqslant C_1(1 + x_1)^{-\frac{1}{\alpha}},$$
$$(3.10b) \qquad |H_\varepsilon(\mu_\varepsilon)| \leqslant C_2(1 + x_1)^{-1-\frac{1}{\alpha}},$$
$$(3.10c) \qquad |H_0'(\mu_\varepsilon)| \leqslant C_2(1 + x_1)^{-1}.$$

The proof in postponed to Appendix B.

In Appendix A, we explain how to define Ψ_ε in $\Omega_\varepsilon \setminus \mathcal{V}_\varepsilon$ in such a way that $\Psi_\varepsilon(\Omega_\varepsilon \setminus \mathcal{V}_\varepsilon)$ defines a domain D which is independent of ε (see Figure 8).

Notice that $\ell_\varepsilon \nearrow +\infty \ (= \ell_0)$ as $\varepsilon \searrow 0^+$ and therefore that we have the nice inclusion properties

$$\varepsilon' > \varepsilon \Rightarrow \omega_{\varepsilon'} \subset \omega_\varepsilon \qquad \forall \varepsilon, \varepsilon' \geqslant 0.$$

PROPOSITION 5. *The following convergence property holds:*

(3.11a) $$\|\Psi_\varepsilon^{-1} - \Psi_0^{-1}\|_{C^1(\overline{\omega}_\varepsilon)} \to 0 \ as \ \varepsilon \to 0^+.$$

Moreover, if $\alpha > 1$, we also have

(3.11b) $$\|\Psi_\varepsilon^{-1} - \Psi_0^{-1}\|_{C^2(\overline{R}_\varepsilon)} \to 0 \ as \ \varepsilon \to 0^+.$$

Proof. We focus on the convergence on the rectangle R_ε, the rest the proof being given in Appendix A.

For all $\varepsilon \geqslant 0$, recall that the expression of the function Ψ_ε^{-1} in R_ε is given in (3.8).

We have $\mu_\varepsilon' = H_\varepsilon(\mu_\varepsilon)$, $\mu_\varepsilon'' = H_0'(\mu_\varepsilon)H_\varepsilon(\mu_\varepsilon)$, and $\mu_\varepsilon''' = H_0''(\mu_\varepsilon)H_\varepsilon(\mu_\varepsilon)^2 + H_0'(\mu_\varepsilon)^2 H_\varepsilon(\mu_\varepsilon)$ for all $\varepsilon \geqslant 0$. Since the functions H_0 and H_0' are bounded and uniformly continuous on the compact $[\delta, 0]$ (because $\alpha > 0$), and the same holds true for H_0'' if $\alpha > 1$, the conclusion follows from (3.9). $\square$

We can now make explicit the matrix $\mathbb{A}_\varepsilon$, arising in the statement of the Neumann problem (1.8) (at least in the rectangle R_ε), based on formulas (1.9) and (3.8):

(3.12) $$\mathbb{A}_\varepsilon(x) = \mathrm{Id} + x_2 H_0'(\mu_\varepsilon(x_1)) \begin{pmatrix} 0 & -1 \\ -1 & x_2 H_0'(\mu_\varepsilon(x_1)) \end{pmatrix} \qquad (\varepsilon \geqslant 0, \ x \in R_\varepsilon).$$

We claim the following.

LEMMA 3.

1. *The following convergence result holds true:*

   (3.13a) $$\|\mathbb{A}_\varepsilon - \mathbb{A}_0\|_{C^0(\overline{\omega}_\varepsilon)} \to 0 \ as \ \varepsilon \to 0^+.$$

   If $\alpha > 1$, we also have

   (3.13b) $$\|\mathbb{A}_\varepsilon - \mathbb{A}_0\|_{C^1(\overline{R}_\varepsilon)} \to 0 \ as \ \varepsilon \to 0^+.$$

2. *There exist two constants $0 < \lambda_1 < \lambda_2$, independent of $\varepsilon \geqslant 0$, such that*

   (3.14) $$\lambda_1 |X|^2 \leqslant \mathbb{A}_\varepsilon(x) X \cdot X \leqslant \lambda_2 |X|^2 \qquad \forall X \in \mathbb{R}^2, \ \forall x \in \omega_\varepsilon.$$

Proof. The convergences (3.13) are a straightforward consequence of Proposition 5.

The definition (1.9) of $\mathbb{A}_\varepsilon$ ($\varepsilon \geqslant 0$) entails that $\mathbb{A}_\varepsilon(x)$ is positive definite for every $\varepsilon \geqslant 0$ and every $x \in \omega_\varepsilon$. Since the eigenvalues depend continuously on the matrix, it suffices to prove (3.14) for $\varepsilon = 0$ to get the conclusion of the lemma. We would be done if $\overline{\omega}_0$ were compact. The way out consists in computing the expression of the eigenvalues of $\mathbb{A}_0$ in R_0. Indeed, we get

$$\lambda_j(x) = F_j(|x_2 H_0'(\mu_\varepsilon(x_1))|), \quad j = 1, 2,$$

where

$$F_1(X) := 1 + \frac{1}{2}X \left[X - \sqrt{X^2 + 4} \right], \quad F_2(X) := 1 + \frac{1}{2}X \left[X + \sqrt{X^2 + 4} \right],$$

and $F_1(X) \geqslant \lambda_1 := F_1((\alpha + 1)\kappa|\delta|^\alpha) > 0$ and $F_2(X) \leqslant \lambda_2 := F_2((\alpha + 1)\kappa|\delta|^\alpha)$ for every $X \in [0, (\alpha + 1)\kappa|\delta|^\alpha]$. The proof is now complete. $\square$

4. Functional framework. The domain ω_ε being bounded for $\varepsilon > 0$ and $-\operatorname{div}(\mathbb{A}_\varepsilon \nabla \cdot)$ being an elliptic operator, problem (1.8) is a well-posed Neumann problem for every $\varepsilon > 0$, the solution being uniquely defined in $H^1(\omega_\varepsilon)$, up to an additive constant (and provided the compatibility condition is satisfied). In order to study the well-posedness of this system when $\varepsilon = 0$, we need to introduce a suitable functional framework since the domain ω_0 is infinite in the x_1 direction. More precisely, for every $\varepsilon \geqslant 0$ and every $\beta \in \mathbb{R}$, we introduce on ω_ε and $\partial \omega_\varepsilon$, respectively, the measures

$$\mathrm{d}\nu_\beta(x) := \begin{cases} (1+x_1)^\beta \, \mathrm{d}x & \text{if } x \in R_\varepsilon, \\ \mathrm{d}x & \text{if } x \in D, \end{cases} \quad \text{and} \quad \mathrm{d}\nu_\beta^S(x) := \begin{cases} (1+x_1)^\beta \, \mathrm{d}x_1 & \text{if } x \in \gamma_\varepsilon^R, \\ \mathrm{d}s & \text{if } x \in \gamma_\varepsilon^D, \end{cases}$$

where $\gamma_\varepsilon^R := \{(x_1, 1) : 0 < x_1 < \ell_\varepsilon\}$ and $\gamma_\varepsilon^D := \gamma_\varepsilon \setminus \gamma_\varepsilon^R$.

Since the cases $\beta = -2$ and $\beta = 2$ will play a particular role, we set

$$\mathrm{d}m = \mathrm{d}\nu_{-2}, \quad \mathrm{d}m^{-1} = \mathrm{d}\nu_2, \quad \mathrm{d}\sigma = \mathrm{d}\nu_{-2}^S, \quad \text{and} \quad \mathrm{d}\sigma^{-1} = \mathrm{d}\nu_2^S.$$

We will denote by m (respectively, σ) the density function such that $\mathrm{d}m = m(x)\mathrm{d}x$ (respectively, $\mathrm{d}\sigma = \sigma(s)\mathrm{d}s$).

Throughout, $L^1(\omega_\varepsilon, \mathrm{d}\nu_\beta)$, $L^2(\omega_\varepsilon, \mathrm{d}\nu_\beta)$, $L^1(\gamma_\varepsilon, \mathrm{d}\nu_\beta^S)$, and $L^2(\gamma_\varepsilon, \mathrm{d}\nu_\beta^S)$ stand for the Lebesgue spaces of integrable and square-integrable functions, respectively, for the measures $\mathrm{d}\nu_\beta$ and $\mathrm{d}\nu_\beta^S$.

For every $\varepsilon \geqslant 0$ and every $\beta \in \mathbb{R}$, we define the weighted Sobolev spaces

$$H^1(\omega_\varepsilon, \mathrm{d}\nu_\beta) := \left\{ u \in L^2(\omega_\varepsilon, \mathrm{d}\nu_\beta) : \partial_{x_i} u \in L^2(\omega_\varepsilon), \, i = 1, 2 \right\}.$$

In the particular case $\beta = 2$, we also set

$$(4.1) \qquad H_N^1(\omega_\varepsilon, \mathrm{d}m) = \left\{ u \in H^1(\omega_\varepsilon, \mathrm{d}m) : \int_{\omega_\varepsilon} u \, \mathrm{d}m = 0 \right\}.$$

This space is well-defined (regarding the L^1 condition for $\varepsilon = 0$), as will be verified in Lemma 6 below.

Since ω_ε is bounded for $\varepsilon > 0$, the space $H^1(\omega_\varepsilon, \mathrm{d}\nu_\beta)$ coincides with the classical Sobolev space $H^1(\omega_\varepsilon)$ (with equivalent norms) for every $\beta \in \mathbb{R}$. However, the use of the weight is more convenient as it will allow us to obtain estimates (in the trace theorems, for the continuity and the coercivity) involving constants which are uniform with respect to $\varepsilon \geqslant 0$.

The introduction of the space $H_N^1(\omega_\varepsilon, \mathrm{d}m)$ is motivated by the following definition of solutions.

DEFINITION 1 (finite energy solution). *For $\varepsilon \geqslant 0$, let $f_\varepsilon \in L^2(\omega_\varepsilon, \mathrm{d}m^{-1})$ and $g_\varepsilon \in L^2(\gamma_\varepsilon, \mathrm{d}\sigma^{-1})$ satisfying the compatibility condition*

$$(4.2) \qquad \int_{\omega_\varepsilon} f_\varepsilon \, \mathrm{d}x + \int_{\gamma_\varepsilon} g_\varepsilon \, \mathrm{d}s = 0.$$

Then, a function $u_\varepsilon \in H_N^1(\omega_\varepsilon, \mathrm{d}m)$ is called a finite energy solution to system (1.8) if

$$(4.3) \qquad \int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla v \, \mathrm{d}x = \int_{\omega_\varepsilon} f_\varepsilon v \, \mathrm{d}x + \int_{\gamma_\varepsilon} g_\varepsilon v \, \mathrm{d}s \qquad \forall v \in H_N^1(\omega_\varepsilon, \mathrm{d}m).$$

The corresponding Dirichlet energy is defined by

$$(4.4) \qquad\qquad E_\varepsilon := \int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, \mathrm{d}x.$$

Remark 2. When $\varepsilon = 0$, we have $f_0/m \in L^2(\omega_0, \mathrm{d}m)$ and $g_0/\sigma \in L^2(\gamma_0, \mathrm{d}\sigma)$. We can rewrite (4.2) as:

$$\int_{\omega_0} \frac{f_0}{m} \, \mathrm{d}m + \int_{\gamma_0} \frac{g_0}{\sigma} \, \mathrm{d}\sigma = 0,$$

which indeed makes sense according to Lemma 6 stated below.

Remark 3. Since f_ε and g_ε satisfy the compatibility condition (4.2), we can equivalently replace $H_N^1(\omega_\varepsilon, \mathrm{d}m)$ by $H^1(\omega_\varepsilon, \mathrm{d}m)$ in the statement of problem 4.3 for every $\varepsilon \geqslant 0$.

Remark 4. It can be easily checked that for data satisfying in the physical domain the (classical) conditions $F_0 \in L^2(\Omega_0)$ and $G_0 \in L^2(\Gamma_0)$, we have necessarily $f_0 \in L^2(\omega_0, \mathrm{d}\nu_{2+2/\alpha})$ and $g_0 \in L^2(\gamma_0, \mathrm{d}\nu_{1+1/\alpha})$. It is also worth noticing that $L^2(\omega_0, \mathrm{d}\nu_{2+2/\alpha}) \subset L^2(\omega_0, \mathrm{d}m^{-1})$ for all $\alpha > 0$, while $L^2(\gamma_0, \mathrm{d}\nu_{1+1/\alpha}) \subset L^2(\gamma_0, \mathrm{d}\sigma^{-1})$ for $\alpha \leqslant 1$. In particular, (nonzero) constant functions are in $L^2(\gamma_0, \mathrm{d}\sigma^{-1})$ only for $\alpha < 2$.

In the rest of this section, we collect some useful results about the functional space $H_N^1(\omega_\varepsilon, \mathrm{d}m)$ (trace theorems, Poincaré inequality, extension operator from ω_ε to ω_0), paying very careful attention to ensure that the constants appearing in these continuity estimates are independent of $\varepsilon \geqslant 0$. These results will be used in section 5 to study the well-posedness of the variational problem (4.3) for $\varepsilon = 0$.

Note that $H^1(\omega_0, \mathrm{d}m)$ contains functions like $x \mapsto \ln(1 + |x|)$, which tend to $+\infty$ as $|x| \to +\infty$. However, we have the following density result.

LEMMA 4. *The space*

$$(4.5) \qquad\qquad \mathcal{E}(\omega_0) = \{u|_{\omega_0} : u \in C_0^\infty(\mathbb{R}^2)\}$$

is dense in $H^1(\omega_0, \mathrm{d}m)$.

Proof. For every integer $n \geqslant 1$, define the cutoff function χ_n on ω_0 by setting $\chi_n(x) = 1$ in D and, for every $x = (x_1, x_2) \in R_0$,

$$\chi_n(x) = \begin{cases} 1 & \text{if } x_1 \leqslant n, \\ 1 - \ln\left(\dfrac{1 + x_1}{1 + n}\right) & \text{if } n < x_1 \leqslant N(n), \\ 0 & \text{if } x_1 > N(n), \end{cases}$$

where

$$N(n) := (n+1)e - 1.$$

Let u be in $H^1(\omega_0, \mathrm{d}m)$ and set $u_n = u\chi_n$. We have

$$\int_{\omega_0} |\nabla(u - u_n)|^2 \, \mathrm{d}x = \int_{\omega_0} |\nabla u - \chi_n \nabla u - u \nabla \chi_n|^2 \, \mathrm{d}x$$

$$\leqslant 2 \left(\int_{\omega_0} (1 - \chi_n) |\nabla u|^2 \, \mathrm{d}x + \int_{\omega_0} u^2 |\nabla \chi_n|^2 \, \mathrm{d}x \right).$$

Let $\mathcal{O}_n^+ = \{x_1 > n\} \cap R_0$ and $\mathcal{O}_n^- = \omega_0 \setminus \overline{\mathcal{O}_n^+}$. We deduce from the last estimate that

$$\int_{\omega_0} |\nabla(u - u_n)|^2 \, dx \leqslant 2 \left(\int_{\mathcal{O}_n^+} |\nabla u|^2 \, dx + \int_{\mathcal{O}_n^+} u^2 \, dm \right),$$

and hence $\|\nabla(u - u_n)\|_{L^2(\omega_0)}$ goes to 0 as n goes to $+\infty$. Since $\|u - u_n\|_{L^2(\omega_0, dm)}$ obviously goes to 0 as well, we get that $\|u - u_n\|_{H^1(\omega_0, dm)}$ tends to 0.

Now, given $\eta > 0$, fix n large enough such that

(4.6) $$\|u - u_n\|_{H^1(\omega_0, dm)} \leqslant \eta,$$

and let M be an integer larger that $N(n)$. Classical density results for the standard Sobolev space $H^1(\mathcal{O}_M^-, dm)$ on the bounded domain $\mathcal{O}_M^-$ ensure the existence of $v \in C_0^\infty(\mathbb{R}^2)$ such that

(4.7) $$\|u_n - v\|_{H^1(\mathcal{O}_M^-, dm)} \leqslant \eta.$$

In particular, this implies that on the rectangle $\mathfrak{R} :=]N(n), M[\times]0, 1[$ we have

(4.8) $$\|v\|_{H^1(\mathfrak{R}, dm)} \leqslant \eta.$$

Set then $u^* = \theta v \in C_0^\infty(\mathbb{R}^2)$, where $\theta \in C_0^\infty(\mathbb{R})$ is a (one dimensional) cutoff function satisfying $0 \leqslant \theta(x_1) \leqslant 1$ for all $x_1 \in \mathbb{R}$, $\theta(x_1) = 1$ for $x_1 < N(n)$, $\theta(x_1) = 0$ for $x_1 > M$. Then, we have

$$\begin{aligned} \|u_n - u^*\|_{H^1(\omega_0, dm)}^2 &= \|u_n - u^*\|_{H^1(\mathcal{O}_{N(n)}^-, dm)}^2 + \|u^*\|_{H^1(\mathcal{O}_{N(n)}^+, dm)}^2 \\ &= \|u_n - v\|_{H^1(\mathcal{O}_{N(n)}^-, dm)}^2 + \|\theta v\|_{H^1(\mathfrak{R}, dm)}^2 \\ &\leqslant \|u_n - v\|_{H^1(\mathcal{O}_M^-, dm)}^2 + \|\theta v\|_{H^1(\mathfrak{R}, dm)}^2. \end{aligned}$$

Using (4.7) and (4.8), the last inequality shows that

$$\|u_n - u^*\|_{H^1(\omega_0, dm)} \leqslant C\eta$$

for some constant $C > 0$ (depending only on θ). Combining this estimate with (4.6) yields

$$\|u - u^*\|_{H^1(\omega_0, dm)} \leqslant (C + 1)\eta$$

which concludes the proof, since η is arbitrary. $\square$

The following lemma explains why the case $\beta = 2$ plays a particular role in the analysis.

LEMMA 5. *For any real number β, we have the following continuous embedding:*

$$H^1(\omega_0, \, d\nu_\beta) \hookrightarrow H^1(\omega_0, dm).$$

Proof. The result would be obvious if ω_0 were bounded. For the sake of simplicity and without loss of generality, we only show that

$$H^1(R_0, \, d\nu_\beta) \hookrightarrow H^1(R_0, dm),$$

which is equivalent to proving that the continuous embedding,

$$H^1(R_0, \, d\nu_\beta) \hookrightarrow L^2(R_0, dm),$$

holds true. Let n be an integer greater than 3 and let u be in $C^1(\overline{\mathfrak{R}_n})$ where, for every $k > 0$, $\mathfrak{R}_k$ denotes the rectangle $\{0 < x_1 < k\} \cap R_0$. Define the cutoff function χ in R_0 by

$$\chi(x) = \begin{cases} 1 & \text{if } 0 \leqslant x_1 < 1, \\ 2 - x_1 & \text{if } 1 \leqslant x_1 < 2, \\ 0 & \text{if } 2 \leqslant x_1, \end{cases}$$

and set $v = u\chi$ and $w = u(1 - \chi)$. We have

$$(4.9) \qquad \int_{\mathfrak{R}_n} u^2 \, \mathrm{d}m \leqslant 2 \left(\int_{\mathfrak{R}_2} v^2 \, \mathrm{d}m + \int_{\mathfrak{R}_n} w^2 \, \mathrm{d}m \right).$$

On the one hand,

$$\int_{\mathfrak{R}_2} v^2 \, \mathrm{d}m \leqslant C \int_{\mathfrak{R}_2} v^2 \, \mathrm{d}\nu_\beta \leqslant C \int_{\mathfrak{R}_2} u^2 \, \mathrm{d}\nu_\beta,$$

where the constant C depends only on β. On the other hand, for every $x \in \mathfrak{R}_n$,

$$|w(x)|^2 = 2 \int_0^{x_1} w(s, x_2) \partial_{x_1} w(s, x_2) \, \mathrm{d}s,$$

and hence

$$\int_{\mathfrak{R}_n} |w|^2 \, \mathrm{d}m = 2 \int_0^1 \int_0^n \int_0^{x_1} w(s, x_2) \partial_{x_1} w(s, x_2)(1 + x_1)^{-2} \, \mathrm{d}s \, \mathrm{d}x_1 \, \mathrm{d}x_2.$$

Invoking Fubini's theorem, we get

$$\int_{\mathfrak{R}_n} |w|^2 \, \mathrm{d}m = 2 \int_0^1 \int_0^n w(s, x_2) \partial_{x_1} w(s, x_2) \left(\int_s^n (1 + x_1)^{-2} \, \mathrm{d}x_1 \right) \mathrm{d}s \, \mathrm{d}x_2,$$

and the Cauchy–Schwarz inequality leads to

$$\int_{\mathfrak{R}_n} |w|^2 \, \mathrm{d}m \leqslant 2 \left(\int_{\mathfrak{R}_n} |\nabla w|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{\mathfrak{R}_n} u^2 \left(\int_s^n (1 + x_1)^{-2} \, \mathrm{d}x_1 \right)^2 \mathrm{d}s \, \mathrm{d}x_2 \right)^{1/2}.$$

Noticing that for every $n \geqslant 3$,

$$\left(\int_s^n (1 + x_1)^{-2} \, \mathrm{d}x_1 \right)^2 \leqslant (1 + s)^{-2},$$

and hence

$$\left(\int_{\mathfrak{R}_n} |w|^2 \, \mathrm{d}m \right)^{1/2} \leqslant 2 \left(\int_{\mathfrak{R}_n} |\nabla w|^2 \, \mathrm{d}x \right)^{1/2}$$

$$= 2 \left(\int_{\mathfrak{R}_n} |(1 - \chi)\nabla u - u\nabla\chi|^2 \, \mathrm{d}x \right)^{1/2}$$

$$\leqslant 2^{3/2} \left(\int_{\mathfrak{R}_n} |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} + 2^{3/2} \left(\int_{\mathfrak{R}_2} u^2 \, \mathrm{d}x \right)^{1/2}$$

$$\leqslant 2^{3/2} \left(\int_{\mathfrak{R}_n} |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} + C \left(\int_{\mathfrak{R}_2} u^2 \, \mathrm{d}\nu_\beta \right)^{1/2},$$

where the constant C depends on β only. In (4.9), we get

$$\int_{\mathfrak{R}_n} u^2 \, \mathrm{d}m \leqslant C \left(\int_{\mathfrak{R}_n} |\nabla u|^2 \, \mathrm{d}x + \int_{\mathfrak{R}_2} u^2 \, \mathrm{d}\nu_\beta \right)^{1/2},$$

where $C = C(\beta)$. Since $C^1(\overline{\mathfrak{R}_n})$ is dense in $H^1(\mathfrak{R}_n)$, we deduce that this estimate still holds true for every $u \in H^1(\mathfrak{R}_n)$.

Let now u be any function in $H^1(\omega_0, \mathrm{d}\nu_\beta)$ and denote it by $u_n = u|_{\mathfrak{R}_n} \in H^1(\mathfrak{R}_n)$. Applying the last estimate to u_n and letting n go to $+\infty$, we obtain the claimed result. $\quad\square$

Remark 5. Let β_1 and β_2 be two real numbers such that $\beta_1 \geqslant \beta_2$. It is obvious to check that

$$H^1(\omega_0, \mathrm{d}\nu_{\beta_1}) \hookrightarrow H^1(\omega_0, \mathrm{d}\nu_{\beta_2}).$$

Then, it follows from Lemma 5 that for every $\beta \leqslant -2$, we have $H^1(\omega_0, \mathrm{d}\nu_\beta) = H^1(\omega_0, \mathrm{d}m)$.

The next result shows in particular that the average of functions of $H^1(\omega_0, dm)$ can be considered, and therefore the space $H^1_N(\omega_\varepsilon, dm)$ introduced in (4.1) is well defined for $\varepsilon = 0$.

LEMMA 6. *For every $\beta < -3/2$, we have the following continuous embedding:*

$$(4.10) \qquad L^2(\omega_0, \mathrm{d}m) \hookrightarrow L^1(\omega_0, \mathrm{d}\nu_\beta) \quad and \quad L^2(\gamma_0, \mathrm{d}\sigma) \hookrightarrow L^1(\gamma_0, \mathrm{d}\nu_\beta^S).$$

Proof. One can simply observe that for every u in $L^2(\omega_0, \mathrm{d}m)$, we have

$$\int_{R_0} |u(x)|(1+x_1)^\beta \, \mathrm{d}x = \int_{R_0} |u(x)|(1+x_1)^{-1}(1+x_1)^{\beta+1} \, \mathrm{d}x$$

$$\leqslant \|u\|_{L^2(R_0, \mathrm{d}m)} \left(\int_{R_0} (1+x_1)^{2\beta+2} \, \mathrm{d}x \right)^{1/2}.$$

The conclusion follows for the first embedding in (4.10). The second embedding is proved exactly the same way. $\quad\square$

In order to establish a convergence result as ε tends to 0, we need to be able to extend functions defined on ω_ε to ω_0 in such a way that the extension operator is uniformly bounded with respect to ε. The following result provides the existence of such an operator.

LEMMA 7 (extension operator). *For every $\varepsilon > 0$ small enough, there exists an extension operator*

$$T_\varepsilon : H^1(\omega_\varepsilon, dm) \to H^1(\omega_0, dm),$$

such that

$$\|T_\varepsilon u\|_{H^1(\omega_0, dm)} \leqslant \sqrt{3} \|u\|_{H^1(\omega_\varepsilon, dm)}.$$

Proof. Since $D \subset \omega_\varepsilon$ for every $\varepsilon \geqslant 0$, it is sufficient to define T_ε from $H^1(R_\varepsilon, dm)$ to $H^1(R_0, dm)$.

For every $\varepsilon > 0$, set

$$\ell_\varepsilon^* := \ell_\varepsilon(2 - e^{-1}) + (1 - e^{-1}),$$

and let χ_ε be the cutoff function defined in R_0 as follows:

$$\chi_\varepsilon(x) = \begin{cases} 1 & \text{if } x_1 < \ell_\varepsilon, \\ 1 + \ln\left(1 + \dfrac{\ell_\varepsilon - x_1}{\ell_\varepsilon + 1}\right) & \text{if } \ell_\varepsilon \leqslant x_1 < \ell_\varepsilon^*, \\ 0 & \text{if } \ell_\varepsilon^* \leqslant x_1. \end{cases}$$

Note that $0 \leqslant \chi_\varepsilon(x) \leqslant 1$ for every $x \in R_0$. Moreover, for $\varepsilon > 0$ small enough, the quantity $\ell_\varepsilon^- := 2\ell_\varepsilon - \ell_\varepsilon^*$ is positive and for every $x \in R_0$ such that $x_1 > \ell_\varepsilon^-$ we have

(4.11)
$$|\chi_\varepsilon'(2\ell_\varepsilon - x_1)|^2 = \sigma(x_1) = (1 + x_1)^{-2}.$$

For every $u \in H^1(R_\varepsilon, \mathrm{d}m)$, we define $T_\varepsilon u$ in R_0 as follows:

$$(T_\varepsilon u)(x) = \begin{cases} u(x_1, x_2) & \text{if } x_1 \leqslant \ell_\varepsilon, \\ \chi_\varepsilon(x_1)u(2\ell_\varepsilon - x_1, x_2) & \text{if } \ell_\varepsilon \leqslant x_1 < \ell_\varepsilon^*, \\ 0 & \text{if } \ell_\varepsilon^* \leqslant x_1. \end{cases}$$

It can be easily verified that $T_\varepsilon u \in H^1_{\mathrm{loc}}(R_0)$. Moreover, we have

$$\int_{R_0} |\nabla(T_\varepsilon u)|^2 \, \mathrm{d}x$$

$$= \int_{R_\varepsilon} |\nabla u_\varepsilon|^2 \, \mathrm{d}x$$

$$+ \int_{R_0 \cap \{\ell_\varepsilon < x_1 < \ell_\varepsilon^*\}} |\partial_{x_1}\chi_\varepsilon(x)u(2\ell_\varepsilon - x_1, x_2) - \chi_\varepsilon(x)\partial_{x_1}u(2\ell_\varepsilon - x_1, x_2)|^2 \, \mathrm{d}x$$

$$+ \int_{R_0 \cap \{\ell_\varepsilon < x_1 < \ell_\varepsilon^*\}} |\partial_{x_2}u(2\ell_\varepsilon - x_1, x_2)|^2 \, \mathrm{d}x.$$

Applying the change of variables $x_1' = 2\ell_\varepsilon - x_1$ in the last two integrals of the right-hand side and using the Cauchy–Schwarz inequality, we get by using (4.11) that

$$\int_{R_0} |\nabla(T_\varepsilon u)|^2 \, \mathrm{d}x \leqslant \int_{R_\varepsilon} |\nabla u_\varepsilon|^2 \, \mathrm{d}x + 2 \int_{R_0 \cap \{\ell_\varepsilon^- < x_1 < \ell_\varepsilon\}} u^2 \, \mathrm{d}m + 2 \int_{R_0 \cap \{\ell_\varepsilon^- < x_1 < \ell_\varepsilon\}} |\nabla u|^2 \, \mathrm{d}x.$$

Consequently

(4.12)
$$\int_{R_0} |\nabla(T_\varepsilon u)|^2 \, \mathrm{d}x \leqslant 3\|u\|^2_{H^1(R_\varepsilon, \mathrm{d}m)}.$$

On the other hand, we also have

$$\int_{R_0} |T_\varepsilon u|^2 \, \mathrm{d}m \leqslant \int_{R_\varepsilon} |u_\varepsilon|^2 \, \mathrm{d}m + \int_{R_0 \cap \{\ell_\varepsilon^- < x_1 < \ell_\varepsilon\}} |\chi_\varepsilon(2\ell_\varepsilon - x_1, x_2)|^2 |u(x)|^2 \, \mathrm{d}m$$

$$\leqslant 2\|u\|^2_{L^2(R_\varepsilon, \mathrm{d}m)}.$$

The announced estimate then follows immediately by combining (4.12) and the last inequality. $\quad\square$

LEMMA 8 (uniform trace mapping). *Let $\mathcal{E}(\omega_0)$ be the functional space defined by (4.5). Then, the mapping*

$$u \in \mathcal{E}(\omega_0) \mapsto u|_{\gamma_0} \in L^2(\gamma_0, \mathrm{d}\sigma)$$

can be uniquely extended as a linear continuous operator:

$$\Lambda_0 : u \in H^1(\omega_0, \mathrm{d}m) \to L^2(\gamma_0, \mathrm{d}\sigma).$$

Moreover, denoting by Λ_ε the usual trace mapping from $H^1(\omega_\varepsilon)$ into $L^2(\gamma_\varepsilon)$ for $\varepsilon > 0$, there exists a constant $C > 0$ (independent of $\varepsilon \geqslant 0$) such that, for every $\varepsilon \geqslant 0$,

$$(4.13) \qquad \|\Lambda_\varepsilon(u)\|_{L^2(\gamma_\varepsilon, \mathrm{d}\sigma)} \leqslant C\|u\|_{H^1(\omega_\varepsilon, \mathrm{d}m)}.$$

Proof. Classically, it is sufficient to prove the existence of Λ_0 defined as an application from $H^1(R_0, \mathrm{d}m)$ into $L^2(\gamma_0^R, \mathrm{d}\sigma)$. For every $u \in \mathcal{E}(R_0)$, we have

$$
\begin{aligned}
|u(x_1, 1)|^2 &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}(|u(x_1, s)|^2 s)\mathrm{d}s \\
&= 2\int_0^1 \partial_{x_2} u(x_1, s)u(x_1, s)s\mathrm{d}s + \int_0^1 |u(x_1, s)|^2\,\mathrm{d}s.
\end{aligned}
$$

Multiplying both sides of this equality by $(1 + x_1)^{-2}$, integrating from 0 to $+\infty$ with respect to x_1, and using the Cauchy–Schwarz inequality, we get

$$\int_{\gamma_0^R} u^2(x_1, 1)\,\mathrm{d}\sigma \leqslant 2\left(\int_{R_0} |\partial_{x_2} u|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{R_0} u^2\,\mathrm{d}\nu_\beta\right)^{1/2} + \int_{R_0} u^2\,\mathrm{d}m,$$

where $\beta = -4$. According to Remark 5, we deduce that there exists a constant $C > 0$ such that

$$\int_{\gamma_0^R} u^2(x_1, 1)\,\mathrm{d}\sigma \leqslant C\|u\|_{H^1(R_0, \mathrm{d}m)}^2.$$

We conclude the existence of Λ_0 by recalling the density of $\mathcal{E}(\omega_0)$ into $H^1(\omega_0, \mathrm{d}m)$ proved in Lemma 4.

To get the uniform estimate (4.13), we write that, for every $u \in H^1(\omega_\varepsilon)$ and using Lemma 7,

$$\|\Lambda_\varepsilon(u)\|_{L^2(\gamma_\varepsilon, \mathrm{d}\sigma)} \leqslant \|\Lambda_0(T_\varepsilon u)\|_{L^2(\gamma_0, \mathrm{d}\sigma)} \leqslant \tilde{C}\|T_\varepsilon u\|_{H^1(\omega_0, \mathrm{d}m)} \leqslant C\|u_\varepsilon\|_{H^1(\omega_\varepsilon, \mathrm{d}m)},$$

where $\tilde{C}$ and C are positive constants independent of $\varepsilon \geqslant 0$. The proof is now complete. $\square$

LEMMA 9 (Poincaré–Wirtinger inequality). *There exists a constant $C > 0$ (independent of ε) such that for every $\varepsilon \geqslant 0$*

$$(4.14) \qquad \|u\|_{L^2(\omega_\varepsilon, \mathrm{d}m)} \leqslant C\|\nabla u\|_{L^2(\omega_\varepsilon)} \quad \forall\, u \in H_N^1(\omega_\varepsilon, \mathrm{d}m).$$

Proof. The result is proved in two steps. Using a direct calculation, we first show that this inequality holds on the (finite or semi-infinite) strip $R_\varepsilon :=]0, \ell_\varepsilon[\times]0, 1[$ with $C = \sqrt{2}$:

$$(4.15) \qquad \|u\|_{L^2(R_\varepsilon, \mathrm{d}m)} \leqslant \sqrt{2}\|\nabla u\|_{L^2(R_\varepsilon)} \quad \forall\, u \in H_N^1(R_\varepsilon, \mathrm{d}m).$$

Next, we prove by contradiction that inequality (4.15) implies (4.14).

Let u be a function in $C^1(\overline{R_\varepsilon})$ or $\mathcal{E}(R_0)$ if $\varepsilon = 0$. For every $x = (x_1, x_2)$ and $x' = (x_1', x_2')$ in R_ε, we have

$$u(x_1, x_2) - u(x_1', x_2') = u(x_1, x_2) - u(x_1', x_2) + u(x_1', x_2) - u(x_1', x_2')$$

$$= \int_{x_1'}^{x_1} \partial_{x_1} u(s, x_2)\, \mathrm{d}s + \int_{x_2'}^{x_2} \partial_{x_2} u(x_1', s)\, \mathrm{d}s.$$

Multiplying by $\left[(1 + x_1)^2 m(R_\varepsilon)\right]^{-1}$ and integrating the last equation with respect to x_1 from 0 to ℓ_ε, we get

$$(4.16) \quad \frac{1}{m(R_\varepsilon)} \int_0^{\ell_\varepsilon} (1 + x_1)^{-2} u(x_1, x_2)\, \mathrm{d}x_1 - u(x_1', x_2')$$

$$= \frac{1}{m(R_\varepsilon)} \int_0^{\ell_\varepsilon} \left(\int_{x_1'}^{x_1} \partial_{x_1} u(s, x_2)\, \mathrm{d}s \right) (1 + x_1)^{-2}\, \mathrm{d}x_1 + \int_{x_2'}^{x_2} \partial_{x_2} u(x_1', s)\, \mathrm{d}s.$$

Applying Fubini's theorem to the first term of the right-hand side, we get

$$\int_0^{\ell_\varepsilon} \left(\int_{x_1'}^{x_1} \partial_{x_1} u(s, x_2)\, \mathrm{d}s \right) (1 + x_1)^{-2}\, \mathrm{d}x_1 = \int_{x_1'}^{\ell_\varepsilon} \partial_{x_1} u(s, x_2) \left(\int_s^{\ell_\varepsilon} (1 + x_1)^{-2}\, \mathrm{d}x_1 \right) \mathrm{d}s.$$

Integrating now (4.16) with respect to x_2 from 0 to 1, we deduce that

$$\left| u(x') - \frac{1}{m(R_\varepsilon)} \int_{R_\varepsilon} u(x)\, \mathrm{d}m \right|$$

$$\leqslant \frac{1}{m(R_\varepsilon)} \int_{R_\varepsilon} |\partial_{x_1} u(s, x_2)|(1 + s)^{-1}\, \mathrm{d}s\, \mathrm{d}x_2 + \int_0^1 |\partial_{x_2} u(x_1', s)|\, \mathrm{d}s.$$

According to the Cauchy–Schwarz inequality, we get

$$\left| u(x') - \frac{1}{m(R_\varepsilon)} \int_{R_\varepsilon} u(x)\, \mathrm{d}m \right|^2 \leqslant 2 \left[\frac{1}{m(R_\varepsilon)} \|\partial_{x_1} u\|_{L^2(R_\varepsilon)}^2 + \int_0^1 |\partial_{x_2} u(x_1', s)|^2\, \mathrm{d}s \right],$$

and then, multiplying by $(x_1' + 1)^{-2}$ and integrating with respect to x' on ω_ε, we obtain

$$\int_{R_\varepsilon} \left| u(x') - \frac{1}{m(R_\varepsilon)} \int_{R_\varepsilon} u(x)\, \mathrm{d}m \right|^2 \mathrm{d}m(x') \leqslant 2 \left[\|\partial_{x_1} u\|_{L^2(R_\varepsilon)}^2 + \|\partial_{x_2} u\|_{L^2(R_\varepsilon)}^2 \right],$$

which shows that (4.15) holds true.

Now, we show by contradiction that (4.14) also holds. If not, there would exist two sequences $(\varepsilon_n)_{n \geqslant 1} \searrow 0$ and $(u_n)_{n \geqslant 1}$, with $u_n \in H_N^1(\omega_n, \mathrm{d}m)$ (for the sake of clarity, we set $\omega_n := \omega_{\varepsilon_n}$ throughout the proof), such that

(4.17a) $\|u_n\|_{L^2(\omega_n, \mathrm{d}m)} = 1,$

(4.17b) $\|\nabla u_n\|_{L^2(\omega_n)} \to 0 \quad$ as $n \to +\infty.$

On the one hand, setting $\overline{u_n} := \frac{1}{m(R_n)} \int_{R_n} u_n\, \mathrm{d}m$, the function defined on $R_n := R_{\varepsilon_n}$ by $u_n - \overline{u_n}$ obviously satisfies $v_n \in H_N^1(R_n, \mathrm{d}m)$ and, thanks to (4.15) and (4.17b),

$$\|u_n - \overline{u_n}\|_{L^2(R_n, \mathrm{d}m)} \leqslant \sqrt{2} \|\nabla u_n\|_{L^2(R_n)} \to 0 \quad \text{as } n \to +\infty,$$

and thus

$$\|u_n - \overline{u_n}\|_{H^1(R_n, \, \mathrm{d}m)} \to 0 \quad \text{as } n \to +\infty. \tag{4.18}$$

Moreover, using the Cauchy–Schwarz inequality and (4.17a), we have

$$|\overline{u_n}| = \frac{1}{m(R_n)} \int_{R_n} |u_n| \, \mathrm{d}m \leqslant \frac{1}{\sqrt{m(R_n)}} \leqslant \frac{1}{\sqrt{m(R_1)}} < \infty.$$

Consequently, there exists a constant $U_R \in \mathbb{R}$ such that the sequence of real numbers $(\overline{u_n})_{n \geqslant 1}$ converges (up to a subsequence) to U_R. According to (4.18), this shows that

$$\|u_n - U_R\|_{H^1(R_n, \, \mathrm{d}m)} \to 0 \quad \text{as } n \to +\infty. \tag{4.19}$$

On the other hand, on the domain $D = \omega_n \setminus R_n$, we immediately get from (4.17a) and (4.17b), using the compactness of the injection from $H^1(D)$ into $L^2(D)$, that (up to a subsequence) there exists a constant $U_D \in \mathbb{R}$ such that

$$\|u_n - U_D\|_{H^1(D)} \to 0 \quad \text{as } n \to +\infty. \tag{4.20}$$

The continuity of the trace of $u_n \in H_N^1(\omega_n, \, \mathrm{d}m)$ through the interface $\partial D \cap \partial R_n$ implies that $U_R = U_D := U$. Since $u_n \in H_N^1(\omega_n, \, \mathrm{d}m)$, this common value U is necessarily zero, as

$$0 = \int_{\omega_n} u_n \, \mathrm{d}m = \int_D u_n \, \mathrm{d}m + \int_{R_n} u_n \, \mathrm{d}m \to (m(D) + m(R_0))U \quad \text{as } n \to +\infty.$$

But this fact contradicts (4.17a), (4.19), and (4.20). $\qquad \square$

5. Some abstract well-posedness and convergence results. We are now in position to prove the well-posedness of the Neumann problem in the unbounded domain ω_0 and a convergence result as ε tends to 0. Applying the Riesz representation theorem, we immediately get by the Poincaré–Wirtinger inequality (see Lemma 9) the following well-posedness and uniqueness result.

THEOREM 2. *For every $f_0 \in L^2(\omega_0, \, \mathrm{d}m^{-1})$ and every $g_0 \in L^2(\gamma_0, \, \mathrm{d}\sigma^{-1})$ satisfying the compatibility condition*

$$\int_{\omega_0} f_0 \, \mathrm{d}x + \int_{\gamma_0} g_0 \, \mathrm{d}s = 0,$$

there exists a unique finite energy solution $u_0 \in H_N^1(\omega_0, \, \mathrm{d}m)$ (in the sense of Definition 1) to problem (4.3) when $\varepsilon = 0$.

Remark 6 (regularity of the solutions). Investigating the maximal regularity for the solution u_0 in terms of weighted Sobolev spaces is out of the range of our study. However, we can mention the following very basic result: In case $g_0 \in H_{\mathrm{loc}}^{1/2}$, then it is classical to verify that $u_0|_{\mathfrak{R}_n} \in H^2(\mathfrak{R}_n)$ for every $n \geqslant 0$ (recall that $\mathfrak{R}_n :=]0, n[\times]0, 1[$). Moreover, still for every $n \geqslant 0$, the function $u_0^n := u_0|_{\mathfrak{R}_n}$ satisfies

$$- \mathrm{div}(\mathbb{A}_0 \nabla u_0^n) = f_0 \text{ in } L^2(\mathfrak{R}_n) \quad \text{and} \quad \mathbb{A}_0 \nabla u_0^n \cdot n = g_0 \text{ in } H^{1/2}(\gamma_0^n),$$

where γ_0^n is the upper boundary of $\mathfrak{R}_n$.

As already mentioned in the beginning of the previous section, the existence and uniqueness of a solution $u_\varepsilon \in H_N^1(\omega_\varepsilon, \, \mathrm{d}m)$ for problem (4.3) when $\varepsilon > 0$ is classical. So, let us now investigate the convergence of u_ε as $\varepsilon \to 0^+$.

Notice that all functions $f_\varepsilon \in L^2(\omega_\varepsilon, \mathrm{d}m^{-1})$ and $g_\varepsilon \in L^2(\gamma_\varepsilon, \mathrm{d}\sigma^{-1})$ can be seen as functions of $L^2(\omega_0, \mathrm{d}m^{-1})$ and $L^2(\gamma_0, \mathrm{d}\sigma^{-1})$, respectively, by setting $f_\varepsilon := 0$ in $\omega_0 \setminus \omega_\varepsilon$ and $g_\varepsilon := 0$ on $\gamma_0 \setminus \gamma_\varepsilon$.

THEOREM 3. *For every $\varepsilon \geqslant 0$, let $f_\varepsilon \in L^2(\omega_\varepsilon, \mathrm{d}m^{-1})$ and $g_\varepsilon \in L^2(\gamma_\varepsilon, \mathrm{d}\sigma^{-1})$ be given such that the compatibility condition (4.2) is satisfied and denote by $u_\varepsilon \in H_N^1(\omega_\varepsilon, \mathrm{d}m)$ the unique solution to problem (4.3).*

Assume that

$$(5.1) \qquad f_\varepsilon \to f_0 \ \text{in} \ L^2(\omega_0, \mathrm{d}m^{-1}) \quad \text{and} \quad g_\varepsilon \to g_0 \ \text{in} \ L^2(\gamma_0, \mathrm{d}\sigma^{-1}).$$

Then, under the convergence result (3.13a), we have

$$(5.2\mathrm{a}) \qquad \|\nabla(u_0 - u_\varepsilon)\|_{L^2(\omega_\varepsilon)} \to 0 \quad as \ \varepsilon \to 0^+,$$

$$(5.2\mathrm{b}) \qquad \left| \int_{\omega_0} \mathbb{A}_0 \nabla u_0 \cdot \nabla u_0 \, \mathrm{d}x - \int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, \mathrm{d}x \right| \to 0 \quad as \ \varepsilon \to 0^+.$$

Proof. Throughout this proof, C will denote a constant that may change from line to line, but that is independent of ε.

Taking $v = u_\varepsilon \in H^1(\omega_\varepsilon, \mathrm{d}m)$ in (4.3) (see Remark 3), we get that

$$\int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, \mathrm{d}x \leqslant \|f_\varepsilon\|_{L^2(\omega_0, \mathrm{d}m^{-1})} \|u_\varepsilon\|_{L^2(\omega_\varepsilon, \mathrm{d}m)} + \|g_\varepsilon\|_{L^2(\gamma_0, \mathrm{d}\sigma^{-1})} \|u_\varepsilon\|_{L^2(\gamma_\varepsilon, \mathrm{d}\sigma)}.$$

Since $\mathbb{A}_\varepsilon$ is positive definite uniformly (with respect to ε) according to Lemma 3, and since the continuity of the trace operator and the Poincaré–Wirtinger constants are uniform with respect to $\varepsilon \geqslant 0$ as well (as asserted in Lemmas 8 and 9), we obtain that

$$\|\nabla u_\varepsilon\|_{L^2(\omega_\varepsilon)}^2 \leqslant C(\|f_\varepsilon\|_{L^2(\omega_0, \mathrm{d}m^{-1})} + \|g_\varepsilon\|_{L^2(\gamma_0, \mathrm{d}\sigma^{-1})}) \|\nabla u_\varepsilon\|_{L^2(\omega_\varepsilon)},$$

and therefore, using again Lemma 9,

$$(5.3) \qquad \|u_\varepsilon\|_{H^1(\omega_\varepsilon, \mathrm{d}m)} \leqslant C \quad \forall \varepsilon \geqslant 0.$$

Specifying now $v = u_0|_{\omega_\varepsilon} - u_\varepsilon \in H^1(\omega_\varepsilon, \mathrm{d}m)$ in (4.3), we get

$$(5.4) \qquad \int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot (\nabla u_0 - \nabla u_\varepsilon) \, \mathrm{d}x = \int_{\omega_\varepsilon} f_\varepsilon (u_0 - u_\varepsilon) \, \mathrm{d}x + \int_{\gamma_\varepsilon} g_\varepsilon (u_0 - u_\varepsilon) \, \mathrm{d}s.$$

On the other hand, taking $v = u_0 - T_\varepsilon u_\varepsilon \in H^1(\omega_0, \mathrm{d}m)$ in (4.3) when $\varepsilon = 0$, where T_ε is the extension operator introduced in Lemma 7, we get

$$(5.5) \quad \int_{\omega_0} \mathbb{A}_0 \nabla u_0 \cdot (\nabla u_0 - \nabla T_\varepsilon u_\varepsilon) \, \mathrm{d}x = \int_{\omega_0} f_0 (u_0 - T_\varepsilon u_\varepsilon) \, \mathrm{d}x + \int_{\gamma_0} g_0 (u_0 - T_\varepsilon u_\varepsilon) \, \mathrm{d}s.$$

Setting $\omega_\varepsilon^c := \omega_0 \setminus \omega_\varepsilon = [\ell_\varepsilon, +\infty[\times]0, 1[$ and $\gamma_\varepsilon^c = \gamma_0 \setminus \gamma_\varepsilon$ and subtracting (5.4) from (5.5), we obtain

$$
\begin{aligned}
&\int_{\omega_\varepsilon} \mathbb{A}_0 (\nabla u_0 - \nabla u_\varepsilon) \cdot (\nabla u_0 - \nabla u_\varepsilon) \, \mathrm{d}x \\
&\qquad = -\int_{\omega_\varepsilon} (\mathbb{A}_0 - \mathbb{A}_\varepsilon) \nabla u_\varepsilon \cdot (\nabla u_0 - \nabla u_\varepsilon) \, \mathrm{d}x - \int_{\omega_\varepsilon^c} \mathbb{A}_0 \nabla u_0 \cdot (\nabla u_0 - \nabla T_\varepsilon u_\varepsilon) \, \mathrm{d}x \\
&\qquad\quad + \int_{\omega_\varepsilon} (f_0 - f_\varepsilon)(u_0 - u_\varepsilon) \, \mathrm{d}x + \int_{\gamma_\varepsilon} (g_0 - g_\varepsilon)(u_0 - u_\varepsilon) \, \mathrm{d}s \\
&\qquad\quad + \int_{\omega_\varepsilon^c} f_0 (u_0 - T_\varepsilon u_\varepsilon) \, \mathrm{d}x + \int_{\gamma_\varepsilon^c} g_0 (u_0 - T_\varepsilon u_\varepsilon) \, \mathrm{d}s.
\end{aligned}
$$

(5.6)

Lemma 7 together with the estimate (5.3) ensure that $\|T_\varepsilon u_\varepsilon\|_{H^1(\omega_\varepsilon, \mathrm{d}m)} \leqslant C$ for every $\varepsilon \geqslant 0$. On the other hand, taking into account the convergence result (3.13a) and the hypothesis (5.1) in (5.6), we get

$$\int_{\omega_\varepsilon} \mathbb{A}_0(\nabla u_0 - \nabla u_\varepsilon) \cdot (\nabla u_0 - \nabla u_\varepsilon)\, \mathrm{d}x \longrightarrow 0 \quad \text{as } \varepsilon \to 0^+,$$

and (5.2a) follows with (3.14).

To prove now (5.2b), we write that

$$\int_{\omega_0} \mathbb{A}_0 \nabla u_0 \cdot \nabla u_0\, \mathrm{d}x - \int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon\, \mathrm{d}x = \int_{\omega_\varepsilon} \mathbb{A}_0(\nabla u_0 - \nabla u_\varepsilon) \cdot (\nabla u_0 + \nabla u_\varepsilon)\, \mathrm{d}x$$
$$+ \int_{\omega_\varepsilon} (\mathbb{A}_0 - \mathbb{A}_\varepsilon) \nabla u_\varepsilon \cdot \nabla u_\varepsilon\, \mathrm{d}x,$$

and the conclusion follows, invoking again the same aforementioned boundedness and convergence arguments. $\square$

6. Application to the collision problem. The weak formulation of system (1.10) reads, for every $\varepsilon \geqslant 0$,

$$(6.1) \qquad \int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla v\, \mathrm{d}x = \int_{\Gamma_\varepsilon} g_\varepsilon v\, \mathrm{d}s \qquad \forall\, v \in H_N^1(\omega_\varepsilon, \mathrm{d}m),$$

where $g_\varepsilon = D\Psi_\varepsilon^{-1} \tau \cdot e_1$. This quantity can be made explicit on $\gamma_\varepsilon^R = \{(x_1, 1) : 0 < x_1 < \ell_\varepsilon\}$ using the expression (3.8) of Ψ_ε^{-1}. Thus we get

$$g_\varepsilon = H_\varepsilon(\mu_\varepsilon) \quad \text{on } \gamma_\varepsilon^R.$$

Notice that although the expression of the matrix $\mathbb{A}_\varepsilon$ depends on α (i.e., on the nature of the cusp) and on $\varepsilon \geqslant 0$, this dependence is somehow irrelevant regarding the well-posedness of problem (6.1) because, as asserted by Lemma 3, the matrix is always uniformly elliptic and therefore the left-hand side of (6.1) always defines a symmetric, elliptic bilinear form on $H_N^1(\omega_\varepsilon, \mathrm{d}m)$, according to Lemma 9.

All of the relevant information regarding the well-posedness of problem (6.1) is carried by the boundary data g_ε. Considering Theorem 2, a sufficient condition for problem (6.1) to be well-posed when $\varepsilon = 0$ is $g_0 \in L^2(\gamma_0, \mathrm{d}\sigma^{-1})$ while, according to Theorem 3, the convergence of the solution u_ε toward u_0 and of the Dirichlet energy E_ε toward E_0 is ensured if $g_\varepsilon \to g_0$ in $L^2(\gamma_0, \mathrm{d}\sigma^{-1})$. These conditions are easy to check and lead us to distinguish two cases, a subcritical case $\alpha < 2$ and a supercritical case $\alpha \geqslant 2$. Let us emphasize that the critical value 2 is nothing but the dimension, and this is in agreement with the results of [2, 18].

6.1. The subcritical case $\alpha < 2$.

THEOREM 4. *When $\alpha < 2$, the following assertions hold true:*
   1. *Well-posedness of the limit problem: The Neumann boundary value problem (6.1) is well-posed for $\varepsilon = 0$. In particular, the corresponding Dirichlet energy E_0 is finite.*
   2. *Convergence of solutions: $\|\nabla u_\varepsilon - \nabla u_0\|_{L^2(\omega_\varepsilon)} \to 0$ as $\varepsilon \to 0^+$, where u_ε and u_0 are the solutions to problem (6.1) for $\varepsilon > 0$ and $\varepsilon = 0$, respectively.*
   3. *Convergence of the Dirichlet energy: The Dirichlet energy E_ε corresponding to problem (6.1) with $\varepsilon > 0$ tends to E_0, the finite Dirichlet energy of the problem when $\varepsilon = 0$.*

Considering the implication of this result for the physical problem of collision, we deduce that the added mass (1.4) is bounded uniformly in $\varepsilon \geqslant 0$. Using this estimate in (1.5), we get that the velocity of the solid is bounded from below and therefore we have the following.

COROLLARY 2. *When $\alpha < 2$, the solid meets the cavity's wall in finite time with nonzero velocity (real shock case).*

Proof of Theorem 4. As already mentioned, the first point of the theorem is a straightforward consequence of Theorem 2. Indeed, applying Theorem 2 with $f_0 = 0$ and $g_0 = H_0(\mu_0)$, we get existence and uniqueness of a solution if $g_0 \in L^2(\gamma_0, \mathrm{d}\sigma^{-1})$. Considering (3.7), we deduce that:

$$g_0 \underset{x_1=+\infty}{\sim} \kappa^{-\frac{1}{\alpha}} \alpha^{-1-\frac{1}{\alpha}} x_1^{-1-\frac{1}{\alpha}},$$

and requiring g_0 to be in $L^2(\gamma_0, \mathrm{d}\sigma^{-1})$ leads to $\alpha < 2$.

The two remaining points result from Theorem 3. It suffices to prove that $g_\varepsilon := H_\varepsilon(\mu_\varepsilon)$ (extended by 0 on $]\ell_\varepsilon, +\infty[$) converges to $g_0 := H_0(\mu_0)$ in $L^2(\gamma_0, \mathrm{d}\sigma^{-1})$. According to Lemma 2, we get that g_ε tends to g_0 a.e. on $\mathbb{R}_+$ and that

$$|g_\varepsilon - g_0|^2 \leqslant C(1 + x_1)^{-2-\frac{2}{\alpha}}$$

for some constant $C > 0$ independent of $\varepsilon \geqslant 0$. The conclusion follows from the dominated convergence theorem. $\square$

6.2. The supercritical case $\alpha \geqslant 2$. When $\alpha \geqslant 2$, the boundary term $g_0 := H_0(\mu_0)$ in problem (6.1) is not anymore in $L^2(\gamma_0, \mathrm{d}\sigma^{-1})$, preventing us from reasoning as in the previous section.

Actually, we already know, from Proposition 1, that problem (6.1) (when $\varepsilon = 0$) does not admit a finite energy solution in this case and from Proposition 2 that $E_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$.

In order to derive the first term in the asymptotic expansion of E_ε when ε goes to 0 in this case, we proceed as follows. For every $\varepsilon \geqslant 0$ we seek an ansatz u_ε^s to u_ε, containing all the information about the asymptotic behavior at infinity of u_ε as $\varepsilon \to 0$ (which corresponds to the blow up of the Dirichlet energy in the physical domain). Equivalently, in the physical domain, this ansatz contains all the information about the appearance of the cusp singularity at the contact point. In particular, u_ε^s will be shown to satisfy

$$(6.2) \qquad \int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon^s \cdot \nabla u_\varepsilon^s \mathrm{d}x \to +\infty \quad \text{as} \quad \varepsilon \to 0^+.$$

This is why we call this ansatz the singular part of the solution (which is a slight abuse of language since u_ε is smooth and has finite Dirichlet energy for every $\varepsilon > 0$), and we will refer to $u_\varepsilon^r := u_\varepsilon - u_\varepsilon^s$ as the regular part of u_ε.

The ansatz u_ε^s will be derived by adapting to our semi-infinite strip the multiscale expansion method used in [2, 18] to obtain the singular behavior near the contact point (in the physical domain). More precisely, for every $\varepsilon \geqslant 0$, the ansatz u_ε^s will be constructed such that the following properties hold true:

- The function u_ε^s is smooth, supported in R_ε, and extended by 0 in D, and the quantity $\mathbb{A}_\varepsilon \nabla u_\varepsilon^s \cdot n$ vanishes on the boundary $\partial \omega_\varepsilon \setminus \gamma_\varepsilon$. This implies in particular that for every $\varepsilon > 0$, the function u_ε^s satisfies the weak formulation (4.3) with source terms $(f_\varepsilon^s, g_\varepsilon^s) \in L^2(\omega_\varepsilon, \mathrm{d}m^{-1}) \times L^2(\gamma_\varepsilon, \mathrm{d}\sigma^{-1})$, where $f_\varepsilon^s := -\operatorname{div}(\mathbb{A}_\varepsilon \nabla u_\varepsilon^s)$ in ω_ε and $g_\varepsilon^s := \mathbb{A}_\varepsilon \nabla u_\varepsilon^s \cdot n$ on γ_ε.

- If we set
  (6.3)
  $$f_\varepsilon^r := -\operatorname{div}(\mathbb{A}_\varepsilon \nabla u_\varepsilon^r) = -f_\varepsilon^s \text{ in } \omega_\varepsilon \quad \text{and} \quad g_\varepsilon^r := \mathbb{A}_\varepsilon \nabla u_\varepsilon^r \cdot n = g_\varepsilon - g_\varepsilon^s \text{ on } \gamma_\varepsilon,$$

  then for $\varepsilon = 0$, $f_0^r \in L^2(\omega_0, \mathrm{d}m^{-1})$, $g_0^r \in L^2(\gamma_0, \mathrm{d}\sigma^{-1})$, and the following convergences hold true:

  $$\|f_\varepsilon^r - f_0^r\|_{L^2(\omega_\varepsilon, \mathrm{d}m^{-1})} \to 0 \quad \text{and} \quad \|g_\varepsilon^r - g_0^r\|_{L^2(\gamma_\varepsilon, \mathrm{d}\sigma^{-1})} \to 0 \text{ as } \varepsilon \to 0^+.$$

Then, according to Lemma 6 and since f_ε^r and g_ε^r satisfy the compatibility condition for every $\varepsilon > 0$ (this follows from the fact that u_ε^s is smooth and $\int_{\omega_\varepsilon} g_\varepsilon \, \mathrm{d}s = 0$), we can pass to the limit in (4.2) to get

$$\int_{\omega_0} f_0^r \, \mathrm{d}x + \int_{\gamma_0} g_0^r \, \mathrm{d}s = 0.$$

Consequently, for every $\varepsilon \geqslant 0$, u_ε^r is a solution (in the sense of Definition 1) of problem (4.3) with the source terms $(f_\varepsilon^r, g_\varepsilon^r)$ given by (6.3). On the other hand, the properties of f_ε^r and g_r^ε ensure, thanks to Theorem 3, that $u_0^r \in H_N^1(\omega_0, \mathrm{d}m)$ and that

$$\|\nabla u_\varepsilon^r - \nabla u_0^r\|_{L^2(\omega_\varepsilon)} = \|\nabla u_\varepsilon - \nabla(u_\varepsilon^s + u_0^r)\|_{L^2(\omega_\varepsilon)} \to 0 \quad \text{as} \quad \varepsilon \to 0^+.$$

Considering the limit problem ($\varepsilon = 0$), this construction will provide a natural solution

$$u_0 = u_0^s + u_0^r,$$

which does not belong to $H^1(\omega_0, \mathrm{d}m)$, leading us to supplement Definition 1 with the following.

DEFINITION 2 (infinite energy solution). *Let* $u_0^r \in H_N^1(\omega_0, \mathrm{d}m)$ *be a solution to system* (1.8) *(for* $\varepsilon = 0$*) with volume source term* $f_0^r \in L^2(\omega_0, \mathrm{d}m^{-1})$ *and boundary data* $g_0^r \in L^2(\omega_0, \mathrm{d}\sigma^{-1})$ *satisfying the compatibility condition*

(6.4) $$\int_{\omega_0} f_0^r \mathrm{d}x + \int_{\gamma_0} g_0^r \mathrm{d}s = 0.$$

Let u_0^s *be a smooth function, supported in* R_0 *(and extended by* 0 *in* D*) with infinite Dirichlet energy, and such that* $\mathbb{A}_0 \nabla u_0^s \cdot n = 0$ *on* $\partial \omega_0 \setminus \gamma_0$ *and denote*

$$f_0^s := -\operatorname{div}(\mathbb{A}_0 \nabla u_0^s) \text{ in } \omega_0 \quad \text{and} \quad g_0^s := \mathbb{A}_0 \nabla u_0^s \cdot n \text{ on } \gamma_0.$$

Then the function

$$u_0 := u_0^s + u_0^r,$$

set in ω_0 *is called an infinite energy solution to system* (1.8) *(for* $\varepsilon = 0$*) with volume source term* $f_0 := f_0^r + f_0^s$ *and boundary data* $g_0 := g_0^r + g_0^s$.

Remark 7. The (smooth) function u_0^s is required to be supported in R_0 in order to ensure that the expression of the volume source term f_0^s makes sense. Indeed, the entries of the matrix $\mathbb{A}_0$ are only supposed to be continuous in D whereas they are C^1 in R_0.

We can now give the expression of the ansatz u_ε^s: Let χ be a smooth cutoff function defined in ω_0 such that $\chi = 1$ in $]1, +\infty[\times]0, 1[$, χ is independent of x_2 in

$]0, 1[\times]0, 1[$, and $\chi = 0$ in D. For every $\varepsilon \geqslant 0$, we define the function u_ε^s in ω_ε by setting

$$(6.5) \qquad u_\varepsilon^s(x) = \chi(x) \left[-\int_0^{x_1} \mu_\varepsilon(s) \, \mathrm{d}s + \frac{1}{2} x_2^2 \left[H_\varepsilon(\mu_\varepsilon(x_1)) - \mu_\varepsilon(x_1) H_0'(\mu_\varepsilon(x_1)) \right] \right],$$

for every $x \in \omega_\varepsilon$. The following lemma, the proof of which is given in Appendix B, asserts that the Dirichlet energy indeed blows up as ε goes to 0^+.

LEMMA 10. *The Dirichlet energy of u_ε^s behaves as follows when $\varepsilon \to 0^+$:*

$$(6.6) \qquad \int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon^s \cdot \nabla u_\varepsilon^s \mathrm{d}x \underset{\varepsilon=0}{\sim} \begin{cases} \frac{1}{3} \kappa^{-1} |\ln(\varepsilon)| & \text{if } \alpha = 2, \\[2mm] \frac{1}{3} \varepsilon^{\frac{3}{1+\alpha}-1} \kappa^{-\frac{3}{1+\alpha}} \dfrac{3\pi/(1+\alpha)}{\sin(3\pi/(1+\alpha))} & \text{if } \alpha > 2. \end{cases}$$

THEOREM 5. *When $\alpha \geqslant 2$, the following assertions hold true:*
1. *There exists a function $u_0^r \in H^1(\omega_0, \mathrm{d}m)$ such that*

$$\|\nabla u_\varepsilon - \nabla(u_\varepsilon^s + u_0^r)\|_{L^2(\omega_\varepsilon)} \to 0 \qquad \text{as } \varepsilon \to 0^+.$$

2. *The function $u_0 := u_0^s + u_0^r$ is an infinite energy solution to system (1.10) (with $\varepsilon = 0$), in the sense of Definition 2.*
3. *The Dirichlet energy of u_ε behaves as the Dirichlet energy of u_ε^s as $\varepsilon \to 0^+$.*

Proof. For the sake of clarity, we provide a constructive proof to explain how to obtain the ansatz (6.5). Our method can be seen as an adaptation for every $\varepsilon \geqslant 0$ of the multiscale expansion method used in [18] in the case $\varepsilon = 0$.

First, recall that the system (1.8) under consideration reads

$$(6.7a) \qquad\qquad -\operatorname{div}(\mathbb{A}_\varepsilon \nabla u_\varepsilon) = 0 \quad \text{in } \omega_\varepsilon,$$
$$(6.7b) \qquad\qquad \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot n = g_\varepsilon \quad \text{on } \gamma_\varepsilon,$$
$$(6.7c) \qquad\qquad \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot n = 0 \quad \text{on } \partial\omega_\varepsilon \setminus \gamma_\varepsilon,$$

where we have set

$$g_\varepsilon(x) = D\Psi_\varepsilon^{-1} \tau \cdot e_1 = H_\varepsilon(X_\varepsilon^1), \qquad\qquad X_\varepsilon^1 := \mu_\varepsilon(x_1).$$

On the rectangle R_ε, we have

$$-\operatorname{div}(\mathbb{A}_\varepsilon \nabla u_\varepsilon)(x)$$
$$(6.8) \qquad = -\partial_{x_1^2}^2 u_\varepsilon(x) + 2x_2 H_0'(X_1^\varepsilon) \partial_{x_1 x_2}^2 u_\varepsilon(x) - \left[1 + \left(x_2 H_0'(X_1^\varepsilon) \right)^2 \right] \partial_{x_2^2}^2 u_\varepsilon(x)$$
$$\qquad\qquad + H_0'(\mu_\varepsilon) \partial_{x_1} u_\varepsilon - x_2 \left[2H_0'(X_1^\varepsilon)^2 + H_0''(X_1^\varepsilon) H_\varepsilon(X_1^\varepsilon) \right] \partial_{x_2} u_\varepsilon(x).$$

We seek an approximate solution to system (6.7) in R_ε in the form

$$(6.9) \qquad \hat{u}_\varepsilon^s(x) = v_\varepsilon(X_1^\varepsilon) + H_\varepsilon(X_1^\varepsilon) V_\varepsilon(X_1^\varepsilon, x_2) \quad \forall x = (x_1, x_2) \in R_\varepsilon,$$

where the functions v_ε and V_ε need to be determined. With (6.8) we get

$$(6.10a) \quad -\operatorname{div}(\mathbb{A}_\varepsilon \nabla \hat{u}_\varepsilon^s)(x) = -H_\varepsilon(X_1^\varepsilon) \left[\partial_{x_2^2}^2 V_\varepsilon(X_1^\varepsilon, x_2) + H_\varepsilon(X_1^\varepsilon) v_\varepsilon''(X_1^\varepsilon) \right] + \hat{f}_\varepsilon^s(x),$$

where
(6.10b)
$$\hat{f}_\varepsilon^s(x) := -H_\varepsilon(X_1^\varepsilon)\Big\{ H_0''(X_1^\varepsilon)H_\varepsilon(X_1^\varepsilon)V_\varepsilon(X_1^\varepsilon, x_2) + x_2^2 H_0'(X_1^\varepsilon)^2 \partial_{x_2}^2 V_\varepsilon(X_1^\varepsilon, x_2)$$
$$+ 2H_0'(X_1^\varepsilon)H_\varepsilon(X_1^\varepsilon)[\partial_{x_1}V_\varepsilon(X_1^\varepsilon, x_2) - x_2\partial_{x_1 x_2}^2 V_\varepsilon(X_1^\varepsilon, x_2)]$$
$$- x_2 H_0''(X_1^\varepsilon)H_\varepsilon(X_1^\varepsilon)\partial_{x_2}V_\varepsilon(X_1^\varepsilon, x_2) + H_\varepsilon(X_1^\varepsilon)^2 \partial_{x_1}^2 V_\varepsilon(X_1^\varepsilon, x_2)\Big\}.$$

On γ_ε^R, i.e., for $x = (x_1, 1)$, $0 < x_1 < \ell_\varepsilon$, or equivalently $\delta < X_1^\varepsilon < 0$, we have

(6.10c) $\quad \mathbb{A}_\varepsilon \nabla \hat{u}_\varepsilon^s \cdot n(x_1, 1) = H_\varepsilon(X_1^\varepsilon)\Big[-v_\varepsilon'(X_1^\varepsilon)H_0'(X_1^\varepsilon) + \partial_{x_2}V_\varepsilon(X_1^\varepsilon, 1)\Big] + \hat{r}_\varepsilon^s(x),$

where
(6.10d)
$$\hat{r}_\varepsilon^s(x) := H_\varepsilon(X_1^\varepsilon)\Big\{ H_0'(X_1^\varepsilon)^2[\partial_{x_2}V_\varepsilon(X_1^\varepsilon, 1) - V_\varepsilon(X_1^\varepsilon, 1)] - H_\varepsilon(X_1^\varepsilon)H_0'(X_1^\varepsilon)\partial_{x_1}V_\varepsilon(X_1^\varepsilon, 1)\Big\}.$$

On the lower boundary $\{(x_1, 0) : 0 < x_1 < \ell_\varepsilon\}$, we have

(6.10e) $\qquad\qquad \mathbb{A}_\varepsilon \nabla \hat{u}_\varepsilon^s \cdot n(x_1, 0) = H_\varepsilon(X_1^\varepsilon)\partial_{x_2}V_\varepsilon(X_1^\varepsilon, 0).$

Finally, on the vertical right boundary $\{(\ell_\varepsilon, x_2) : 0 < x_2 < 1\}$ we get

(6.10f) $\qquad\qquad \mathbb{A}_\varepsilon \nabla \hat{u}_\varepsilon^s \cdot n(\ell_\varepsilon, x_2) = \varepsilon[v_\varepsilon'(0) + \varepsilon\partial_{x_1}V_\varepsilon(0, x_2)].$

The function $\hat{u}_\varepsilon^s$ is supposed to be an ansatz for u_ε, so in view of the expressions (6.10), we seek the functions v_ε and V_ε in order to cancel the "leading" (i.e., less decreasing) terms in (6.10a). We also want the Neumann boundary conditions for $\hat{u}_\varepsilon$ to approximate "at best" the boundary conditions of u_ε. This leads to the following one dimensional Neumann system that must be satisfied for every fixed $\delta < X_1^\varepsilon < 0$:

(6.11a) $\qquad\qquad -\partial_{x_2}^2 V_\varepsilon(X_1^\varepsilon, \cdot) = H_\varepsilon(X_1^\varepsilon)v_\varepsilon''(X_1^\varepsilon) \qquad$ on $]0, 1[,$

(6.11b) $\qquad\qquad \partial_{x_2}V_\varepsilon(X_1^\varepsilon, 1) = 1 + v_\varepsilon'(X_1^\varepsilon)H_0'(X_1^\varepsilon),$

(6.11c) $\qquad\qquad \partial_{x_2}V_\varepsilon(X_1^\varepsilon, 0) = 0.$

The compatibility condition, necessary for this system to admit solutions, reads

$$H_\varepsilon(X_1^\varepsilon)v_\varepsilon''(X_1^\varepsilon) + 1 + v_\varepsilon'(X_1^\varepsilon)H_0'(X_1^\varepsilon) = 0, \quad \delta < X_1^\varepsilon < 0.$$

It can been rewritten as

$$\frac{d^2}{dx_1^2}\big(v_\varepsilon(\mu_\varepsilon(x_1))\big) = -\mu_\varepsilon'(x_1).$$

We choose the solution of this ODE that vanishes at $x_1 = 0$:

(6.12) $\qquad\qquad\qquad v_\varepsilon(\mu_\varepsilon(x_1)) = -\int_0^{x_1} \mu_\varepsilon(s)\, \mathrm{d}s.$

We deduce that

$$-H_\varepsilon(X_1^\varepsilon)v_\varepsilon''(X_1^\varepsilon) = 1 + v_\varepsilon'(X_1^\varepsilon)H_0'(X_1^\varepsilon) = \frac{H_\varepsilon(X_1^\varepsilon) - X_1^\varepsilon H_0'(X_1^\varepsilon)}{H_\varepsilon(X_1^\varepsilon)},$$

whence we infer the expression of V_ε:

$$(6.13) \qquad V_\varepsilon(x_1, x_2) = \frac{1}{2} x_2^2 \left[\frac{H_\varepsilon(X_1^\varepsilon) - X_1^\varepsilon H_0'(X_1^\varepsilon)}{H_\varepsilon(X_1^\varepsilon)} \right],$$

and then the expression of $\hat{u}_\varepsilon^s$ in R_ε thanks to (6.9):

$$(6.14) \qquad \hat{u}_\varepsilon^s(x_1, x_2) = -\int_0^{x_1} \mu_\varepsilon(s)\, ds + \frac{1}{2} x_2^2 \left[H_\varepsilon(X_1^\varepsilon) - X_1^\varepsilon H_0'(X_1^\varepsilon) \right].$$

Reconsidering now the expressions (6.10a) and (6.10c), we have by construction $-\operatorname{div}(\mathbb{A}_\varepsilon \nabla \hat{u}_\varepsilon^s) = \hat{f}_\varepsilon^s$ in ω_ε, $\mathbb{A}_\varepsilon \nabla \hat{u}_\varepsilon^s \cdot n = \hat{r}_\varepsilon^s + g_\varepsilon := \hat{g}_\varepsilon^s$ on γ_ε^R, and $\mathbb{A}_\varepsilon \nabla \hat{u}_\varepsilon^s \cdot n = 0$ on the lower and right boundaries of R_ε (according, respectively, to (6.10e) and (6.10f)). Recalling that χ is the cutoff function introduced above Lemma 10, we can define u_ε^s in the whole domain ω_ε by setting

$$u_\varepsilon^s(x) := \chi(x) \hat{u}_\varepsilon^s(x) \qquad (x \in \omega_\varepsilon),$$

and we recover the announced expression (6.5).

It remains to verify now that u_ε^s indeed carries the singular part of u_ε. Straightforward computation leads to:

$$\begin{aligned}
-\operatorname{div}(\mathbb{A}_\varepsilon \nabla u_\varepsilon^s) &= f_\varepsilon^s && \text{in } \omega_\varepsilon, \\
\mathbb{A}_\varepsilon \nabla u_\varepsilon^s \cdot n &= g_\varepsilon^s && \text{on } \gamma_\varepsilon, \\
\mathbb{A}_\varepsilon \nabla u_\varepsilon^s \cdot n &= 0 && \text{on } \partial\omega_\varepsilon \setminus \gamma_\varepsilon,
\end{aligned}$$

where

$$(6.15\text{a}) \qquad f_\varepsilon^s = [-\operatorname{div}(\mathbb{A}_\varepsilon \nabla \chi)\hat{u}_\varepsilon^s - 2\mathbb{A}_\varepsilon : (\nabla \chi \otimes \nabla \hat{u}_\varepsilon^s)] + \chi \hat{f}_\varepsilon^s \quad \text{in } \omega_\varepsilon,$$

$$(6.15\text{b}) \qquad g_\varepsilon^s = [\hat{u}_\varepsilon^s(\mathbb{A}_\varepsilon \nabla \chi) \cdot n] + \chi \hat{g}_\varepsilon^s \qquad\qquad\qquad \text{on } \gamma_\varepsilon.$$

The function $u_\varepsilon^r := u_\varepsilon - u_\varepsilon^s$ satisfies

$$(6.16\text{a}) \qquad -\operatorname{div}(\mathbb{A}_\varepsilon \nabla u_\varepsilon^r) = f_\varepsilon^r \quad \text{in } \omega_\varepsilon,$$

$$(6.16\text{b}) \qquad \mathbb{A}_\varepsilon \nabla u_\varepsilon^r \cdot n = g_\varepsilon^r \quad \text{on } \gamma_\varepsilon,$$

$$(6.16\text{c}) \qquad \mathbb{A}_\varepsilon \nabla u_\varepsilon^r \cdot n = 0 \quad \text{on } \partial\omega_\varepsilon \setminus \gamma_\varepsilon,$$

where

$$(6.17) \qquad f_\varepsilon^r = -f_\varepsilon^s \text{ in } \omega_\varepsilon \quad \text{and} \quad g_\varepsilon^r = g_\varepsilon - g_\varepsilon^s \text{ on } \gamma_\varepsilon.$$

For every $\varepsilon > 0$ the functions f_ε^s and g_ε^s satisfy the compatibility condition (they are defined as being, respectively, the divergence and the flux across the boundary of the smooth vector field $\mathbb{A}_\varepsilon \nabla u_\varepsilon^s$). Moreover, $\int_{\gamma_\varepsilon} g_\varepsilon\, ds = 0$ for every $\varepsilon \geqslant 0$. We deduce that f_ε^r and g_ε^r satisfy the compatibility condition as well (for every $\varepsilon > 0$).

The following lemma ensures that the ansatz function u_ε^s does the job it has been designed for.

LEMMA 11. *The function f_0^r belongs to $L^2(\omega_0, dm^{-1})$, the function g_0^r belongs to $L^2(\gamma_0, d\sigma^{-1})$, and they satisfy the compatibility condition (6.4). Moreover, we have the following convergence results:*

$$(6.18) \qquad \|f_\varepsilon^r - f_0^r\|_{L^2(\omega_\varepsilon, dm^{-1})} \to 0 \quad \text{and} \quad \|g_\varepsilon^r - g_0^r\|_{L^2(\gamma_\varepsilon, d\sigma^{-1})} \to 0 \text{ as } \varepsilon \to 0^+.$$

The proof is postponed to Appendix B.

To complete the proof of the two firsts points of the theorem, it suffices now to apply Theorem 3.

Finally, the last remaining point results from the second triangular inequality,

$$\left| \left(\int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, \mathrm{d}x \right)^{\frac{1}{2}} - \left(\int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon^s \cdot \nabla u_\varepsilon^s \, \mathrm{d}x \right)^{\frac{1}{2}} \right| \leqslant \left(\int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon^r \cdot \nabla u_\varepsilon^r \, \mathrm{d}x \right)^{\frac{1}{2}},$$

together with Lemma 10 and again Theorem 3. The proof of the theorem is now complete. □

7. Back to miscellaneous cases of section 2. In section 2 we state some results allowing comparison of the Dirichlet energy for different configurations, some of them not being covered by our general study. We show in this section that the method used to determine the asymptote of the Dirichlet energy in the previous section can be adapted to the case where the bottom of the solid is locally flat (referred to as "the flat case" in what follows).

7.1. The flat case. We shall now focus on the cases depicted on the left of Figure 6, i.e., where the bottom of the solid is locally flat. Being more specific, we consider the case where the function H_ε has the following expression (hereafter, in addition to the already defined constants, δ' is a negative number such that $\delta < \delta' < 0$):

$$H_\varepsilon(\xi_1) = \begin{cases} \hat{H}_\varepsilon(\xi_1 - \delta') & \text{if } \delta \leqslant \xi_1 < \delta', \\ \varepsilon & \text{if } \delta' \leqslant \xi_1 < 0, \end{cases}$$

where $\hat{H}_\varepsilon(\xi_1) := \kappa |\xi_1|^{1+\alpha} + \varepsilon$ with $\alpha > 2$. Denoting $\hat{\delta} := \delta - \delta' < 0$, we can define, associated with $\hat{H}_\varepsilon$ and $\hat{\delta}$, the functions $\hat{\rho}_\varepsilon$, $\hat{\mu}_\varepsilon$, $\hat{\Psi}_\varepsilon$, and $\hat{\Psi}_\varepsilon^{-1}$ based on formulas (3.3), (3.6), (3.2), and (3.8). We will also need the constant $\hat{\ell}_\varepsilon := \lim_{\xi_1 \to 0^+} \hat{\rho}_\varepsilon(\xi_1)$. Observe now that, corresponding to H_ε and using the very same formula as above, we get

$$\rho_\varepsilon(\xi_1) = \begin{cases} \hat{\rho}_\varepsilon(\xi_1 - \delta') & \text{if } \delta \leqslant \xi_1 < \delta,' \\ \frac{1}{\varepsilon}(\xi_1 - \delta') + \hat{\ell}_\varepsilon & \text{if } \delta' \leqslant \xi_1 < 0, \end{cases}$$

with $\ell_\varepsilon := \rho_\varepsilon(0) = \hat{\ell}_\varepsilon - \frac{\delta'}{\varepsilon}$ ($\ell_0 = +\infty$). Notice that the function $\rho_{\varepsilon=0}$ is only defined for $\delta \leqslant \xi_1 < \delta'$. For $\varepsilon \geqslant 0$, the inverse of ρ_ε defined on $[0, \ell_\varepsilon[$ reads

$$\mu_\varepsilon(x_1) = \begin{cases} \hat{\mu}_\varepsilon(x_1) + \delta' & \text{if } 0 \leqslant x_1 < \hat{\ell}_\varepsilon, \\ \varepsilon(x_1 - \hat{\ell}_\varepsilon) + \delta' & \text{if } \hat{\ell}_\varepsilon \leqslant x_1 < \ell_\varepsilon. \end{cases}$$

We deduce that, for every $\varepsilon \geqslant 0$,

(7.1)
$$H_\varepsilon(\mu_\varepsilon(x_1)) = \begin{cases} \hat{H}_\varepsilon(\hat{\mu}_\varepsilon(x_1)) & \text{if } 0 \leqslant x_1 < \hat{\ell}_\varepsilon, \\ \varepsilon & \text{if } \hat{\ell}_\varepsilon \leqslant x_1 < \ell_\varepsilon, \end{cases}$$
$$H_\varepsilon^{(k)}(\mu_\varepsilon(x_1)) = \begin{cases} \hat{H}_\varepsilon^{(k)}(\hat{\mu}_\varepsilon(x_1)) & \text{if } 0 \leqslant x_1 < \hat{\ell}_\varepsilon, \\ 0 & \text{if } \hat{\ell}_\varepsilon \leqslant x_1 < \ell_\varepsilon, \end{cases}$$

where $k \geqslant 1$ stands for the derivative of order k (if well-defined). On the rectangle R_ε, we get

$$\Psi_\varepsilon^{-1} = \hat{\Psi}_\varepsilon^{-1} + \begin{pmatrix} \delta' \\ 0 \end{pmatrix} \quad \text{in } \hat{R}_\varepsilon =]0, \hat{\ell}_\varepsilon[\times]0, 1[\quad \text{and}$$

$$\Psi_\varepsilon^{-1}(x) = \varepsilon \begin{pmatrix} x_1 - \hat{\ell}_\varepsilon \\ x_2 \end{pmatrix} + \begin{pmatrix} \delta' \\ 0 \end{pmatrix} \quad \text{in } [\hat{\ell}_\varepsilon, \ell_\varepsilon[\times]0, 1[.$$

According to formula (1.9), we deduce that

(7.2)
$$\mathbb{A}_\varepsilon = \begin{cases} \hat{\mathbb{A}}_\varepsilon & \text{in } \hat{R}_\varepsilon :=]0, \hat{\ell}_\varepsilon[\times]0, 1[, \\ \varepsilon \mathrm{Id} & \text{in } [\hat{\ell}_\varepsilon, \ell_\varepsilon[\times]0, 1[. \end{cases}$$

Notice once more that $\hat{R}_{\varepsilon=0} = R_{\varepsilon=0} =]0, +\infty[\times]0, 1[$. Considering (7.2), we deduce that the conclusions of Lemma 3 still hold true. This convergence result is required in Theorem 3, while all the results of section 4 are completely independent of the change of variables. We can now jump directly to the computation of the ansatz function. The general expression (6.5) leads to, in our case,

$$u_\varepsilon^s(x) = \begin{cases} \hat{u}_\varepsilon^s(x) - \delta' \chi(x) \left[x_1 + \frac{1}{2} x_2^2 \hat{H}_0'(\hat{\mu}_\varepsilon(x_1)) \right] & \text{if } x \in \hat{\omega}_\varepsilon, \\ - \int_0^{\hat{\ell}_\varepsilon} \hat{\mu}_\varepsilon(s) \, \mathrm{d}s + \frac{\varepsilon}{2}(x_2^2 - x_1^2 + 2x_1 \ell_\varepsilon - \ell_\varepsilon^2) & \text{if } x \in [\hat{\ell}_\varepsilon, \ell_\varepsilon[\times]0, 1[, \end{cases}$$

where $\hat{u}_\varepsilon^s$ is the ansatz for the problem corresponding to $\hat{H}_\varepsilon$ and χ the cutoff function defined above identity (6.5). The mirror image of Lemma 10 is the following.

LEMMA 12. *The Dirichlet energy of u_ε^s behaves as follows when $\varepsilon \to 0^+$:*

$$\int_{\omega_\varepsilon} \mathbb{A}_\varepsilon \nabla u_\varepsilon^s \cdot \nabla u_\varepsilon^s \, \mathrm{d}x \underset{\varepsilon=0}{\sim} \frac{|\delta'|^3}{3} \varepsilon^{-1}.$$

Unlike the other cases, neither κ nor α appear in the expression of the leading term in the asymptotic expansion of the energy. They would probably play a role in lower order terms only. This observation lead us to think that the technical condition $\alpha > 2$ has to be understood as a regularity assumption for the solid's boundary and is not related to the "strength" of the cusp of the fluid domain when $\varepsilon = 0$.

We can now claim the following.

PROPOSITION 6. *Theorem 5 is true for the flat case. In particular, the Dirichlet energy behaves as follows when ε goes to 0:*

(7.3)
$$E_\varepsilon \underset{\varepsilon=0}{\sim} \frac{|\delta'|^3}{3} \varepsilon^{-1}.$$

Proof. It suffices to verify that Lemma 11 and more precisely that the expressions (B.5) are in the appropriate function spaces. There is a subtlety here because the decay properties (3.10) are not true in our case. However, with (7.1) and (B.5) and since there is at least one derivative of H_0 in every product arising in the right-hand side of the expressions (B.5), it can be verified that we still get enough decay rate to get the conclusion. □

Considering the problem of collision, we can apply Lemma 1 to get the following.

PROPOSITION 7. *In the flat case, the solid reaches the cavity's bottom in finite time with null velocity (smooth landing case).*

7.2. Other cases. Combining (7.3) with the results of section 2, we can easily deduce the following:

- In the situation depicted in the right of Figure 6, the solid will collide with the outer boundary in finite time with null velocity (smooth landing case).
- In both configurations in the right of Figure 7, the solid will behave the same way when approaching the outer boundary and reach it in finite time. It is not possible (without further computations) to determine whether the velocity is null or not at the touching time.

Appendix A. Construction of the change of variables. In this section, we aim to construct the diffeomorphisms Ψ_ε $(0 \leqslant \varepsilon \leqslant \varepsilon^*)$ earlier introduced in section 3.

Recall the definitions (3.1) of $\mathcal{V}_\varepsilon$ $(0 \leqslant \varepsilon \leqslant \varepsilon^*)$. Since we need to emphasize the dependence on δ, we denote it rather $\mathcal{V}_\varepsilon(\delta)$ in this section and we recall that Ψ_ε has already been defined in $\mathcal{V}_\varepsilon(\delta)$ in section 3.

To simplify the construction, there is no loss of generality in assuming that (up to a rescaling) $H_0(\delta) = 1$.

Let us define $\hat{\Psi}_\varepsilon := \Psi_\varepsilon + \delta e_1$ and notice that, on the left vertical boundary of $\mathcal{V}_\varepsilon(\delta)$, we have

$$\hat{\Psi}((\delta, \xi_2)) = \left(\delta, \frac{\xi_2}{1 + \varepsilon}\right) \qquad (0 < \xi_2 < 1 + \varepsilon).$$

Starting from this observation, our leading idea is to extend $\hat{\Psi}_\varepsilon$ (rather than Ψ_ε) as a perturbation of the identity in $\Omega_\varepsilon \setminus \overline{\mathcal{V}}_\varepsilon(\delta)$. We proceed in several steps.

Step 1. We introduce an open cover $(U_k)_{1 \leqslant k \leqslant 3}$ of Ω_ε (see Figure 9), such that

1. $\Omega_\varepsilon \subset U_1 \cup U_2 \cup U_3$ for every $\varepsilon \geqslant 0$ small enough;
2. there exists $\delta'' < \delta' < \delta$ such that

$$\mathcal{V}_\varepsilon(\delta') \subset U_1 \subset \mathcal{V}_\varepsilon(\delta'') \quad \text{and} \quad U_k \cap \mathcal{V}_\varepsilon(\delta') = \varnothing \quad (k = 2, 3)$$

for every $\varepsilon \geqslant 0$ small enough;
3. $\Gamma_\varepsilon \cap U_3 = \varnothing$ for every $\varepsilon \geqslant 0$ small enough.

Consider $(\chi_k)_{1 \leqslant k \leqslant 3}$ a partition of unity subordinated to the open cover U_k $(k = 1, 2, 3)$ and let us define $\hat{\Psi}_\varepsilon^k$ $(k = 1, 2, 3)$, three functions, respectively, defined in U_1, U_2, and U_3 and out of which we are going to build $\hat{\Psi}_\varepsilon$.

Step 2. (construction of $\hat{\Psi}_\varepsilon^1$). For every $\varepsilon \geqslant 0$, we define the function

$$(A.1a) \qquad \hat{\rho}_\varepsilon(\xi_1) = \delta + \int_\delta^{\xi_1} F_\varepsilon(t) \, \mathrm{d}t \qquad (\delta'' < \xi_1 < 0),$$

where the function $F_\varepsilon \in C^0([\delta'', 0[)$ is given by

$$(A.1b) \qquad F_\varepsilon(t) = \begin{cases} 1 & \text{if } \xi_1 < \delta', \\ \frac{\varepsilon}{1+\varepsilon}\left(a\,t^2 + b\,t + c\right) + 1 & \text{if } \delta' < \xi_1 \leqslant \delta, \\ H_\varepsilon(t)^{-1} & \text{if } \xi_1 > \delta, \end{cases}$$

with

$$(A.1c) \qquad a = -\frac{3}{(\delta - \delta')^2}, \qquad b = \frac{2\,(\delta + 2\,\delta')}{(\delta - \delta')^2}, \qquad c = -\frac{\delta'(2\delta + \delta')}{(\delta - \delta')^2}.$$

We set $\hat{\ell}_\varepsilon := \hat{\rho}_\varepsilon(0)$ $(\varepsilon > 0)$ and $\hat{\ell}_0 := +\infty$.
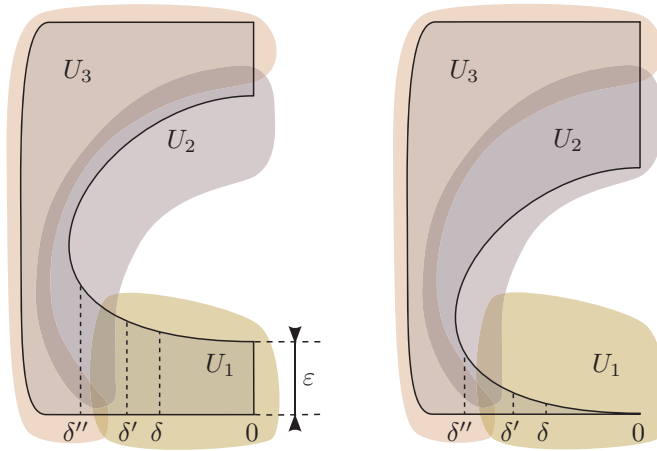
FIG. 9. *The open cover* $(U_k)_{1\leqslant k\leqslant 3}$ *of* Ω_ε *(for* $\varepsilon > 0$ *and* $\varepsilon = 0$*).*
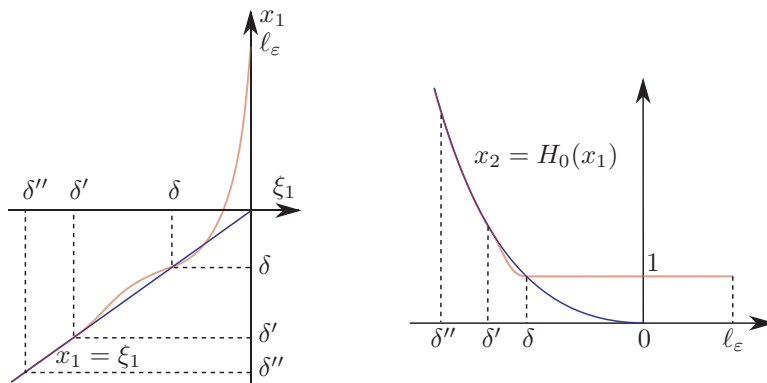

FIG. 10. *Graph of the functions* $\hat{\rho}_\varepsilon$ *(left) and* $\hat{H}_0$ *(right).*

For $\varepsilon \geqslant 0$ small enough, F_ε is positive and we denote by $\hat{\mu}_\varepsilon :]\delta'', \hat{\ell}_\varepsilon[\to]\delta'', 0[$ the inverse of $\hat{\rho}_\varepsilon$.

Define now a C^1 function $\hat{H}$ on $]\delta'', 0]$, as on Figure 10, satisfying in particular

$$\hat{H}_0(\xi_1) = \begin{cases} H_0(\xi_1) & \text{if } \xi_1 > \delta, \\ 1 & \text{if } \xi_1 < \delta'. \end{cases}$$

The function $\hat{\Psi}_\varepsilon^1$ is defined on $\mathcal{V}_\varepsilon(\delta'')$ as follows:

(A.2a)
$$\Psi_\varepsilon^1(\xi) = \begin{pmatrix} \hat{\rho}_\varepsilon(\xi_1) \\ \kappa_\varepsilon(\xi)\xi_2 - \varepsilon\chi_2(\xi) \end{pmatrix},$$

where

(A.2b)
$$\kappa_\varepsilon(\xi) = \frac{\hat{H}_0(\hat{\rho}_\varepsilon(\xi_1)) + \varepsilon\chi_2(\xi)}{H_\varepsilon(\xi_1)}.$$

Step 3. We introduce $\hat{\Psi}_\varepsilon^2 = \mathrm{Id} - \varepsilon e_2$ and $\hat{\Psi}_\varepsilon^3 = \mathrm{Id}$ ($\varepsilon \geqslant 0$) and we claim that the function $\hat{\Psi}_\varepsilon$ defined by

$$\hat{\Psi}_\varepsilon := \sum_{k=1}^3 \chi_k \hat{\Psi}_\varepsilon^k,$$

fulfilled the requirements. More precisely, setting $\Psi_\varepsilon = \hat{\Psi}_\varepsilon - \delta e_1$ ($\varepsilon \geqslant 0$), we have the following.

PROPOSITION 8. *The function Ψ_ε enjoys the following properties:*
1. *For every $\varepsilon \geqslant 0$, the set $D := \Psi_\varepsilon(\Omega_\varepsilon \setminus \overline{\mathcal{V}}_\varepsilon(\delta))$ is independent of ε.*
2. *For every $\varepsilon \geqslant 0$, Ψ_ε is C^1, invertible, and Ψ_ε^{-1} is C^1.*
3. $\|\Psi_\varepsilon^{-1} - \Psi_0^{-1}\|_{C^1(\overline{D})} \to 0$ *as $\varepsilon \to 0^+$.*

Proof. Since $\hat{\Psi}_\varepsilon$ and Ψ_ε only differ in a translation, the proof is carried out with $\hat{\Psi}_\varepsilon$ instead of Ψ_ε.

The first point is easily verified by computing the image of the boundary of $\Omega_\varepsilon \setminus \overline{\mathcal{V}}_\varepsilon(\delta)$. Indeed, denoting

$$\hat{\mathcal{V}}_\varepsilon := \{(x_1, x_2) \in \Omega_\varepsilon \ : \ \delta'' < x_1 < \delta, \, 0 < \xi_2 < H_\varepsilon(\xi_1)\},$$

we have

$$\hat{\Psi}_\varepsilon(\Gamma_\varepsilon \cap \hat{\mathcal{V}}_\varepsilon) = \{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ \delta'' < x_1 < \delta, \, x_2 = \hat{H}_0(x_1)\},$$
$$\hat{\Psi}_\varepsilon(\Gamma_\varepsilon \cap U_2) = \Gamma \cap U_2,$$
$$\hat{\Psi}_\varepsilon(\Gamma'_\varepsilon \cap (U_3 \cup \hat{\mathcal{V}}_\varepsilon)) = \Gamma' \cap (U_3 \cup \hat{\mathcal{V}}_\varepsilon).$$

For the second and third points of the proposition, we proceed as follows:
a. We prove that $\hat{\Psi}_\varepsilon : \mathcal{V}_\varepsilon(\delta') \mapsto R'_\varepsilon$, where

$$R'_\varepsilon := \{(x_1, x_2) \in \mathbb{R}^2 \ : \ \delta' < x_1 < \ell_\varepsilon, \, 0 < x_2 < \hat{H}_0(x_1)\},$$

is a C^1 diffeomorphism by studying its inverse, which can be made explicit. With the expression of this inverse, we also prove rather easily that $\|\hat{\Psi}_\varepsilon^{-1} - \hat{\Psi}_0^{-1}\|_{C^1(\overline{R'_\varepsilon})} \to 0$ as $\varepsilon \to 0^+$.
b. By noticing that $\hat{\Psi}_\varepsilon$ is a C^1, ε-perturbation of the identity in $\Omega_\varepsilon \setminus \mathcal{V}_\varepsilon(\delta')$, we prove that $\hat{\Psi}_\varepsilon$ is also a C^1 diffeomorphism from $\Omega_\varepsilon \setminus \mathcal{V}_\varepsilon(\delta')$ onto its image and that $\|\hat{\Psi}_\varepsilon^{-1} - \hat{\Psi}_0^{-1}\|_{C^1(\overline{\Psi_\varepsilon(\Omega_\varepsilon \setminus \mathcal{V}_\varepsilon(\delta'))})} \to 0$ as $\varepsilon \to 0^+$.
c. We get the conclusion of the proposition by remarking that

(A.3) $$\det D\hat{\Psi}_\varepsilon \neq 0 \quad \text{in } \Omega_\varepsilon$$

and

(A.4) $$\hat{\Psi}_\varepsilon(\mathcal{V}_\varepsilon(\delta')) \cap \hat{\Psi}_\varepsilon(\Omega_\varepsilon \setminus \mathcal{V}_\varepsilon(\delta')) = \varnothing.$$

The inverse of $\hat{\Psi}_\varepsilon$ in $\mathcal{V}_\varepsilon(\delta')$ is

(A.5) $$\hat{\Psi}_\varepsilon^{-1}(x) = \begin{pmatrix} \hat{\mu}_\varepsilon(x_1) \\ \frac{H_\varepsilon(\hat{\mu}_\varepsilon(x_1))}{\hat{H}_0(x_1)} x_2 \end{pmatrix}.$$

Following the lines of the proof of Proposition 5, we verify that $\hat{\Psi}_\varepsilon : \mathcal{V}_\varepsilon(\delta') \mapsto R'_\varepsilon$ is indeed a C^1 diffeomorphism and that $\|\hat{\Psi}_\varepsilon^{-1} - \hat{\Psi}_0^{-1}\|_{C^1(\overline{R'_\varepsilon})} \to 0$ as $\varepsilon \to 0^+$.

Let us denote now $\mathcal{U}_\varepsilon := \Omega_\varepsilon \setminus \mathcal{V}_\varepsilon(\delta')$. With our construction, we get, for every $\xi \in \mathcal{U}_\varepsilon$,

$$(A.6) \qquad\qquad \hat{\Psi}_\varepsilon(\xi) = \xi + \varepsilon F_\varepsilon(\xi),$$

where

$$F_\varepsilon(\xi) := \left[\chi_1(\xi) \frac{\chi_2(\xi) - 1}{H_\varepsilon(\xi_1)} \xi_2 - \chi_2(\xi) - \chi_1(\xi)\chi_2(\xi) \right].$$

Since F_ε is Lipschitz continuous uniformly in ε for every ε small enough, we deduce that $\hat{\Psi}_\varepsilon$ is one-to-one in $\mathcal{U}_\varepsilon$ for every ε small enough. Finally $\hat{\Psi}_\varepsilon$ is a bijection from $\mathcal{U}_\varepsilon$ onto its image. From expression (A.6), according to the local inversion theorem, we get that $\hat{\Psi}_\varepsilon^{-1}$ is also C^1. Using once again (A.6), it is clear that $\|\hat{\Psi}_\varepsilon - \hat{\Psi}_0\|_{C^1(\overline{\mathcal{U}}_\varepsilon)} \to 0$ as $\varepsilon \to 0^+$ with $\hat{\Psi}_0 = \mathrm{Id}$.

It remains to address the convergence of $\hat{\Psi}_\varepsilon^{-1}$ into $\hat{\Psi}_0^{-1} = \mathrm{Id}$ in $\hat{\Psi}(\overline{\mathcal{U}}_\varepsilon)$. On the one hand, we have

$$(A.7) \qquad \|\hat{\Psi}_\varepsilon^{-1} - \mathrm{Id}\|_{L^\infty(\hat{\Psi}_\varepsilon(\overline{\mathcal{U}}_\varepsilon))} = \|\hat{\Psi}_\varepsilon - \mathrm{Id}\|_{L^\infty(\overline{\mathcal{U}}_\varepsilon)} = \varepsilon \|F_\varepsilon\|_{L^\infty(\overline{\mathcal{U}}_\varepsilon)}.$$

On the other hand, we have

$$(A.8) \qquad\qquad D\hat{\Psi}_\varepsilon(\xi) = \mathrm{Id} + \varepsilon DF_\varepsilon(\xi)$$

and thus, since DF_ε is clearly uniformly bounded with respect to ε in $L^\infty(\overline{\mathcal{U}}_\varepsilon)$ by some constant C, its inverse is given via the Neumann series

$$(D\hat{\Psi}_\varepsilon(\xi))^{-1} = \mathrm{Id} + \varepsilon G_\varepsilon(\xi)$$

with

$$G_\varepsilon(\xi) = -\sum_{k=0}^{+\infty} (-\varepsilon)^k DF_\varepsilon(\xi)^{k+1}$$

provided ε is small enough, or more precisely for $\|\varepsilon DF_\varepsilon\|_{L^\infty(\overline{\mathcal{U}}_\varepsilon)} < 1$. For such ε, we can write that

$$\begin{aligned}
\|D\hat{\Psi}_\varepsilon^{-1} - \mathrm{Id}\|_{L^\infty(\hat{\Psi}_\varepsilon(\overline{\mathcal{U}}_\varepsilon))} &= \|(D\hat{\Psi}_\varepsilon \circ \hat{\Psi}_\varepsilon^{-1})^{-1} - \mathrm{Id}\|_{L^\infty(\hat{\Psi}_\varepsilon(\overline{\mathcal{U}}_\varepsilon))} \\
&= \|(D\hat{\Psi}_\varepsilon)^{-1} - \mathrm{Id}\|_{L^\infty(\overline{\mathcal{U}}_\varepsilon)} \\
&= \varepsilon \|G_\varepsilon\|_{L^\infty(\overline{\mathcal{U}}_\varepsilon)} \\
&\leq \varepsilon \|DF_\varepsilon\|_{L^\infty(\overline{\mathcal{U}}_\varepsilon)} \\
(A.9) \qquad\qquad &\leqslant C\varepsilon.
\end{aligned}$$

Gathering the estimates (A.7) and (A.9), we finally get

$$\|\hat{\Psi}_\varepsilon^{-1} - \hat{\Psi}_0^{-1}\|_{C^1(\hat{\Psi}_\varepsilon(\overline{\mathcal{U}}_\varepsilon))} \to 0 \text{ as } \varepsilon \to 0^+.$$

We prove (A.3) by direct computation, using (A.5) for $\xi \in \mathcal{V}_\varepsilon(\delta')$ and (A.8) for $\xi \in \Omega_\varepsilon \setminus \mathcal{V}_\varepsilon(\delta')$. Notice in particular that (A.8) remains true "up to the boundary" between $\mathcal{V}_\varepsilon(\delta')$ and $\Omega_\varepsilon \setminus \mathcal{V}_\varepsilon(\delta')$. The nonoverlapping property (A.4) is easily verified and the proof is now complete. $\quad\square$

Appendix B. Proofs of technical results.

Proof of Lemma 2. One easily checks that the function $F_\varepsilon := \mu_\varepsilon - \mu_0$ is positive on $[0, \ell_\varepsilon]$, which leads to

$$|\mu_\varepsilon| \leqslant |\mu_0|,$$

and (3.10a) follows from (3.7). Moreover, we have $F'_\varepsilon = H_0(\mu_0)[G_\varepsilon - 1]$, where

$$G_\varepsilon := \frac{H_\varepsilon(\mu_\varepsilon)}{H_0(\mu_0)}.$$

Straightforward computations lead to

$$G'_\varepsilon = G_\varepsilon[H'_0(\mu_\varepsilon) - H'_0(\mu_0)],$$

which is a positive function on $[0, \ell_\varepsilon]$. Since $G_\varepsilon(0) = 1 + \varepsilon/H_0(\delta) > 1$, we deduce that $F'_\varepsilon > 0$ and then that

$$\|\mu_\varepsilon - \mu_0\|_{C^0([0,\ell_\varepsilon])} = -\mu_0(\ell_\varepsilon),$$

and (3.9) follows with (3.7) and (3.4).

Still from (3.7) and (3.4), we infer that

$$G_\varepsilon(\ell_\varepsilon) \underset{\varepsilon=0}{\sim} \left[\alpha \frac{\pi/(\alpha+1)}{\sin(\pi/(\alpha+1))} \right]^{1+\frac{1}{\alpha}},$$

whence we deduce that, on $[0, \ell_\varepsilon]$, we have

$$|H_\varepsilon(\mu_\varepsilon)| \leqslant CH_0(\mu_0)$$

for some constant $C > 0$ dependent on α only. Combining this estimate again with (3.7) and (3.4), we get (3.10b).

Finally, since $|\mu_\varepsilon| \leqslant |\mu_0|$, we have

$$|H'_0(\mu_\varepsilon)| \leqslant |H'_0(\mu_0)|,$$

and (3.10c) follows, using (3.7) and (3.4). The proof is now complete. □

Proof of Lemma 1. Let us recall that the Cauchy problem (1.5) we are dealing with can be rewritten as

(B.1a) $$\varepsilon'(t) = \varepsilon'_0 F(\varepsilon(t), \varepsilon^*), \quad t > 0,$$
(B.1b) $$\varepsilon(t)|_{t=0} = \varepsilon^*$$

with

$$F(\varepsilon, \varepsilon^*) := \sqrt{\frac{m_s + m_f(\varepsilon^*)}{m_s + m_f(\varepsilon)}},$$

and $\varepsilon^* > 0$ and $\varepsilon'_0 < 0$ are given. As already mentioned, it is proved in [4] that the function

$$\varepsilon \in]0, \varepsilon^*[\mapsto m_f(\varepsilon) \in \mathbb{R}^+$$

is analytic and hence the function $F(\cdot, \varepsilon^*) :]0, \varepsilon^*[\to \mathbb{R}^+$ has the same regularity.

The hypothesis $E_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$ entails that $F(\cdot, \varepsilon^*) \to 0$ as $\varepsilon \to 0^+$ and therefore that the velocity of the solid tends to 0 when approaching the outer boundary; real shock cannot occur in this case.

The hypothesis $E_\varepsilon = O(\varepsilon^{-\beta})$ means that there exists $0 < \varepsilon^\dagger < \varepsilon^*$ and $C > 0$ such that

$$E_\varepsilon < C\varepsilon^{-\beta} \quad (0 < \varepsilon < \varepsilon^\dagger).$$

This estimate entails that, for a different positive constant still denoted by C,

(B.2) $$F(\varepsilon, \varepsilon^*) > C\varepsilon^{\frac{\beta}{2}} \quad (0 < \varepsilon < \varepsilon^\dagger).$$

On the other hand, the function $F(\cdot, \varepsilon^*)$ is bounded from below on the compact $[\varepsilon^\dagger, \varepsilon^*]$ by some constant $c > 0$. We deduce that $\varepsilon'(t) < c\varepsilon_0'$ as long as $\varepsilon(t) > \varepsilon^\dagger$. To simplify, let us relabel $t = 0$ the time for which $\varepsilon(t) = \varepsilon^\dagger$ (this time being no greater than $\varepsilon_\eta/(c\varepsilon_0')$).

Using now the estimate (B.2) in the Cauchy problem, now restated as

$$\varepsilon'(t) = \varepsilon_0' F(\varepsilon(t), \varepsilon^*), \quad t > 0,$$
$$\varepsilon(t)|_{t=0} = \varepsilon^\dagger,$$

we deduce that (changing again the value of the positive constant C)

$$\varepsilon(t) \leqslant \left[C\varepsilon_0' t + (\varepsilon^\dagger)^{-\frac{\beta}{2}+1} \right]^{\frac{2}{2-\beta}},$$

and therefore, the solid meets the cavity's wall in finite time. □

Proof of Lemma 10. The Dirichlet energy of u_ε^s is decomposed as follows:

$$\int_{\omega_\varepsilon} |\nabla u_\varepsilon^s(x)|^2 \, \mathrm{d}x = \int_{\omega_\varepsilon \backslash]1,\ell_\varepsilon[\times]0,1[} |\nabla u_\varepsilon^s(x)|^2 \, \mathrm{d}x + \int_1^{\ell_\varepsilon} \int_0^1 |\nabla u_\varepsilon^s(x_1, x_2)|^2 \, \mathrm{d}x_2 \mathrm{d}x_1.$$

The first integral in the right-hand side is uniformly bounded for $\varepsilon \geqslant 0$, so let us focus on the latter.

In $]1, \ell_\varepsilon[\times]0, 1[$, we have

$$\partial_{x_1} u_\varepsilon^s(x) = -\mu_\varepsilon(x_1) \left[1 + \frac{1}{2} x_2^2 H_0''(\mu_\varepsilon(x_1)) H_\varepsilon(\mu_\varepsilon(x_1)) \right],$$
$$\partial_{x_2} u_\varepsilon^s(x) = x_2 \left[H_\varepsilon(\mu_\varepsilon(x_1)) - \mu_\varepsilon(x_1) H_0'(\mu_\varepsilon(x_1)) \right].$$

Expending $|\partial_{x_1} u_\varepsilon^s(x)|^2 + |\partial_{x_2} u_\varepsilon^s(x)|^2$, and using the estimates (3.10), we obtain that the only remaining term which is not uniformly bounded (with respect to $\varepsilon \geqslant 0$) by a function in $L^1(\omega_0)$ is

$$\int_1^{\ell_\varepsilon} \int_0^1 |\mu_\varepsilon(x_1)|^2 \, \mathrm{d}x_2 \mathrm{d}x_1 = \int_1^{\ell_\varepsilon} |\mu_\varepsilon(x_1)|^2 \, \mathrm{d}x_1.$$

The change of variables $\xi_1 = \mu_\varepsilon(x_1)$ leads to

$$\int_1^{\ell_\varepsilon} |\mu_\varepsilon(x_1)|^2 \, \mathrm{d}x_1 = \int_{\mu_\varepsilon(1)}^0 \frac{\xi_1^2 \, \mathrm{d}\xi_1}{\kappa |\xi_1|^{1+\alpha} + \varepsilon}.$$

Setting now $\zeta = \kappa^{\frac{1}{\alpha+1}} \xi_1 / \varepsilon^{\frac{1}{\alpha+1}}$, we can transform the expression above into

$$\int_{\mu_\varepsilon(1)}^0 \frac{\xi_1^2 \, d\xi_1}{\kappa |\xi_1|^{1+\alpha} + \varepsilon} = \kappa^{-\frac{3}{\alpha+1}} \varepsilon^{\frac{3}{1+\alpha}-1} \int_{\beta(\varepsilon)}^0 \frac{\zeta^2 \, d\zeta}{|\zeta|^{1+\alpha} + 1},$$

where $\beta(\varepsilon) := \kappa^{-\frac{1}{1+\alpha}} \mu_\varepsilon(1) \varepsilon^{-\frac{1}{1+\alpha}}$. We have now to distinguish, according to the value of α,

(B.3) $\quad \displaystyle\int_{\beta(\varepsilon)}^0 \frac{\zeta^2 \, d\zeta}{|\zeta|^{1+\alpha} + 1} = \begin{cases} \int_{\beta(\varepsilon)}^0 \frac{d\zeta}{|\zeta|+1} + \int_{\beta(\varepsilon)}^0 \frac{|\zeta|-1}{|\zeta|^3+1} \, d\zeta & \text{if } \alpha = 2, \\ \int_{-\infty}^0 \frac{\zeta^2 \, d\zeta}{|\zeta|^{1+\alpha}+1} - \int_{-\infty}^{\beta(\varepsilon)} \frac{\zeta^2 \, d\zeta}{|\zeta|^{1+\alpha}+1} & \text{if } \alpha > 2. \end{cases}$

We deduce that, when $\alpha = 2$,

$$\int_{\beta(\varepsilon)}^0 \frac{\zeta^2 \, d\zeta}{|\zeta|^{1+\alpha} + 1} \underset{\varepsilon=0}{\sim} |\ln|\beta(\varepsilon)|| \underset{\varepsilon=0}{\sim} \frac{1}{3} |\ln(\varepsilon)|.$$

When $\alpha > 2$, since the second term in (B.3) tends to 0 with ε, we are led to compute the value of

$$\int_0^{+\infty} \frac{x^2 \, dx}{x^{1+\alpha} + 1}.$$

Yet another change of variable, namely, $y = x^3$, allows us to do that. We finally get

$$\int_0^{+\infty} \frac{x^2 \, dx}{x^{1+\alpha} + 1} = \frac{1}{3} \int_0^{+\infty} \frac{dy}{y^{\frac{1+\alpha}{3}} + 1} = \frac{1}{3} \frac{3\pi/(1+\alpha)}{\sin(3\pi/(1+\alpha))}.$$

The proof is then complete. □

Proof of Lemma 11. We observe first that $f_\varepsilon^r = 0$ in D and $g_\varepsilon^r = 0$ in γ_ε^D for every $\varepsilon \geqslant 0$ because of the cutoff function χ and hence ω_ε can be replaced by R_ε and γ_ε by γ_ε^R in (6.18). Recall that (identities (6.17) and (6.15))

(B.4a) $\qquad f_\varepsilon^r = [\text{div}(\mathbb{A}_\varepsilon \nabla \chi) \hat{u}_\varepsilon^s + 2\mathbb{A}_\varepsilon : (\nabla \chi \otimes \nabla \hat{u}_\varepsilon^s)] - \chi \hat{f}_\varepsilon^s \quad$ in ω_ε,

(B.4b) $\qquad g_\varepsilon^r = [-\hat{u}_\varepsilon^s (\mathbb{A}_\varepsilon \nabla \chi) \cdot n + (1-\chi) g_\varepsilon] - \chi \hat{r}_\varepsilon^s \qquad$ on γ_ε.

Using the expressions (6.12) and (6.13) of v_ε and V_ε in (6.10b) and (6.10d), we get, for every $x \in R_\varepsilon$,

(B.5a)

$$\hat{f}_\varepsilon^s(x_1, x_2) = x_2^2 \left[\frac{3}{2} H_\varepsilon(X_1^\varepsilon)^2 H_0''(X_1^\varepsilon) + \frac{1}{2} X_1^\varepsilon H_\varepsilon(X_1^\varepsilon)^2 H_0'''(X_1^\varepsilon) \right.$$

$$\left. - 3 X_1^\varepsilon H_\varepsilon(X_1^\varepsilon) H_0'(X_1^\varepsilon) H_0''(X_1^\varepsilon) - 3 H_\varepsilon(X_1^\varepsilon) H_0'(X_1^\varepsilon)^2 + 3 X_1^\varepsilon H_0'(X_1^\varepsilon)^2 \right],$$

and for every $x \in \gamma_\varepsilon^R$,

(B.5b) $\quad \hat{r}_\varepsilon^s(x) = \left[\frac{1}{2} X_1^\varepsilon H_\varepsilon(X_1^\varepsilon) H_0'(X_1^\varepsilon) H_0''(X_1^\varepsilon) + H_\varepsilon(X_1^\varepsilon) H_0'(X_1^\varepsilon)^2 - X_1^\varepsilon H_0'(X_1^\varepsilon)^3 \right].$

Observing that $|H_0''(\mu_\varepsilon)|$ and $|H_0'''(\mu_\varepsilon)|$ are uniformly bounded (because $\alpha > 2$), we deduce, according to the estimates (3.10), that the functions

$$x \mapsto |\hat{f}_\varepsilon^s(x)|^2 (1 + x_1)^2 \quad \text{and} \quad x \mapsto |\hat{r}_\varepsilon^s(x)|^2 (1 + x_1)^2,$$

are uniformly (in $\varepsilon \geqslant 0$) bounded by a function belonging to $L^1(R_0)$ and $L^1(\gamma_0^R)$, respectively. Invoking again Lemma 2 and applying the dominated convergence theorem, we get that

$$\|\hat{f}_\varepsilon^s - \hat{f}_0^s\|_{L^2(R_\varepsilon, \mathrm{d}m^{-1})} \to 0 \quad \text{and} \quad \|\hat{r}_\varepsilon^s - \hat{r}_0^s\|_{L^2(\gamma_\varepsilon^R, \mathrm{d}\sigma^{-1})} \to 0 \text{ as } \varepsilon \to 0^+.$$

Using Proposition 5, we deduce that $-\operatorname{div}(\mathbb{A}_\varepsilon \nabla \chi)$ converges uniformly in $]0, 1[\times]0, 1[$ to $-\operatorname{div}(\mathbb{A}_0 \nabla \chi)$ and since, in addition, $\hat{u}_\varepsilon^s$ converges in $C^1([0, 1]^2)$ to $\hat{u}_0^s$, we get, according to the identities (B.4), that

$$\|f_\varepsilon^r - f_0^r\|_{L^2(R_\varepsilon, \mathrm{d}m)} \to 0 \text{ as } \varepsilon \to 0^+.$$

We proceed similarly to show the second convergence result.

We know (see above Lemma 11) that f_ε^r and g_ε^r satisfy the compatibility condition (4.2) for every $\varepsilon > 0$. Lemma 6 and the convergence results above allow us to pass to the limit in (4.2). The proof is now complete. ∎

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