Optimal and Equilibrium Retrial Rates in Single-Server Multi-orbit Retrial Systems

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Abstract. We consider a single-server retrial system with one and several classes of customers. In the case of several classes, each class has its own orbit for retrying customers. The retrials from the orbits are generated with constant retrial rates. In the single class case, we are interested in finding an optimal retrial rate. Whereas in the multi-class case, we use game theoretic framework and find equilibrium retrial rates. Our performance criteria balance the number of retrials per retrying customer with the number of unhappy customers.

Keywords: retrial queueing system, constant retrial rate, optimal retrial rate, stability region, multi-class system

1 Introduction

We study the optimization of the retrial rate in retrial systems with constant retrial rate and possibly several classes of customers. More specifically, we consider a retrial queueing system consisting of a service facility with only one server and no waiting space, and one or several orbits for retrying customers. If more than one class of customers exist in the system, each class has its own orbit. We consider a so-called constant retrial rate when the rate of retrials does not depend on the number of customers in orbit. We also consider an auxiliary case of the single class with additional exogenous input of non-retrying customers.

As a performance measure, we choose a linear combination of the expected number of unsuccessful attempts of a generic orbit customer and the expected number of jobs in orbit. In the single-class case, we obtain an explicit expression for the optimal retrial rate. In the multi-class case, we formulate the problem of choosing retrial rate in the game theoretic framework and, using a balance-rate argument, obtain an explicit approximation which performs very well. Furthermore, if there are just two classes and one class exhibit non-stable behaviour, the approximation becomes exact for the class with stable behaviour.

The retrial queueing systems with constant retrial rate has several applications. In [15] the retrial systems with constant retrial rate have been introduced and applied to telephone exchange centers. The model of [15] was extended in [1], [2] for more complex settings, such as multiple servers and waiting places. Retrial systems under consideration can be applied to model ALOHA-type multiple access systems, see e.g., [10], [11]. TCP transfers have been modelled by constant retrial rate queues in [8] and [9]. We expect that the analytical results of the present work will help to tune the retransmission rate in the ALOHA-type systems and the value of the retransmission timeout in TCP. The retrial systems with constant retrial rate have also been applied to model logistic systems [19].

The optimization and game theoretic problems in retrial queues have been studied, e.g., in [12–14, 16–18, 21]. In all papers, except [18], the system has only single class of customers and the decision makers are individual customers. In [18] a retrial system with two classes is analyzed. However, in [18] each customer retries independently and hence the rate of retrials grows with the number of orbit customers. In [12, 21], the retrial rate is constant, but each customer takes his own decision and the decision consists in entering the system or reneging. Also, the objective function proposed in this work has not been analysed in the past.

The main contributions of this work are: explicit expressions for the optimal retrial rate in the single-class model and in the auxiliary model with exogenous input, and accurate approximation of the equilibrium retrial rates in the multi-class case. In fact, in the case of two classes, when one class is unstable, the proposed approximation becomes exact and provides best reply action for the stable class.

The paper is organized as follows: in the next section the models are formally presented. Then, in Section 3, the auxiliary model (with an exogenous input) is thoroughly investigated. In Section 4 we introduce multi-class retrial system with constant retrial rate and illustrate its connection with the auxiliary system. In Section 5, extensive numerical results are presented and discussed. We conclude the paper with Section 6.

2 Setting

We consider a single-server bufferless retrial queueing system with K classes of customers where class-i customers follow a Poisson arrival process with rate λ_i, have i.i.d. exponential service times with rate μ_i and, provided the server is busy upon arrival, join orbit $i = 1, \dots, K$. Orbit i works like a single-server queueing system with (exponential) constant rate retrial time (with rate μ_{0i}) regardless of the orbit size. Thus, there are K different orbits with retrial customers, and the ith orbit behaves like the orbit of a single-class system. We denote this original system by Σ.

Also consider the following auxiliary retrial system $\tilde{\Sigma}$ with two Poisson inputs with rates λ (input 1) and λ_0 (input 2), respectively. The blocked customers from input 1 join an orbit and then attempt to enter server after the exponentially distributed retrial time with rate μ_0, while the blocked customers from input 2 leave the system forever. The system has the exponentially distributed service times with rate μ. Denote by N the stationary orbit size in system $\tilde{\Sigma}$ (if exists) and by γ the number of the unsuccessful attempts which a blocked customer makes in stationary regime of system $\tilde{\Sigma}$ before a successful attempt. In this work, we will solve the following optimization problem: select a value of retrial rate μ_0 (denoted hereafter μ_0^*) in system $\tilde{\Sigma}$ which, for fixed parameters λ, λ_0, μ, provides the minimum of function

$$F(\mu_0) := c_1 E\gamma + c_2 EN, \tag{1}$$

where c_i are some positive constants.

The motivation for such a choice of penalty function (1) is as follows. On the one hand, we do not want to create too many requests for service (e.g., avoiding Denial of Service attack). This corresponds to minimization of $E\gamma$. On the other hand, we would like to keep as few as possible customers in unhappy situation. This corresponds to minimization of EN. This motivation is supported below by explicit expressions (7) and (8) which show that indeed $E\gamma$ increases and EN decreases, as μ_0 increases.

In spite of the difference between the models Σ and $\tilde{\Sigma}$, we will show that the analysis of a simpler model $\tilde{\Sigma}$, allows to select a near-optimal value of the retrial rate in the system Σ. We note that in the multi-class multi-orbit case, we consider the performance optimization from the game theoretic viewpoint.

3 Analysis of the Auxiliary System

We consider the two-dimensional process $\{X(t) := (M(t), N(t)), t \geq 0\}$, where $M(t)$ is the number of customers in the server (that is $M(t) \in \{0, 1\}$), and $N(t)$ is the number of customers in orbit, at instant t, in stationary system $\tilde{\Sigma}$. We use notation P_{mn} for the corresponding stationary probabilities of the process $\{X(t)\}$ (so $m = 0$, 1 and $n \geq 0$). A close model has been recently analyzed in paper [12]. To adapt below some required results from [12] to our model, we first show an equivalence between our model and the model from [12]. Authors in [12] present a retrial queueing system with Poisson input with a rate λ. Customers enter a retrial queue (orbit) with probability r whenever the server is busy and with probability 1 whenever the server is idle. The time required to find a costumer from the retrial orbit is assumed exponentially distributed with rate α. Now we illustrate some relations which synchronize this model with our model $\tilde{\Sigma}$ with extra input. A retrial rate α from [12] is replaced to our μ_0, input rate λ from [12] is replaced to our sum rate $\lambda + \lambda_0$. If a customer from extra input (with a rate λ_0) in a system $\tilde{\Sigma}$ faces a busy server, we detect a loss. Thus, we can conclude, that arrival from total input (with a rate $\lambda + \lambda_0$) would be lost with probability $\lambda_0/(\lambda + \lambda_0)$ if a server is busy. This value is the analogue of parameter r from [12]. From this point of view, both systems could be considered equivalent.

Returning to our system $\tilde{\Sigma}$, denote as P_1 and P_0 stationary probabilities of busy and idle server, respectively. The explicit values for P_1 and P_0 could easily be obtained after some calculation using Kolmogorov's equations technique and generation function method. Note that balance equations for stationary probabilities in considered system is equivalent to equations from [12] (see (3.7) – (3.9) there). The idle and busy probabilities have the following form:

$$P_0 = \frac{\mu - \lambda}{\mu + \lambda_0}, \tag{2}$$

$$P_1 = \frac{\lambda + \lambda_0}{\mu + \lambda_0}, \tag{3}$$

which is equivalent to (3.3), (3.4) in [12]. Using also Kolmogorov's equations technique, it is rather easy to get the formula for the probability of empty system as follows:

$$P_{00} = P_0 - \frac{\lambda}{\mu_0} P_1. \tag{4}$$

Note that P_1 is the busy probability which (by PASTA property) is also the blocking probability of an arriving primary customer. It follows from (2)-(4) that

$$P_{00} = 1 - \frac{(\lambda + \lambda_0)}{(\mu + \lambda_0)} - \frac{\lambda}{\mu_0} \cdot \frac{(\lambda + \lambda_0)}{(\mu + \lambda_0)}. \tag{5}$$

It is easy to check by (2), (3) that

$$P_b = P_1 = \frac{\lambda + \lambda_0 P_0}{\mu}. \tag{6}$$

To explain (6), we note that the the effective input rate in the stationary system $\tilde{\Sigma}$ (that is the limiting time-average rate of the customers entering the server) is $\lambda_e := \lambda + \lambda_0 P_0$, and, it is well-known that then the stationary busy probability of the server equals $P_1 = \lambda_e/\mu$. The mean orbit size can be expressed in the following form (for detail see statement (3.13) in [12], and also [3],[7]):

$$EN = \frac{\lambda(\lambda + \lambda_0)(\lambda + \lambda_0 + \mu + \mu_0)}{\mu(\mu + \lambda_0)(\mu_0 - \rho(\lambda + \lambda_0 + \mu_0))}, \tag{7}$$

where $\rho := \lambda/\mu$.

With no loss of generality, we assume that each blocked primary costumer joins the end of the orbit queue, and only head (top) customer in the orbit queue attempts to enter server until success. Denote γ_n the total number of the unsuccessful attempts of the nth retrying customer. Then variables $\{\gamma_n, \, n \geq 1\}$ constitute an i.i.d. sequence (with generic element γ). If a customer completes service, then a primary customer occupies server with the probability

$$p := \int_0^\infty (\lambda + \lambda_0) e^{-(\lambda+\lambda_0)\, x} \mu_0 \, e^{-\mu_0 \, x} dx = \frac{\lambda + \lambda_0}{\lambda + \lambda_0 + \mu_0},$$

and thus p is the probability of the unsuccessful attempt of the head orbital customer to capture free server after the departure. Note that the average number of (certainly) unsuccessful attempts during service time is equal to

$$\int_0^\infty \sum_{k=0}^\infty k \frac{(\mu_0 x)^k}{k!} e^{-\mu_0 x} \mu e^{-\mu x} dx = \frac{\mu_0}{\mu}.$$

Now assume that there is only (top) orbital customer. Then he attempts to access the server after joining orbit, and thus the average number of the unsuccessful attempts during service time is μ_0/μ. Otherwise, assume a few orbital customers, and a customer, already being in orbit, becomes top after starting service (of orbital customer). Again the average number of his unsuccessful attempts during service time is μ_0/μ. Thus, anyway $E\gamma \geq \mu_0/\mu$. Then each new unsuccessful attempt to capture empty server after the departure gives on average μ_0/μ unsuccessful attempts (of the top orbital customer) during next service time. This immediately yields the following expression

$$E\gamma = \frac{\mu_0}{\mu} + \frac{\mu_0}{\mu} \cdot \sum_{k=1}^\infty p^k = \frac{\lambda + \lambda_0 + \mu_0}{\mu}. \tag{8}$$

Now we obtain from (7), (8) (also see (1)) that

$$F(\mu_0) = c_1 \cdot \frac{\lambda + \lambda_0 + \mu_0}{\mu} + c_2 \cdot \frac{\lambda(\lambda + \lambda_0)(\lambda + \lambda_0 + \mu + \mu_0)}{\mu(\mu + \lambda_0)(\mu_0 - \rho(\lambda + \lambda_0 + \mu_0))}. \tag{9}$$

Next one can check that

$$\frac{d(EN)}{d\mu_0} = \frac{-\rho(\lambda + \lambda_0)}{(\mu_0(1-\rho) - \rho(\lambda + \lambda_0))^2}, \tag{10}$$

implying

$$\frac{dF}{d\mu_0} = \frac{c_1}{\mu} - \frac{c_2 \lambda(\lambda + \lambda_0)}{\mu(\mu_0 - \rho(\lambda + \lambda_0 + \mu_0))^2}. \tag{11}$$

Then, after some algebra, equation $F'(\mu_0) = 0$ gives the following explicit expression for the optimizer:

$$\mu_0^* = \frac{\mu\sqrt{c_2/c_1}\sqrt{\lambda(\lambda + \lambda_0)} + \lambda(\lambda + \lambda_0)}{\mu - \lambda}. \tag{12}$$

In particular, if we set $\lambda_0 = 0$, we obtain a nice explicit expression for the optimal value of the retrial rate in the single-class model:

$$\mu_0^* = \lambda \frac{\mu\sqrt{c_2/c_1} + \lambda}{\mu - \lambda}. \tag{13}$$

To the best of our knowledge, this is a new result.

To guarantee μ_0^* to be well-defined, we must define stability condition. It is shown in [12] (Proposition 3.1) that the system $\tilde{\Sigma}$ is stable if and only if

$$\rho(\lambda + \lambda_0 + \mu_0) < \mu_0. \tag{14}$$

By the noted above equivalence of the systems, it is also stability criteria of the original system Σ. Note that (14) can be written as

$$\rho < 1 - P^{(1)}, \tag{15}$$

and implies that $\rho < 1$. Also we note that for $\lambda_0 = 0$ condition (14) coincides with stability criterion of the single-orbit retrial M/M/1-type system obtained in [4]. (We remark that stability analysis of the retrial system with constant retrial rate is also presented in [3], [20].)

4 Multi-class Mutli-orbit System

In this section we discuss the relations between original K-orbit system and the auxiliary two-class system $\tilde{\Sigma}$. In particular we show how to apply to original system the analytical results obtained for the auxiliary system $\tilde{\Sigma}$. The key link connecting both systems (provided that *all orbits in Σ are stable*) is the following balance relation between the *effective input rate* $\lambda_0 P_0$ in the system $\tilde{\Sigma}$ and given input rates λ_i, $i = 2, \cdots, K$ in the system Σ:

$$\lambda_0 P_0 = \sum_{i=2}^{K} \lambda_i, \qquad i = 2, \ldots, K. \tag{16}$$

It establishes a balance between the total workload arriving in both systems and implies the equality of limiting fractions of the server busy times in both systems. In turn, it implies the equality of the stationary busy probabilities in both systems. However, the effective input of λ_0-customers (entering the server) in the system $\tilde{\Sigma}$ is a result of a (state-depending) thinning of the λ_0-input, while, in the system Σ, the inter-arrival times of the i-th class customers finally entering server are a complicated combination of the original (exponential) interarrival times and a random sum of the retrial times between the unsuccessful attempts until a successful attempt occurs. It is remarkable that, in spite of this difference between the two systems, the optimal value μ_0^* found analytically for the system $\tilde{\Sigma}$ allows to approximate the required optimal retrial rate in the system Σ with a high precision, see Section 5. It is important to stress that the inputs of class-i customers with $i \geq 2$, from the point of view of class-1 customers, can be considered as a single Poisson input with the summary rate $\lambda_0 P_0$. Thus, the analysis of original K-orbit system with arbitrary K is reduced to the analysis of the auxiliary system $\tilde{\Sigma}$. Note that the stability conditions found in [6, 5] allow to select the corresponding parameters in such a way that the 1st class orbits in both systems are always stable, while the 2nd class orbit in system Σ may be either stable or unstable. Now we focus on two-orbit system Σ and recall that the 1-class orbit in the system Σ is assumed to be stable. Thus, we will distinguish the following possible cases.

The 2nd stable orbit: symmetric case. Assume that two classes are equivalent with the same input rate $\lambda_1 = \lambda_2 := \lambda$, the same service rate $\mu_1 = \mu_2 = \mu$

and the same retrial rate $\mu_{01} = \mu_{02} := \mu_0$. Then the 2nd orbit (in system Σ) is stable, and the input of the 2nd class of customers is interpreted as λ_0-input in system $\tilde{\Sigma}$. For this symmetric case balance equation (16) becomes

$$\lambda_0 P_0 = \lambda. \tag{17}$$

Then it follows from (2) that

$$\lambda_0 = \frac{\mu\lambda}{\mu - 2\lambda}, \tag{18}$$

and (12) gives us

$$\mu_0^* = \sqrt{\lambda\Big(\lambda + \frac{\mu\lambda}{\mu - 2\lambda}\Big)} \cdot \frac{\Big(\sqrt{c_2/c_1} \cdot \mu + \sqrt{\lambda[\lambda + \mu\lambda/(\mu - 2\lambda)]}\Big)}{(\mu - \lambda)}. \tag{19}$$

The 2nd stable orbit: non-symmetric case. Consider the system Σ with non-equivalent classes, that is, $\lambda_1 \neq \lambda_2$, $\mu_{01} \neq \mu_{02}$, and the same service rate μ for both classes. According to (16), $\lambda_0 = \lambda_2/P_0$, and we obtain (see (2))

$$\lambda_0 = \frac{\mu\lambda_2}{\mu - (\lambda_1 + \lambda_2)}. \tag{20}$$

Then (12) implies

$$\mu_0^* = \sqrt{\lambda_1\Big(\lambda_1 + \frac{\mu\lambda_2}{\mu - (\lambda_1 + \lambda_2)}\Big)} \cdot \frac{\Big(\sqrt{c_2/c_1} \cdot \mu + \sqrt{\lambda_1(\lambda_1 + \mu\lambda_2/[\mu - (\lambda_1 + \lambda_2)])}\Big)}{(\mu - \lambda_1)}.$$

Our purpose is to find the optimal μ_{01} for the fixed μ_{02}. Furthermore, we note that, by (20), λ_0 is independent of μ_{02}. Thus, our approximation can be interpreted as equilibrium rate in game theoretic framework. It follows from [5, 6] that the necessary stability conditions of original system are

$$\lambda_i P_b < (1 - P_b)\mu_{0i}, \;\; i = 1, 2, \tag{21}$$

where $P_b := (\lambda_1 + \lambda_2)/\mu$ is the *busy probability.* We note that (21) is stability criteria in the symmetric case [5]. Thus, if the 2nd orbit in Σ is stable, then

$$\lambda_2 \frac{(\lambda_1 + \lambda_2)}{\mu} < \Big(1 - \frac{\lambda_1 + \lambda_2}{\mu}\Big)\mu_{02}, \tag{22}$$

and the following *stability measure* of the 2nd orbit,

$$\Gamma_2 := \big(\mu - (\lambda_1 + \lambda_2)\big)\mu_{02} - \lambda_2(\lambda_1 + \lambda_2), \tag{23}$$

is positive (whenever the 2nd orbit is stable). Moreover, as Γ_2 increases, the system Σ moves "deeper" in the stability region of the 2nd orbit.

The 2nd unstable orbit. Now assume that the 2nd orbit is unstable (with a fixed rate μ_{02}). In this case the systems Σ, $\tilde{\Sigma}$ become identical for class-1 customers, and parameter λ_0 is defined as

$$\lambda_0 = \lambda_2 + \mu_{02}, \tag{24}$$

implying

$$\mu_0^* = \sqrt{\lambda_1(\lambda_1 + \lambda_2 + \mu_{02})} \cdot \frac{\left(\sqrt{c_2/c_1} \cdot \mu + \sqrt{\lambda_1(\lambda_1 + \lambda_2 + \mu_{02})}\right)}{(\mu - \lambda_1)}. \tag{25}$$

This gives the best reply of the 1st class for any action of the 2nd class. We note that since the 2nd orbit is unstable, the objective function of the 2nd class takes infinite value. Nevertheless, the objective function of the 1st class is well-defined and takes finite values. This is one more confirmation that the constant retrial rate can be protective [7].

Remark 1. The equivalence of both systems from the point of view of class-1 customers follows from discussion in the work [6] where more detailed analysis of unstable orbits can be found.

5 Simulation results

In this section we compare analytical results for the auxiliary $\tilde{\Sigma}$ with the numerical results for the corresponding two-orbit system Σ. More exactly, we compare the objective function $F(\mu_0)$ (see (1)) with the empirical function

$$\hat{F}(\mu_0) = c_1\hat{\gamma} + c_2\hat{N}, \tag{26}$$

where $\hat{\gamma}$, $\hat{N}$ are the sample means of the number of unsuccessful attempts and the 1st orbit size, respectively, in the original system Σ, based on 500 independent replications.

1. Consider *symmetric case*, then the 2nd orbit is stable as well. In Table 1, simulation results for $\lambda_1 = \lambda_2 = \lambda = 1$, and $\mu = 3$ are given. Note that μ_0^* is obtained from (19), $\hat{\mu}_0^*$ is the estimated retrial rate (for both orbits in the system Σ) which provides the minimum of function $\hat{F}$, and ε is the relative error of $\hat{\mu}_0^*$.

As Table 1 shows, the best fit (the smallest relative error $\varepsilon = 3.9 \times 10^{-9}$) is obtained if $c_1/c_2 = 25$ (and $\mu = 3$). It means that a small number of the unsuccessful attempts and, as a result a rarity of attacks on server, is more important than a small orbit size. Moreover, as μ increases (from $\mu = 3$ to $\mu = 30$), then evidently, the system moves deeper into stability region and the relative error increases.

For all values of c_1/c_2 the smallest relative error is obtained for the most slow rate $\mu = 3$ because in this case the orbits are the most saturated, and the system Σ is better described by the analytical results obtained for the system

Table 1. Simulation results for symmetric Σ.

c_1/c_2	μ	μ_0^*	$\hat{\mu}_0^*$	ε	F	$\hat{F}$	λ_0
1	3	5	4.6	0.080	4.333	5.232	3
1	8	2.079	1.8	0.134	0.805	0.789	1.333
1	30	1.576	1.4	0.112	0.175	0.153	1.111
25	3	2.6	2.6	0.000	60.333	62.821	3
25	8	0.682	0.6	0.121	10.551	10.671	1.333
25	30	0.369	0.4	0.083	2.284	2.306	1.111
1/25	3	17	15.6	0.082	20.333	31.396	3
1/25	8	9.062	6.8	0.25	3.408	3.322	1.333
1/25	30	7.516	5.2	0.202	0.625	0.455	1.111

$\tilde{\Sigma}$. Also we note that the worst results are obtained for $c_1/c_2 = 1/25$, in which case the orbit size is more important than the number of attempts. In the most of cases (except $c_1/c_2 = 25$, $\mu = 30$) the estimated optimal value $\hat{\mu}_0^*$ satisfies

$$0.9 \cdot \mu_0^* \le \hat{\mu}_0^* \le \mu_0^*. \tag{27}$$

This approximation can be used in practice to select a near optimal retrial rate in the original system, based on analytical result obtained for the system $\tilde{\Sigma}$. Figure 1 shows a difference between F and $\hat{F}$ (depending on μ_0) for $c_1 = c_2 = 1$.

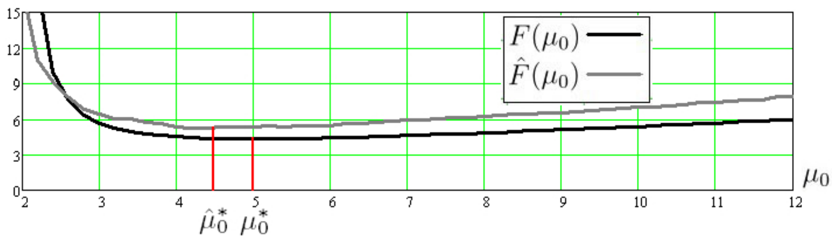

Fig. 1. $F(\mu_0)$ vs. $\hat{F}(\mu_0)$ for $c_1 = c_2 = 1$, $\mu = 3$, $\mu_0^* = 5$, $\hat{\mu}_0^* = 4.6$.

2. Consider *asymmetric case.* We take $\lambda_1 = \lambda_2 = 1$ and $\mu = 3$, implying $\lambda_0 = 3$ in auxiliary system $\tilde{\Sigma}$. Figure 2 shows the function $F(\mu_{01})$, μ_{01} denotes the orbit rate in $\tilde{\Sigma}$. If classes are not equivalent, then we *assume* that the violation of conditions (21) for $i = 2$ implies instability of the 2nd orbit. (Note that this assumption is confirmed by simulation of all considered scenarios.) For fixed parameters $\lambda_1 = \lambda_2 = 1$, $\mu = 3$ and the values $\mu_{02} = 5$, 4, 3, 2.1, we obtain the

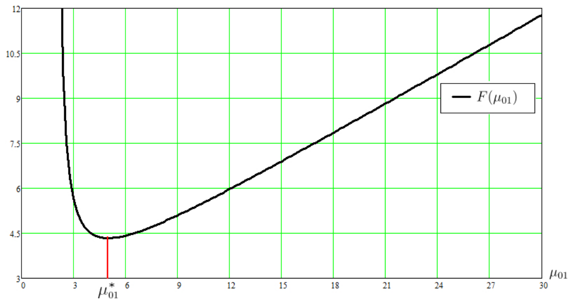

Fig. 2. Function $F(\mu_{01})$, $\lambda_0 = 3$, $\mu = 3$, $\mu_{01}^* = 5$.

following values of the stability measure $\varGamma_2$: 3, 2, 1, 0.1, respectively. The positivity of $\varGamma_2$ implies the fulfilment of condition (21) and, consequently, stability of the system. Note that, for $\mu_{02} = 2.1$, the 2nd stable orbit is near the stability region border and empties very rarely. On Figure 3, functions $F(\mu_{01})$, $\hat{F}(\mu_{01})$ are presented for a few values of μ_{02}. Note that we investigate the behavior of $\hat{F}$ in system $\varSigma$ for various μ_{01}, having μ_{02} fixed and $\varGamma_2 > 0$.

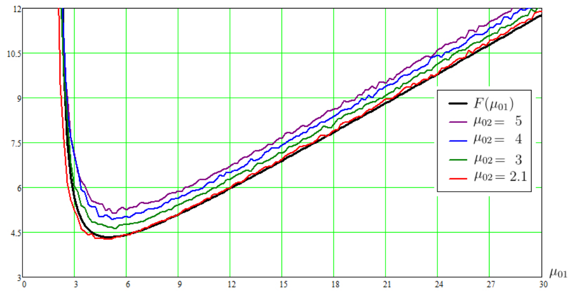

Fig. 3. $F(\mu_{01})$ vs. $\hat{F}(\mu_{01})$, $\lambda_1 = \lambda_2 = 1$, $\lambda_0 = 3$, $\mu = 3$.

The curves in Figure 3 become closer as the 2nd orbit becomes *less stable* (*more saturated*). For instance, if $\mu_{02} = 2.1$ (then $\varGamma_2 = 0.1$), then the curves almost coincide. This proximity becomes the equality when the 2nd orbit is unstable, as we show in the next point. (Also see *Remark 1* in Section 4.)

3. Finally, consider *asymmetric case with unstable 2nd orbit*. Thus, the value of μ_0^* is given by (25). Figure 4 shows F and $\hat{F}$ with parameters $\lambda_1 = 0.3$, $\lambda_2 = 3$, $\mu = 4$, $\mu_{02} = 5$. In this case, the class-2 input is Poisson with rate $\lambda_2 + \mu_{02}$ and, as a result, functions F and $\hat{F}$ coincide. Thus, if the 2nd orbit is unstable, the value $\mu_0^* = 2.37$ *becomes optimal*, best response, for the 1st class customers in system $\varSigma$.

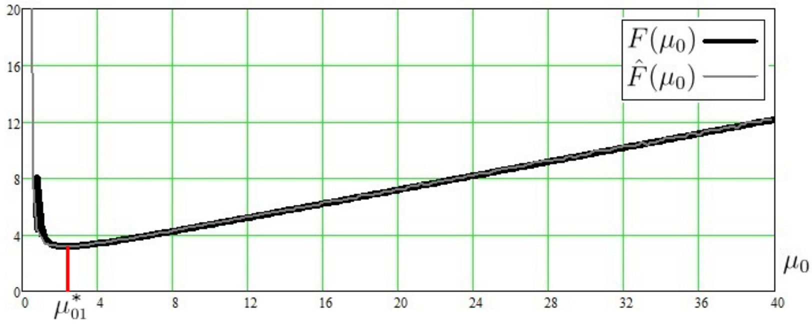

Fig. 4. $F(\mu_0)$ vs. $\hat{F}(\mu_0)$, $c_1 = c_2 = 1$, $\mu = 4$, $\mu_{01}^* = 2.37$.

Remark 2. It is worth mentioning that, as simulation shows, the estimated (optimal) parameter $\hat{\mu}_0^*$ in the systems with $K = 5, 10, 20, 50$ classes satisfies condition (27) as well.

6 Conclusion

For a Markovian multi-class single-server retrial system with constant retrial rates depending on class of customers, we find an optimal (equilibrium) retrial rate of a fixed class customers using optimization and game theoretic frameworks and a balance between the number of retrials (per customer) and the number of orbital customers. To address this problem, we study a more simple auxiliary system with exogenous Poisson input, which, in spite of the difference with the original system, allows to predict optimal retrial rate in the original system with a remarkable accuracy.

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