A uniformly stable finite difference space semi-discretization for the internal stabilization of the plate equation in a square

Karim Ramdani, Takeo Takahashi, Marius Tucsnak

A uniformly stable finite difference space semi-discretization for the internal stabilization of the plate equation in a square

Karim Ramdani[1], Takéo Takahashi[2], and Marius Tucsnak[3]

[1] IECN (Université Henri Poincaré) and INRIA ramdani@loria.fr
[2] IECN (Université Henri Poincaré) and INRIA takahash@iecn.u-nancy.fr
[3] IECN (Université Henri Poincaré) and INRIA tucsnak@iecn.u-nancy.fr

Summary. We propose a finite difference space semi-discretization of the stabilized Bernoulli-Euler plate equation in a square. The scheme studied yields a uniform exponential decay rate with respect to the mesh size.

1 Statement of the main result

Consider a square plate $\Omega = (0,\pi) \times (0,\pi)$ subject to a feedback force distributed on a rectangular subdomain $\mathcal{O} = [a,b] \times [c,d]$ of Ω. If $\chi_{\mathcal{O}}$ denotes the characteristic function of $\mathcal{O}$, the stabilization problem considered reads:

$$\begin{cases} \ddot{w}(t) + \Delta^2 w(t) + \chi_{\mathcal{O}}\ \dot{w}(t) = 0, & x \in \Omega,\ t > 0, \\ w(t) = \Delta w(t) = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x), \quad \dot{w}(x,0) = w_1(x), & x \in \Omega. \end{cases} \tag{1}$$

It is well known (cf. [3]) that the energy $E(t) = ||\dot{w}(t)||^2_{L^2(\Omega)} + ||\Delta w(t)||^2_{L^2(\Omega)}$ of system (1) decreases exponentially. The aim of this paper is to propose a space semi-discretization of this internal stabilization problem that ensures an exponential decay of the discretized energy $E_h(t)$ which is *uniform* with respect to the mesh size. This is not a trivial issue because of the possible appearance during the approximation process of high frequency spurious modes that cannot be damped by the feedback term. The appearance of such spurious modes in the approximation by finite differences or finite elements of control problems has been emphasized in several works (see, for instance [1], [2], [6] and the review paper [7]). Various solutions to overcome this difficulty have been proposed in the literature. The one followed in this paper is the one based on the introduction of an artificial numerical viscosity term.

Let us now precise the numerical scheme proposed. Given $N_1 \in \mathbb{N}$, denote by $h = \pi/(N_1 + 1)$, and assume that there exist integers $a(h), b(h), c(h), d(h)$ in $\{1, \ldots, N_1\}$ such that

$$a = a(h)h, \qquad b = b(h)h, \qquad c = c(h)h, \qquad d = d(h)h. \tag{2}$$

Let $w_{j,k}$ denote for all $j,k \in \{0, N_1+1\}$ the approximation of the solution w of system (1) at the point $x_{j,k} = (jh, kh)$. We use the standard finite difference approximation of the laplacian, by setting for all $j,k \in \{1,\dots,N_1\}$:

$$\Delta w(jh, kh) \approx \frac{1}{h^2}\left(w_{j+1,k} + w_{j-1,k} + w_{j,k+1} + w_{j,k-1} - 4w_{j,k}\right).$$

Set $V_h = \mathbb{R}^{(N_1)^2}$ and let $w_h \in V_h$ be the vector whose components are the $w_{j,k}$ for $1 \leq j,k \leq N_1$. In order to satisfy the boundary conditions in (1), we impose that

$$\forall k \in \{0,\dots,N_1+1\} : \begin{cases} w_{0,k} = w_{k,0} = w_{N_1+1,k} = w_{k,N_1+1} = 0 \\ w_{-1,k} = -w_{1,k},\ w_{N_1+2,k} = -w_{N_1,k}, \\ w_{k,-1} = -w_{k,1},\ w_{k,N_1+2} = -w_{k,N_1}. \end{cases} \tag{3}$$

The matrix A_{0h} representing the discretization of the bilaplacian with hinged boundary conditions is defined via its square root $A_{0h}^{\frac{1}{2}}$ given by

$$\left(A_{0h}^{\frac{1}{2}} w_h\right)_{j,k} = -\frac{1}{h^2}\left(w_{j+1,k} + w_{j-1,k} + w_{j,k+1} + w_{j,k-1} - 4w_{j,k}\right),$$

for all $1 \leq j,k \leq N_1$. The finite-difference space semi-discretization for system (1) studied in this paper reads then

$$\begin{cases} \ddot{w}_{j,k} + \left(A_{0h} w_h\right)_{j,k} + \left(\chi_{\mathcal{O}}\, \dot{w}_h\right)_{j,k} + h^2\left(A_{0h}\dot{w}_h\right)_{j,k} = 0, & 1 \leq j,k \leq N_1, \\ w_{j,k}(0) = w_{0h}, \quad \dot{w}_{j,k}(0) = w_{1h}, & 1 \leq j,k \leq N_1, \end{cases} \tag{4}$$

In the above equations, w_{0h} and w_{1h} are suitable approximations of the initial data w_0 and w_1 on the finite-difference grid and $\chi_{\mathcal{O}}\, \dot{w}_h$ denotes the vector of V_h whose components are the $\dot{w}_{j,k}$ if j and k are such that $x_{j,k} \in \mathcal{O}$, and 0 otherwise. The numerical viscosity term $h^2 A_{0h}\dot{w}_h$ in (4) is introduced in order to damp the high frequency modes. Our main result is the following.

Theorem 1. *The family of systems defined by* (3)-(4) *is uniformly exponentially stable, i.e. there exist constants* $C,\ \alpha,\ h^* > 0$ *(independent of* h*,* w_{0h} *and* w_{1h}*) such that :*

$$\|\dot{w}_h(t)\|^2 + \left\|A_{0h}^{\frac{1}{2}} w_h(t)\right\|^2 \leq Ce^{-\alpha t}\left(\|w_{1h}\|^2 + \left\|A_{0h}^{\frac{1}{2}} w_{0h}\right\|^2\right),$$

for all $h \in (0, h^*)$ *and all* $t > 0$.

In the above theorem and in the remaining part of this paper, we denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^m$ for various values of m. The proof of theorem 1 is based on the following frequency domain characterization for the uniform exponential stability of a sequence of semigroups (see [4, p.162]).

Theorem 2. *Let $(\mathbb{T}_h)_{h>0}$ be a family of semigroups of contractions on the Hilbert space V_h and A_h be the corresponding infinitesimal generators. The family $(\mathbb{T}_h)_{h>0}$ is uniformly exponentially stable if and only if the two following conditions are satisfied:*
i) For all $h>0$, $i\mathbb{R} \subset \rho(A_h)$, where $\rho(A_h)$ denotes the resolvent set of A_h,
ii) $\sup_{h>0,\omega\in\mathbb{R}} \|(i\omega - A_h)^{-1}\| < +\infty$.

2 Proof of Theorem 1

2.1 Abstract second and first order formulations

Let $U_h = \mathbb{R}^{(b(h)-a(h)+1)\times(d(h)-c(h)+1)}$ be the discretized input space, where the integers $a(h), b(h), c(h)$ and $d(h)$ are defined by (2). If $B_0 \in \mathcal{L}(L^2(\mathcal{O}), L^2(\Omega))$ denotes the restriction operator defined by $B_0 u = \chi_{\mathcal{O}}\, u$ for all $u \in L^2(\mathcal{O})$, we introduce its finite-difference approximation $B_{0h} \in \mathcal{L}(U_h, V_h)$ by setting for all $u_h \in U_h$: $(B_{0h}\, u_h)_{j,k} = u_{j,k}$ if j and k are such that $x_{j,k} \in \mathcal{O}$, and 0 otherwise. The adjoint $B_{0h}^* \in \mathcal{L}(V_h, U_h)$ of B_{0h} is then defined for all $w_h \in V_h$ by $(B_{0h}^* w_h)_{j,k} = w_{j,k}$ for all j,k such that $x_{j,k} \in \mathcal{O}$.

The finite-difference semi-discretization (3)-(4) admits the following abstract second order formulation:

$$\begin{cases} \ddot{w}_{j,k} + (A_{0h} w_h)_{j,k} + (B_{0h} B_{0h}^* \dot{w}_h)_{j,k} + h^2 (A_{0h} \dot{w}_h)_{j,k} = 0, & 1 \le j,k \le N_1, \\ w_{j,k} = \left(A_{0h}^{\frac{1}{2}} w_h\right)_{j,k} = 0, & j,k = 0, N_1+1, \\ w_{j,k}(0) = w_{0h}, \quad \dot{w}_{j,k}(0) = w_{1h}, & 0 \le j,k \le N_1+1. \end{cases} \tag{5}$$

It can be easily checked that the sequence $(\|B_{0h}\|_{\mathcal{L}(U_h,V_h)})$ is bounded and that the eigenvalues of $A_{0h}^{\frac{1}{2}}$ are

$$\lambda_{p,q,h} = \frac{4}{h^2}\left[\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right)\right], \text{ for } 1 \le p,q \le N_1. \tag{6}$$

A corresponding sequence of normalized eigenvectors is given by the vectors $\varphi_{p,q,h} = \left(\varphi_{p,q,h}^{j,k}\right)_{1\le j,k\le N_1}$, with components $\varphi_{p,q,h}^{j,k} = \frac{2h}{\pi}\sin(jph)\sin(kqh)$.

In order to apply theorem 2, we write system (5) as a first order system. Let us then introduce the space $X_h = V_h \times V_h$, which will be endowed with the norm $\|(\varphi_h, \psi_h)\|_{X_h}^2 = \|\varphi_h\|^2 + \left\|A_{0h}^{\frac{1}{2}}\psi_h\right\|^2$. Setting $z_h = \begin{bmatrix} w_h \\ \dot{w}_h \end{bmatrix}$, equations (5) can be easily written in the equivalent form

$$\dot{z}_h(t) = A_h z_h(t), \quad z_h(0) = z_{0h},$$

where $z_{0h} = \begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix}$ and $A_h \in \mathcal{L}(X_h)$ is defined by

$$A_h = \begin{bmatrix} 0 & I \\ -A_{0h} & -h^2 A_{0h} - B_{0h}B_{0h}^* \end{bmatrix}. \tag{7}$$

It will be useful to introduce the operator $A_{1h} = \begin{bmatrix} 0 & I \\ -A_{0h} & 0 \end{bmatrix} \in \mathcal{L}(X_h)$ such that

$$A_h = A_{1h} - \begin{bmatrix} 0 & 0 \\ 0 & h^2 A_{0h} + B_{0h}B_{0h}^* \end{bmatrix}. \tag{8}$$

We will also need in the sequel the spectral basis of the operator A_{1h}. Moreover, it will be more convenient to number the eigenelements of A_{1h} using only one index m instead of the couple (p, q). To achieve this, let us first rearrange the sequence of eigenvalues $\lambda_{p,q} = p^2 + q^2,\ p, q \in \mathbb{N}^*$, of the continuous problem in nondecreasing order to obtain a new sequence $(\Lambda_m)_{m\in\mathbb{N}^*}$. Then, if

$$\Lambda_m = \lambda_{p,q} = p^2 + q^2, \qquad \forall\, m \in \mathbb{N}^*,\ \forall p, q \in \mathbb{N}^*, \tag{9}$$

then we set for all $1 \leq m \leq (N_1)^2$, and for all $1 \leq p, q \leq N_1$:

$$\Lambda_{m,h} = \lambda_{p,q,h}, \qquad \varphi_{m,h} = \varphi_{p,q,h}. \tag{10}$$

Let then $N_2(h) = (N_1)^2 = \left(\frac{\pi}{h} - 1\right)^2$ be the number of nodes of the finite-difference grid. If we extend the definition of $\Lambda_{m,h}$ and $\varphi_{m,h}$ to the values $m \in \{-1, \ldots, -N_2(h)\}$ by setting

$$\Lambda_{m,h} = -\Lambda_{-m,h}, \qquad \varphi_{m,h} = \varphi_{-m,h}, \tag{11}$$

then it can be easily checked that the eigenvalues of A_{1h} are $i\Lambda_{m,h}$, where $1 \leq |m| \leq N_2(h)$, and that an orthonormal basis of X_h formed by eigenvectors of A_{1h} is given by

$$\Phi_{m,h} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\dfrac{i}{\Lambda_{m,h}} \varphi_{m,h} \\ \varphi_{m,h} \end{bmatrix}, \qquad 1 \leq |m| \leq N_2(h), \tag{12}$$

We are now in position to apply theorem 2.

2.2 Checking the assumptions of theorem 2

To prove condition $i)$ in theorem 2, we use a contradiction argument. Suppose that there exist $\begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} \in X_h$ and $\omega \in \mathbb{R}$ such that: $A_h \begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} = i\omega \begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix}$. Then, by using the definition (7) of A_h, we easily obtain that $\psi_h = i\omega\varphi_h$ and that

$$\left[\omega^2 - A_{0h} - i\omega(h^2 A_{0h} + B_{0h}B_{0h}^*)\right]\varphi_h = 0.$$

By taking the imaginary part of the inner product of this last relation with φ_h, we get that $\varphi_h = 0$, and thus $\psi_h = 0$. Therefore, for all $\omega \in \mathbb{R}$, $i\omega$ cannot be an eigenvalue of A_h. Thus, condition $i)$ in theorem 2 holds true.

Now, we check condition $ii)$ of theorem 2. Once again, we use a contradiction argument. Let us thus assume the existence for all $n \in \mathbb{N}$ of $h_n \in (0, h^*)$, $\omega_n \in \mathbb{R}$, $z_n = \begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix} \in X_{h_n}$ such that

$$\|z_n\|^2 = \left\| A_{0h_n}^{\frac{1}{2}} \phi_n \right\|^2 + \|\psi_n\|^2 = 1 \qquad \forall\, n \in \mathbb{N} \tag{13}$$

$$\|i\omega_n z_n - A_{h_n} z_n\| \to 0. \tag{14}$$

To obtain a contradiction, the idea is to decompose z_n into a low frequency part and a high frequency part. Then, thanks to the numerical viscosity introduced in the scheme, we prove that the high frequency part tends to 0. Finally, we conclude by using a result on the uniform observability of low frequency packets of eigenvectors.

More precisely, for $0 < \varepsilon < 1$ and $h \in (0, h^*)$, we define the integer

$$M(h) = \max \Big\{ m \in \{1, \ldots, N_2(h)\} \mid h^2 (\Lambda_m)^2 \le \varepsilon \Big\}, \tag{15}$$

where the sequence $(\Lambda_m)_{m \in \mathbb{N}^*}$ defined in (9) constitutes the sequence of eigenvalues of the continuous problem. The eigenvalues $\Lambda_{m,h}$ for $1 \le |m| \le M(h)$ correspond to "low frequencies" and will be damped to zero by the feedback control term $B_{0h} B_{0h}^* \dot{w}_h$. The eigenvalues $\Lambda_{m,h}$ for $|m| > M(h)$ correspond to "high frequencies" and will be damped by the numerical viscosity term. To get the desired contradiction, we follow several steps.

Step 1

Let us prove the two relations

$$h_n^2 \left\| A_{0h_n}^{\frac{1}{2}} \psi_n \right\|^2 + \|B_{0h_n}^* \psi_n\|^2 \to 0, \tag{16}$$

$$\lim_{n \to \infty} \left\| A_{0h_n}^{\frac{1}{2}} \phi_n \right\|^2 = \lim_{n \to \infty} \|\psi_n\|^2 = \frac{1}{2}. \tag{17}$$

Relation (16) follows directly from (14) by taking the inner product in X_{h_n} of $i\omega_n z_n - A_{h_n} z_n$ by z_n and by considering only the real part. By using (14), (16), (8) and the fact that the operators B_{0h_n} are uniformly bounded we obtain that

$$\left\| i\omega_n z_n - A_{1h_n} z_n + \begin{bmatrix} 0 \\ h_n^2 A_{0h_n} \psi_n \end{bmatrix} \right\| \to 0. \tag{18}$$

It can be easily that the sequence (ω_n) is bounded away from zero for n large enough(use a contradiction argument). Therefore, taking the inner product in X_{h_n} of (18) by $\dfrac{1}{\omega_n} \begin{bmatrix} \phi_n \\ -\psi_n \end{bmatrix}$ and by considering the imaginary part, we obtain that $\lim_{n \to \infty} \left\| A_{0h_n}^{\frac{1}{2}} \phi_n \right\|^2 - \|\psi_n\|^2 = 0$. This last relation and (13) yield (17). Step 1 is thus complete.

In order to state the second step, let us introduce the modal decomposition of z_n on the spectral basis of $(\Phi_{m,h_n})_{1\leq|m|\leq N_2(h_n)}$ of A_{1h_n}. For all $n \in \mathbb{N}$, there exist complex coefficients $(c_m^n)_{1\leq|m|\leq N_2(h_n)}$ such that

$$z_n = \begin{bmatrix}\phi_n \\ \psi_n\end{bmatrix} = \sum_{1\leq|m|\leq N_2(h_n)} c_m^n \Phi_{m,h_n}. \tag{19}$$

The normalization condition (13) reads then

$$\sum_{1\leq|m|\leq N_2(h_n)} |c_m^n|^2 = 1. \tag{20}$$

<u>*Step 2*</u>

In this step, we prove that the following relations holds true

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{m=1}^{N_2(h_n)} \left(c_m^n + c_{-m}^n\right) \varphi_{m,h_n}, \tag{21}$$

$$\sum_{M(h_n)<m\leq N_2(h_n)} \left|c_m^n + c_{-m}^n\right|^2 \to 0, \tag{22}$$

$$\sum_{1\leq|m|\leq M(h_n)} \left|\omega_n - \Lambda_{m,h_n}\right|^2 \ |c_m^n|^2 \to 0. \tag{23}$$

Note that, roughly speaking, relations (21) and (22) show that the projection of ψ_n on the high frequencies tends to 0 as n tends to $+\infty$. Relation (21) follows directly by taking the second component in (19) and by using (12). On the other hand, by using (19) and the fact that $\Phi_{m,h}$ is an eigenvector of A_{1h} associated to the eigenvalue $i\Lambda_{m,h}$, we have

$$i\omega_n z_n - A_{1h_n} z_n = \sum_{1\leq|m|\leq N_2(h_n)} i\left(\omega_n - \Lambda_{m,h_n}\right) c_m^n \Phi_{m,h_n} \tag{24}$$

From (16) and (21) it follows that

$$h_n^2 \left\|A_0^{\frac{1}{2}}\psi_n\right\|^2 = \sum_{m=1}^{N_2(h_n)} h_n^2 \Lambda_{m,h_n}^2 \left|c_m^n + c_{-m}^n\right|^2 \to 0. \tag{25}$$

Using the expression (6) of $\lambda_{p,q,h}$ and that $\lambda_{p,q} = p^2 + q^2$, it can be easily checked that $\frac{4}{\pi^2}\lambda_{p,q} \leq \lambda_{p,q,h} \leq \lambda_{p,q}$, for all $1 \leq p, q \leq N_1$, or equivalently

$$\frac{4}{\pi^2}\Lambda_m \leq \Lambda_{m,h} \leq \Lambda_m \qquad \forall\ 1 \leq m \leq N_2(h). \tag{26}$$

Relations (25), (26) and (15) imply (22). On the other hand, relations (26) and (25) clearly imply that there exists a constant C independent of h such that

$$h_n^4 \sum_{m=1}^{M(h_n)} \Lambda_{m,h_n}^4 \left|c_m^n + c_{-m}^n\right|^2 \leq C\varepsilon \sum_{m=1}^{M(h_n)} h_n^2 \Lambda_{m,h_n}^2 \left|c_m^n + c_{-m}^n\right|^2 \to 0. \tag{27}$$

On the other hand, a simple calculation shows that

$$\begin{bmatrix} 0 \\ h_n^2 A_{0h_n} \psi_n \end{bmatrix} = \sum_{1 \leq |m| \leq N_2(h_n)} \frac{h_n^2}{2} \Lambda_{m,h_n}^2 \left(c_m^n + c_{-m}^n\right) \Phi_{m,h_n}, \tag{28}$$

Relations (27) and (28) imply that

$$\begin{bmatrix} 0 \\ h_n^2 A_{0h_n} \psi_n \end{bmatrix} - \sum_{M(h_n) < |m| \leq N_2(h_n)} \frac{h_n^2}{2} \Lambda_{m,h_n}^2 \left(c_m^n + c_{-m}^n\right) \Phi_{m,h_n} \to 0 \tag{29}$$

By using (18), (24) and (29) it follows that

$$\sum_{1 \leq |m| \leq N_2(h_n)} i\left(\omega_n - \Lambda_{m,h_n}\right) c_m^n \Phi_{m,h_n} + \sum_{M(h_n) < |m| \leq N_2(h_n)} \frac{h_n^2}{2} \Lambda_{m,h_n}^2 \left(c_m^n + c_{-m}^n\right) \Phi_{m,h_n} \to 0.$$

Since the family (Φ_{m,h_n}) is orthogonal, the above relation implies (23).
Step 3
Consider the set

$$\mathcal{F} = \left\{ n \in \mathbb{N} \mid \exists\, m(n) \in \mathbb{Z},\ 1 \leq |m(n)| \leq M(h_n),\ \text{and } |\omega_n - \Lambda_{m(n),h_n}| < \frac{1}{8} \right\}.$$

In other words, $\mathcal{F}$ is constituted by those integers n such that ω_n is located in the "low frequency band". We distinguish then two cases:

First Case: **The set $\mathcal{F}$ is finite.** Then, for the sake of simplicity, we can suppose, without loss of generality, that $\mathcal{F}$ is empty. In this case, all the elements of the sequence (ω_n) are located in the "high frequency band". By using relation (23) in Step 2 and the above relation, we obtain that $< \sum_{1 \leq |m| \leq M(h_n)} |c_m^n|^2 \to 0$, i.e. that the low-frequency part of ψ_n tends to 0. Thus, the above relation, (21) and (22) in Step 2 imply that

$$\psi_n \to 0 \ \text{ in } \ H,$$

which contradicts (17).

Second case: **The set $\mathcal{F}$ is infinite.** Then, for the sake of simplicity, we can suppose, without loss of generality, that $\mathcal{F} = \mathbb{N}$. In this case, all the sequence ω_n is located in the "low frequency band". For all $n \in \mathbb{N}$, we introduce the set $\mathcal{F}_n = \left\{ m \in \mathbb{Z} \mid 1 \leq |m| \leq M(h_n) \text{ and } |\omega_n - \Lambda_{m,h_n}| < \frac{1}{8} \right\}$.

Note that $\mathcal{F}_n$ is never empty (since it always contains $m(n)$) and represents the collection of low frequency eigenvalues located near ω_n. Set then $\widetilde{\psi}_n = \frac{1}{\sqrt{2}} \sum_{m\in\mathcal{F}_n} c_m^n \ \varphi_{m,h_n}$. The definition of $\mathcal{F}_n$, together with relation (23) of Step 2 imply that

$$\sum_{m\in\{1,\ldots,N_2(h_n)\}\setminus\mathcal{F}_n} |c_m^n|^2 \to 0. \tag{30}$$

Using now relations (21) and (22) of Step 2, we see that (30) exactly states that

$$\|\psi_n - \widetilde{\psi}_n\| \to 0. \tag{31}$$

The above relation implies that $\|B_{0h_n}^*(\psi_n - \widetilde{\psi}_n)\| \to 0$. This relation together with relation (16) of Step 1 show that

$$\|B_{0h_n}^* \widetilde{\psi}_n\| \to 0. \tag{32}$$

But on the other hand, applying lemma 3.2 in [5] on the uniform observability of low frequency packets of eigenvectors (note that $I_{h_n}(\omega_n) = \mathcal{F}_n$) , we get the existence of $\delta > 0$ such that for all $n \in \mathbb{N}$, we have

$$\|B_{0h_n}^* \widetilde{\psi}_n\|^2 > \delta^2 \sum_{m\in\mathcal{F}_n} |c_m^n|^2. \tag{33}$$

Gathering (30), (32) and (33), we finally obtain that $\widetilde{\psi}_n \to 0$ in H. By using (31), we obtain that $\psi_n \to 0$ which contradicts (17). The proof of theorem 1 is now complete.

References

1. Glowinski, R., Li, C.H, Lions, J-L. : A numerical approach to the exact boundary controllability of the wave equation. I. Dirichlet controls: description of the numerical methods. Japan J. Appl. Math., **7**, 1–76 (1990)
2. Infante, J. A., Zuazua, E. : Boundary observability for the space semi-discretizations of the 1-D wave equation. M2AN Math. Model. Numer. Anal., **33**, 2, 407–438 (1999)
3. Jaffard, S. : Contrôle interne exact des vibrations d'une plaque rectangulaire. Port. Math., **47**, 4, 423–429 (1990)
4. Liu, Z., Zheng, S.: Semigroups associated with dissipative systems, Chapman & Hall/CRC, Boca Raton (1999)
5. Ramdani, K., Takahashi, T., Tucsnak, M. : Internal stabilization of the plate equation in a square : the continuous and the semi-discretized problems. J. Math. Pures Appl., To appear
6. Tcheugoué Tébou, L. R., Zuazua, E. : Uniform exponential long time decay for the space semi-discretization of a locally damped wave equation via an artificial numerical viscosity. Numer. Math., **95**, 3, 563–598 (2003)
7. Zuazua, E. : Propagation, observation and control of waves approximated by finite difference methods. SIAM. Rev., **47**, 197–243 (2005)