Uniqueness results for inverse Robin problems with bounded coefficient

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Abstract

In this paper we address the uniqueness issue in the classical Robin inverse problem on a Lipschitz domain $\Omega \subset \mathbb{R}^n$, with L^∞ Robin coefficient, L^2 Neumann data and conductivity of class $W^{1,r}(\Omega)$, $r > n$. We show that uniqueness of the Robin coefficient on a subpart of the boundary, given Cauchy data on the complementary part, does hold in dimension $n = 2$ but needs not hold in higher dimension. We also raise on open issue on harmonic gradients which is of interest in this context.

Keywords: Robin inverse problem, holomorphic Hardy–Smirnov classes, elliptic regularity, unique continuation.

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1. Introduction

This study deals with uniqueness issues for the classical Robin inverse boundary value problem. Mathematically speaking, the inverse Robin problem for

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an elliptic partial differential equation on a domain consists in finding the ratio between the normal derivative and the trace of the solution (the so-called Robin coefficient) on a subset of the boundary, granted the Cauchy data (*i.e.* the normal derivative and the trace of the solution) on the complementary subset. In this paper, we deal primarily with L^∞ Robin coefficients and L^2 Neumann data, for isotropic conductivity equations of the type $\mathrm{div}\,(\sigma\,\mathrm{grad}\,u) = 0$ on Lipschitz domains $\Omega \subset \mathbb{R}^n$, with Sobolev-smooth real-valued strictly elliptic conductivity σ of class $W^{1,r}(\Omega)$, $r > n$. An anisotropic analog to our uniqueness result is discussed in a separate section.
The Robin inverse problem arises for example when considering non-destructive testing of corrosion in an electrostatic conductor. In this case, data consist of surface measurements of both the current and the voltage on some (accessible) part of the boundary of the conductor, while the complementary (inaccessible) part of the boundary is subject to corrosion. Non-destructive testing consists in quantifying corrosion from the data. Robin boundary condition can be regarded as a simple model for corrosion [33]. Indeed, as was proved in [16], such boundary conditions arise when considering a thin oscillating coating surrounding a homogeneous background medium such that the thickness of the layer and the wavelength of the oscillations tend simultaneously to 0. A mathematical framework for corrosion detection can then be described as follows. We consider a conductivity equation in an open domain Ω, as a generalization of Laplace equation to non-homogeneous media, the boundary of which is divided into two parts. The first part Γ is characterized by a homogeneous Robin condition with functional coefficient λ. A non vanishing flux is imposed on the second part Γ_0 of the boundary. This provides us with a well-posed forward problem, that is, there uniquely exists a solution in Ω meeting the prescribed boundary conditions. The inverse problem consists in recovering the unknown Robin coefficient λ on Γ from measurements of the trace of the solution on Γ_0. Further motivation to solve the Robin problem are indicated in [39] and its bibliography.
A basic question is uniqueness: is the coefficient λ on Γ uniquely defined by the available Cauchy data on Γ_0 as soon as the latter has positive measure? In other words, can we find two different Robin coefficients that produce the same measurements? The answer naturally depends on the smoothness assumed for λ.
On smooth domains, for the Laplace operator at least, uniqueness of the inverse Robin problem for (piecewise) continuous λ has been known for decades to hold in all dimensions. The proof is for example given in [33], and in [23]

for the Helmholtz equation. It relies on a strong unique continuation property (Holmgren's theorem), *i.e.* on the fact that a harmonic function in Ω, the trace and normal derivative of which both vanish on a non-empty open subset of the boundary $\partial\Omega$, vanishes identically.

This argument no longer works for functions λ that are merely bounded. In this case we meet the following weaker unique continuation problem: *does a harmonic function, the trace and normal derivative of which both vanish on a subset of $\partial\Omega$ with positive measure, vanish identically?* A famous counterexample in [14] shows that such a unique continuation result is false in dimension 3 and higher. In dimension 2, a proof that such a unique continuation property holds for the Laplace equation can be found in [5] when the solution is assumed to be C^1 up to the boundary and Ω is the unit disk.

In this work, we prove more generally that this unique continuation result still holds for a $W^{3/2,2}$ solution to a conductivity equation with $W^{1,r}$-conductivity σ, $r > 2$, in a bounded simply connected Lipschitz domain $\Omega \subset \mathbb{R}^2$. This enables us to conclude to uniqueness in the inverse Robin problem. Our proof relies on two devices:

– A factorization result for the complex derivative of a solution to an isotropic conductivity equation, where one factor is holomorphic and the other is smoothly invertible. This factorization implicitly appears in [13], but we shall have to work out its regularity on a Lipschitz domain. The holomorphic factor in fact belongs to a Hardy–Smirnov class, hence is uniquely defined by its boundary values on a boundary subset of positive measure.

– A Rolle-type theorem for $W^{1,2}$ Sobolev functions on the real line.

Our uniqueness result for the Robin inverse problem generalizes that of [18] established in smoother cases and under the restriction that the imposed flux is non negative. The proof therein is based on positivity and monotonicity arguments established in [19], and does not use complex analysis. We also turn the counterexample of [14] into a counterexample to uniqueness in the Robin problem in dimension 3, and raise an intriguing issue on harmonic gradients vanishing on a boundary subset of positive measure which governs uniqueness in higher dimension under mild smoothness assumptions on the sets where the Cauchy data and the Robin coefficient are defined.

The paper is organized as follows. In section 2, we set some notation and we recall several results from the theory of Sobolev spaces. In Section 3, we introduce the isotropic conductivity PDE and associated Robin problems. In Section 4, we state our uniqueness results for such equations on Lipschitz domains in dimension 2. We also give a counterexample in higher dimension. Section 5 is a review of holomorphic Hardy spaces on the disk and their generalization into Smirnov spaces on Lipschitz domains, in connection with the Dirichlet problem for harmonic functions. Proofs of the results in Section 4 are provided in Section 6, along with the necessary factorization and regularity properties of solutions to the 2D Neumann problem which are of interest in their own right. Surprisingly perhaps, these seem not to have appeared before in the literature. In section 7, we indicate how our uniqueness result for the isotropic Robin problem implies a corresponding result in the anisotropic case. For this, we rely on the method of isothermal coordinates initiated in [50] and pursued in [10, 49], allowing us to transform an anisotropic equation in the plane into an isotropic one. Section 8 contains concluding remarks.

2. Notation and preliminaries on Sobolev spaces

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers. With superscript "t" to mean "transpose", we write $x = (x_1, \cdots, x_n)^t$ to indicate the coordinates of $x \in \mathbb{R}^n$, and we identify $\mathbb{C}$ with $\mathbb{R}^2$ on putting $z = x_1 + ix_2$.
For $1 \leq p \leq \infty$, $k > 0$ an integer and $E \subset \mathbb{R}^n$ a Lebesgue measurable set, we let $L^p(E)$ be the space of $\mathbb{R}^k$-valued measurable functions on E such that

$$\begin{aligned} \|f\|^p_{L^p(E)} &= \int_E |f|^p \, dm_n < \infty \qquad \text{if } p < \infty, \\ \|f\|_{L^\infty(E)} &= \text{ess sup}_E \, |f| < +\infty, \end{aligned} \tag{1}$$

where m_n stands for Lebesgue measure. In (1) above, $|f|$ designates the Euclidean norm of f and the notation is irrespective of k, which should cause no confusion.
In Section 2.1 we recall some properties of Sobolev spaces. We turn in Section 2.2 to classical definitions of non tangential convergence and maximal functions, while Section 2.3 is specifically devoted to the planar case.

2.1. Sobolev spaces

For $\Omega \subset \mathbb{R}^n$ an open set, we let $W^{1,p}(\Omega)$ be the familiar Sobolev space of complex-valued functions in $L^p(\Omega)$ whose first order derivatives again lie in $L^p(\Omega)$. A complete norm on $W^{1,p}(\Omega)$ is given by

$$\begin{aligned} \|f\|^p_{W^{1,p}(\Omega)} &= \|f\|^p_{L^p(\Omega)} + \|\nabla f\|^p_{L^p(\Omega)} \qquad \text{if } p < \infty, \\ \|f\|_{W^{1,\infty}(\Omega)} &= \max\big(\|f\|_{L^\infty(\Omega)}, \|\nabla f\|_{L^\infty(\Omega)}\big), \end{aligned} \tag{2}$$

where ∇f is the gradient of f defined as $\nabla f = (\partial_{x_1} f, \cdots, \partial_{x_n} f)^t$, with ∂_{x_j} to indicate the derivative with respect to x_j.
When $n = 1$, we simply write f' instead of $\partial_{x_1} f$. Throughout, differentiation is given in the distributional sense: $\int_\Omega \partial_{x_j} f \varphi dm_n = -\int_\Omega f \partial_{x_j} \varphi dm_n$ whenever $\varphi \in \mathcal{D}(\Omega)$, the space of complex-valued C^∞ smooth functions with compact support in Ω.
When $n = 2$, which is the main (but not the sole) concern of this paper, it is often convenient to use the complex differential operators:

$$\partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \qquad \bar{\partial} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \tag{3}$$

so that $df = \partial f dz + \bar{\partial} f d\bar{z}$. When f is holomorphic: $\bar{\partial} f = 0$, we also write f' instead of $\partial f = df/dz$.
We put $W^{1,p}_{loc}(\Omega)$ for the space of functions whose restriction to any relatively compact open subset Ω_0 of Ω lies in $W^{1,p}(\Omega_0)$. The space $W^{2,p}(\Omega)$ is comprised of L^p-functions whose distributional derivatives of the first order lie in $W^{1,p}(\Omega)$, with norm $\|f\|^p_{W^{2,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \sum_j \|\partial_{x_j} f\|^p_{W^{1,p}(\Omega)}$. The definition of $W^{2,p}_{loc}(\Omega)$ parallels that of $W^{1,p}_{loc}(\Omega)$.
For emphasis, we use at places a subscript "$\mathbb{R}$", as in $W^{1,p}_{\mathbb{R}}(\Omega)$, to single out the real subspace of real-valued functions. The same symbol (*e.g.* "C") is used many times to mean different constants. We write $A \sim B$ to abbreviate $CA \leq B \leq C'A$, where C, C' are constants.
If $n = 1$, then $W^{1,p}(\Omega)$ is just the space of locally absolutely continuous functions with derivative in $L^p(\Omega)$. The corresponding characterization when $n > 1$ is more subtle [56, Thm. 2.1.4], but in any case $W^{1,\infty}(\Omega)$ identifies with Lipschitz-continuous functions on Ω [48, Sec. V.6.2].
An open set $\Omega \subset \mathbb{R}^n$ is called Lipschitz if, in a neighborhood of each boundary point, it is isometric to the epigraph of a Lipschitz function [31, Def. 1.2.1.1]. When Ω is bounded and Lipschitz, each member of $W^{1,p}(\Omega)$ is the restriction to Ω of a function in $W^{1,p}(\mathbb{R}^n)$ (the extension theorem [48, Ch. VI, Thm 5]), and the space of restrictions $(\mathcal{D}(\mathbb{R}^n))_{|\Omega}$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$ [2, Thm 3.22]. Here and below, the subscript "$|E$" indicates restriction to a set E. If moreover $p > n$ then $W^{1,p}(\Omega)$ embeds continuously in the space of Hölder-continuous functions on Ω with exponent $1 - n/p$; when $p = n$ such an embedding holds in every $L^\ell(\Omega)$, $1 \leq \ell < \infty$, and if $p < n$ then $W^{1,p}(\Omega)$ embeds continuously in L^{p_*} with $p_* = np/(n-p)$ (the Sobolev embedding theorem [2, Thms 4.12, 4.39]). In addition, for $p \leq n$ and $\ell < p_*$ ($p_* = \infty$

if $p = n$), the previous embeddings are compact (the Rellich-Kondrachov theorem [2, Thm 6.3]).
Also, a distribution g on Ω whose first derivatives lie in $L^p(\Omega)$ does belong to $W^{1,p}(\Omega)$ [25, Thm 6.74][1], and if Ω is connected while $E \subset \Omega$ is such that $m_n(E) > 0$, then

$$\|g - g_E\|_{L^p(\Omega)} \leq C\| \nabla g \|_{L^p(\Omega)}, \qquad \text{where } g_E := \frac{1}{m_n(E)} \int_E g \, dm_n, \tag{4}$$

for some $C = C(p, \Omega, E)$ (the Poincaré inequality, apply [56, Thm 4.2.1] with $L(u) = u_E$). The Sobolev embedding theorem entails that $W^{1,p}(\Omega)$ is an algebra for $p > n$ [2, Thm 4.39], in particular if $f \in W^{1,p}(\Omega)$ and F is entire then $F(f) \in W^{1,p}(\Omega)$ with norm bounded in terms of Ω, p, F, and $\|f\|_{W^{1,p}(\Omega)}$.
For $1 < p < \infty$, the space $W^{\theta,p}(\Omega)$ of fractional order $\theta \in (0,1)$ consists of those $f \in L^p(\Omega)$ for which

$$\|f\|^p_{W^{\theta,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+\theta p}} dm_n(x) dm_n(y) < \infty. \tag{5}$$

The space $W^{1+\theta,p}(\Omega)$ is comprised of $f \in L^p(\Omega)$ whose derivatives of the first order lie in $W^{\theta,p}(\Omega)$, with norm $\|f\|^p_{W^{1+\theta,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \sum_j \|\partial_{x_j} f\|^p_{W^{\theta,p}(\Omega)}$.
When Ω is bounded and Lipschitz, $W^{\theta,p}(\Omega)$ may also be defined *via* real interpolation between $L^p(\Omega)$ and $W^{1,p}(\Omega)$ where it corresponds to the Besov space $B^{\theta,p,p}(\Omega)$; that is, using standard notation for the interpolation functor, it holds that $W^{\theta,p}(\Omega) = [L^p(\Omega), W^{1,p}(\Omega)]_{\theta,p}$, see [2, Sec. 7.32 & Thm 7.47].
A slightly different, but equivalent interpolation method is that of trace spaces of J.-L. Lions [1, Ch. 7]. If $\mathrm{d}(x, \partial\Omega)$ denotes Euclidean distance from $x \in \mathbb{R}^n$ to the boundary of Ω, there is $C = C(\Omega, \theta, p)$ such that for all $f \in L^p(\Omega)$ with $|\nabla f| \in L^p_{loc}(\Omega)$,

$$\|f\|_{W^{\theta,p}(\Omega)} \leq C\Big(\|\mathrm{d}(.,\partial\Omega)^{1-\theta} \, \nabla f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\Big). \tag{6}$$

In fact, [35, Thm 4.1] asserts that the left and right hand sides of (6) are equivalent when f is harmonic (with constants depending only on Ω), and one can check that the portion of proof yielding (6) (which rests on trace space interpolation) does not depend on harmonicity.

[1]The proof given there for bounded C^1-smooth Ω carries over to the Lipschitz case.

Recall the basic property of interpolation: if A is linear and continuous both $X \to X'$ and $Y \to Y'$ where (X, X') and (Y, Y') are interpolation pairs of Banach spaces, then A is continuous $[X, Y]_{\theta,p} \to [X', Y']_{\theta,p}$ [2, Thm 7.23]. From this, a fractional version of the Sobolev embedding theorem is easily obtained [25, Cor. 4.5.3]. Namely, if $\theta p > n$ then $W^{\theta,p}(\Omega)$ embeds continuously in Hölder-continuous functions with exponent $\theta - n/p$; if $\theta p = n$, such an embedding holds in $L^\ell(\Omega)$ for $1 \leq \ell < \infty$; if $\theta p < n$, then $W^{\theta,p}(\Omega)$ embeds continuously in L^{p^*} with $p^* = np/(n - \theta p)$.
When Ω is Lipschitz and bounded, its boundary $\partial\Omega$ is a compact $(n-1)$-dimensional Lipschitz manifold on which $L^p(\partial\Omega)$, $W^{1,p}(\partial\Omega)$, and $W^{\theta,p}(\partial\Omega)$ are defined as before, only with area measure $d\Sigma$ instead of dm_n and Lipschitz-continuous test functions rather than smooth ones [31, Sec. 1.3.3]. For $1 < p < \infty$, each $f \in W^{1,p}(\Omega)$ has a trace on $\partial\Omega$, denoted again by f or sometimes $\mathrm{tr}_{\partial\Omega} f$ for emphasis, whose pointwise definition Σ-a.e. rests on the extension theorem and the fact that non-Lebesgue points of f have 1–Hausdorff measure zero [56, Ch. 4, Rmk 4.4.5]. In particular, $\mathrm{tr}_{\partial\Omega} f$ coincides with the limit of f at points of $\partial\Omega$ where this limit exists. The function $\mathrm{tr}_{\partial\Omega} f$ lies in $W^{1-1/p,p}(\partial\Omega)$ [2, Thm 7.47], [31, Sec. 1.3.3], and the trace operator defines a continuous surjection from $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\partial\Omega)$ with continuous right inverse [31, Thm 1.5.1.3]. The subspace $W_0^{1,p}(\Omega)$ of functions whose trace is zero coincides with the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$ [31, Cor. 1.5.1.6]. If Ω is connected, a variant of the Poincaré inequality involving the trace is: for $p > 1$ and $\Gamma \subset \partial\Omega$ a subset of strictly positive measure $\Sigma(\Gamma) > 0$, there is $C > 0$ depending only on p, Ω and Γ such that for all $g \in W^{1,p}(\Omega)$

$$\left\| g - \int_\Gamma \mathrm{tr}_{\partial\Omega}\, g \, d\Sigma \right\|_{L^p(\Omega)} \leq C \|\nabla g\|_{L^p(\Omega)}. \tag{7}$$

This follows from the continuity of the trace operator, the Rellich–Kondrachov theorem and [56, Lem. 4.1.3].
We need mention Sobolev spaces of negative order in connection with duality of trace spaces: if $1 < p < \infty$ and $1/p + 1/q = 1$ then, since $W^{1/q,p}(\partial\Omega)$ embeds in $L^p(\partial\Omega)$, each $g \in L^q(\partial\Omega)$ gives rise via $h \mapsto \int_{\partial\Omega} g\bar{h} d\Sigma$ to a member of $(W^{1/q,p}(\partial\Omega))'$, the dual space of $W^{1/q,p}(\partial\Omega)$. As $W^{1/q,p}(\partial\Omega)$ is reflexive (for it is uniformly convex), we see as in [2, Sec. 3.13, 3.14] that the completion $W^{-1/q,q}(\partial\Omega)$ of $L^q(\partial\Omega)$ with respect to the norm

$$\|g\|_{W^{-1/q,q}(\partial\Omega)} := \sup_{\|h\|_{W^{1/q,p}(\partial\Omega)}=1} \left| \int_{\partial\Omega} g\bar{h} \, d\Sigma \right|$$

can be identified with $(W^{1/q,p}(\partial\Omega))'$. This we use when $p = q = 2$ only.

2.2. Non tangential maximal function

For $\xi \in \partial\Omega$, each $\alpha > 1$ defines a nontangential region of approach to ξ from Ω given by

$$R_\alpha^\Omega(\xi) = \{x \in \Omega : \ |x - \xi| < \alpha \, \mathrm{d}(x, \partial\Omega)\}. \tag{8}$$

When Ω is Lipschitz and bounded, $R_\alpha^\Omega(\xi)$ contains a nonempty open truncated cone with vertex ξ, whose aperture and height are independent of ξ [31, Thm 1.2.2.2]. Subsequently, whenever h is $\mathbb{R}^k$-valued on Ω, we define its nontangential maximal function (associated with α) to be

$$\mathcal{M}_\alpha h(\xi) = \sup_{x \in R_\alpha^\Omega(\xi)} |h(x)|, \qquad \xi \in \partial\Omega, \tag{9}$$

which is well-defined with values in $[0, +\infty]$. Also, we say that h defined on Ω converges nontangentially to a at $\xi \in \partial\Omega$ if, for every $\alpha > 1$,

$$\lim_{x \to \xi, \, x \in R_\alpha^\Omega(\xi)} h(x) = a. \tag{10}$$

2.3. Planar case

In dimension $n = 2$, $\partial\Omega$ is a curve and tangential differentiation produces a total derivative. This makes for specific notation as follows. A simply connected Lipschitz domain $\Omega \subset \mathbb{R}^2$ has a rectifiable Jordan curve as boundary and we write Λ (instead of Σ) for arclength measure on $\partial\Omega$. We let τ and n respectively indicate the unit tangent and (outwards pointing) normal vector fields on $\partial\Omega$, which are well defined in $L^\infty(\partial\Omega) \times L^\infty(\partial\Omega)$ [31, Sec. 1.5.1]. Here, τ is oriented so that (n, τ) is a positive frame Λ-a.e.

By what we said before, $W^{1,p}(\partial\Omega)$ consists of absolutely continuous functions with respect to Λ whose derivative lies in $L^p(\partial\Omega)$. We shall write $\partial_\tau h$ instead of $dh/d\Lambda$. If φ is smooth on a neighborhood of $\partial\Omega$ in $\mathbb{R}^2$, then the restriction $\psi = \varphi_{|\partial\Omega}$ belongs to $W^{1,\infty}(\partial\Omega)$ and $\partial_\tau \psi = \nabla\varphi.\tau$. Using duality, one can extend the definition of tangential derivative to less smooth classes of functions, but at this point we restrict the discussion to $p = 2$ which is enough for our purposes[2]. For $f \in L^2(\partial\Omega)$, define $\partial_\tau f \in (W^{1,2}(\partial\Omega))'$ to be the linear form $h \mapsto -\int_{\partial\Omega} f \partial_\tau h d\Lambda$, $h \in W^{1,2}(\partial\Omega)$. This generalizes the previous definition

[2]Appealing to [41, Ch. II, Thm 1.1] instead of [42, Ch. I, Thm 6.2], the same reasoning shows that ∂_τ is continuous $W^{1/q,p}(\partial\Omega) \to W^{-1/p,p}(\partial\Omega)$ for $1 < p < \infty$, $1/p + 1/q = 1$.

of ∂_τ when $f \in W^{1,2}(\partial\Omega)$, for in this case integration by parts shows that the linear form just mentioned extends to a member of $(L^2(\partial\Omega))' \sim L^2(\partial\Omega)$ which is just $\partial_\tau f$ in the former sense. Thus, by interpolation, ∂_τ is continuous from $W^{1/2,2}(\partial\Omega)$ into the space $(W^{1/2,2}(\partial\Omega))' \sim W^{-1/2,2}(\partial\Omega)$. Indeed, from [42, Ch.I, Thm 6.2]:

$$[(L^2(\partial\Omega))', (W^{1,2}(\partial\Omega))']_{1/2,2} = ([W^{1,2}(\partial\Omega), L^2(\partial\Omega)]_{1/2,2})' = (W^{1/2,2}(\partial\Omega))' .$$

3. Conductivity equation and Robin inverse problem

In Section 3.1 we introduce the conductivity equation under study. Sections 3.2 and 3.3 are dedicated to the associated forward Neumann and Robin problems. Section 3.4 concerns the inverse Robin problem.

3.1. The conductivity equation

The conductivity equation with unknown real-valued function u is

$$\nabla \cdot (\sigma \nabla u) = 0 , \tag{11}$$

where "$\nabla \cdot X$" means "divergence of the vector field X". Except in Section 7, we assume that the conductivity σ is a real-valued function on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ satisfying

$$\sigma \in W^{1,r}_{\mathbb{R}}(\Omega), \quad r > n, \tag{12}$$

$$0 < c \leq \sigma \leq 1/c < +\infty \quad \text{for some constant } c. \tag{13}$$

The fact that σ is real means that the conduction is isotropic. Condition (13) above means that (11) is strictly elliptic. Condition (12) is less restrictive than Lipschitz-regularity, but still it implies some Hölder-smoothness. Note that, since $r > n$, the space $W^{1,r}(\Omega)$ consists of multipliers on $W^{1,2}(\Omega)$, see [58] or [31, Thm 1.4.4.2].

As (12) and (13) together imply that $1/\sigma \in W^{1,r}_{\mathbb{R}}(\Omega)$, our assumptions are thus to the effect that multiplication by (the restriction to $\partial\Omega$ of) $1/\sigma$ is an isomorphism on $W^{1/2,2}_{\mathbb{R}}(\partial\Omega)$. By duality, it follows that multiplication by $1/\sigma$ is an isomorphism on $W^{-1/2,2}_{\mathbb{R}}(\partial\Omega)$. This entails that each $u \in W^{1,2}_{\mathbb{R}}(\Omega)$ solving for (11) has a well-defined normal derivative on $\partial\Omega$, denoted by $\partial_n u \in W^{-1/2,2}_{\mathbb{R}}(\partial\Omega)$. The standard definition is the weak one: if J designates a right inverse to the trace operator $W^{1,2}_{\mathbb{R}}(\Omega) \to W^{1/2,2}_{\mathbb{R}}(\partial\Omega)$ and $\langle \ , \ \rangle$ the duality pairing on $W_{\mathbb{R}}^{-1/2,2}(\partial\Omega) \times W_{\mathbb{R}}^{1/2,2}(\partial\Omega)$, then $h \mapsto \int_\Omega \sigma\nabla u.\nabla(J(h))\, d\, m_n$ is a continuous linear form on $W_{\mathbb{R}}^{1/2,2}(\partial\Omega)$ which can be represented uniquely as $\langle \phi, h\rangle$ for some $\phi \in W_{\mathbb{R}}^{-1/2,2}(\partial\Omega)$. Since division by σ is an isomorphism of $W_{\mathbb{R}}^{-1/2,2}(\partial\Omega)$, we may set $\partial_n u = \phi/\sigma \in W_{\mathbb{R}}^{-1/2,2}(\partial\Omega)$ and then it holds that

$$\langle \sigma\partial_n u\,,\, \mathrm{tr}_{\partial\Omega}\psi\rangle = \int_\Omega \sigma\nabla u.\nabla\psi\, dm_n, \qquad \psi \in W_{\mathbb{R}}^{1,2}(\Omega). \tag{14}$$

Indeed, (14) holds by construction when $\psi \in \mathrm{Ran}\, J$, hence it is enough to check it when $\psi \in W_{0,\mathbb{R}}^{1,2}(\Omega)$ in order to get it for all $\psi \in W_{\mathbb{R}}^{1,2}(\Omega)$. By density, we are left to prove that $\int_\Omega \sigma\nabla u.\nabla\psi\, dm_n = 0$ whenever $\psi \in \mathcal{D}_{\mathbb{R}}(\Omega)$ which is nothing but the distributional meaning of (11). Comparing (11) and (14) with the classical Green formula, it is natural to call $\partial_n u$ the (exterior) normal derivative of u on $\partial\Omega$. Checking (14) against $\psi \equiv 1$, we observe in particular that

$$\langle \partial_n u\,,\, \sigma\rangle = 0. \tag{15}$$

3.2. The Neumann problem

The Neumann problem in $W^{1,2}(\Omega)$ for the conductivity equation (11) is: given $g \in W_{\mathbb{R}}^{-1/2,2}(\partial\Omega)$ such that $\langle g\,,\, \sigma\rangle = 0$, to find $u \in W_{\mathbb{R}}^{1,2}(\Omega)$ such that

$$\left\{ \begin{array}{l} \nabla\cdot(\sigma\,\nabla u) = 0 \ \text{ in } \ \Omega, \\ \partial_n u = g \ \text{ on } \ \partial\Omega. \end{array} \right. \tag{16}$$

Note that the vanishing of $\langle g\,,\, \sigma\rangle$ is necessary by (15). A solution to (16) exists and is unique up to an additive constant. To check this well-known fact, simply observe that $f \mapsto \langle \sigma g, \mathrm{tr}_{\partial\Omega} f\rangle$ is a continuous linear form on $W_{\mathbb{R}}^{1,2}(\Omega)/\mathbb{R}$ (the quotient space of $W_{\mathbb{R}}^{1,2}(\Omega)$ modulo constants), a Hilbert norm on which is given by $\|\nabla f\|_{L^2(\Omega)}$ in view of (4). As $\|\sigma^{1/2}\nabla f\|_{L^2(\Omega)}$ is an equivalent norm by (13), we see upon denoting by $\check{u} \in W_{\mathbb{R}}^{1,2}(\Omega)/\mathbb{R}$ the equivalence class of $u \in W_{\mathbb{R}}^{1,2}(\Omega)$ that there is a unique $\check{u}$ to meet (14) with $\partial_n u$ replaced by g, thanks to the Lax-Milgram theorem [15, Cor. V.8]. As pointed out earlier, this is equivalent to u solving (16). Such a u is called an energy solution to the Neumann problem.

3.3. The forward Robin problem

The forward Robin problem is an implicit variation of the Neumann problem where the solution to (11) and its normal derivative have to satisfy an affine relation with functional coefficients on the boundary. In particular, the normal derivative is sought to be a function rather than a distribution on $\partial\Omega$. Below we consider a rather simple form of the problem, arising naturally in the setting of non-destructive control, where the affine relation has L^2 right-hand side and bounded coefficient. More general versions with right-hand side in $L^p(\partial\Omega)$, $p \in (1, 2]$, are studied in [39].

Throughout we assume that $\partial\Omega$ is partitioned into measurable subsets Γ and Γ_0 of strictly positive arclength:

$$\partial\Omega = \Gamma \cup \Gamma_0, \quad \Gamma \cap \Gamma_0 = \emptyset, \quad \Sigma(\Gamma) > 0, \quad \Sigma(\Gamma_0) > 0. \tag{17}$$

We put for simplicity

$$L^\infty_+(\Gamma) := \{\lambda \in L^\infty_{\mathbb{R}}(\Gamma)\,,\ \lambda \geq 0 \text{ a.e. on } \Gamma\,,\ \lambda \not\equiv 0\}. \tag{18}$$

Given $\lambda \in L^\infty_+(\Gamma)$ and $g \in L^2(\Gamma_0)$, the forward Robin problem consists in seeking $u \in W^{1,2}(\Omega)$ such that

$$\begin{cases} \nabla \cdot (\sigma\, \nabla u) = 0 \ \text{ in } \ \Omega, \\ \partial_n u = g \ \text{ on } \ \Gamma_0, \\ \partial_n u + \lambda u = 0 \ \text{ on } \ \Gamma. \end{cases} \tag{19}$$

As $\mathrm{tr}_{\partial\Omega} u \in W^{1/2,2}_{\mathbb{R}}(\partial\Omega) \subset L^2_{\mathbb{R}}(\partial\Omega)$, boundary conditions make sense in that g concatenated with $(-\lambda \mathrm{tr}_{\partial\Omega} u)_{|\Gamma}$ defines a member of $L^2_{\mathbb{R}}(\partial\Omega) \subset W^{-1/2,2}_{\mathbb{R}}(\partial\Omega)$. Replacing g and λ by g/σ and λ/σ respectively, which is possible by (13), solving (19) is tantamount to obtain $u \in W^{1,2}_{\mathbb{R}}(\Omega)$ satisfying

$$\begin{cases} \nabla \cdot (\sigma\, \nabla u) = 0 \ \text{ in } \ \Omega \\ \sigma\, \partial_n u = g \ \text{ on } \ \Gamma_0 \\ \sigma\, \partial_n u + \lambda u = 0 \ \text{ on } \ \Gamma. \end{cases} \tag{20}$$

In view of (14), problem (20) is equivalent to the following weak formulation: to find u in $W^{1,2}(\Omega)$ such that

$$\int_\Omega \sigma\, \nabla u \cdot \nabla \psi \, dm_n + \int_\Gamma \lambda u \psi \, d\Sigma = \int_{\Gamma_0} g\psi \, d\Sigma, \qquad \psi \in W^{1,2}_{\mathbb{R}}(\Omega). \tag{21}$$

As soon as $\sigma \in L^\infty_{\mathbb{R}}(\Omega)$, well-posedness of problem (21), that is, existence and uniqueness of a solution $u \in W^{1,2}(\Omega)$, follows at once from the Lax-Milgram theorem and Lemma 3.1 below. Further, as a consequence of (21), it holds that

$$\int_\Gamma \lambda u \, d\Sigma = \int_{\Gamma_0} g \, d\Sigma\,.$$

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\sigma \in L^\infty_{\mathbb{R}}(\Omega)$ satisfy (13), and $\lambda \in L^\infty_+(\Gamma)$ for some $\Gamma \subset \partial\Omega$ such that $\Sigma(\Gamma) > 0$. Then,*

$$u \mapsto \left(\int_\Omega \sigma \, |\nabla u|^2 \, dm_n + \int_\Gamma \lambda \, u^2 \, d\Sigma \right)^{1/2}$$

is an equivalent norm on $W^{1,2}(\Omega)$.

Proof. We must show that there exist two constants $c, C > 0$ such that

$$c \, \|\psi\|^2_{W^{1,2}(\Omega)} \leq \int_\Omega \sigma \, |\nabla \psi|^2 \, dm_n + \int_\Gamma \lambda \, \psi^2 \, d\Sigma \leq C \, \|\psi\|^2_{W^{1,2}(\Omega)} \, , \quad \psi \in W^{1,2}_{\mathbb{R}}(\Omega).$$

The right inequality follows from the boundedness of σ, λ, together with the continuity of the trace operator and the embedding $W^{1/2,2}(\partial\Omega) \to L^2(\partial\Omega)$. To prove the left inequality we can replace Γ by a subset on which $\lambda \geq \varepsilon > 0$, and then the result drops out from (7), the Schwarz inequality, and the fact that σ is bounded away from 0 by (13). □

3.4. The inverse Robin problem

Associated to the forward Robin problem (20) is the inverse Robin problem, which consists in finding the unknown impedance λ in $L^\infty_+(\Gamma)$ from the knowledge of u and g on Γ_0. Note that a solution u to (20) uniquely exists in $W^{1,2}(\Omega)$, as was pointed out before Lemma 3.1 above. In the setting of nondestructive control, Γ_0 represents that part of the boundary $\partial\Omega$ which is accessible to pointwise measurement or imposition of u and $\partial_n u$.
In this work, we consider the uniqueness issue as to whether λ is uniquely determined by g and $u_{|\Gamma_0}$ when Ω is a bounded contractible Lipschitz domain. For general partitions of the boundary like (17), it will turn out that the answer is “yes” when $n = 2$ and “no” when $n \geq 3$. Pointing out this structural difference between the planar and the higher dimensional cases is the main purpose of the present article.

4. Uniqueness results

The two uniqueness theorems in Section 4.1 are the main results of this work. Section 4.2 provides a counterexample in dimension 3.

4.1. Inverse Robin problem in dimension 2: uniqueness results

In this section we investigate the planar case: $\Omega \subset \mathbb{R}^2$, in particular it is understood throughout that $n = 2$ in (12) and we write Λ instead of Σ in (17).

Theorem 4.1. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded simply connected Lipschitz domain and that* (17) *holds. Let σ satisfy* (12)-(13) *and $g \in L^2(\Gamma_0)$ be such that $g \not\equiv 0$. Suppose $\lambda_1, \lambda_2 \in L^\infty_+(\Gamma)$ are such that the corresponding solutions $u_1, u_2 \in W^{1,2}_{\mathbb{R}}(\Omega)$ to problem (20) satisfy ${u_1}_{|\Gamma_0} = {u_2}_{|\Gamma_0}$. Then $\lambda_1 = \lambda_2$.*

Theorem 4.1 will be a consequence of the following unique continuation result which is proved in Section 6.

Theorem 4.2. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded simply connected Lipschitz domain and that σ satisfies* (12)-(13)*. Let $u \in W^{1,2}_{\mathbb{R}}(\Omega)$ be a solution to* (11) *in Ω such that $\partial_n u \in L^2(\partial\Omega)$. If both u and $\partial_n u$ vanish on a subset $\gamma \subset \partial\Omega$ of strictly positive measure, then $u \equiv 0$ in Ω.*

Proof. (*Theorem 4.1*) By assumption, u_1 and u_2 have the same Cauchy data on $\Gamma_0 \subset \partial\Omega$ with $\Lambda(\Gamma_0) > 0$, so Theorem 4.2 implies that $u_1 \equiv u_2$ in $\overline{\Omega}$ whence $(\lambda_1 - \lambda_2)\, u_1 = 0$ on Γ by the Robin boundary condition. Assume for a contradiction that $\lambda_1 \neq \lambda_2$ a.e. on Γ. Then, there exists a subset $\gamma \subset \Gamma$, $\Lambda(\gamma) > 0$, such that $\lambda_1 - \lambda_2 \neq 0$ on γ, and of necessity u_1 vanishes on γ by what precedes. In turn $\partial_n u_1 = -\lambda_1\, u_1/\sigma$ vanishes identically on γ, therefore Theorem 4.2 implies that $u_1 \equiv 0$ in Ω. Consequently $\partial_n u_1 = 0$ a.e. on $\partial\Omega$, thereby contradicting the assumption that $g \not\equiv 0$ in Γ_0. □

The proof of Theorem 4.2 (see Section 6 and Theorem 6.1) ultimately rests on the fact that, in dimension 2, a harmonic gradient (*i.e.* the conjugate of a holomorphic function if we identify $\mathbb{C}$ with $\mathbb{R}^2$) which has nontangential limit zero on a subset of $\partial\Omega$ of positive measure is identically zero (see Section 5). This is no longer true in higher dimension, as illustrated in the next section.

4.2. Examples of non uniqueness in higher dimension

An initial example was constructed in [54, Thm 1] of a nonconstant harmonic function on a half space in $\mathbb{R}^3$, with Hölder-continuous derivatives up to the boundary, whose gradient vanishes on a boundary set E with $m_2(E) > 0$, see also [6]. In [14], this construction was refined to the effect that there is a nonzero harmonic function on a half space, C^1-smooth up to the boundary,

that vanishes together with its normal derivative on a boundary set E with $m_2(E) > 0$. In fact, such examples can be constructed on any open subset of $\mathbb{R}^n$, $n \geq 3$, whose boundary is a $C^{1,\varepsilon}$ manifold [53]. This shows that Theorem 4.2 does not hold in dimension strictly bigger than 2, and casts doubt on whether an analog to Theorem 4.1 can hold in higher dimension. Indeed, the example below shows that it cannot, already for harmonic functions on smooth domains.
Hereafter, we denote by $\mathbb{B}^3 \subset \mathbb{R}^3$ the open unit ball and by $\mathbb{S}^2$ the boundary sphere (recall the definition (18) of $L^\infty_+(\Gamma)$).

Example 4.1. *Let u be a nonzero harmonic function in $\mathbb{B}^3$, of class C^1 on $\overline{\mathbb{B}^3}$, such that $u_{|E} = (\partial_n u)_{|E} = 0$ where $E \subset \mathbb{S}^2$, with $\Sigma(E) > 0$. If in problem (19) we set:*

$$\Gamma_0 = \{\xi \in \mathbb{S}^2,\ u^2(\xi) + \partial_n u^2(\xi) \neq 0\}, \qquad g = \partial_n u_{|\Gamma_0},$$

then $\Gamma := \mathbb{S}^2 \setminus \Gamma_0$ contains E hence it has strictly positive Σ-measure, but clearly λ can be arbitrary in $L^\infty_+(\Gamma)$ since $u_{|\Gamma} = \partial_n u_{|\Gamma} \equiv 0$.

Example 4.1 shows that a solution to (19) may be associated to all Robin functions. This is an extreme example of non uniqueness which, however, is not fully satisfactory in that it is highly non generic and will be destroyed by small perturbations of the Neumann boundary data g on Γ_0. The theorem below gives another example of non uniqueness which is easily seen to be stable under $L^p(\Gamma_0)$-small perturbations of g, for $p > 2$.

Theorem 4.3. *Set $\sigma \equiv 1$ on $\mathbb{B}^3$. Then, there is a partition $\mathbb{S}^2 = \Gamma \cup \Gamma_0$ of the form* (17)*, along with functions $g \in L^2_{\mathbb{R}}(\Gamma_0)$ and $\lambda_1 \neq \lambda_2 \in L^\infty_+(\Gamma)$ such that the corresponding solutions u_1, u_2 to* (19) *on $\mathbb{B}^3$, though distinct, satisfy $(u_1)_{|\Gamma_0} = (u_2)_{|\Gamma_0}$.*

Proof. Let $\Gamma_0 \subset \mathbb{S}^2$, $\Sigma(\Gamma_0) > 0$, have the property that there is a nonzero harmonic function u in $\mathbb{B}^3$, of class C^1 on $\overline{\mathbb{B}^3}$, with $u_{|\Gamma_0} = (\partial_n u)_{|\Gamma_0} = 0$. Such a Γ_0 exists by [14]. Let $h \in L^\infty_{\mathbb{R}}(\mathbb{S}^2)$ be such that $0 < c < -h < C$ on $\Gamma = \mathbb{S}^2 \setminus \Gamma_0$ for some constants c, C, and moreover $\int_{\mathbb{S}^2} h d\Sigma = 0$. In addition, we pick c large enough that $h < -|\partial_n u| - 1$ on Γ. Let v be a solution to the Neumann problem (16) where $\sigma \equiv 1$, $\Omega = \mathbb{B}^3$, and $\partial_n v = h$. On the sphere, the "Riesz tranform" mapping the normal derivative of a harmonic function w in $\mathbb{B}^3$ to its tangential gradient vector field is continuous in L^p-norm for $1 < p < \infty$; this follows easily by dominated convergence from the fact that, for each $\alpha > 1$, $\|\mathcal{M}_\alpha \nabla w\|_{L^p(\mathbb{S}^2)} \leq C_\alpha \|\partial_n w\|_{L^p(\mathbb{S}^2)}$, see [27, Thm 2.6]. Therefore $v_{|\mathbb{S}^2} \in W^{1,p}_{\mathbb{R}}(\mathbb{S}^2)$ for all $p \in (1, \infty)$, hence it is bounded by the Sobolev embedding theorem. Thus, upon adding a positive constant to v, we may assume that $v > |u| + 1$ on Γ and that the function $v\partial_n u - hu$ does not identically vanish on Γ. Now, letting $\lambda_1 = -h/v$ and $\lambda_2 = -(h+\partial_n u)/(u+v)$ on Γ, we have that $\lambda_1, \lambda_2 \in L^\infty_+(\Gamma)$, $\lambda_1 \not\equiv \lambda_2$, while the functions $u_1 = v$ and $u_2 = v + u$ coincides together with their normal derivatives on Γ_0, as desired. □

Counterexamples similar to the one in Theorem 4.3 can be constructed in any dimension greater than 3.

Remark 4.1. *Whenever $\sigma \in W^{1,\infty}_{\mathbb{R}}(\Omega)$ and γ contains an open subset of $\partial\Omega$, it is not difficult to deduce from the unique continuation result in [30] that the analog of Theorem 4.2 holds for any $n \geq 2$. However, Example 4.1 shows that assuming Γ_0 open cannot rescue a higher dimensional analog of Theorem 4.1. The situation becomes more interesting if we assume that the interiors of Γ_0 and Γ fill $\mathbb{S}^2$ up to a set of Σ-measure zero. Then, proving or disproving the analog of Theorem 4.1 when $n \geq 3$ is tantamount to decide if a solution to* (16) *that vanishes together with its normal derivative on some $E \subset \partial\Omega$, with $\Sigma(E) > 0$, can be such that $\partial_n u/u$ is (essentially) bounded and nonnegative in a neighborhood of E in $\partial\Omega$. This question seems to be open, even for harmonic functions in a ball.*

5. Hardy-Smirnov classes of holomorphic functions

In Section 5.1 we review Hardy spaces and conjugate functions on the disk, as well as conformal maps onto simply connected Lipschitz domains. This we use in Section 5.2 to discuss Smirnov spaces on Lipschitz domains, in particular of exponent 2. There, we bridge classical material from complex analysis with known results from elliptic regularity theory to characterize Smirnov functions in terms of Sobolev smoothness (Theorem 5.1). Roughly speaking, Smirnov spaces consist of holomorphic functions with Lebesgue integrable boundary values with respect to arclength, and as such they are basic to solve Dirichlet and Neumann problems for the Laplace equation in dimension 2. In Section 5.3, we dwell on this connection to prove well-posedness of the Dirichlet problem with $W^{1,2}$-data which we could not find in the literature (Proposition 5.2). This well-posedness and the fact that a nonzero Smirnov function cannot vanish on a boundary subset of positive measure are fundamental to the proof of Theorem 4.2 in Section 4.

5.1. Hardy spaces of the disk

We set $\mathbb{D}(\xi,\rho)$ and $\mathbb{T}(\xi,\rho)$ to designate the disk and the circle of radius ρ, centered at ξ in the complex plane. When $\xi = 0$ we simply write $\mathbb{D}_\rho$ and $\mathbb{T}_\rho$, and if $\rho = 1$ we omit subscripts. Arclength on $\mathbb{T}_\rho$ will be denoted by m, irrespective of ρ, which should cause no confusion. Thus, $dm(\rho e^{i\theta}) = \rho d\theta$. Given a function f on $\mathbb{D}$ and $\rho \in [0,1)$, we write f_ρ to mean the function on $\mathbb{D}$ given by $f_\rho(z) = f(\rho z)$.

For $p \in [1,\infty)$, the Hardy space H^p consists of functions f which are holomorphic in the unit disk and satisfy the growth condition:

$$\|f\|_{H^p} = \sup_{0<\rho<1} \Big(\int_{\mathbb{T}_\rho} |f(\xi)|^p \, dm(\xi) \Big)^{1/p} < +\infty. \tag{22}$$

The space H^∞ is comprised of bounded holomorphic functions endowed with the *sup* norm. Note that $\|f_\rho\|_{L^p(\mathbb{T})}$ is non-decreasing with ρ by subharmonicity of $|f|^p$, see [45, Thm 17.6], hence the sup in (22) is really a limit as $\rho \to 1^-$.

It is well-known (see [26, Ch. 2] or [28, Ch. 2]) that each $f \in H^p$ has a non-tangential limit $f(\xi)$ at m-a.e. $\xi \in \mathbb{T}$, which makes for a definition of f on the unit circle. The map $f \mapsto f_{|\mathbb{T}}$ is an isometry from H^p onto the closed subspace of $L^p(\mathbb{T})$ consisting of functions whose Fourier coefficients of strictly negative index do vanish. This allows us to regard H^p both as a space of holomorphic functions on $\mathbb{D}$ and as a space of L^p-functions on $\mathbb{T}$, upon identifying f with $f_{|\mathbb{T}}$. Every $f \in H^p$ can be represented as the Cauchy as well as the Poisson integral of its non-tangential limit:

$$f(z) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{f(\xi)}{\xi - z} d\xi, \qquad f(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2} f(\xi) \, dm(\xi), \qquad z \in \mathbb{D}.$$

Hereafter, the Poisson integral of a function $\psi \in L^1(\mathbb{T})$ will be abbreviated as $P[\psi]$. If $f \in H^p$, $1 \le p < \infty$, then $\|(f_\rho)_{|\mathbb{T}} - f_{|\mathbb{T}}\|_{L^p(\mathbb{T})} \to 0$ as $\rho \to 1^-$. As for the nontangential maximal function, it holds if $p > 1$ and $f \in H^p$ that, for any $\alpha > 1$,

$$\|\mathcal{M}_\alpha f\|_{L^p(\mathbb{T})} \le C \|f\|_{L^p(\mathbb{T})} \tag{23}$$

where the constant C depends on α and p [28, Ch. II, Thm 3.1].

Clearly $H^p \subset L^p(\mathbb{D})$, moreover one can see from the Cauchy formula that if $f \in H^p$ and $\varepsilon > 0$ then the derivative f' satisfies $\|f'_\rho\|_{L^p(\mathbb{T})} \leq C(1-\rho)^{-1-\varepsilon}$ where C depends only on ε and p [26, Thm 5.5]. Thus, using Fubini's theorem to evaluate the right hand side of (6), we deduce that H^p embeds in $W^{\theta,p}(\mathbb{D})$ for $\theta \in (0, 1/p)$ and we get by the Sobolev embedding theorem that

$$\|f\|_{L^\lambda(\mathbb{D})} \leq C\|f\|_{H^p}, \qquad p \leq \lambda < 2p, \tag{24}$$

where $C = C(p, \lambda)$. When $p = 2$ these estimates can be sharpened, for in this case Green's formula yields that $\|(1-|z|^2)^{1/2}\nabla f(z)\|_{L^2(\mathbb{D})} \sim \|f - f(0)\|_{H^2}$ [28, Ch. VI, Lem. 3.2], hence it follows from (6) that $H^2 \subset W^{1/2,2}(\mathbb{D})$ and subsequently, by the Sobolev embedding theorem, that $\|f\|_{L^4(\mathbb{D})} \leq C\|f\|_{H^2}$. In fact, since both sides of (6) are equivalent quantities when f is harmonic (see discussion after (6)), H^2 is precisely the space of holomorphic functions in $W^{1/2,2}(\mathbb{D})$ with equivalence of norms. A nonzero $f \in H^p$ is such that $\log|f_{|\mathbb{T}}| \in L^1(\mathbb{T})$ [26, Thm 2.2], in particular, *a nonzero H^p-function cannot vanish on a subset of $\mathbb{T}$ of strictly positive measure*[3]. Conversely, if $h \in L^p(\mathbb{T})$ is non-negative and $\log h \in L^1(\mathbb{T})$, then

$$E_h(z) = \exp\left\{\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\log h(\xi)\, dm(\xi)\right\}, \qquad z \in \mathbb{D}, \tag{25}$$

belongs to H^p and satisfies $|(E_h)_{|\mathbb{T}}| = h$. A function of the form (25) is called *outer*, and it is characterized among H^p functions by the fact that $\log|E_h|$ (which is harmonic in $\mathbb{D}$ since E_h has no zeros there) is the Poisson integral of its nontangential limit. Each nonzero $f \in H^p$ can be factored as $f = JE_{|f|}$ where J is *inner*, meaning that $J \in H^\infty$ and $|J_{|\mathbb{T}}| \equiv 1$ [26, Thm 2.8] [28, Ch. II, Cor. 5.7]. Conversely, every product JE_h where J is inner and h as in (25) is a member of H^p. The multiplicative decomposition $f = JE_{|f|}$ is called the inner-outer factorization of f. We shall need that if $f \in H^p$, $g \in H^q$, and $|f_{|\mathbb{T}}g_{|\mathbb{T}}| \in L^r$ for some $p, q, r \geq 1$, then $fg \in H^r$. Indeed, one has inner-outer factorizations $f = J_1E_{|f|}$ and $g = J_2E_{|g|}$, so that $fg = J_1J_2E_{|f|}E_{|g|} = J_1J_2E_{|fg|}$; now, J_1J_2 is inner and $E_{|fg|} \in H^r$ since $\log|f_{|\mathbb{T}}g_{|\mathbb{T}}| = \log|f_{|\mathbb{T}}| + \log|g_{|\mathbb{T}}| \in L^1(\mathbb{T})$ and $|fg|_{|\mathbb{T}} \in L^r(\mathbb{T})$, whence $fg \in H^r$. Every real-valued harmonic function u on $\mathbb{D}$ has a *harmonic conjugate*, that is, a real-valued harmonic function v on $\mathbb{D}$ such that $u + iv$ is holomorphic;

[3]More generally, it is a theorem of Privalov that no nonzero meromorphic function on $\mathbb{D}$ has nontangential limit zero on a set of strictly positive measure on $\mathbb{T}$, see [44, Sec. 6.1].

this follows by simple connectedness of $\mathbb{D}$ from the fact that $\Delta u = 0$ makes $-\partial_{x_2} u dx_1 + \partial_{x_1} u dx_2$ an exact differential. The conjugate function is defined up to an additive constant, and we customarily normalize it so that $v(0) = 0$. When $1 \leq p \leq \infty$ and real $\psi \in L^p(\mathbb{T})$, then $u(z) = P[\psi](z)$ is harmonic on $\mathbb{D}$ and it is a theorem of Fatou that it has nontangential limit ψ a.e. on $\mathbb{T}$. Also, it holds that $\|u\|_{L^p(\mathbb{T}_\rho)} \leq \|\psi\|_{L^p(\mathbb{T})}$ for $0 \leq \rho < 1$. Under the stronger assumption that $1 < p < \infty$, then $v = P[\widetilde{\psi}]$ where

$$\widetilde{\psi}(e^{i\theta}) := \lim_{\varepsilon \to 0} \frac{1}{2\,\pi} \int_{\varepsilon < |\theta - t| < \pi} \frac{\psi(e^{it})}{\tan(\frac{\theta - t}{2})}\, dm(t) \tag{26}$$

is called the *conjugate function* of ψ. It is a theorem of M. Riesz that the conjugation operator $\psi \mapsto \widetilde{\psi}$ is an isomorphism of $L^p(\mathbb{T})$ when $1 < p < \infty$. Thus, we see that if $\psi \in L^p_{\mathbb{R}}(\mathbb{T})$ and $1 < p < \infty$, then there exists $g \in H^p$ (namely $g = u + iv$) such that $\operatorname{Re} g = \psi$ on $\mathbb{T}$ [28, Ch. III]. Such a g is unique up to addition of a pure imaginary constant, and if we normalize it so that $\operatorname{Im} g(0) = 0$, then $\|g\|_{H^p} \leq C \|\psi\|_{L^p(\mathbb{T})}$ with $C = C(p)$.
When $\psi \in L^1(\mathbb{T})$, the conjugate function $\widetilde{\psi}$ is still defined pointwise almost everywhere *via* (26) but it may no longer belong to $L^1(\mathbb{T})$.
For $p \in (1, \infty)$, a non-negative function $\mathfrak{w} \in L^1(\mathbb{T})$ is said to satisfy Muckenhoupt condition A_p if

$$\{\mathfrak{w}\}_{A_p} := \sup_I \Big(\frac{1}{m(I)} \int_I \mathfrak{w}\, dm \Big) \Big(\frac{1}{m(I)} \int_I \mathfrak{w}^{-1/(p-1)} dm \Big)^{p-1} < +\infty, \tag{27}$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$. A theorem of Hunt, Muckenhoupt and Wheeden [28, Ch. VI, Thm 6.2][4] asserts that $\mathfrak{w}$ satisfies condition A_p if and only if there is $C > 0$ independent of ϕ for which

$$\int_{\mathbb{T}} |\widetilde{\phi}|^p\, \mathfrak{w}\, dm \leq C \int_{\mathbb{T}} |\phi|^p\, \mathfrak{w}\, dm, \qquad \phi \in L^1(\mathbb{T}), \tag{28}$$

and also that (28) is equivalent to

$$\int_{\mathbb{T}} |M\phi|^p\, \mathfrak{w}\, dm \leq C_1 \int_{\mathbb{T}} |\phi|^p\, \mathfrak{w}\, dm \tag{29}$$

[4] The proof given there on the half-plane easily carries over to the disk.

for $C_1 > 0$ and where $M\phi$ is the Hardy-Littlewood maximal function of ϕ:

$$M\phi(\xi) = \sup_{I \ni \xi} \frac{1}{m(I)} \int_I |\phi|\, dm, \qquad \xi \in \mathbb{T}, \tag{30}$$

the supremum being taken over all subarcs of $\mathbb{T}$ that contain ξ. In (28), the assumption $\phi \in L^1(\mathbb{T})$ is just a means to ensure that $\widetilde{\phi}$ is well defined m-a.e. and the constants C, C_1 can be chosen to depend only on $\{\mathfrak{w}\}_{A_p}$.
Condition A_2 is fundamental to function theory on Lipschitz (and more generally chord-arc[5]) domains, as was first pointed out in the seminal work [36], see also [34, 55, 57]. Recall from the Riemann mapping theorem that to each simply connected domain $\Omega \subset \mathbb{C}$ there is a conformal map φ from $\mathbb{D}$ onto Ω, which is unique if we impose for instance $\varphi(0) \in \Omega$ and $\arg \varphi'(0) \in [0, 2\pi)$. The precise normalization is unimportant in what follows.

Lemma 5.1. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected Lipschitz domain and $\varphi : \mathbb{D} \to \Omega$ a conformal map. Then φ extends homeomorphically from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ and preserves nontangential regions of approach in that, to every $\alpha, \beta > 1$, there are $\alpha', \beta' > 1$ such that:*

$$R_\alpha^\Omega(\varphi(\xi)) \subset \varphi\left(R_{\alpha'}^{\mathbb{D}}(\xi)\right) \quad \text{and} \quad \varphi\left(R_\beta^{\mathbb{D}}(\xi)\right) \subset R_{\beta'}^\Omega(\varphi(\xi)), \qquad \xi \in \mathbb{T}. \tag{31}$$

The derivative φ' as well as its reciprocal $1/\varphi'$ lie in H^p for some $p > 1$, and it holds for any measurable $E \subset \partial\Omega$ that $\Lambda(E) = \int_{\varphi^{-1}(E)} |\varphi'| dm$. Moreover, the weights $|\varphi'_{|\mathbb{T}}|$ and $1/|\varphi'_{|\mathbb{T}}|$ satisfy condition A_2.

Proof. Since $\partial\Omega$ is a Jordan curve, φ extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ mapping $\mathbb{T}$ to $\partial\Omega$ by Carathéodory's theorem [44, Thm 2.6]. To prove (31), we follow the argument (attributed to F. Gehring) outlined in [34, Prop. 1.1] for conformal maps from a half-plane onto unbounded chord-arc domains. Observe first that Ω is *a fortiori* chord-arc since it is Lipschitz, in particular Ω is a quasi-disk[6] [44, Prop. 7.7], and consequently φ extends to a quasi-conformal homeomorphism[7]of $\mathbb{C}$ [44, Thm 5.17]. Such a map is quasi-symmetric [9, Def. 3.2.1, Thm3.5.3], meaning that there is an increasing

[5]A Jordan domain Ω is chord-arc (or Lavrentiev) if $\Lambda(J(\xi_1, \xi_2)) \leq M|\xi_1 - \xi_2|$ whenever $\xi_1, \xi_2 \in \partial\Omega$, where $J(\xi_1, \xi_2)$ is the smaller arc of $\partial\Omega$ between ξ_1 and ξ_2 and M is a constant.

[6]Same definition as a chord-arc domain except that Λ gets replaced by "diameter".

[7]An orientation-preserving homeomorphism $\varphi \in W^{1,2}_{loc}(\mathbb{C})$ is quasi-conformal if $\|\bar{\partial}\varphi/\partial\varphi\|_{L^\infty(\mathbb{C})} < 1$, see [9, Def. 2.5.2, Thm 2.5.4].

homeomorphism η of $[0,\infty)$ such that

$$\left|\frac{\varphi(z_0)-\varphi(z_1)}{\varphi(z_0)-\varphi(z_2)}\right| \leq \eta\left(\left|\frac{z_0-z_1}{z_0-z_2}\right|\right), \qquad z_0,z_1,z_2\in\mathbb{C}. \tag{32}$$

Now, fix $\beta>1$, $z_1\in\mathbb{T}$ and let z_0 range over $R^{\mathbb{D}}_{\beta}(z_1)$. If we choose $z_2\in\mathbb{T}$ such that $|\varphi(z_0)-\varphi(z_2)|=\mathrm{d}(\varphi(z_0),\partial\Omega)$, then it follows from (32) that

$$\frac{|\varphi(z_0)-\varphi(z_1)|}{\mathrm{d}(\varphi(z_0),\partial\Omega)} \leq \eta\left(\left|\frac{z_0-z_1}{z_0-z_2}\right|\right) \leq \eta\left(\frac{|z_0-z_1|}{\mathrm{d}(z_0,\mathbb{T})}\right) \leq \eta(\beta), \tag{33}$$

hence $\varphi(z_0)\in R^{\Omega}_{\beta'}(\varphi(z_1))$ with $\beta'=\eta(\beta)$. This proves the second inclusion in (31) and the first follows in the same manner, replacing φ by its inverse which is also quasi-conformal [9, Thm 3.7.7].
Next, since $\partial\Omega$ is rectifiable, φ' lies in H^1 and $\Lambda(E)=\int_{\varphi^{-1}(E)}|\varphi'|dm$ for every measurable $E\subset\partial\Omega$ [44, Thm 6.8]. The fact that $|\varphi'_{|\mathbb{T}}|$ meets A_2 is a consequence of [55, Prop. 15] (which deals more generally with local chord arc graphs), see also [40, Sec. 2] and the references therein or [29, Ch. VII, Thm 4.2] for a proof when Ω is star-shaped. The fact that $|\varphi'_{|\mathbb{T}}|$ satisfies condition A_2 implies that it belongs to $L^{1+\delta}(\mathbb{T})$ for some $\delta>0$ [28, Ch. VI, Cor. 6.10], hence it holds in fact that $\varphi'\in H^p$ for some $p>1$. Clearly $\{|\varphi'_{|\mathbb{T}}|\}_{A_2}=\{1/|\varphi'_{|\mathbb{T}}|\}_{A_2}$. As Ω is chord-arc, it is in particular a Smirnov domain, meaning that φ' is outer [44, Sec. 7.3, 7.4]. Hence $1/\varphi'$ is also outer and since $1/|\varphi'_{|\mathbb{T}}|\in L^{1+\delta}(\mathbb{T})$ for some $\delta>0$ because it satisfies A_2, we find that in turn $1/\varphi'\in H^p$ for some $p>1$. □

5.2. Smirnov classes of a Lipschitz plane domain

On an arbitrary simply connected domain Ω (whose boundary contains more than one point), there are at least two generalizations of the Hardy space H^p of the disk. One which goes by the name of Hardy space, but is of no concern to us here, requires $|f|^p$ to have a harmonic majorant on Ω. The other, which is the one we are interested in, is the so-called Smirnov space, denoted as $\mathcal{S}^p(\Omega)$. It consists of functions f, holomorphic in Ω, for which there is a sequence of relatively compact Jordan domains $\Delta_n\subset\Omega$ with rectifiable boundary such that each compact $K\subset\Omega$ is contained in Δ_n for $n\geq n(K)$ and

$$\sup_{n\in\mathbb{N}}\|f\|_{L^p(\partial\Delta_n)}<\infty. \tag{34}$$

By the maximum principle $\mathcal{S}^\infty(\Omega)$ consists of bounded holomorphic functions on Ω. When $1 \le p < \infty$ it is not immediately clear that $\mathcal{S}^p(\Omega)$ is a Banach space, but this is nevertheless true and there is in fact a fixed sequence Δ_n such that (34) holds for all $f \in \mathcal{S}^p(\Omega)$. Such a sequence can be taken to be $\varphi(\mathbb{D}_{\rho_n})$ where $\rho_n \to 1^-$ and φ is a conformal map from $\mathbb{D}$ onto Ω [26, Thm 10.1]. Consequently f belongs to $\mathcal{S}^p(\Omega)$ if and only if $(f \circ \varphi)(\varphi')^{1/p}$ belongs to H^p, and $\|(f \circ \varphi)(\varphi')^{1/p}\|_p$ will serve as a norm on $\mathcal{S}^p(\Omega)$ [26, Ch. 10, Sec. 1, Cor. to Thm 10.1].
As soon as $\partial\Omega$ is rectifiable, so that $\varphi' \in H^1$, the previous characterization together with Lemma 5.1 and the discussion in Section 5.1 imply that each $f \in \mathcal{S}^p(\Omega)$ has nontangential limits a.e. on $\partial\Omega$ with respect to arclength, and that the boundary function thus defined lies in $L^p(\partial\Omega)$. Moreover, *this boundary function cannot vanish on a set of positive arclength unless* $f \equiv 0$, and its norm in $L^p(\partial\Omega)$ coincides with $\|f\|_{\mathcal{S}^p(\Omega)}$, thereby identifying $\mathcal{S}^p(\Omega)$ with a closed subspace of $L^p(\partial\Omega)$. Again f is recovered from its boundary function by a Cauchy integral [26, Thm 10.4], but the (analog of the) Poisson representation may now fail.
Our interest in Smirnov spaces is here limited to $\mathcal{S}^2(\Omega)$ for Ω a bounded simply connected Lipschitz domain. Theorem 5.1 below gives two alternative descriptions of this space. We mention that the analog of point (i) for unbounded chord-arc domains is contained in [34, Thm 2.2]. First, we need a lemma:

Lemma 5.2. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected Lipschitz domain, and φ map $\mathbb{D}$ conformally onto Ω. If $f \in \mathcal{S}^2(\Omega)$, then $f \circ \varphi \in H^1$.*

Proof. Set for simplicity $F = f \circ \varphi$. Since $f \in \mathcal{S}^2(\Omega)$, we have that $F(\varphi')^{1/2} \in H^2$, and we know from Lemma 5.1 that $1/\varphi'$ lies in H^1. Therefore, by the Schwarz inequality and the monotonicity of $\rho \mapsto \|g_\rho\|_{L^p(\mathbb{T})}$ for holomorphic g, we get that

$$\left(\int_{\mathbb{T}_\rho} |F| dm\right)^2 \le \int_{\mathbb{T}_\rho} |F|^2 |\varphi'| dm \int_{\mathbb{T}_\rho} |1/\varphi'| dm \le \left\|F(\varphi')^{1/2}\right\|_{H^2}^2 \|1/\varphi'\|_{H^1}.$$

□

Theorem 5.1. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected Lipschitz domain.*

(i) For each $\alpha > 1$, the space $\mathcal{S}^2(\Omega)$ coincides with holomorphic functions f in Ω such that $\mathcal{M}_\alpha f \in L^2(\partial\Omega)$ and $f \mapsto \|\mathcal{M}_\alpha f\|_{L^2(\partial\Omega)}$ is an equivalent norm on $\mathcal{S}^2(\Omega)$.

(ii) $\mathcal{S}^2(\Omega)$ *is the closed subspace of* $W^{1/2,2}(\Omega)$ *consisting of holomorphic functions, with equivalence of norms.*

Proof. Let φ map $\mathbb{D}$ conformally onto Ω, put ψ for the inverse map, and pick $\alpha > 1$. By (31), there is $\beta > 1$ such that $\mathcal{M}_\alpha f \leq (\mathcal{M}_\beta F) \circ \psi$ for $f \in \mathcal{S}^2(\Omega)$ and $F = f \circ \varphi$. From Lemma 5.2 we get that $F \in H^1$, hence it is the Poisson integral of $F_{|\mathbb{T}}$. It is known, however, that $\mathcal{M}_\beta F \leq C M F_{|\mathbb{T}}$ pointwise on $\mathbb{T}$ for some constant C depending only on β [28, Ch. I, Thm 4.2][8]. Consequently,

$$\begin{aligned} \int_{\partial\Omega} (\mathcal{M}_\alpha f)^2 \, d\Lambda &\leq \int_{\partial\Omega} (\mathcal{M}_\beta F \circ \psi)^2 \, d\Lambda \\ &= \int_{\mathbb{T}} (\mathcal{M}_\beta F)^2 \, |\varphi'| dm \leq C^2 \int_{\mathbb{T}} (MF)^2 |\varphi'| dm, \end{aligned} \tag{35}$$

where the change of variable is justified by Lemma 5.1. Now, as $|\varphi'|$ satisfies condition A_2, we get in view of (29) that

$$\int_{\mathbb{T}} (MF)^2 |\varphi'| \, dm \leq C_1^2 \int_{\mathbb{T}} |F|^2 |\varphi'| \, dm = C_1^2 \|f\|^2_{\mathcal{S}^2(\Omega)} \tag{36}$$

for some C_1 depending only on $\{|\varphi'|\}_{A_2}$. From (35) and (36), it follows that

$$\|\mathcal{M}_\alpha f\|_{L^2(\partial\Omega)} \leq C_2 \|f\|_{\mathcal{S}^2(\Omega)} \tag{37}$$

with $C_2 = C_2(\Omega, \alpha)$. Conversely, assume that f is holomorphic in Ω with $\|\mathcal{M}_\alpha f\|_{L^2(\partial\Omega)} < \infty$. Whenever $\delta \in (1, \infty)$ and $z_0 \in \Omega$, it is a famous estimate for harmonic functions on Lipschitz domains (in any dimension) that

$$\|\mathcal{M}_\delta (f - f(z_0))\|_{L^2(\partial\Omega)} \sim \|\mathrm{d}(.,\partial\Omega)^{1/2} \nabla f\|_{L^2(\Omega)} \tag{38}$$

where the constants depend only on Ω, δ and z_0 [24, Thm 1, Cor. 1]. Assume first that $f(\varphi(0)) = 0$, in which case it follows from (38) that

$$\|\mathcal{M}_\delta f\|_{L^2(\partial\Omega)} \leq C \|\mathcal{M}_\alpha f\|_{L^2(\partial\Omega)} < \infty, \tag{39}$$

where $C = C(\alpha, \delta, \Omega)$. Pick $\delta = \eta(2)$, where η is as in (32); note that indeed $\eta(2) > 1$, since η is strictly increasing and $\eta(1) \geq 1$. Now, the argument in (33) can be reversed (set $\beta = 2$ there) so that, if $z_1 \in \mathbb{T}$, then $R_\delta^\Omega(\varphi(z_1)) \supset \varphi(R_2^{\mathbb{D}}(z_1))$. Hence $\mathcal{M}_2 F \leq (\mathcal{M}_\delta f) \circ \varphi$, $F = f \circ \varphi$, and for $\rho \in [0, 1)$ we get from the obvious inequality $|F(\rho e^{i\theta})| \leq \mathcal{M}_2 F(e^{i\theta})$ that

$$\int_{\mathbb{T}} |F_\rho|^2 |\varphi'| \, dm \leq \int_{\mathbb{T}} |\mathcal{M}_2 F|^2 |\varphi'| \, dm \leq \int_{\mathbb{T}} |(\mathcal{M}_\delta f) \circ \varphi|^2 |\varphi'| \, dm = \|\mathcal{M}_\delta f\|^2_{L^2(\partial\Omega)} \, .$$

[8]The proof given there on the half-plane carries over immediately to the disk.

In view of (39), the previous inequality shows that $\|F_\rho(\varphi')^{1/2}\|_{H^2}$ is bounded independently of ρ, so there is a sequence $\rho_k \to 1^-$ such that $F_{\rho_k}(\varphi')^{1/2}$ converges weakly in H^2 to some function G. Since $F_{\rho_k}(z) = F(\rho_k z)$ converges to $F(z)$ locally uniformly in $\mathbb{D}$, passing to the weak limit in the Cauchy formula yields $G = F(\varphi')^{1/2}$. As the norm of the weak limit cannot exceed the lim inf of the norms, we deduce on using (39) that

$$\|f\|_{\mathcal{S}^2(\Omega)} = \|F(\varphi')^{1/2}\|_{H^2} \le \|\mathcal{M}_\delta f\|_{L^2(\partial\Omega)} \le C\|\mathcal{M}_\alpha f\|_{L^2(\partial\Omega)}. \tag{40}$$

Finally, if $f(\varphi(0)) \neq 0$, we apply (40) to ψf which has the same $\mathcal{S}^2(\Omega)$-norm as f and does vanish at $\varphi(0)$ (that $\|\psi f\|_{\mathcal{S}^2(\Omega)} = \|f\|_{\mathcal{S}^2(\Omega)}$ is clear from the relation $(\psi f)\circ\varphi(z) = zf(\varphi(z))$). Since $\mathcal{M}_\alpha \psi f \le \mathcal{M}_\alpha f$ because $|\psi| \le 1$, we get that $\|f\|_{\mathcal{S}^2(\Omega)} \le C\|\mathcal{M}_\alpha f\|_{L^2(\partial\Omega)}$ where $C = C(\alpha, \Omega)$, thereby proving (i). As for (ii), since holomorphic functions are harmonic, we know from [35, Thm 4.1] that

$$\|f\|_{L^2(\mathbb{D})} + \|\mathrm{d}(.,\partial\Omega)^{1/2}\nabla f\|_{L^2(\Omega)} \sim \|f\|_{W^{1/2,2}(\Omega)}, \qquad f \in \mathcal{S}^2(\Omega), \tag{41}$$

where the constants depend only on Ω. Pick $z_0 \in \Omega$. As $f(z_0)$ is the mean of f over some disk $\mathbb{D}_{z_0,\rho_0} \subset \Omega$, we get that $|f(z_0)| \le C_3\|f\|_{L^2(\Omega)}$ where $C_3 = C_3(z_0, \Omega)$. From this, together with (41) and (38), it follows that

$$\begin{aligned}\|\mathcal{M}_\alpha f\|_{L^2(\partial\Omega)} &\le |f(z_0)| + \|\mathcal{M}_\alpha(f - f(z_0))\|_{L^2(\partial\Omega)} \\ &\le C_4\big(\|f\|_{L^2(\mathbb{D})} + \|\mathrm{d}(.,\partial\Omega)^{1/2}\nabla f\|_{L^2(\Omega)}\big) \sim \|f\|_{W^{1/2,2}(\Omega)}.\end{aligned} \tag{42}$$

Conversely, the Schwarz inequality implies that

$$\|f\|^2_{L^2(\Omega)} = \int_{\mathbb{D}} |(f\circ\varphi)|^2|\varphi'|^2 dm_2 \le \|(f\circ\varphi)(\varphi')^{1/2}\|^2_{L^4(\Omega)}\|\varphi'\|^2_{L^2(\Omega)},$$

and since H^2 embeds in $L^4(\mathbb{D})$ (see discussion after (24)) while $(\varphi')^{1/2} \in H^2$ by Lemma 5.1, we get that $\|f\|_{L^2(\Omega)} \le C_5\|f\|_{\mathcal{S}^2(\Omega)}$ where $C_5 = C_5(\Omega, \varphi)$. From this together with (41), (38), and the inequality $|f(z_0)| \le C_3\|f\|_{L^2(\Omega)}$ already mentioned, we obtain:

$$\begin{aligned}\|f\|_{W^{1/2,2}(\Omega)} &\sim \|f\|_{L^2(\Omega)} + \|\mathrm{d}(.,\partial\Omega)^{1/2}\nabla f\|_{L^2(\Omega)} \\ &\le C_6\big(\|f\|_{\mathcal{S}^2(\Omega)} + \|\mathcal{M}_\alpha f\|_{L^2(\partial\Omega)} + |f(z_0)|\big) \\ &\le C_7\big(\|f\|_{\mathcal{S}^2(\Omega)} + \|\mathcal{M}_\alpha f\|_{L^2(\partial\Omega)}\big).\end{aligned} \tag{43}$$

Now, point (ii) follows from (42), (43) and point (i). □

5.3. Smirnov spaces and Dirichlet problems for the Laplacian

Let $\mathfrak{t} \in L^\infty(\partial\Omega)$ be the tangent vector field to $\partial\Omega$ written in complex form: $\mathfrak{t} = \tau_{x_1} + i\tau_{x_2}$ Λ-a.e. in $\partial\Omega$.

Proposition 5.1. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected Lipschitz domain and $H \in W^{3/2,2}_{\mathbb{R}}(\Omega)$ be a harmonic function. Then $\partial H \in \mathcal{S}^2(\Omega)$ and*

$$\partial_\tau H = 2Re\,(\partial H\,\mathfrak{t})\ . \tag{44}$$

In particular, $tr_{\partial\Omega} H \in W^{1,2}_{\mathbb{R}}(\partial\Omega)$.

Proof. Since H is harmonic, ∂H is holomorphic, and $\partial H \in W^{1/2,2}(\Omega)$ because $H \in W^{3/2,2}_{\mathbb{R}}(\Omega)$. Thus, $\partial H \in \mathcal{S}^2(\Omega)$ by Theorem 5.1 (*ii*). Also, by the Sobolev embedding theorem, H is continuous on $\overline{\Omega}$. Let φ map $\mathbb{D}$ conformally onto Ω. Lemma 5.1 implies that $u := H \circ \varphi$ is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Moreover, the complex chain rule [4, Ch. 1, Sec. C] gives us, since $\bar\partial\varphi = 0$, that

$$\begin{aligned} du &= (\partial H \circ \varphi)\,\varphi'\,dz + (\bar\partial H \circ \varphi)\,\overline{\varphi'}\,d\bar z \\ &= 2\mathrm{Re}\Big((\partial H \circ \varphi)\,\varphi'\,dz\Big), \end{aligned} \tag{45}$$

where we used that $\bar\partial H = \overline{\partial H}$. Now, to say that $\partial H \in \mathcal{S}^2(\Omega)$ is equivalent to say that $(\partial H \circ \varphi)(\varphi')^{1/2} \in H^2$, and Lemma 5.1 implies that $(\varphi')^{1/2} \in H^2$, hence $F := (\partial H \circ \varphi)\varphi' \in H^1$. In particular, F_ρ converges to $F_{|\mathbb{T}}$ in $L^1(\mathbb{T})$ as $\rho \to 1^-$ and therefore, by integration, we get from (45) upon setting $\varphi(e^{i\theta_j}) = \zeta_j \in \partial\Omega$, $j = 1, 2$, that

$$\begin{aligned} H(\zeta_1) - H(\zeta_2) &= \lim_{\rho\to1^-}\left(u(\rho e^{i\theta_1}) - u(\rho e^{i\theta_2})\right) = \lim_{\rho\to1^-} 2\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(F(\rho e^{i\theta}) i e^{i\theta}\right)\rho d\theta \\ &= 2\int_{\theta_1}^{\theta_2} \mathrm{Re}\left((\partial H \circ \varphi)(e^{i\theta})\varphi'(e^{i\theta}) i e^{i\theta}\right) d\theta. \end{aligned} \tag{46}$$

Since $\varphi'(e^{i\theta}) i e^{i\theta}/|\varphi'(e^{i\theta})| = \mathfrak{t}(\varphi(e^{i\theta}))$ and $d\Lambda = |\varphi'(e^{i\theta})|d\theta$ by Lemma 5.1, we may rewrite (46) as

$$H(\zeta_1) - H(\zeta_2) = 2\int_{[\zeta_1,\zeta_2]} \mathrm{Re}\,(\partial H\mathfrak{t})\,d\Lambda,$$

where $[\zeta_1, \zeta_2]$ is the oriented arc from ζ_1 to ζ_2 on $\partial\Omega$. This proves (44). $\square$

Remark 5.1. *That* $\mathrm{tr}_{\partial\Omega}H$ *belongs to* $W^{1,2}_{\mathbb{R}}(\partial\Omega)$ *in Proposition 5.1 depends on the fact that* H *is harmonic, and is* not *a general property of* $W^{3/2,2}(\Omega)$*-functions, see the discussion before [35, Prop. 3.2] for a counterexample credited to G. David.*

In view of Theorem 5.1, the next proposition stands analog in the planar case to a well-known result on the Dirichlet problem obtained in [52, Thm 5.1] for $n \geq 3$. The proof we give here in the planar case is quite different, and uses conformal mapping and the M. Riesz theorem as global tools[9].

Proposition 5.2. *Let* $\Omega \subset \mathbb{C}$ *be a bounded simply connected Lipschitz domain, and* $\psi \in L^2_{\mathbb{R}}(\partial\Omega)$ *be such that* $\int_{\partial\Omega}\psi d\Lambda = 0$*. Then, there is a harmonic function* $U \in W^{3/2,2}_{\mathbb{R}}(\Omega)$ *such that* $\partial_\tau \mathrm{tr}_{\partial\Omega}U = \psi$*. Such a function is unique up to an additive real constant and* $\partial U \in \mathcal{S}^2(\Omega)$ *with* $\|\partial U\|_{\mathcal{S}^2(\Omega)} \leq C\|\psi\|_{L^2(\partial\Omega)}$*, where* C *depends only on* Ω*.*

Proof. As U is continuous on $\overline{\Omega}$ by the Sobolev embedding theorem, and $\mathrm{tr}_{\partial\Omega}U \in W^{1,2}(\partial\Omega)$ by Proposition 5.1, uniqueness follows from the maximum principle for harmonic functions.
Next, let φ map $\mathbb{D}$ conformally onto Ω and Υ be the inverse map. Define $h = (\psi\circ\varphi)|\varphi'|$ on $\mathbb{T}$. By Lemma 5.1 $\|\psi\|_{L^2(\partial\Omega)} = \|h(\varphi')^{-1/2}_{|\mathbb{T}}\|_{L^2(\mathbb{T})}$ and $(\varphi')^{1/2}_{|\mathbb{T}} \in L^\ell(\mathbb{T})$ for some $\ell > 2$. Therefore $h \in L^p(\mathbb{T})$ for some $p > 1$, by Hölder's inequality. Moreover $\int_{\mathbb{T}} h dm = \int_{\partial\Omega}\psi d\Lambda = 0$, hence by the M. Riesz theorem there is $G \in H^p$ such that $G(0) = 0$ and $\mathrm{Re}\, G_{|\mathbb{T}} = h$. Because $\|\psi\|_{L^2(\partial\Omega)} = \|h(\varphi')^{-1/2}_{|\mathbb{T}}\|_{L^2(\mathbb{T})}$ and $1/|\varphi'|$ meets condition A_2 by Lemma 5.1, we get from (28) that $\|G_{|\mathbb{T}}(\varphi')^{-1/2}_{|\mathbb{T}}\|_{L^2(\mathbb{T})} \leq C\|\psi\|_{L^2(\partial\Omega)}$ with $C = C(\varphi)$. Therefore, as $G \in H^p$ while $(\varphi')^{-1/2} \in H^2$ by Lemma 5.1, the product $H = G(\varphi')^{-1/2}$ lies in H^2 and $\|H\|_{H^2} \leq C\|\psi\|_{L^2(\partial\Omega)}$. Since $H(0) = 0$ the function $H_1(z) = H(z)/(iz)$ in turn lies in H^2 with same norm as H, and consequently

$$F(\zeta) := \frac{H_1(\Upsilon(\zeta))}{(\varphi'(\Upsilon(\zeta)))^{1/2}} = \frac{G(\Upsilon(\zeta))}{i\Upsilon(\zeta)\varphi'(\Upsilon(\zeta))} \in \mathcal{S}^2(\Omega) \tag{47}$$

[9]The restriction to $n \geq 3$ in [52] may be due to the fact that it dwells on the method of layer potentials, where the discrepancy between Riesz and logarithmic potentials makes it cumbersome to treat both in a single stroke.

with $\|F\|_{\mathcal{S}^2(\Omega)} \leq C\|\psi\|_{L^2(\partial\Omega)}$. In view of Lemma 5.1, $\zeta \in \Omega$ converges nontangentially to $\xi \in \partial\Omega$ if, and only if $z = \Upsilon(\zeta) \in \mathbb{D}$ converges nontangentially to $e^{i\theta} = \Upsilon(\xi) \in \mathbb{T}$. Since $ie^{i\theta}\varphi'(e^{i\theta})/|\varphi'(e^{i\theta})| = \mathfrak{t}(\varphi(e^{i\theta}))$, with $\mathfrak{t}$ the tangent vector field in complex form as defined before Proposition 5.1, we see from equation (47) and the definition of G that

$$\operatorname{Re}(F(\xi)\,\mathfrak{t}(\xi)) = \psi(\xi), \qquad \Lambda - \text{a.e. } \xi \in \partial\Omega. \tag{48}$$

Let U be harmonic and real-valued in Ω with $\partial U = F/2$. Clearly U exists, for $F(z)dz + \overline{F(z)}d\bar{z}$ is a closed real-valued differential on the simply connected domain Ω. Moreover, $U \in W^{3/2,2}_{\mathbb{R}}(\Omega)$ because $\partial U \in W^{1/2,2}(\Omega)$ by Theorem 5.1 (ii). Then, it follows from (48) and Proposition 5.1 that $\partial_\tau U = \psi$. $\square$

6. Proof of Theorem 4.2

In Section 6.1, we state Theorem 6.1 which is instrumental for the proof of Theorem 4.2 but is also of independent interest. It is proved in Section 6.2, along with generalizations of results from Section 5.3 to more general conductivity equations, and a version of Rolle's theorem in $W^{1,2}(\mathbb{R})$. Finally, the proof of Theorem 4.2 is given in Section 6.3.

6.1. Factorization and regularity

Theorem 6.1. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain and that σ satisfies* (12)-(13). *Let $u \in W^{1,2}_{\mathbb{R}}(\Omega)$ be a solution to* (11) *which is such that $\partial_n u \in L^2_{\mathbb{R}}(\partial\Omega)$. Then:*

(i) $u \in W^{3/2,2}_{\mathbb{R}}(\Omega)$ and $\partial u = e^{\Psi}\Phi$ where $\Psi \in W^{1,r}(\Omega)$ and $\Phi \in \mathcal{S}^2(\Omega)$. Moreover ∇u converges nontangentially to $\partial_\tau u\,\tau + \partial_n u\,n$ on $\partial\Omega$, and if u gets normalized so that $u(z_0) = 0$ for some $z_0 \in \Omega$, there is a constant C depending only on Ω, z_0, r, $\|\sigma\|_{W^{1,r}(\Omega)}$ and c in (13) *such that*

$$\|u\|_{W^{3/2,2}(\Omega)} \leq C\|\partial_n u\|_{L^2(\partial\Omega)}. \tag{49}$$

(ii) For each $\alpha > 1$, it holds that

$$\|\partial_\tau u\|_{L^2(\partial\Omega)} \sim \|\partial_n u\|_{L^2(\partial\Omega)} \sim \|\mathcal{M}_\alpha \nabla u\|_{L^2(\partial\Omega)}, \tag{50}$$

where constants depend only on Ω, r, $\|\sigma\|_{W^{1,r}(\Omega)}$, c in (13), *and also on α as to the second equivalence.*

(iii) We have that $u \in W^{2,r}_{\mathbb{R},loc}(\Omega)$ *and that*

$$\sum_{j=1,2} \|d(.,\partial\Omega)^{1/2}\,\partial_{x_j}\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{W^{1,2}(\Omega)} \sim \|u\|^2_{W^{3/2,2}(\Omega)}, \tag{51}$$

where constants depend only on Ω, r, $\|\sigma\|_{W^{1,r}(\Omega)}$, *and* c *in* (13).

The proof of Theorem 4.2 dwells on the factorization $\partial u = e^{\Psi}\Phi$ introduced in Theorem 6.1 (i) and on a generalized form of Rolle's theorem given in Proposition 6.1. Roughly speaking, the latter shows that if both u and $\partial_n u$ vanish on a subset of positive measure of $\partial\Omega$, then the full gradient $\nabla u = \partial_\tau u\,\tau + \partial_n u\,n$ also has to vanish on such a set. Consequently, Theorem 6.1 shows that the gradient vanishes everywhere in Ω, because ∂u factors through a holomorphic function of Smirnov class which cannot vanish on a subset of positive measure of $\partial\Omega$ if it is not identically zero.
The regularity results needed to put this approach to work are set forth in Theorem 6.1 points (i)-(ii). Point (ii) is known, even in higher dimension and with less regular σ, provided Ω is starlike [38]. Point (iii) is not used but mentioned for its own sake, as it generalizes to more general conductivities, in the case where $n = 2$, the equivalence between the two hand sides of (6) established for harmonic functions in [35, Thm 4.1].

6.2. Proof of Theorem 6.1

6.2.1. The σ-harmonic conjugate function

When $\Omega \subset \mathbb{R}^2$ is simply connected, we observe that (11) is a compatibility condition for the generalized Cauchy-Riemann system:

$$\begin{cases} \partial_{x_1} v = -\sigma\partial_{x_2} u, \\ \partial_{x_2} v = \sigma\partial_{x_1} u, \end{cases} \tag{52}$$

with unknown real-valued functions u, v. In fact, (11) is equivalent to the Schwarz rule $\partial_{x_2}\partial_{x_1} v = \partial_{x_1}\partial_{x_2} v$ in (52), thus there is a distribution v to meet the latter whenever $u \in W^{1,2}_{\mathbb{R}}(\Omega)$ satisfies (11) [46, Ch. II, Sec. 6, Thm VI]. From (13) and (52) we get that $|\nabla v| \in L^2_{\mathbb{R}}(\Omega)$, hence $v \in W^{1,2}_{\mathbb{R}}(\Omega)$ and $\partial_\tau \mathrm{tr}_{\partial\Omega} v$ exists in $W^{-1/2,2}_{\mathbb{R}}(\partial\Omega)$. The function v is a so-called σ-harmonic conjugate to u, and it is unique up to an additive constant by (4).
Because (52) entails that ∇v is the rotation of $\sigma\nabla u$ by $\pi/2$ on Ω, it may be surmised that $\partial_n u = \partial_\tau \mathrm{tr}_{\partial\Omega} v/\sigma \in W^{-1/2,2}_{\mathbb{R}}(\partial\Omega)$. This is indeed the case, as

follows from the Green formula on Lipschitz domains [43, Ch. 3, Thm 1.1]:

$$\int_\Omega (h\,\partial_{x_i} g + g\,\partial_{x_i} h) dm = \int_{\partial\Omega} g\,h\,n_{x_i}\,d\Lambda\,,\ \ g,h\in W^{1,2}(\Omega)\,,\ \ i=1,2\,, \tag{53}$$

where we have put $n=(n_{x_1},n_{x_2})^t$. In fact, since $\tau=(-n_{x_2},n_{x_1})$, it holds for $\varphi\in\mathcal{D}(\mathbb{R}^2)$ that

$$\int_{\partial\Omega}(\partial_\tau v)\varphi\,d\Lambda = -\int_{\partial\Omega} v\nabla\varphi.\tau\,d\Lambda = \int_{\partial\Omega} v(n_{x_2}\partial_{x_1}\varphi - n_{x_1}\partial_{x_2}\varphi)\,d\Lambda$$

so that, by (53),

$$\int_{\partial\Omega}\partial_\tau\,v\,\varphi\,d\Lambda = \int_\Omega (\partial_{x_2}v\,\partial_{x_1}\varphi - \partial_{x_1}v\,\partial_{x_2}\varphi)\ dm = \int_\Omega \sigma\,\nabla u\,.\,\nabla\,\varphi\,dm. \tag{54}$$

By density, we conclude on comparing (14) and (54) that $\partial_n u = \partial_\tau v/\sigma$, as announced. It is easy to check that v satisfies (11) with σ replaced by $1/\sigma$, so the previous discussion also yields that $\partial_n v = -\sigma\,\partial_\tau u$ on $\partial\Omega$. In fact, the peculiarity of the planar case is that solving the Neumann problem in $W^{1,2}_{\mathbb{R}}(\Omega)$ for the conductivity equation (11), with normal derivative $g\in W^{-1/2,2}_{\mathbb{R}}(\partial\Omega)$, is tantamount to solve the Dirichlet problem in $W^{1,2}_{\mathbb{R}}(\Omega)$ for a conductivity equation having conductivity $1/\sigma$ with tangential derivative $\sigma g\in W^{-1/2,2}_{\mathbb{R}}(\partial\Omega)$, and then compute the σ-harmonic conjugate. In particular uniqueness-up-to-a-constant of energy solutions implies that a solution $u\in W^{1,2}(\Omega)$ to (11) meeting $\partial_\tau u=0$ is a constant.

The functions v and $f=u+iv$, which lie respectively in $W^{1,2}_{\mathbb{R}}(\Omega)$ and $W^{1,2}(\Omega)$, will be instrumental to our analysis. For definiteness, we normalize v (initially defined up to an additive real constant) so that $\int_{\partial\Omega} v\,d\Lambda = 0$. A short computation (see [13, Sec. 3.1]) shows that f satisfies on Ω the conjugate Beltrami equation:

$$\bar\partial f = \nu\overline{\partial f}, \qquad \nu=(1-\sigma)/(1+\sigma). \tag{55}$$

Note that $\|\nu\|_{L^\infty(\Omega)}<1$ and that $\nu\in W^{1,r}_{\mathbb{R}}(\Omega)$ because of (4), (12) and (13). Interior regularity estimates for (55) imply that $f\in W^{2,r}_{loc}(\Omega)$ [12, Cor. 3.3] (see also Section 6.2.2), hence also $u,v\in W^{2,r}_{\mathbb{R},loc}(\Omega)$. In particular, by the Sobolev embedding theorem, $\nabla u,\nabla v$ are locally Hölder continuous on Ω.

Let $\{\Omega_k\}$ be a sequence of open subsets of Ω with smooth boundary such that $\overline{\Omega_k}\subset\Omega_{k+1}$ and $\cup_n\Omega_k=\Omega$. Whenever $u\in W^{1,2}(\Omega)$ satisfies (11) and

$g \in W^{-1/2,2}(\partial\Omega)$, it follows from (14) (and its analog on Ω_k) by means of the Schwarz inequality, and since $\|\nabla u\|_{L^2(\Omega\setminus\Omega_n)} \to 0$ as $k \to \infty$, that

$$\partial_n u = g \iff \lim_{k\to\infty} \int_{\partial\Omega_k} \sigma \nabla u.n\, \psi\, d\Lambda = \langle \sigma g\,,\, \mathrm{tr}_{\partial\Omega}\psi\rangle, \quad \psi \in W^{1,2}_{\mathbb{R}}(\Omega), \tag{56}$$

where n denotes the unit normal on $\partial\Omega_k$, irrespective of k. Elaborating on this, we let $\mathfrak{n}$ indicate the complex number $n_x + in_y$ where $(n_x, n_y)^t = n$, and we observe upon making use of (52) that

$$\sigma\, \partial u\, \mathfrak{n} = \sigma\, \frac{\partial_{x_1} u - i\partial_{x_2} u}{2}\, \mathfrak{n} = (\sigma\, \nabla u + i\, \nabla v)\,.\frac{n}{2}\,,$$

where "." indicates the Euclidean scalar product. In view of (56) and its analog for v (remember v satisfies (11) with σ replaced by $1/\sigma$), we obtain:

$$u \in W^{1,2}(\Omega) \text{ satisfies (11) with } \partial_n u = g \text{ and } \partial_\tau u = -h$$

$$\iff 2 \lim_{k\to\infty} \int_{\partial\Omega_k} \sigma\, \partial u\, \mathfrak{n}\, \psi\, d\Lambda = \langle \sigma\,(g + ih)\,,\, \mathrm{tr}_{\partial\Omega}\psi\rangle\,,\ \psi \in W^{1,2}(\Omega)\,, \tag{57}$$

where we complexified the space of test functions (*i.e.* from $\psi \in W^{1,2}_{\mathbb{R}}(\Omega)$ to $\psi \in W^{1,2}(\Omega)$) upon extending the pairing $\langle\, ,\, \rangle$ in a complex-linear manner.

6.2.2. Factorization of the complex derivative

The lemma below substantially reduces the study of solutions to (11), when $n = 2$ and (12)-(13) hold, to that of harmonic functions.

Lemma 6.1. *Let $u \in W^{1,2}_{\mathbb{R}}(\Omega)$ satisfy* (11) *on a bounded simply connected Lipschitz domain $\Omega \subset \mathbb{R}^2$, with σ subject to* (12) *and* (13)*. Then, there exists a holomorphic function $F \in L^2(\Omega)$, a number $r_1 \in (2, r]$, a function $\Upsilon \in W^{1,r_1}(\Omega)$ with real-valued $tr_{\partial\Omega}\Upsilon$ whose norm is bounded solely in terms of Ω, $\|\sigma\|_{W^{1,r}(\Omega)}$, r, and ellipticity constants in* (13)*, such that $\partial u = e^\Upsilon F$. Moreover $F = \partial H$ where $H \in W^{1,2}_{\mathbb{R}}(\Omega)$ is harmonic in Ω and satisfies on $\partial\Omega$:*

$$\partial_n H = e^{-\Upsilon}\partial_n u, \qquad \partial_\tau H = e^{-\Upsilon}\partial_\tau u. \tag{58}$$

Proof. Let $f = u + iv$ where v is the σ-harmonic conjugate to u. Since $f \in W^{2,r}_{loc}$ is a fortiori locally bounded and satisfies (55) with $\nu \in W^{1,r}_{\mathbb{R}}(\Omega)$ and $\|\nu\|_{L^\infty(\Omega)} < 1$, a short computation as in the proof of [12, Cor. 3.3] or [13, Lem. 5] shows that $w := (1 - \nu^2)^{1/2}\partial f$ satisfies $\bar{\partial} w = (\partial\nu/(1 - \nu^2))\bar{w}$.

As $\partial\nu/(1-\nu^2) \in L^r(\Omega)$ and $r>2$, the Bers similarity principle for pseudo-holomorphic functions (see *e.g.* [12, Prop. 3.2]) entails that there exist $s \in W^{1,r}(\Omega)$, whose norm[10] is bounded in terms of r and $\|\partial\nu/(1-\nu^2)\|_{L^r(\Omega)}$ only, and also a holomorphic function F_1 on Ω such that $w = e^s F_1$. Hence $\partial f = e^{s_1}F_1$, where $s_1 = s - \log(1-\nu^2)^{1/2}$ belongs to $W^{1,r}(\Omega)$ by (12), (13) and (4). Now, it is straightforward to check using (52) that $\partial f = (1+\sigma)\partial u$. Therefore, if we set $\Upsilon_1 = s_1 - \log(1+\sigma)$ and appeal again to (12), (13) and (4), we get that

$$\partial u = e^{\Upsilon_1}F_1, \qquad \Upsilon_1 \in W^{1,r}(\Omega), \quad F_1 \text{ holomorphic in } \Omega\,, \tag{59}$$

where we notice that $\|\Upsilon_1\|_{W^{1,r}(\Omega)}$ is bounded in terms of Ω, r, the constants in (13) and $\|\sigma\|_{W^{1,r}(\Omega)}$. Factorization (59) is not yet what we need, for $\mathrm{tr}_{\partial\Omega}\Upsilon_1$ may not be real-valued. To remedy this, we will trade r for a possibly smaller exponent $r_1 > 2$. Specifically, it follows from [35, Thm 5.1][11] that the Dirichlet problem for harmonic functions with boundary values in $W^{\theta,p}(\partial\Omega)$ is solvable in $W^{\theta+1/p,p}(\Omega)$, as soon as $0<\theta<1$ and $p\in[2,2+\varepsilon)$ where $\varepsilon>0$ depends on the Lipschitz constant of $\partial\Omega$. Since $\mathrm{tr}_{\partial\Omega}\Upsilon_1 \in W^{1-1/r,r}(\partial\Omega)$ and the latter space increases as r decreases, there is $r_1 \in (2,r]$ (depending on Ω and r) and a harmonic function $h \in W^{1,r_1}(\Omega)$ such that $\mathrm{tr}_{\partial\Omega}h = \mathrm{tr}_{\partial\Omega}\mathrm{Im}\Upsilon_1$. Let g be a harmonic conjugate to h, normalized so that $g_E = 0$ for some $E \subset \Omega$ with $m_2(E) > 0$. Since $|\nabla h| = |\nabla g|$ pointwise by the Cauchy-Riemann equations, it follows from (4) that g lies in $W^{1,r_1}(\Omega)$, and so do the holomorphic functions $G := -g + ih$ and e^G since exp is entire. Setting

$$\Upsilon = \Upsilon_1 - G \qquad \text{and} \qquad F = F_1 e^G, \tag{60}$$

we have that $\Upsilon \in W^{1,r_1}(\Omega)$ is real-valued on $\partial\Omega$ and that F is holomorphic, while $\partial u = e^{\Upsilon}F$, as desired.

[10]Reference [12] deals with Dini-smooth Ω but this assumption is not used in the proof of equations (18), (19) *loc. cit.* The similarity principle is called *representation of pseudo-analytic functions of the first kind* in [51, Ch. III, Sec. 4], and later appeared in many works.

[11]The result is stated there for $n \geq 3$ only, which may be confusing, but the proof is valid for $n=2$ as well. In fact, all we need is [35, Thm 5.15, (a),(b)] for Besov spaces, along with interpolation arguments on top of [35, p. 200]. Since that part of the proof of [35, Thm 5.15] depends only on [24], complex interpolation and multiplier theory for singular integral operators, the restriction $n \geq 3$ is easily seen to be superfluous.

Because $u \in W^{1,2}_{\mathbb{R}}(\Omega)$ by assumption and $\Upsilon \in L^\infty(\Omega)$ by the Sobolev embedding theorem, we get that $F \in L^2(\Omega)$. Being holomorphic, F can be written as ∂H for some real-valued harmonic H, and necessarily $H \in W^{1,2}_{\mathbb{R}}(\Omega)$ for its complex derivatives ∂H and $\bar\partial H = \overline{\partial H}$ are in $L^2(\Omega)$. Being harmonic, H satisfies (11) with σ replaced by 1, so we get the following analog to (57):

$$\partial_n H = \gamma \text{ and } \partial_\tau H = -\mu$$

$$\iff 2 \lim_{k\to\infty} \int_{\partial\Omega_k} \partial H \, \mathfrak{n} \, \phi \, d\Lambda = \langle \gamma + i\mu \, , \, \mathrm{tr}_{\partial\Omega}\phi \rangle, \;\; \phi \in W^{1,2}(\Omega). \qquad (61)$$

Substituting $\partial u = e^\Upsilon \partial H$ in (57) and reckoning that $\psi \mapsto \phi = \sigma e^\Upsilon \psi$ is an isomorphism of $W^{1,2}(\Omega)$ because $\sigma e^\Upsilon \in W^{1,r_1}(\Omega)$, we see from (61) that

$$\partial_n H = \mathrm{Re}\Big(e^{-\Upsilon}(\partial_n u - i\partial_\tau u)\Big), \quad \partial_\tau H = -\mathrm{Im}\Big(e^{-\Upsilon}(\partial_n u - i\partial_\tau u)\Big). \qquad (62)$$

Taking into account in (62) that e^Υ is real-valued on $\partial\Omega$ yields (58). □

6.2.3. Proof of Theorem 6.1

Proof. Write $\partial u = e^\Upsilon F$ as in Lemma 6.1, and let $H \in W^{1,2}_{\mathbb{R}}(\Omega)$ be a harmonic function such that $F = \partial H$. Since $e^{-\Upsilon}$ is bounded, being continuous on $\overline{\Omega}$ by the Sobolev embedding theorem, we deduce from (58) that $\partial_n H$ lies in $L^2(\partial\Omega)$. Set G to be a harmonic conjugate to H. By the discussion after (54) (with $\sigma \equiv 1$ throughout), we get that G is, up to an additive constant, the unique harmonic function in $W^{1,2}_{\mathbb{R}}(\Omega)$ such that $\partial_\tau G = \partial_n H$. From Proposition 5.2, we now see that $\partial G \in \mathcal{S}^2(\Omega)$ with $\mathrm{Re}(\partial G \, \mathfrak{t}) = e^{-\Upsilon}\partial_n u/2$ on $\partial\Omega$, where $\mathfrak{t}$ is the tangent vector field to $\partial\Omega$ written in complex form. By the Cauchy-Riemann equations $\partial H = i\partial G$, so that in turn $F \in \mathcal{S}^2(\Omega)$ and $\mathrm{Im}(F\,\mathfrak{t}) = e^{-\Upsilon}\partial_n u/2$ on $\partial\Omega$. Also $H \in W^{3/2,2}_{\mathbb{R}}(\Omega)$ for $\partial H = F$ lies in $W^{1/2,2}(\Omega)$ by Theorem 5.1 (*ii*). Let $\mathfrak{n} = \mathfrak{t}/i$ be the normal vector field on $\partial\Omega$, written in complex form. By definition of complex derivatives (*cf.* (3)), the nontangential convergence of ∇u to $\partial_n u \, n + \partial_\tau u \, \tau$ is equivalent to the nontangential convergence of $\partial u = e^\Upsilon F$ to $\partial_n u \, \overline{\mathfrak{n}}/2 + \partial_\tau u \, \overline{\mathfrak{t}}/2$. Considering the existence of nontangential limits a.e. for Smirnov functions and the continuity of Υ, this is in turn equivalent to $2F = e^{-\Upsilon}(\partial_n u \, \overline{\mathfrak{n}} + \partial_\tau u \, \overline{\mathfrak{t}})$, that is $2F\mathfrak{t} = e^{-\Upsilon}(i\partial_n u + \partial_\tau u)$ on $\partial\Omega$. Taking real and imaginary parts, we are thus left to verify two real equations:

$$2\mathrm{Re}(F\mathfrak{t}) = e^{-\Upsilon}\partial_\tau u \qquad \text{and} \qquad 2\mathrm{Im}(F\mathfrak{t}) = e^{-\Upsilon}\partial_n u.$$

The second of these has already been checked. By (58), the first reduces to $2\mathrm{Re}(\partial H \mathfrak{t}) = \partial_\tau H$ which holds good by (44). That $u \in W^{3/2,2}_{\mathbb{R}}(\Omega)$ follows from the relation $\partial u = e^{\Upsilon} F$, the membership $F \in W^{1/2,2}(\Omega)$, and the fact that e^{Υ} is a multiplier of $W^{1/2,2}(\Omega)$ for it lies in $W^{1,r_1}(\Omega)$ with $r_1 > 2$. Because $\|\Upsilon\|_{W^{1,r_1}(\Omega)}$ depends only on Ω, $\|\sigma\|_{W^{1,r}(\Omega)}$, r, and ellipticity constant c in (13), as asserted by Lemma 6.1, so does the norm of this multiplier. Combining this with Theorem 5.1 (ii), we obtain:

$$\|\nabla u\|_{W^{1/2,2}(\Omega)} \leq C\|\partial H\|_{W^{1/2,2}(\Omega)} = C\|\partial G\|_{W^{1/2,2}(\Omega)} \leq C'\|\partial G\|_{\mathcal{S}^2(\Omega)}, \tag{63}$$

where C' depends only on the above-mentioned parameters. Besides, we get from Proposition 5.2 (applied with $U = G$) and (58) that

$$\|\partial G\|_{\mathcal{S}^2(\Omega)} \leq C''\|\partial_\tau G\|_{L^2(\partial\Omega)} = C''\|\partial_n H\|_{L^2(\partial\Omega)} \leq C'' e^{\|\Upsilon\|_{L^\infty(\partial\Omega)}}\|\partial_n u\|_{L^2(\partial\Omega)}$$

where C'' depends only on Ω. The latter estimate and (63) together yield

$$\|\nabla u\|_{W^{1/2,2}(\Omega)} \leq C_0\|\partial_n u\|_{L^2(\partial\Omega)}, \tag{64}$$

where C_0 depends on the same parameters as C'.
Now, if we normalize u so that $u(z_0) = 0$ for some $z_0 \in \Omega$ and let $\rho_0 > 0$ be such that $\mathbb{D}(z_0, \rho_0) \subset \Omega$, we deduce from (11), (13), the Green formula (which is valid since $u \in W^{2,r}_{\mathbb{R},loc}(\Omega)$) and Hölder's inequality that, for $\rho \in (0, \rho_0)$,

$$\begin{aligned} \left|\frac{d}{d\rho}\left(\frac{1}{\rho}\int_{\mathbb{T}(z_0,\rho)} u\,dm\right)\right| &= \left|\frac{1}{\rho}\int_{\mathbb{T}(z_0,\rho)} \partial_n u\,dm\right| \leq \frac{1}{c\rho}\left|\int_{\mathbb{T}(z_0,\rho)} \sigma\partial_n u\,dm\right| \\ &= \frac{1}{c\rho}\left|\int_{\mathbb{D}(z_0,\rho)} \nabla u.\nabla\sigma\,dm_2\right| \leq \frac{\pi^{1/4}}{c\rho^{1/2}}\|\nabla u\|_{L^4(\Omega)}\|\nabla\sigma\|_{L^2(\Omega)}. \end{aligned} \tag{65}$$

Since $\lim_{\rho\to 0}\int_{\mathbb{T}(z_0,\rho 0)} udm/\rho = u(z_0) = 0$, we infer from (64), (65) and the Sobolev embedding theorem that, for $\rho \in (0, \rho_0)$,

$$\left|\frac{1}{\rho}\int_{\mathbb{T}(z_0,\rho)} u\,dm\right| \leq C_1\rho^{1/2}\|\partial_n u\|_{L^2(\partial\Omega)} \tag{66}$$

where C_1 depends on Ω, $\|\sigma\|_{W^{1,r}(\Omega)}$, r, and c in (13). Integrating (66) yields

$$\left|\frac{1}{\pi\rho_0^2}\int_{\mathbb{D}(z_0,\rho_0)} u\,dm_2\right| \leq C_2\rho_0^{1/2}\|\partial_n u\|_{L^2(\partial\Omega)} \tag{67}$$

where C_2 depends on the same parameters as C_1. Then, (49) follows from (4), (64) and (67).
Finally, remember from (60) the factorization $\partial u = e^{\Upsilon_1} F_1$ where $\Upsilon_1 \in W^{1,r}(\Omega)$ and $F_1 = Fe^{-G}$ with $G \in W^{1,r_1}(\Omega)$, G holomorphic. As $G \in L^\infty(\Omega)$ by the Sobolev embedding theorem, we see that $F_1 \in \mathcal{S}^2(\Omega)$ because F does, so we may set $\Psi = \Upsilon_1$ and $\Phi = F_1$, thereby completing the proof of (i).
In view of (58), the factorization $\partial u = e^{\Upsilon} F$, and the boundedness of e^{Υ}, the proof of (50) reduces to the case where u is harmonic ($i.e.$ $\sigma \equiv 1$), and then it follows from Theorem 5.1 and Proposition 5.2 applied to u and its conjugate function. This shows (ii).
As to (iii), we already mentioned that $u \in W^{2,r}_{\mathbb{R},loc}(\Omega)$ (this is now obvious anyway since $\partial u = e^{\Psi}\Phi$) and we need to prove (51) which is equivalent to

$$\sum_{j=1,2} \|\mathrm{d}(.,\partial\Omega)^{1/2}\, \partial_{x_j}(e^{\Upsilon} F)\|^2_{L^2(\Omega)} + \|e^{\Upsilon} F\|^2_{L^2(\Omega)} \sim \|e^{\Upsilon} F\|^2_{W^{1/2,2}(\Omega)}\,. \tag{68}$$

We already know from (6) that the right hand side of (68) is less than a constant (depending only on Ω) times the left hand side. To prove the reverse inequality, let φ conformally map $\mathbb{D}$ onto Ω, so that $f := (F\circ\varphi)(\varphi')^{1/2} \in H^2$, and recall that $|f(z)| \leq c\|f\|_{H^2}/(1-|z|)^{1/2}$ for $z \in \mathbb{D}$ and some absolute constant c, by a classical inequality of Hardy and Littlewood [26, Thm 5.9]. Since $\mathrm{d}(\varphi(z), \partial\Omega) \leq (1-|z|^2)|\varphi'(z)|$ by standard properties of conformal maps [44, Ch.1, Cor. 1.4], we get that $\|\mathrm{d}(.,\partial\Omega)^{1/2} F\|_{L^\infty(\Omega)} \leq \sqrt{2}c\|F\|_{\mathcal{S}^2(\Omega)}$. Now, by the Leibniz rule and the triangle inequality, the first summand in the left hand side of (68) is bounded above by

$$\sum_{j=1,2} \|\mathrm{d}(.,\partial\Omega)^{1/2}(\partial_{x_j}\Upsilon)e^{\Upsilon} F\|^2_{L^2(\Omega)} + \sum_{j=1,2} \|\mathrm{d}(.,\partial\Omega)^{1/2} e^{\Upsilon}(\partial_{x_j} F)\|^2_{L^2(\Omega)}$$

which is less than

$$\|e^{\Upsilon}\|^2_{L^\infty(\Omega)} \left(2\, c^2\, \|\nabla\Upsilon\|^2_{L^2(\Omega)}\, \|F\|^2_{\mathcal{S}^2(\Omega)} + \sum_{j=1,2} \|\mathrm{d}(.,\partial\Omega)^{1/2}(\partial_{x_j} F)\|^2_{L^2(\Omega)} \right).$$

By (43) and Theorem 5.1 (ii), this quantity is majorized by $c'\|F\|^2_{W^{1/2,2}(\Omega)}$ where $c' = c'(\Omega, r_1, \|\Upsilon\|_{W^{1,r_1}(\Omega)})$, and since $\|e^{\Upsilon} F\|_{W^{1/2,2}(\Omega)} \sim \|F\|_{W^{1/2,2}(\Omega)}$ we are done with the proof. $\square$

6.2.4. A generalized Rolle's theorem

We recall below the 1-dimensional version of a Lusin-type theorem for Sobolev functions, to be found in [56, Thm 3.10.5]. More precisely, we state the case $n = 1$, $\ell = k = 1$ and $p = 2$ of the result just quoted. The latter is in terms of Bessel capacities that we did not introduce, but we use here that the Bessel capacity $B_{0,2}$ is just Lebesgue measure, see [56, Def. 2.6.2].

Lemma 6.2. *[56, Thm 3.10.5] Let $v \in W^{1,2}(\mathbb{R})$ and $\varepsilon > 0$. There exists an open set $\mathcal{U} \subset \mathbb{R}$ and a function $w \in C^1(\mathbb{R})$ such that $m_1(\mathcal{U}) < \varepsilon$ and*

$$w(t) = v(t), \quad w'(t) = v'(t), \qquad \forall t \in \mathbb{R} \setminus \mathcal{U}.$$

We use Lemma 6.2 to prove the following generalization of Rolle's theorem.

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $v \in W^{1,2}(\partial\Omega)$. Assume that $v = 0$ at Λ-a.e. point of a set $B \subset \partial\Omega$ with $\Lambda(B) > 0$. Then, there is $B' \subset B$, with $\Lambda(B') > 0$, such that $\partial_\tau v = 0$ at Λ-a.e. point of B'.*

Proof. As Ω is bounded and Lipschitz, $\partial\Omega$ can be covered with open parallelepiped $Q_1, \cdots, Q_N$ of the form $Q_j = \mathcal{R}_j(Q_{a,b})$, where $\mathcal{R}_j$ is an affine isometry of $\mathbb{R}^2$ and $Q_{a,b} = (-a, a) \times (-b, b)$, with $a, b > 0$, in such a way that

$$\mathcal{R}_j^{-1}(\Omega) \cap Q_{a,b} = \{x \in Q_{a,b} : x_2 > \psi_j(x_1)\},$$

where ψ_j is a Lipschitz function. Denote by P_1 the projection onto the first component in $\mathbb{R}^2$, and for $E \subset \partial\Omega \cap Q_j$ set $E_j = P_1(\mathcal{R}_j^{-1}(E)) \subset (-a, a)$ so that $\Lambda(E) = \int_{E_j} |(1, \psi_j')^t| dm_1$. Since Lipschitz changes of variables preserve Sobolev classes [56, Thm 2.2.2], it holds that $v \in W^{1,2}(\partial\Omega)$ if and only if $v(\mathcal{R}_j(x_1, \psi_j(x_1)))$ belongs to $W^{1,2}((-a, a))$ for all j. Thus, as $\Lambda(B \cap Q_j) > 0$ for at least one j, it is enough to prove the analog of the proposition on the real interval $(-a, a)$ instead of $\partial\Omega$. By the extension theorem we may assume that v is defined over the whole real line. Then, taking ε small enough in Lemma 6.2, we conclude that it is enough to prove Proposition 6.1 when v has continuous derivative. Assume it is the case and let A be the set of accumulation points of B. Note that $m_1(A) = m_1(B) > 0$ because $B \setminus A$ is countable [32]. Moreover, to each $t \in A$, there exists a non-stationary sequence $(t_n) \subset B$, $n \in \mathbb{N}$, such that $t_n \to t$. Without loss of generality, we may assume that (t_n) is monotone, say $t_n < t_{n+1}$ (the case $t_n > t_{n+1}$ is similar) for all $n \in \mathbb{N}$. Since $v(t_n) = v(t_{n+1}) = 0$, there is $s_n \in [t_n, t_{n+1}]$ such that $v'(s_n) = 0$ by Rolle's theorem. Since $s_n \to t$ and v' in continuous, we get that $v'(t) = 0$, as desired. □

6.3. Proof of Theorem 4.2

Proof. From Theorem 6.1 (ii), we get that $u_{|\partial\Omega}$ lies in $W^{1,2}(\partial\Omega)$, hence Proposition 6.1 implies that both $\partial_n u$ and $\partial_\tau u$ vanish on some $E \subset \gamma$ with $\Lambda(E) > 0$. By Theorem 6.1 (i), we now see that $(e^{\Psi}\Phi)_{|\partial\Omega}$ vanishes a.e. on E, and since $e^{-\Psi}$ is bounded, by the Sobolev embedding theorem, we must have that $F_{|\partial\Omega} = 0$ a.e. on E. As F belongs to the Smirnov class $\mathcal{S}^2(\Omega)$, we deduce that $F \equiv 0$, hence $\partial u \equiv 0$ and thus $\nabla u \equiv 0$ since u is real. Therefore u is a constant, and in fact $u \equiv 0$. □

7. The anisotropic case

We consider in this section a conductivity equation of the form (11) where σ is valued in the set of real symmetric matrices and the ellipticity condition (13) is replaced by

$$c\, I_n \leq \sigma \leq c^{-1}\, I_n \quad \text{for some constant } c \in (0, +\infty), \tag{69}$$

with I_n the identity matrix of order n. Isotropic equations correspond to the case where the image of σ consists of scalar matrices; otherwise the conduction is said to be anisotropic. Existence and uniqueness of solutions to the Neumann problem and the forward Robin problem proceed as before provided that the normal derivative gets replaced by $n.\sigma\nabla u$. Questions about uniqueness of the Robin coefficient for the inverse problem may be raised as in Section 3, namely: *given* $0 \not\equiv g \in L^2(\Gamma_0)$ *and* u *the solution to the forward Robin problem with Robin coefficient* $\lambda \in L^\infty_+(\Gamma)$*, subject to the boundary condition* $n.\sigma\nabla u = g$*, does the knowledge of* $u_{|\Gamma_0}$ *determine* λ *uniquely?*
Of course when $n \geq 3$, uniqueness cannot prevail in general as we saw it may not even hold for the ordinary Laplacian. But if $n = 2$ uniqueness does hold: this follows from Theorem 4.2 and the fact that an anisotropic equation on a bounded Lipschitz domain with $W^{1,r}$ coefficients, $r > 2$, is the diffeomorphic image of an isotropic one (on another Lipschitz domain). More precisely, if Θ is a diffeomorphism of $\mathbb{R}^2$ of class C^1 and if we set $\Omega_1 = \Theta(\Omega)$,

a computation shows (see *e.g.* [10]) that u solves for (11) in $W^{1,2}(\Omega)$ if and only if $v = u \circ \Theta^{-1}$ solves for $\nabla \cdot (\tilde{\sigma}\, \nabla v) = 0$ in $W^{1,2}(\Omega_1)$, where

$$\tilde{\sigma}(\Theta(z)) = \frac{1}{|D\Theta(z)|} D\Theta(z)\sigma(z)D\Theta^t(z) \tag{70}$$

and $|D\Theta|$ indicates the determinant of the Jacobian matrix $D\Theta$. Moreover, using a subscript 1 for the unit tangent and normal vectors to Ω_1, it holds by construction that $\partial_{\tau_1} v \circ \Theta = \partial_\tau u/|D\Theta\tau|$, and from the weak formulation of the Neumann problem we get that $(n_1.\tilde{\sigma}\nabla v) \circ \Theta = n.\sigma\nabla u/|D\Theta\tau|$.
Now, since $\sigma = (\sigma_{ij})$ has entries in $W^{1,r}(\Omega)$ and satisfies (69), we can extend it into a symmetric matrix-valued function with entries in $W^{1,r}_{loc}(\mathbb{R}^2)$ meeting (69) and equal to I_2 outside of a compact set; this only requires the extension theorem, continuity of $W^{1,r}$-functions when $r > 2$, and a smooth partition of unity. Denoting this extension by σ again, define the complex function $\mu_1 = (-\sigma_{11} + \sigma_{2,2} - 2i\sigma_{12})/(\sigma_{11} + \sigma_{22} + 2\sqrt{|\sigma|})$. As μ_1 is compactly supported and $|\mu_1| < C < 1$, the solution Θ to the Beltrami equation $\bar{\partial}\Theta = \mu_1 \partial\Theta$ which is $z + O(1/z)$ at infinity is a homeomorphism of $\mathbb{C}$ of class $W^{2,r}_{loc}$ (a fortiori it is C^1-smooth) and $\tilde{\sigma}$ given by (70) satisfies $\tilde{\sigma} = |\sigma \circ \Theta^{-1}|^{1/2}$ [49], see also [50] where this technique was initiated for smoother coefficients and the nice exposition in [10] which deals with bounded coefficients (but less smooth Θ).
Because $\Omega_1 = \Theta(\Omega)$ is Lipschitz and the scalar-valued function $\tilde{\sigma}$ satisfies (12) and (13) (with $n = 2$), we can apply Theorem 4.2 to v on Ω_1. Thus, we deduce from the relations between u and v that if $u \not\equiv 0$ then u and $n.\sigma\nabla u$ cannot vanish together on a subset of positive measure of $\partial\Omega$. The proof of Theorem 4.1 can now be repeated to give us:

Corollary 7.1. *Theorem 4.1 still holds in the anisotropic case when σ is real symmetric 2×2-valued with entries in $W^{1,r}(\Omega)$ and meets (69), $r > 2$, provided the normal derivative $\partial_n u$ in (20) gets replaced by $n.\sigma\nabla u$.*

8. Concluding remarks

In the notation of (19), stable determination with respect to $u_{|\Gamma_0}$, of a smooth Robin coefficient λ has been studied in [7, 17, 20, 22, 47] for the Laplace equation. When $n = 2$, the factorization of ∂u given in Theorem 6.1 may help dealing with this issue for more general conductivities and less smooth λ.

We further mention that stability of the Cauchy problem in dimension 2, for general anisotropic conductivity equations with bounded conductivity, has been extensively studied in [8] using tools from complex analysis.
In this connection, we point out that the factorization technique of Lemma 6.1 enjoys some generalization to the anisotropic case which rests on the method of isothermal coordinates that we recalled in Section 7. This suggests a research path worth exploring when dealing with stability for Sobolev-smooth conductivities.
It is also natural to ask whether results from the present paper remain valid when σ is merely bounded. At present, the derivation of a factorization for ∂u requires some smoothness and it is not even clear if it holds for $\sigma \in L^\infty \cap W^{1,r}$ when $r < 2$. The case where $\sigma \in W^{1,2}(\Omega)$ deserves special mentioning: although the equation may no longer be strictly elliptic and solutions need not even be locally bounded, it is possible to make sense out of the Dirichlet problem for L^p data and the factorization $\partial u = e^\Psi \Phi$ still holds in slightly modified form where Ψ is pure imaginary on $\partial\Omega$ [11]. Therefore we expect some of our theorems to remain valid, at least if Ω is smooth.
Yet another generalization concerns with complex-valued σ, which arise in impedance tomography [21]. In this case (52) becomes a system of complex equations, and the factorization of ∂u has apparently not been investigated.
Since the negative result of [14], weaker unique continuation issues have been raised in dimension 3 and higher. One of them is: does a harmonic function in Ω, the trace of which vanishes on a non-empty open subset $\mathcal{O}$ of $\partial\Omega$ and whose normal derivative vanishes on a subset of positive measure in $\mathcal{O}$, have to vanish identically? This question is still open in general, and we refer the reader to [3, 37] for advances on the subject. In the setting of Robin inverse problems, the issue raised in Remark 4.1 as to whether $\partial_n u/u$ can remain non-negative and bounded in a neighborhood of a set of positive measure where u, $\partial_n u$ both vanish, seems to be more relevant and deserves further study.
Finally, we did not touch upon multiply connected domains Ω, where similar uniqueness properties can be proved.

Acknowledgments

The authors thank the reviewer for his constructive remarks.

[1] R. Adams. *Sobolev spaces.* Academic Press, 1975.

[2] R. Adams and J. Fournier. *Sobolev spaces*. Academic Press, 2003.

[3] V. Adolfsson and L. Escauriaza. $C^{1,\alpha}$ domains and unique continuation at the boundary. *Comm. Pure Appl. Math.*, 50(10):935–969, 1997.

[4] L.V. Ahlfors. *Lectures on quasiconformal mappings*. Cole Advanced Books and Software. Wadsworth and Brooks, 1987.

[5] A.B. Aleksandrov, J. Bourgain, M. Giesecke, V. Havin, and Yu. Vymenets. Uniqueness and free interpolation for logarithmic potentials and the Cauchy problem for the Laplace equation in $\mathbb{R}^2$. *Geom. Funct. Anal.*, 5(3):529–571, 1995.

[6] A.B. Aleksandrov and P.P. Kargaev. Hardy classes of functions that are harmonic in a half-space. *Algebra i Analiz*, 5(2):1–73, 1993. transl. in St Petersburg Math. J., 5(2), pp. 229-286, 1994.

[7] G. Alessandrini, L. Del Piero, and L. Rondi. Stable determination of corrosion by a single electrostatic boundary measurement. *Inverse Problems*, 19(4):973–984, 2003.

[8] G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella. The stability for the Cauchy problem for elliptic equations. *Inverse Problems*, 25(12):123004 (47 p), 2009.

[9] K. Astala, T. Iwaniec, and G. Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*, volume 48 of *Princeton Mathematical Series*. Princeton University Press, 2009.

[10] K. Astala, L. Païvärinta, and M. Lassas. Calderòn's inverse problem for anisotropic conductivity in the plane. *Communications in Partial Differential Equations*, 30(1-2):207–224, 2005.

[11] L. Baratchart, S. Borichev, and S. Chaabi. Pseudo-holomorphic functions at the critical exponent. To appear in J. European Math. Soc. (arXiv:1309.3079, hal-00824224v2), 2016.

[12] L. Baratchart, Y. Fischer, and J. Leblond. Dirichlet/Neumann problems and Hardy classes for the planar conductivity equation. *Complex Var. & Elliptic Equ.*, 59(4):504–538, 2014.

[13] L. Baratchart, J. Leblond, S. Rigat, and E. Russ. Hardy spaces of the conjugate Beltrami equation. *J. Functional Analysis*, 259:384–427, 2010.

[14] J. Bourgain and T. Wolff. A remark on gradients of harmonic functions in dimension ≥ 3. *Colloq. Math.*, 60/61(1):253–260, 1990.

[15] H. Brezis. *Analyse fonctionnelle*. Masson, 1987.

[16] G. Buttazzo and R.V. Kohn. Reinforcement by a thin layer with oscillating thickness. *Appl. Math. Optim.*, 16(3):247–261, 1987.

[17] S. Chaabane, I. Fellah, M. Jaoua, and J. Leblond. Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems. *Inverse Problems*, 20(1):47–59, 2004.

[18] S. Chaabane, J. Ferchichi, and K. Kunisch. Differentiability properties of the L^1-tracking functional and application to the Robin inverse problem. *Inverse Problems*, 20(4):1083–1097, 2004.

[19] S. Chaabane, J. Ferchichi, and K. Kunisch. Differentiability properties of the L^1-tracking functional and application to the Robin inverse problem. Technical Report University of Graz, 285, 2004.

[20] S. Chaabane and M. Jaoua. Identification of Robin coefficients by the means of boundary measurements. *Inverse Problems*, 15(6):1425–1438, 1999.

[21] M. Cheney, D. Isaacson, and J.C. Newell. Electrical impedance tomography. *SIAM Rev.*, 41:85–101, 1999.

[22] J. Cheng, M. Choulli, and J. Lin. Stable determination of a boundary coefficient in an elliptic equation. *Math. Models Methods Appl. Sci.*, 18(1):107–123, 2008.

[23] D. Colton and A. Kirsch. The determination of the surface impedance of an obstacle from measurements of the far field pattern. *SIAM J. Appl. Math.*, 41(1):8–15, 1981.

[24] B.E.J. Dahlberg. Weighted norm inequalities for the Lusin area integral and the nontangential maximal functions for functions harmonic in a Lipschitz domain. *Studia Math.*, 67:297–314, 1980.

[25] F. Demengel and G. Demengel. *Espaces fonctionnels. Utilisation dans la résolution des équations aux dérivées partielles.* EDP Sciences, 2007.

[26] P.L. Duren. *Theory of H^p spaces.* Academic-Press, 1970.

[27] E.B. Fabes, M.Jr. Jodeit, and N.M. Rivière. Potential techniques for boundary value problems on C^1-domains. *Acta Math.*, 141:165–186, 1978.

[28] J.B. Garnett. *Bounded analytic functions.* Springer, 2007.

[29] J.B. Garnett and D.E. Marshall. *Harmonic measure.* Number 2 in New Math. Monographs. Cambridge University Press, 2005.

[30] N. Garofalo and F.-H. Lin. Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J.*, 35:245–268, 1986.

[31] P. Grisvard. *Elliptic problems in non-smooth domains.* Pitman, 1985.

[32] K. Hrbacek and T. Jech. *Introduction to set theory*, volume 220 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker Inc., third edition, 1999.

[33] G. Inglese. An inverse problem in corrosion detection. *Inverse Problems*, 13(4):977–994, 1997.

[34] D.A. Jerison and C.E. Kenig. Hardy spaces, A_∞ and singular integrals on chord-arc domains. *Math. Scand.*, 50:221–247, 1982.

[35] D.A. Jerison and C.E. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. of Functional Analysis*, 130:161–219, 1995.

[36] C.E. Kenig. Weighted H^p spaces on Lipschitz domains. *American Journal of Mathematics*, 102(1):129–163, 1980.

[37] C.E. Kenig and W. Wang. A note on boundary unique continuation for harmonic functions in non-smooth domains. *Potential Anal.*, 8(2):143–147, 1998.

[38] E. Kenig, Carlos and Jill Pipher. The Neumann problem for elliptic equations with non-smooth coefficients. *Invent. Math.*, 113:447–509, 1993.

[39] L. Lanzani and Z. Shen. On the Robin boundary condition for Laplace's equation in Lipschitz domains. *Communications in Partial Differential Equations*, 29(1 & 2):91–109, 2004.

[40] L. Lanzani and E.M. Stein. Szegő and Bergman projections on non-smooth planar domains. *J. Geometric Analysis*, 14(1), 2004.

[41] J.-L. Lions. Sur les espaces d'interpolation; dualité. *Math. Scand.*, 9:147–177, 1960.

[42] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et application*, volume 1. Dunod, 1968.

[43] J. Necǎs. *Les méthodes directes en théorie des équations elliptiques.* Masson, 1967.

[44] Ch. Pommerenke. *Boundary behaviour of conformal maps.* Springer-Verlag, 1992.

[45] W. Rudin. *Real and complex analysis.* McGraw-Hill Book Co., third edition, 1987.

[46] L. Schwartz. *Théorie des distributions.* Hermann, 1978.

[47] E. Sincich. Lipschitz stability for the Robin problem. *Inverse Problems*, 23:1311–1326, 2007.

[48] E.M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Univ. Press, 1970.

[49] Z. Sun and G. Uhlmann. Anisotropic inverse problems in two dimensions. *Inverse Problems*, 19:1001–1010, 2003.

[50] J. Sylvester. An anisotropic inverse boundary value problem. *Comm. Pure Appl. Math.*, 43(2):201–232, 1990.

[51] I.N. Vekua. *Generalized analytic functions.* Addison-Wesley, 1962.

[52] G. Verchota. Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. of Functional Analysis*, 59:572–611, 1984.

[53] W. Wang. A remark on gradients of harmonic functions. *Rev. Mat. Iberoamericana*, 11(2):227–245, 1995.

[54] T.H. Wolff. *Essays on Fourier analysis in honour of Elias M. Stein*, chapter 15. Princeton Univ. Press, 1995.

[55] T.H. Wolff. Counterexamples to two variants of the Helson-Szegö theorem. *J. d'Analyse Math.*, 88:41–62, 2002.

[56] W.P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, 1989.

[57] M. Zinsmeister. Courbes de Jordan vérifiant une condition corde-arc. *Ann. Inst. Fourier*, 32:13–21, 1982.

[58] J.-P. Zolesio. Multiplication dans les espaces de Besov. *Proc. Royal Soc. Edimburgh*, 78A:113–117, 1977.