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Identification of individual demands from market data under uncertainty*

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Abstract

We show that, even under incomplete markets, the equilibrium manifold identifies individual demands everywhere in their domains. Under partial observation of the manifold, we determine maximal subsets of the domains on which identification holds. For this, we assume conditions of smoothness, interiority and regularity. It is crucial that there be date-zero consumption. As a by-product, we develop some duality theory under incomplete markets.

Keywords: Identification; Equilibrium manifold; Incomplete Markets.

JEL classification numbers: D52, D12

Under the hypothesis of general equilibrium, the aggregate demand function cannot be assumed to be observed: while at equilibrium prices aggregate demand is, by definition, equal to aggregate endowment, demand, either individual or aggregate, cannot be observed for out-of-equilibrium prices. One can observe, however,

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equilibrium prices and individual incomes. In this paper we address the problem of identifying aggregate and individual demands from the equilibrium manifold of a two-period economy with financial markets.

For the standard Arrow-Debreu model, positive results have been obtained by Balasko (2004), Chiappori et al. (2002 and 2004), Matzkin (2005) and Carvajal and Riascos (2005). Balasko has been criticized for making very strong observational assumptions: that one can observe equilibrium prices in situations in which the endowment is zero for all individuals but one. Under regularity assumptions, Chiappori et al. obtain local identification of individual demands, but their argument has been criticized by Balasko, who has pointed out that it requires extreme smoothness assumptions. Matzkin determines the largest class of fundamentals for which identification is possible, by excluding translations of the income expansion paths of individual demands. Carvajal and Riascos obtains local (maximal) and global identification of individual demands, by combining the methods of Balasko and Chiappori et al. in a way that avoids their weaknesses: it does not use boundary information, nor does it require analyticity of preferences.

The case of uncertainty is more cumbersome. Under the assumption of additively separable preferences, Kübler et al. (2002) extend the results of Chiappori et al. (2002): they use the implicit function theorem to identify individual demands (locally) from the equilibrium correspondence, and then use Geanakoplos and Polemarchakis (1990) to identify preferences from individual demand functions.[1]

We extend the results of Carvajal and Riascos (2005) -and hence of Balasko (2004) and Chiappori et al. (2002)- to the case of uncertainty. We assume an economy with numèraire assets and show that we can identify individual demands locally. As a corollary, it follows that identification holds globally, if there is global equilibrium information. We extend the idea of Balasko (2004) on how to recover the aggregate demand function from the equilibrium manifold to the case of (possibly incomplete) asset markets, hence we avoid using the implicit function theorem. We then use a slightly different argument from Kübler et al. (2002) to identify individual demands from the aggregate demand function and we also avoid the strong observational assumption of Balasko. In contrast to Kübler et al., our results do not assume that preferences are additively separable across states. If that assumption is made, however, our result suffices to imply, by Geanakoplos and Polemarchakis (1990), that the equilibrium manifold locally identifies individual preferences. As a necessary by-product, we develop some basic duality theory for incomplete markets.[2]

The paper is organized as follows. The next section defines the economy and its equilibria, and introduces the basic assumptions. After that, we introduce the concepts of identification: we first define identification of the aggregate demand from the equilibrium manifold, and then define identification of the profile of individual demands from the aggregate demand. The third section is auxiliary: it gives an alternative setting for the problem and extends the main results of Carvajal and Riascos (2005) to that setting. Section 4 then exploits the auxiliary results to obtain identification of aggregate and individual demands from the equilibrium manofold. The paper contains two appendices: one for the more technical arguments, in the form of lemmata, and one where we extend the standard duality theory to the case of a consumer facing incomplete financial markets.

1 The economy and the concept of equilibrium

1.1 A two-period economy with numèraire assets

Consider the canonical two-period exchange economy with financial assets. In the second period, there are S states of nature that can realize, $s = 1, \ldots, S$, while state $s = 0$ is used to denote the first period of the economy. There are $L \geq 2$ commodities, $l = 1, \ldots, L$, that can be consumed in nonnegative amounts in each state of nature (including the first period of the economy); we use the notation $N = L(S + 1)$, so that we can define the commodity space simply as $\mathbb{R}^N_+$.

There is a finite number of individuals, which we denote by $i = 1, \ldots, I$. Individual i has preferences over consumption plans, which are represented by the utility function $u^i : \mathbb{R}^N_+ \to \mathbb{R}$. Throughout the paper, we restrict preferences to lie within a basic class, according to the following assumption:

Assumption 1. *Each utility function u^i is continuous, monotone and strongly quasi-concave, and has interior upper-contour sets: for all $x \in \mathbb{R}^N_{++}$, $\{\tilde{x} \in \mathbb{R}^N_+ | u^i(\tilde{x}) \geq u^i(x)\} \subseteq \mathbb{R}^N_{++}$.*[3]

There is a finite number, J, of numèraire assets in the economy, which we index by $j = 1, \ldots, J$. Asset j is a contract that promises delivery of an amount v_s^j of commodity 1, contingent on the realization of state of nature $s = 1, \ldots, S$; formally, the asset is a vector $v^j \in \mathbb{R}^S$, so the matrix of income transfers expressed in units of commodity 1 is

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_S \end{pmatrix} = \begin{pmatrix} v_1^1 & \cdots & v_1^J \\ \vdots & \ddots & \vdots \\ v_S^1 & \cdots & v_S^J \end{pmatrix}.$$

The price of commodity l in state s is $p_{s,l}$, the vector of commodity prices at state s is $p_s = (p_{s,l})_{l=1}^L$, and we denote by $p = (p_s)_{s=0}^S$, the vector of commodity prices across states. For simplicity, future commodity prices are denoted by $p_{\mathbf{1}} = (p_s)_{s=1}^S$. The matrix of (nominal) income transfers is, therefore,

$$V(p_{\mathbf{1}}) = \begin{pmatrix} V_1(p_{1,1}) \\ \vdots \\ V_S(p_{S,1}) \end{pmatrix} = \begin{pmatrix} p_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{S,1} \end{pmatrix} V,$$

and the space of income transfers is the column span of the matrix of income transfers, $\langle V(p_{\mathbf{1}}) \rangle$. By our assumption that assets pay only in one commodity, for strictly positive commodity prices the dimension of the space of income transfers is always equal to the rank of matrix V. For simplicity, we impose the following condition:

Assumption 2. *There are no redundant assets: matrix V has full column rank.*

1.2 Equilibrium in financial markets

Let $q \in \mathbb{R}^J$ be the price vector at which assets can be bought in the first period of the economy. For an individual with state-contingent endowment of commodities $w^i = (w_s^i)_{s=0}^S$, which we assume to be strictly positive, define the budget set

$$\mathbf{B}(p, q, w^i) = \{x | \exists z \in \mathbb{R}^J : p_0 \cdot (x_0 - w_0^i) \leq -qz \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}^i) = V(p_{\mathbf{1}})z\},$$

where $x_{\mathbf{1}} = (x_s)_{s=1}^S$ and $w_{\mathbf{1}}^i = (w_s^i)_{s=1}^S$ are future consumption and wealth, respectively, and the notation

$$\rho \boxdot \Delta = \begin{pmatrix} \rho_1 \cdot \Delta_1 \\ \vdots \\ \rho_S \cdot \Delta_S \end{pmatrix}$$

stands for any pair of vectors ρ and Δ in $\mathbb{R}^{LS}$.

Since there are $S+1$ degrees of nominal indeterminacy, we normalize commodity prices, in each state, to lie in $\mathcal{S}^{L-1} = \{p_s \in \mathbb{R}^L_{++} | p_{s,1} = 1\}$, so that commodity 1 acts as numèraire in all states. In this space of prices, it is immediate that the matrix of nominal asset returns is simply V.

A vector of asset prices q allows no arbitrage opportunities if $Vz > 0$ implies $q \cdot z > 0$. With strictly positive commodity prices, it is well known that no-arbitrage is a necessary and sufficient condition for the existence of a maximizer of a continuous and monotone utility function over $\mathbf{B}(\cdot)$. Also, it is well known that q is a no-arbitrage price vector if, and only if, $\pi V = q$ for some $\pi \in \mathbb{R}^S_{++}$.

Let $\mathcal{Q}$ denote the set (positive cone) of no-arbitrage price vectors. We define individual demand functions

$$\mathbf{f}^i : (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}^N_{++} \to \mathbb{R}^N_{++},$$

by

$$\mathbf{f}^i(p, q, w) = \text{argmax} u^i(x) : x \in \mathbf{B}(p, q, w),$$

and the aggregate demand function

$$\mathbf{F} : (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++} \to \mathbb{R}^N_{++}$$

by $\mathbf{F}(p, q, w) = \sum_i \mathbf{f}^i(p, q, w^i)$.

An economy is completely defined by a profile of preferences, a profile of endowments, and a financial market. A financial markets equilibrium for an economy is a triple consisting of an allocation of consumption plans, a vector of commodity prices and a vector of asset prices, such that all consumption plans are individually rational at the given prices, and all markets clear: : given a profile of preferences $u = (u^1, \ldots, u^I)$ and a financial market V, the **financial markets equilibrium manifold** (henceforth simply the equilibrium manifold) is

$$\mathbf{M} = \{(p, q, w) | \mathbf{F}(p, q, w) = \sum_i w^i\}.$$

2 The concepts of identification

We assume that all the individual preferences and the financial market are invariant, and study the response of equilibrium prices to variations on individual endowments.

We also assume that some subset of the equilibrium manifold is observed. We study whether unobserved variables can be uniquely determined (identified) from the observed subset of the manifold.[4] Since, under assumptions 2 and 1, equilibrium exists for every profile of preferences and endowments, our observational assumption is not vacuous.

We first study whether the equilibrium manifold identifies aggregate demand, and whether the aggregate demand identifies the profile of individual demands.

2.1 Identification of the aggregate demand from the equilibrium manifold

Our goal is to prove that for any function that *could be* the aggregate demand, given that it satisfies the properties of an aggregate demand and is consistent with market clearing on the observed equilibria, it is true that it is equal to the *actual* aggregate demand (over some part of its domain, at least).

For a function that maps prices and a profile of (strictly positive) individual endowments into quantities of all commodities in all states

$$\mathbf{\Phi} : (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++} \to \mathbb{R}^{N}_{++},$$

define the following properties:

1. it **satisfies Walras's law and financial feasibility** if for every triple (p, q, w) there exists a portfolio of assets z such that

$$p_0 \cdot (\mathbf{\Phi}_0(p, q, w) - \sum_i w^i_0) = -qz$$

and

$$p_{\mathbf{1}} \boxdot (\mathbf{\Phi}_{\mathbf{1}}(p, q, w) - \sum_i w^i_{\mathbf{1}}) = Vz;$$

2. it **satisfies budget determinacy** if $\mathbf{\Phi}(p, q, w) = \mathbf{\Phi}(\tilde{p}, \tilde{q}, \tilde{w})$ whenever $\mathbf{B}(p, q, w^i) = \mathbf{B}(\tilde{p}, \tilde{q}, \tilde{w}^i)$ for every individual i;

3. it **is consistent with E ⊆ M**, if

$$\{(p, q, w) | \mathbf{\Phi}(p, q, w) = \sum_i w^i\} \supseteq \mathbf{E}.$$

The intuition is that function $\mathbf{\Phi}$ is a "candidate" to be (identified as) the aggregate demand of the economy, and these properties are requirements that one such candidate must satisfy. Walras's law, financial feasibility and budget determinacy are immediate implications of the definition of an aggregate demand (for the class of preferences under consideration); function $\mathbf{\Phi}$ is said to be **admissible** if it is continuous and satisfies these properties. The last property is one of consistency with observed data: if one has observed a set of equilibria $\mathbf{E}$, which is a subset of the equilibrium manifold defined by the profile of preferences u and the financial market V, it should not be that some markets do not clear (according to the candidate $\mathbf{\Phi}$).

By condition 1, the real aggregate demand function, $\mathbf{F}$, is admissible and is consistent with any $\mathbf{E} \subseteq \mathbf{M}$. We will say that a subset of the manifold identifies aggregate demand, over some subset of its domain, if all the candidates that satisfy the admissibility and consistency conditions coincide with the real aggregate demand over the given subset of their domains: we say that $\mathbf{E} \subseteq \mathbf{M}$ **identifies F over**

$$\mathbf{D} \subseteq (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++},$$

if $\mathbf{\Phi}_{|\mathbf{D}} = \mathbf{F}_{|\mathbf{D}}$ for every admissible function $\mathbf{\Phi}$ that is consistent with $\mathbf{E}$.

2.2 Identification of individual demands from the aggregate demand

Again, our goal is to show that any profile of functions that *could be* the individual demands of the economy must be equal to the *actual* profile of demands, at least over some part of their domains. For a profile of functions

$$\phi^i : (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}^N_{++} \to \mathbb{R}^N_{++}, \text{ for } i = 1, \dots, I,$$

define the following, analogous, conditions:

1. the profile **satisfies Walras's law and financial feasibility** if for every individual i and every triple (p, q, w^i), there exists a portfolio of assets z^i such that

$$p_0 \cdot (\phi^i_0(p, q, w^i) - w^i_0) = -qz^i$$

and

$$p_{\mathbf{1}} \boxdot (\phi^i_{\mathbf{1}}(p, q, w^i) - w^i_{\mathbf{1}}) = V z^i;$$

2. it **satisfies budget determinacy** if $\phi^i(p, q, w^i) = \phi^i(\tilde{p}, \tilde{q}, \tilde{w}^i)$ whenever $\mathbf{B}(p, q, w^i) = \mathbf{B}(\tilde{p}, \tilde{q}, \tilde{w}^i)$;

3. it **is consistent with F over D** $\subseteq (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++}$, if $\sum_i \phi^i(p, w^i) = \mathbf{F}(p, q, w)$ at every $(p, q, w) \in \mathbf{D}$.

As before, we think of the profile of functions $(\phi^1, \ldots, \phi^I)$ as a candidate to be (identified as) the real profile of individual demands, and we say that the observation of the aggregate demand identifies individual demands, over some subsets of their domains, if all plausible candidates coincide with the real demand functions over those subsets. Again, plausibility comes from theoretical considerations and from the contrast of a candidate with observed data. The latter contrast is the consistency requirement: for the data at hand, the candidate individual demands should aggregate to the observed aggregate demand.

The theoretical considerations in this case, however, are more than before: in addition to Walras's law, financial feasibility and budget determinacy, we will need to impose a differential condition on the form of Slutsky symmetry. This, we do as follows: given a candidate profile, define associated functions

$$\varphi(p, w^i; \phi^i) = \phi^i \left((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w^i \right),$$

which have as domain the set $\{p \in \mathbb{R}^N_{++} | p_{0,1} = 1\} \times \mathbb{R}^N_{++}$; we say that the candidate profile **satisfies Slutsky symmetry** if, for every individual, it is true that $\varphi(\cdot; \phi^i) \in \mathbf{C}^3$ and that

$$\frac{\partial \varphi_{s,l}(\cdot; \phi^i)}{\partial p_{s',l'}} + (\varphi_{s',l'}(\cdot; \phi^i) - w_{s',l'}) \frac{\partial \varphi_{s,l}(\cdot; \phi^i)}{\partial w_{0,1}} = \frac{\partial \varphi_{s',l'}(\cdot; \phi^i)}{\partial p_{s,l}} + (\varphi_{s,l}(\cdot; \phi^i) - w_{s,l}) \frac{\partial \varphi_{s',l'}(\cdot; \phi^i)}{\partial w_{0,1}},$$

for every pair of commodity-state pairs $(s, l), (s', l') \neq (0, 1)$.[5]

For differentiable preferences this condition is standard when financial markets are complete, but it is not obvious when markets are incomplete. When needed, we impose the following smoothness condition on preferences:

Assumption 3. *For every individual i, $u^i \in \mathbf{C}^4(\mathbb{R}^N_{++})$ and is differentiably strictly monotone and differentiably strongly concave.*

Under this assumption, the fact that $f^i \in \mathbf{C}^3$ follows from Duffie and Shafer (1995), while the symmetry of derivatives follows from propositions 2 and 3 in Appendix 2, as we will see in theorem 5 below. Given this, we say that a profile $(\phi^1, \ldots, \phi^I)$ is **admissible** if it satisfies Walras's law, financial feasibility, budget determinacy and Slutsky symmetry.

The actual profile of demands $(\mathbf{f}^1, \ldots, \mathbf{f}^I)$ is admissible and is consistent with $\mathbf{F}$ over any set $\mathbf{D}$. Thus, given a set $\mathbf{D}$, we say that **aggregate demand over D identifies individual demands over the profile of sets**

$$\mathbf{D}^i \subseteq (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}^N_{++}, \text{ for } i = 1, \ldots, I,$$

if, for each individual i, $\phi^i_{|\mathbf{D}^i} = \mathbf{f}^i_{|\mathbf{D}^i}$ for every admissible profile of functions $(\phi^1, \ldots, \phi^I)$ that is consistent with $\mathbf{F}$ over $\mathbf{D}$.

3 Auxiliary concepts and results

The next section will contain the main results of the paper: the equilibrium manifold identifies the aggregate demand function, which in turn identifies individual demands. We obtain those results by first extending the results of Carvajal and Riascos (2005) to the case of uncertainty, in a setting in which all individual budgets are expressed in present value (using no-arbitrage considerations). Since this setting deals with prices that are not (directly) observable in real life, we consider this extension as an intermediary step for the results in the next section.

3.1 No-arbitrage equilibrium

Take p to denote present-value prices of all commodities at all states, as defined, for instance, in Magill and Shafer (1991). Given an individual's strictly positive endowment of commodities w^i, define the budget set

$$B(p, w) = \{x | p \cdot (x - w) \leq 0 \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle\}.$$

Since in this formulation there is only one degree of nominal indeterminacy, here we normalize prices to lie in $\mathcal{S}^{N-1} = \{p \in \mathbb{R}^N_{++} | p_{0,1} = 1\}$, so that the first commodity in the first period acts as numèraire of the economy. Individual i's optimal demand for commodities in this setting is $f^i(p, w^i) = \text{argmax}_{B(p,w^i)} u^i(x)$, and the aggregate demand function is $F(p, w) = \sum_i f^i(p, w^i)$.[6] The **no-arbitrage equilibrium manifold** is $M = \{(p, w) | F(p, w) = \sum_i w^i\}$.

3.2 Regularity

We will show that in this no-arbitrage setting, under a regularity assumption, aggregate demand identifies individual demands locally. Our proof is somewhat similar to the one presented by Kübler et al. (2002), but simpler: it does not require separability of preferences; it does not require us to claim uniqueness of the solution to a system of partial differential equations; and it requires a weaker regularity condition than the one used by Kübler et al.:

Assumption 4. *For every individual i, and for every pair (p, w), there exists a pair of state-commodity pairs $(s, l), (s', l') \neq (0, 1)$, such that*

$$\frac{\partial^2 F_{s,l}}{\partial (w^i_{0,1})^2}(p, w) \neq 0$$

and

$$\begin{vmatrix} \frac{\partial^2 F_{s,l}}{\partial (w^i_{0,1})^2}(p, w) & \frac{\partial^2 F_{s',l'}}{\partial (w^i_{0,1})^2}(p, w) \\ \frac{\partial^3 F_{s,l}}{\partial (w^i_{0,1})^3}(p, w) & \frac{\partial^3 F_{s'l'}}{\partial (w^i_{0,1})^3}(p, w) \end{vmatrix} \neq 0.$$

This assumption is weaker than its analogous used in Kübler et al. (2002), because it is not assumed state-by-state, and because it is not assumed for asset demands. What the assumption requires is that income effects be different enough for at least two commodity-dates, in the sense that the vectors of second and third derivatives are linearly independent. This is necessary, because it will be the combination of Slutsky symmetry and this linear independence what allows us to pin down individual demands from the variation of aggregate demand with respect to endowments.[7]

3.3 From the equilibrium manifold to aggregate demand under no arbitrage

3.3.1 Definition of identification

For any function $\Phi : \mathcal{S}^{N-1} \times \mathbb{R}^{NI}_{++} \to \mathbb{R}^{N}_{++}$, we again define the standard conditions:

1. it is **admissible** if it is continuous and satisfies:
   (a) **Walras's law**: $p \cdot \Phi(p, w) = p \cdot \sum_i w^i$;
   (b) **financial feasibility**: $p_{\mathbf{1}} \boxdot (\Phi_{\mathbf{1}}(p, w) - \sum_i w^i_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle$;
   (c) **budget determinacy**: $\Phi(p, w) = \Phi(\hat{p}, \hat{w})$ whenever $B(p, w^i) = B(\hat{p}, \hat{w}^i)$ for every individual i.
2. it **is consistent with** $E \subseteq M$, if $\{(p, w) | \Phi(p, w) = \sum_i w^i\} \supseteq E$.

Under assumption 1, function F is admissible and is consistent with any $E \subseteq M$. Hence, given $D \subseteq \mathcal{S}^{N-1} \times \mathbb{R}^{NI}_{++}$, we say that $E \subseteq M$ **identifies** F **over** D if $\Phi_{|D} = F_{|D}$ for every admissible function Φ that is consistent with E.

3.3.2 Identification results

For any $E \subseteq M$, define

$$D_E = \{(p, w) | \exists (\tilde{p}, tildew) \in E : B(p, w^i) = B(\tilde{p}, \tilde{w}^i) \text{ for all } i\}.$$

The first auxiliary generalizes the idea of Balasko (2004) to less-than-global observation under uncertainty.

Theorem 1. *Under assumptions 1 and 2,*

1. *local identification holds: set $E \subseteq M$ identifies F over any $D \subseteq \overline{D_E}$;*
2. *global identification holds: M identifies F over $\mathcal{S}^{N-1} \times \mathbb{R}^{NI}_{++}$;*
3. *if $E \subseteq M$ is compact and identifies F over D, then $D \subseteq D_E$.*

Proof. The proofs of the first two parts are as in Carvajal and Riascos (2005), theorem 2 and corollary 1, and are therefore omitted.

For the third part, the argument of theorem 4 in Carvajal and Riascos (2005) does not apply when markets are incomplete, se we include the proof here.[8] By lemma 2 in Appendix 2, D_E is closed. Define

$$\delta(p, w) = \min_{(\tilde{p}, \tilde{w}) \in E} ||(p, w) - (\tilde{p}, \tilde{w})||,$$

the *distance-to-E* function;[9] by closedness of D_E, it follows that $\delta(p, w) = 0$ if, and only if, $(p, w) \in E$. Define also

$$\mathcal{D}(p, w) = \{(\tilde{p}, \tilde{w}) \in \mathcal{S}^{N-1} \times \mathbb{R}^{NI}_{++} | B(\tilde{p}, \tilde{w}^i) = B(p, w^i) \text{ for all } i\},$$

and let

$$\Delta(p, w) = \min\{\inf_{\mathcal{D}(p,w)} \delta(\tilde{p}, \tilde{w}), 1\},$$

which is continuous (because δ is continuous), and satisfies the following properties: if $B(p, w^i) = B(\tilde{p}, \tilde{w}^i)$ for all i, then $\Delta(p, w) = \Delta(\tilde{p}, \tilde{w})$; if $\Delta(p, w) = 0$, then $(p, w) \in D_E$; [10] and if $(p, w) \in D_E$, then $\Delta(p, w) = 0$. These properties imply that function

$$\Phi(p, w) = F(p, w) + \Delta(p, w) \frac{F_{0,2}(p, w)}{2} \begin{pmatrix} p_{0,2} \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is admissible[11] and is consistent with E. Now, suppose that $D \nsubseteq D_E$ and let $(\bar{p}, \bar{w}) \in D \setminus D_E$. By identification over D, $\Phi(\bar{p}, \bar{w}) = F(\bar{p}, \bar{w})$, so $\Delta(\bar{p}, \bar{w}) = 0$ and, hence, $(\bar{p}, \bar{w}) \in D_E$, a contradiction. □

3.4 From aggregate demand to individual demands under no arbitrage

3.4.1 Definition of identification

For a profile of functions $\varphi^i : \mathcal{S}^{N-1} \times \mathbb{R}^N_{++} \to \mathbb{R}^N_{++}$, for $i = 1, \ldots, I$, define the following conditions:

1. it is **admissible** if it satisfies:

   (a) **Walras's law and financial feasibility**: $p \cdot \varphi^i(p, w^i) = p \cdot w^i$ and $p_{\mathbf{1}} \boxdot (\varphi^i_{\mathbf{1}}(p, w^i) - w^i_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle$, for every individual i;

   (b) **budget determinacy**: $\varphi^i(p, w^i) = \varphi^i(\tilde{p}, \tilde{w}^i)$ whenever $B(p, w^i) = B(\tilde{p}, \tilde{w}^i)$.

   (c) **Slutsky symmetry**: for every individual i, $\varphi^i \in \mathbf{C}^3$, and

   $$\frac{\partial \varphi^i_{s,l}}{\partial p_{s',l'}}(\cdot) + (\varphi^i_{s',l'}(\cdot) - w_{s',l'})\frac{\partial \varphi^i_{s,l}}{\partial w_{0,1}}(\cdot) = \frac{\partial \varphi^i_{s',l'}}{\partial p_{s,l}}(\cdot) + (\varphi^i_{s,l}(\cdot) - w_{s,l})\frac{\partial \varphi^i_{s',l'}}{\partial w_{0,1}}(\cdot),$$

   for every pair of commodity-state pairs $(s, l), (s', l') \neq (0, 1)$;[12]

2. it is **consistent with F over** $D \subseteq \mathcal{S}^{N-1} \times \mathbb{R}^{NI}_{++}$ if $\sum_i \varphi^i(p, w^i) = F(p, w)$ at every $(p, w) \in D$.

Profile $(f^1, \ldots, f^I)$ satisfies Walras's law, financial feasibility and budget determinacy; under assumption 3, it follows from propositions 2 and 3 in Appendix 1 that the profile also satisfies Slutsky symmetry and is, therefore, admissible. It is also immediate that $(f^1, \ldots, f^I)$ is consistent with F over any D. As before, we say that **aggregate demand over D identifies individual demands over the profile of sets**

$$D^i \subseteq \mathcal{S}^{N-1} \times \mathbb{R}^N_{++}, \text{ for } i = 1, \ldots, I,$$

if $\varphi^i_{|D^i} = f^i_{|D^i}$, for each individual i, for every admissible profile of functions $(\varphi^1, \ldots, \varphi^I)$ that is consistent with F over D.

3.4.2 Identification results

Theorem 2. *Under assumptions 1, 2, 3 and 4,*

1. *local identification holds: given D, denote, for each i,*

   $$D^i = \{(p, w^i) | \exists (\tilde{p}, \tilde{w}) \in D_0 : B(\tilde{p}, \tilde{w}^i) = B(p, w^i)\},$$

   where D_0 is the interior of D; then, aggregate demand over D identifies individual demands over $(\bar{D}^1, \ldots, \bar{D}^I)$;

2. *global identification holds: aggregate demand over the whole of its domain identifies individual demands over the whole of theirs.*

Proof. The proof follows the same argument as in Carvajal and Riascos (2005), theorems 5 and 6, so details are omitted. □

It is important to notice, however, that in the proofs of theorems 5 and 6 in Carvajal and Riascos (2005), for a given p, one does only need matrix $\Delta_{(s,l),(s',l')}$ to be nonsingular at some w^i in the relevant domain (because γ^i depends on prices only). In this sense, assumption 4 is stronger than needed.

4 Identification results

We now use the identification results obtained for the no-arbitrage formulation to get identification results in the (observable) setting of current-value commodity prices and asset prices. The technique we use is to translate objects between the two settings using an equivalence identity for budget sets: it is well known that if $p \in \mathbb{R}^N_{++}$ and $\pi = (\pi_0, \pi_{\mathbf{1}}) \in \{1\} \times \mathbb{R}^S_{++}$, then

$$\mathbf{B}(p, \pi_{\mathbf{1}} V(p_{\mathbf{1}}), w) = B((\pi_s p_s)_{s=0}^S, w).$$

By substitution, then, for every $p \in \mathcal{S}^{N-1}$ and $\pi = (\pi_0, \pi_{\mathbf{1}}) \in \{1\} \times \mathbb{R}^S_{++}$, one has that

$$\mathbf{B}((\pi_s^{-1} p_s)_{s=0}^S, \mathbf{1}^\mathsf{T} V(p_{\mathbf{1}}), w) = B(p, w).$$

An immediate implication of this equivalence is that, for all individuals, $\varphi(\cdot; \mathbf{f}^i) = f^i$. Similarly, if for every function

$$\mathbf{\Phi} : (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++} \to \mathbb{R}^N_{++},$$

we define

$$\Phi(p, w; \mathbf{\Phi}) = \mathbf{\Phi}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w),$$

then the equivalence implies that $\Phi(\cdot, \mathbf{F}) = F$.

4.1 From the equilibrium manifold to the aggregate demand

We follow the same strategy as before: we first show that the equilibrium manifold identifies the aggregate demand function.

Theorem 3. *Under assumptions 1 and 2,*

1. *local identification holds: let* $\mathbf{E} \subseteq \mathbf{M}$ *and define*

$$\mathbf{D_E} = \{(p, q, w) | \exists (\tilde{p}, \tilde{q}, \tilde{w}) \in \mathbf{E} : B(p, q, w^i) = B(\tilde{p}, \tilde{q}, \tilde{w}^i) \text{ for all } i\};$$

   then, $\mathbf{E}$ *identifies* $\mathbf{F}$ *over* $\overline{\mathbf{D_E}}$*;*

2. *global identification holds: the equilibrium manifold,* $\mathbf{M}$*, identifies* $\mathbf{F}$ *over the whole of its domain.*

Proof. For the first part, by continuity, it suffices to prove that $\mathbf{E}$ identifies $\mathbf{F}$ over $\mathbf{D_E}$. Let $\mathbf{\Phi}$ be admissible and consistent with $\mathbf{E}$. Fix $(p, q, w) \in \mathbf{D_E}$. By definition, there exists $(\tilde{p}, \tilde{q}, \tilde{w}) \in \mathbf{E}$ such that $B(p, q, w^i) = B(\tilde{p}, \tilde{q}, \tilde{w}^i)$, for all i. Define

$$E_{\mathbf{E}} = \{(p, w) | ((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) \in \mathbf{E}\}.$$

Since $q \in \mathcal{Q}$, for some $\pi \in \{1\} \times \mathbb{R}_{++}^S$ it is true that $\tilde{q} = \pi_{\mathbf{1}} V$ and, then, $((\pi_s \tilde{p}_s)_{s=0}^S, \tilde{w}) \in E_{\mathbf{E}}$. By lemmas 3 and 4 and the first part of theorem 1, it follows that $\Phi((\pi_s \tilde{p}_s)_{s=0}^S, \tilde{w}; \mathbf{\Phi}) = F((\pi_s \tilde{p}_s)_{s=0}^S, \tilde{w})$. Now,

$$\mathbf{\Phi}(p, q, w) = \mathbf{\Phi}(\tilde{p}, \tilde{q}, \tilde{w}) = \Phi((\pi_s \tilde{p}_s)_{s=0}^S, \tilde{w}; \mathbf{\Phi})$$

and

$$\mathbf{F}(p, q, w) = \mathbf{F}(\tilde{p}, \tilde{q}, \tilde{w}) = F((\pi_s \tilde{p}_s)_{s=0}^S, \tilde{w}),$$

so $\mathbf{\Phi}(p, q, w) = \mathbf{F}(p, q, w)$.

For the second part, it suffices to show that $\mathbf{D_M} \supseteq (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI}$. Fix any (p, q, w), and define $(\tilde{p}, \tilde{q}, \tilde{w}) = (p, q, (f^i(p, q, w^i))_{i=1}^I)$. It is immediate that $(\tilde{p}, \tilde{q}, \tilde{w}) \in \mathbf{M}$ and $B(p, q, w^i) = B(\tilde{p}, \tilde{q}, \tilde{w}^i)$ for every i, so $(p, q, w) \in \mathbf{D_M}$. □

We now identify the largest domain on which, given $\mathbf{E} \subseteq \mathbf{M}$, identification is possible.

Theorem 4. *Under assumptions 1 and 2, if* $\mathbf{E} \subseteq \mathbf{M}$ *is compact and identifies* $\mathbf{F}$ *over* $\mathbf{D}$*, then* $\mathbf{D} \subseteq \overline{\mathbf{D_E}}$*.*

Proof. By lemma 6, we only need to show that $\mathbf{D} \subseteq \mathbf{D_E}$. Define on $\mathcal{Q}$ the correspondence

$$\Pi(q) = \{\pi \in \{1\} \times \mathbb{R}^S_{++} | \pi_{\mathbf{1}} V = q\},$$

and let π be a continuous selection from Π.[13] Let $E_{\mathbf{E}}$ be defined as in the proof of theorem 3. By lemma 3, $E_{\mathbf{E}} \subseteq M$. Now, for every admissible (in the no-arbitrage sense) function Φ that is consistent with $E_{\mathbf{E}}$, define

$$\mathbf{\Phi}(p, q, w; \Phi) = \Phi((\pi_s(q)p_s)_{s=0}^S, w),$$

which is admissible in the setting of financial markets. Also, for $(p, q, w) \in \mathbf{E}$, by definition, one has that $((\pi_s(q)p_s)_{s=0}^S, w) \in E_{\mathbf{E}}$, so

$$\mathbf{\Phi}(p, q, w; \Phi) = \Phi((\pi_s(q)p_s)_{s=0}^S, w) = \sum_i w^i,$$

which implies that $\mathbf{\Phi}(\cdot; \Phi)$ is consistent with $\mathbf{E}$.

Since $\mathbf{E}$ identifies $\mathbf{F}$ over $\mathbf{D}$, it follows that $\mathbf{\Phi}_{|\mathbf{D}}(\cdot; \Phi) = \mathbf{F}_{|\mathbf{D}}$. Now, let

$$D = \{(p, w) | ((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) \in \mathbf{D}\},$$

and let $(p, w) \in D$. By construction,

$$\mathbf{\Phi}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w; \Phi) = \mathbf{F}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w).$$

Since $\pi_{\mathbf{1}}(\sum_{s=1}^S V_s(p_{s,1}))V = \sum_{s=1}^S V_s(p_{s,1})$,

$$\begin{aligned} B(p, w^i) &= \mathbf{B}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w^i) \\ &= \mathbf{B}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \pi_{\mathbf{1}}(\sum_{s=1}^S V_s(p_{s,1}))V, w^i) \\ &= B((\pi_s(\sum_{s=1}^S V_s(p_{s,1}))\frac{1}{p_{s,1}} p_s)_{s=0}^S, w^i), \end{aligned}$$

so

$$\begin{aligned}\Phi(p,w) &= \Phi((\pi_s(\sum_{s=1}^{S} V_s(p_{s,1}))\frac{1}{p_{s,1}}p_s)_{s=0}^{S}, w) \\ &= \mathbf{\Phi}((\frac{1}{p_{s,1}}p_s)_{s=0}^{S}, \sum_{s=1}^{S} V_s(p_{s,1}), w; \Phi) \\ &= \mathbf{F}((\frac{1}{p_{s,1}}p_s)_{s=0}^{S}, \sum_{s=1}^{S} V_s(p_{s,1}), w) \\ &= F(p,w),\end{aligned}$$

which means that $E_{\mathbf{E}}$ identifies F over D.

By part 3 of theorem 1 and lemma 6, it follows that $D \subseteq D_{E_{\mathbf{E}}}$, so if $(p,q,w) \in \mathbf{D}$ then $((\pi_s(q)p_s)_{s=0}^{S}, w) \in D_{E_{\mathbf{E}}}$, which implies that $B((\pi_s(q)p_s)_{s=0}^{S}, w^i) = B(\tilde{p}, \tilde{w}^i)$, for all i, for some $(\tilde{p}, \tilde{w}) \in E_{\mathbf{E}}$. Since $(\tilde{p}, \tilde{w}) \in E_{\mathbf{E}}$, it is also true that

$$((\frac{1}{\tilde{p}_{s,1}}\tilde{p}_s)_{s=0}^{S}, \sum_{s=1}^{S} V_s(\tilde{p}_{s,1}), \tilde{w}) \in \mathbf{E}.$$

But, then,

$$\mathbf{B}(p,q,w^i) = B((\pi_s(q)p_s)_{s=0}^{S}, w^i) = B(\tilde{p}, \tilde{w}^i) = \mathbf{B}((\frac{1}{\tilde{p}_{s,1}}\tilde{p}_s)_{s=0}^{S}, \sum_{s=1}^{S} V_s(\tilde{p}_{s,1}), \tilde{w}^i),$$

so $(p,q,w) \in \mathbf{D}_{\mathbf{E}}$. □

4.2 From aggregate demand to individual demands

We first argue that the imposition of Slutsky symmetry as one of the theoretical conditions that have to be satisfied by profiles of candidates to be identified as individual demands is correct.

Theorem 5. *Under assumptions 1 and 3, the profile of individual demand functions, $(\mathbf{f}^1, \dots, \mathbf{f}^I)$, satisfies Slustky symmetry.*

Proof. This follows from propositions 2 and 3 in Appendix 1, given the equivalence $\varphi(\cdot; \mathbf{f}^i) = f^i$. □

Theorem 6. *Under assumptions 1, 2, 3 and 4,*

1. *local identification holds: aggregate demand over* $\mathbf{D}$ *identifies individual demands over the profile of sets*

   $$\mathbf{D}^i = \{(p, q, w^i) | \exists (\tilde{p}, \tilde{q}, \tilde{w}) \in \mathbf{D}_0 : \mathbf{B}(\tilde{p}, \tilde{q}, \tilde{w}^i) = \mathbf{B}(p, q, w^i)\}, \text{ for } i = 1, \ldots, I,$$

   where $\mathbf{D}_0$ *denotes the interior of* $\mathbf{D}$*;*

2. *global identification holds: aggregate demand over the whole of its domain identifies individual demands over the whole of theirs.*

Proof. Suppose that the first part is not true. Then, we can find an admissible profile $(\phi^1, \ldots, \phi^I)$, an individual, i', and a point $(\bar{p}, \bar{q}, \bar{w}^{i'}) \in \mathbf{D}^{i'}$ such that the profile is consistent with $\mathbf{F}$ over $\mathbf{D}$, and, still, $\phi^{i'}(\bar{p}, \bar{q}, \bar{w}^{i'}) \neq \mathbf{f}^{i'}(\bar{p}, \bar{q}, \bar{w}^{i'})$. By definition, then, we can find $(\tilde{p}, \tilde{q}, \tilde{w}) \in \mathbf{D}_0$ such that $\mathbf{B}(\tilde{p}, \tilde{q}, \tilde{w}^{i'}) = \mathbf{B}(\bar{p}, \bar{q}, \bar{w}^{i'})$, which implies that $\phi^{i'}(\tilde{p}, \tilde{q}, \tilde{w}^{i'}) \neq \mathbf{f}^{i'}(\tilde{p}, \tilde{q}, \tilde{w}^{i'})$; if we let $\pi \in \{1\} \times \mathbb{R}^S_{++}$ be such that $\pi_{\mathbf{1}} V = \tilde{q}$, then,

$$\varphi((\pi_s^{-1}\tilde{p}_s)_{s=0}^S, \tilde{w}^{i'}; \phi^{i'}) \neq f^{i'}((\pi_s^{-1}\tilde{p}_s)_{s=0}^S, \tilde{w}^{i'}).$$

Consider now the profile of functions $(\varphi(\cdot; \phi^1), \ldots, \varphi(\cdot; \phi^I))$, each of which is continuously defined over $\mathcal{S}^{N-1} \times \mathbb{R}^N_{++}$ into $\mathbb{R}^N_{++}$. By lemma 7 in appendix 1, this profile is admissible and consistent with

$$D = \{(p, w) | ((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) \in \mathbf{D}\}.$$

By construction, $((\pi_s^{-1}\tilde{p}_s)_{s=0}^S, \tilde{w}) \in D_0$, which implies, by the first part of theorem 2, that

$$\varphi((\pi_s^{-1}\tilde{p}_s)_{s=0}^S, \tilde{w}^{i'}; \phi^{i'}) = f^{i'}((\pi_s^{-1}\tilde{p}_s)_{s=0}^S, \tilde{w}^{i'}),$$

a contradiction.

The second part follows from the first part, for $\mathbf{D} = (\mathcal{S}^{L-1})^{S+1} \times \mathcal{Q} \times \mathbb{R}^{LI}_{++}$. □

5 Concluding remarks

We have shown that, under the equilibrium hypothesis, enough information on how prices respond to income shocks can pin down, in a unique manner, unobserved individual information.

When there is global information about the equilibria, given unobserved preferences and observed asset structure, one can identify the aggregate demand function globally. It is a remarkable property of the competitive model that under mild assumptions the roots of a function (the aggregate excess demand) contain as much information as the function itself. Less than global knowledge will, obviously, give less comprehensive information about the aggregate demand. These results obtain as a consequence of simple properties of the model, namely (i) Walras's law, (ii) the fact that identical profiles of budget sets imply identical aggregate demand, and (iii) no-trade equilibria immediately inform about aggregate demand.

Then we show that aggregate demand can be used to recover, uniquely, individual demands. This requires that income effects be different across commodities, which in turn requires that observed domains allow for perturbations of all possible endowments, which is indeed a restrictive assumption (in particular, as it requires that some income effects do not vanish). Again, local information gives local identification, while the same is true for global information.

These results do not require that preferences be separable across states, but, if one is willing to assume that they are, then they imply that equilibrium prices contain the same information as the profile of individual preferences (by Geanakoplos and Polemarchakis, 1990). Namely, when going from preferences to equilibrium prices, we first optimize, then aggregate and then solve for market clearing; the results imply that after all these transformations we still have essentially the same information as at the beginning, which contrasts with the anything-goes intuition that was derived from the Sonnenschein-Mantel-Debreu literature.

From a more practical perspective, identification results are important for the unambiguous determination of the welfare effects of economic policy, something that is desirable (see Geanakoplos and Polemarchakis 1986) but far from obvious. Identification of Pareto improving policies has been shown to fail when price effects across commodities are unknown (see Geanakoplos and Polemarchakis 1990) and when only a finite data set is available for a nonstationary economy (see Carvajal and Polemarchakis, 2008). Also, identification, or lack thereof, in the presence of production, or for stationary economies, remains an open problem.

Appendix 1: Duality in Incomplete Markets

For the purposes of this appendix, we maintain an individual fixed, assume that her preferences satisfy assumptions 1 and 3, but ignore her superindex for simplicity.

Let U be the range of function u. For each second-period wealth $w_{\mathbf{1}}$ and a feasible utility level $\mu \in U$, define the sets

$$D(w_{\mathbf{1}}, \mu) = \{p \in \mathcal{S}^{N-1} | \exists x \in \mathbb{R}^N_{++} : u(x) = \mu \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle\}$$

and

$$\mathbf{D}(w_{\mathbf{1}}) = \{(p, m) \in \mathcal{S}^{N-1} \times \mathbb{R}_{++} | \exists x \in \mathbb{R}^N_{++} : \sum_{s=0}^{S} p_s \cdot x_s \leq m \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle\}.$$

Notice that $D(w_{\mathbf{1}}, \mu)$ is diffeomorphic to the set strictly positive, non-numéraire prices, $((p_{0,2}, \ldots, p_{0,L}), p_{\mathbf{1}}) \in \mathbb{R}^{N-1}_{++}$, for which one can find a consumption plan $x \in \mathbb{R}^N_{++}$ such that $u(x) = \mu$ and $p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}})$; moreover, by assumption 3, the implicit function theorem guarantees that the latter set is open. Also, $\mathbf{D}(w_{\mathbf{1}})$ is diffeomorphic to the set of pairs of strictly positive non-numéraire prices and nominal income levels, $(((p_{0,2}, \ldots, p_{0,L}), p_{\mathbf{1}}), m) \in \mathbb{R}^{N-1}_{++} \times \mathbb{R}_{++}$, for which there exists $x \in \mathbb{R}^N_{++}$ such that $\sum_{s=0}^{S} p_s \cdot x_s \leq m$ and $p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}})$, which is nonempty and open.

For $(w_{\mathbf{1}}, \mu)$ such that $D(w_{\mathbf{1}}, \mu) \neq \varnothing$, define the Hicksian demand function

$$h(p; w_{\mathbf{1}}, \mu) = \text{argmin} p \cdot x : u(x) \geq \mu \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle,$$

and the expenditure function $e(p; w_{\mathbf{1}}, \mu) = p \cdot h(p; w_{\mathbf{1}}, \mu)$, both over the domain $D(w_{\mathbf{1}}, \mu)$. By assumption 1, $h(p; w_{\mathbf{1}}, \mu)$ is well defined into $\mathbb{R}^N_{++}$. Also, define the conditional individual demand function

$$\tilde{f}(p, m; w_{\mathbf{1}}) = \text{argmax} u(x) : p \cdot x \leq m \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle,$$

with domain $\mathbf{D}(w_{\mathbf{1}})$.[14]

Proposition 1 (Duality)**.** *Under condition 1*

1. *for every w and every p, it is true that $u(f(p, w)) \in U$, $p \in D(w_{\mathbf{1}}, u(f(p, w)))$ and $h(p; w_{\mathbf{1}}, u(f(p, w))) = f(p, w)$;*

2. *for every $p \in D(w_{\mathbf{1}}, \mu)$, $f(p, h(p; w_{\mathbf{1}}, \mu)) = h(p; w_{\mathbf{1}}, \mu)$;*

3. *for every $p \in D(w_{\mathbf{1}}, \mu)$, $(p, e(p, w, \mu)) \in \mathbf{D}(w_{\mathbf{1}})$ and $\tilde{f}(p, e(p, w, \mu); w_{\mathbf{1}}) = h(p; w_{\mathbf{1}}, \mu)$.*

Proof. Part 1 is straightforward, by strict monotonicity of u. Given that u is continuous, for parts 2 and 3 it suffices to prove that $u(h(p;w_{\mathbf{1}},\mu))=\mu$. For this, suppose not: $u(h(p;w_{\mathbf{1}},\mu))>\mu$. Let $x=h(p;w_{\mathbf{1}},\mu)-(\epsilon,0,\ldots,0)$, where $\epsilon>0$. By construction, $x_{\mathbf{1}}=h_{\mathbf{1}}(p;w_{\mathbf{1}},\mu)$, from where $p_{\mathbf{1}}\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})\in\langle V(p_{\mathbf{1}})\rangle$, and $p\cdot x<e(p;w_{\mathbf{1}},\mu)$, whereas for ϵ small enough $x\in\mathbb{R}^N_+$ and, by continuity, $u(x)\geqslant\mu$, which is a contradiction. □

Proposition 2 (Shepard's Lemma)**.** *Under conditions 1 and 3, for every pair* $(w_{\mathbf{1}},\mu)$ *function* $e(\cdot;w_{\mathbf{1}},\mu)$ *is differentiable and* $\partial_p e(p;w_{\mathbf{1}},\mu)=h(p;w_{\mathbf{1}},\mu)$.

Proof. This is an immediate consequence of the Duality Theorem (see Mas-Colell et al., 1995, proposition 3.F.1), since set

$$\{x|u^i(x)\geq\mu \text{ and } p_{\mathbf{1}}\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})\in\langle V(p_{\mathbf{1}})\rangle\}$$

is closed and has function $e(p;w_{\mathbf{1}},\mu)$ as support. □

Under conditions 1 and 3, it can be argued in the same way as fact 5 in Duffie and Shafer (1985) that every conditional demand function $\tilde{f}(\cdot;w_{\mathbf{1}})$ is differentiable.

Proposition 3 (Slutsky Equation in incomplete markets)**.** *Assume conditions 1 and 3. Fix* $(p,w)\in\mathcal{S}^{N-1}\times\mathbb{R}^N_+$, *and let* $\mu=u(f(p,w))$. *Then,* $h(\cdot;w_{\mathbf{1}},\mu)$ *is differentiable and*

$$\frac{\partial h_{s,l}(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}}=\frac{\partial f_{s,l}(p,w)}{\partial p_{s',l'}}+\frac{\partial f_{s,l}(p,w)}{\partial w_{0,1}}(f_{s',l'}(p,w)-w_{s',l'})$$

for every pair of commodity-state pairs $(s,l),(s',l')\neq(0,1)$.

Proof. That $h(\cdot;w_{\mathbf{1}},\mu)$ is differentiable follows from proposition 1, since $\tilde{f}(\cdot;w_{\mathbf{1}})$ is differentiable. Also from proposition 1, we have that $h(p;w_{\mathbf{1}},\mu)=\tilde{f}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})$, which implies that

$$\frac{\partial h_{s,l}(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}}=\frac{\partial\tilde{f}_{s,l}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial p_{s',l'}}+\frac{\partial\tilde{f}_{s,l}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial m}\frac{\partial e(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}}.$$

By proposition 2, then,

$$\frac{\partial h_{s,l}(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}}=\frac{\partial\tilde{f}_{s,l}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial p_{s',l'}}+\frac{\partial\tilde{f}_{s,l}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial m}h_{s',l'}(p;w_{\mathbf{1}},\mu),$$

while, since $f(p,w)=\tilde{f}(p,p\cdot w;w_{\mathbf{1}})$, then

$$\frac{\partial f_{s,l}(p,w)}{\partial p_{s',l'}}=\frac{\partial\tilde{f}_{s,l}(p,p\cdot w;w_{\mathbf{1}})}{\partial p_{s',l'}}+\frac{\partial\tilde{f}_{s,l}(p,p\cdot w;w_{\mathbf{1}})}{\partial m}w_{s',l'}.$$

Under monotonicity, at $\mu = u^i(f^i(p, w))$, one has that $e(p; w_{\mathbf{1}}, \mu) = p \cdot w$ and, hence, that

$$\frac{\partial f_{s,l}(p, w)}{\partial p_{s',l'}} = \frac{\partial \tilde{f}_{s,l}(p, e(p; w_{\mathbf{1}}, \mu); w_{\mathbf{1}})}{\partial p_{s',l'}} + \frac{\partial \tilde{f}_{s,l}(p, e(p; w_{\mathbf{1}}, \mu); w_{\mathbf{1}})}{\partial m} w_{s',l'}.$$

Substitution gives that

$$\frac{\partial h_{s,l}(p; w_{\mathbf{1}}, \mu)}{\partial p_{s',l'}} = \frac{\partial f_{s,l}(p, w)}{\partial p_{s',l'}} + \frac{\partial \tilde{f}_{s,l}(p, e(p; w_{\mathbf{1}}, \mu); w_{\mathbf{1}})}{\partial m} (h_{s',l'}(p; w_{\mathbf{1}}, \mu) - w_{s',l'}).$$

By proposition 1, since $\mu = u(f(p, w))$,

$$\frac{\partial h_{s,l}(p; w_{\mathbf{1}}, \mu)}{\partial p_{s',l'}} = \frac{\partial f_{s,l}(p, w)}{\partial p_{s',l'}} + \frac{\partial \tilde{f}_{s,l}(p, p \cdot w; w_{\mathbf{1}})}{\partial m} (f_{s',l'}(p, w) - w_{s',l'}).$$

Also, notice that

$$\frac{\partial f_{s,l}(p, w)}{\partial w_{0,1}} = \frac{\partial \tilde{f}_{s,l}(p, p \cdot w; w_{\mathbf{1}})}{\partial m},$$

so substitution completes the proof. □

Appendix 2: lemmata

Lemma 1. *The budget correspondence in the no-arbitrage setting, $B : \mathcal{S}^{N-1} \times \mathbb{R}^N_{++} \rightrightarrows \mathbb{R}^N_+$, is continuous.*

Proof. That B is upper hemicontinuous follows from the fact that the rank of every submatrix of $V(p_{\mathbf{1}})$ is equal to the rank of the corresponding submatrix of V. For lower hemicontinuity, notice that the correspondence defined by $B(p, w) \cap \mathbb{R}^N_{++}$ is lower hemicontinuous, and, hence, that B is lower hemicontinuous, by proposition 2.3 in Michael (1956). □

Lemma 2. *If $E \subseteq M$ is compact, then D_E is closed.*[15]

Proof. Take a sequence $(p_n, w_n)_{n=1}^\infty$ in D_E that converges to some $(p, w) \in \mathcal{S}^{N-1} \times \mathbb{R}^{NI}_{++}$. By definition, there exists a sequence $(\tilde{p}_n, \tilde{w}_n)_{n=1}^\infty$ in E such that, $B(p_n, w_n^i) = B(\tilde{p}_n, \tilde{w}_n^i)$ for all individuals. Since E is compact, there is a convergent subsequence $(\tilde{p}_{n(k)}, \hat{w}_{n(k)})_{k=1}^\infty$ that converges to some $(\tilde{p}, \tilde{w}) \in E \subseteq \mathcal{S}^{N-1} \times \mathbb{R}^{NI}_{++}$. [16]

By lower hemicontinuity of B (lemma 1), for every $x \in B(p, w^i)$ there exists a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R}^N_+$ such that, $x_n \in B(p_n, w_n^i)$ and $x_n \to x$. It follows by construction that $x_n \in B(\tilde{p}_n, \tilde{w}_n^i)$ and, hence, by upper hemicontinuity, $x \in B(\tilde{p}, \tilde{w}^i)$ and $B(p, w^i) \subseteq B(\tilde{p}, \tilde{w}^i)$. A similar analysis yields $B(\tilde{p}, \tilde{w}^i) \subseteq B(p, w^i)$, which implies that $B(\tilde{p}, \tilde{w}^i) = B(p, w^i)$. Since the latter is true for all individuals, it follows that $(p, w) \in D_E$. □

Lemma 3. *For* $\mathbf{E} \subseteq \mathbf{M}$*, let* $E_{\mathbf{E}}$ *be defined as in the proof of theorem 3. Then,* $E_{\mathbf{E}} \subseteq M$*, and, moreover,* $E_{\mathbf{M}} = M$*.*

Proof. Let $(p, w) \in E_{\mathbf{E}}$. By definition,

$$((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) \in \mathbf{E}.$$

Define $\pi = (1, (p_{s,1})_{s=1}^S)$, and notice that

$$\sum_i w^i = \sum_i \mathbf{f}^i((\pi_s^{-1} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1})), w^i) = \sum_i f^i(p, w^i) = F(p, w),$$

which shows that $(p, w) \in M$. To see that $E_{\mathbf{M}} = M$, it now suffices to show that $M \subseteq E_{\mathbf{M}}$. Let $(p, w) \in M$. By construction,

$$\begin{aligned}
\sum_i w^i &= \sum_i f^i(p, w^i) \\
&= \sum_i \mathbf{f}^i((\frac{1}{p_{s,1}} p_s)_{s=0}^S, (\sum_{s=1}^S V_s(p_{s,1})), w^i) \\
&= \mathbf{F}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, (\sum_{s=1}^S V_s(p_{s,1})), w).
\end{aligned}$$

□

Lemma 4. *For any function* $\mathbf{\Phi}$*,*

1. *if it is admissible (in financial markets), then so is* $\Phi(\cdot; \mathbf{\Phi})$ *(in the no-arbitrage setting);*

2. *if it is consistent with* $\mathbf{E} \subseteq \mathbf{M}$*, then* $\Phi(\cdot; \mathbf{\Phi})$ *is consistent with* $E_{\mathbf{E}}$*.*

Proof. For the first part, continuity and budget determinacy are immediate. Now, let $(p, w) \in \mathcal{S}^{N-1} \times \mathbb{R}^N_{++}$. Since $\mathbf{\Phi}$ satisfies Walras's law and financial feasibility, we can fix some z such that

$$p_0 \cdot (\mathbf{\Phi}_0((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) - \sum_i w_0^i) = -(\sum_{s=1}^S V_s(p_{s,1}))z.$$

and

$$(\frac{1}{p_{s,1}} p_s)_{s=1}^S \boxdot (\mathbf{\Phi}_{\mathbf{1}}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) - \sum_i w_{\mathbf{1}}^i) = Vz.$$

The latter implies that

$$p_{\mathbf{1}} \boxdot (\mathbf{\Phi}_{\mathbf{1}}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) - \sum_i w_{\mathbf{1}}^i) = V(p_{\mathbf{1}})z,$$

so summing up over s gives

$$p \cdot (\mathbf{\Phi}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) - \sum_i w^i) = 0,$$

which means that $\Phi(\cdot; \mathbf{\Phi})$ satisfies Walras's law. Now, by definition,

$$p_{\mathbf{1}} \boxdot (\Phi_{\mathbf{1}}(p, w; \mathbf{\Phi}) - \sum_i w_{\mathbf{1}}^i) = p_{\mathbf{1}} \boxdot (\mathbf{\Phi}_{\mathbf{1}}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) - \sum_i w_{\mathbf{1}}^i),$$

so $\Phi(\cdot; \mathbf{\Phi})$ satisfies financial feasibility.

For the second part, let $(p, w) \in E_{\mathbf{E}}$. It is immediate that

$$((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) \in \mathbf{E},$$

and then, since $\mathbf{\Phi}$ is consistent with $\mathbf{E}$, we have that

$$\mathbf{\Phi}((\frac{1}{p_{s,1}} p_s)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w) = \sum_i w^i.$$

□

Lemma 5. *The budget correspondence* $\mathbf{B} : (\mathcal{S}_{++}^{L-1})^(S+1) \times \mathcal{Q} \times \mathbb{R}_{++}^N \rightrightarrows \mathbb{R}_+^N$*, is continuous.*

Proof. Define the function π as in the proof of theorem 4. Upper hemicontinuity, follows again from the fact that the rank of every submatrix of $V(p_{n,\mathbf{1}})$ is constant. For lower hemicontinuity, fix (p, q, w), $x \in \mathbf{B}(p, q, w)$ and a sequence $(p_n, q_n, w_n)_{n=1}^\infty$ that converges to (p, q, w); then, $x \in B((\pi_s(q)p_s)_{s=0}^S, w)$ and $((\pi_s(q_n)p_{n,s})_{s=0}^S, w_n)$ converges to $((\pi_s(q)p_s)_{s=0}^S, w)$, so, by lower hemicontinuity of B, there exists a sequence $(x_n)_{n=1}^\infty$ such that $x_n \in B((\pi_s(q_n)p_{n,s})_{s=0}^S, w_n)$ and $x_n \to x$; it follows that $\mathbf{B}$ is lower hrmicontinuous, since $B((\pi_s(q_n)p_{n,s})_{s=0}^S, w_n) = \mathbf{B}(p_n, q_n, w_n)$. □

Lemma 6. *Given* $\mathbf{E} \subseteq \mathbf{M}$*, let* $E_{\mathbf{E}}$ *be as in the proof of theorem 3. If* $\mathbf{E}$ *is compact, then* $E_{\mathbf{E}} \subseteq M$ *is compact and* $\mathbf{D}_{\mathbf{E}}$ *is closed.*

Proof. That $E_{\mathbf{E}}$ is compact is straightforward. For closedness of $\mathbf{D_E}$, let $(p_n, q_n, w_n)_{n=1}^{\infty}$ in $\mathbf{D_E}$ converge to (p, q, w). By definition, there exists a sequence $(\tilde{p}_n, \tilde{q}_n, \tilde{w}_n)_{n=1}^{\infty}$ in $\mathbf{E}$ such that $\mathbf{B}(p_n, q_n, w_n^i) = \mathbf{B}(\tilde{p}_n, \tilde{q}_n, \tilde{w}_n^i)$ for all individuals. Since $\mathbf{E}$ is compact, some subsequence $(\tilde{p}_{n(k)}, \tilde{q}_{n(k)}, \tilde{w}_{n(k)})_{k=1}^{\infty}$ converges to some $(\tilde{p}, \tilde{q}, \tilde{w}) \in \mathbf{E}$. Let $x \in \mathbf{B}(p, q, w^i)$. By lower hemicontinuity, there exists a sequence $(x_{n(k)})_{k=1}^{\infty}$ such that $x_{n(k)} \in \mathbf{B}(p_{n(k)}, q_{n(k)}, w_{n(k)}^i)$ and which converges to x. By construction, $x_{n(k)}) \in \mathbf{B}(\tilde{p}_{n(k)}, \tilde{q}_{n(k)}, \tilde{w}_{n(k)}^i)$, and then, by upper hemicontinuity, $(x_{n(k)})_{k=1}^{\infty}$ has further subsequence that converges to some $\bar{x} \in \mathbf{B}(\tilde{p}, \tilde{q}, \tilde{w}^i)$. Since $x_{n(k)}$ itself converges to x, it follows that $x \in \mathbf{B}(\tilde{p}, \tilde{q}, \tilde{w}^i)$, and hence that $\mathbf{B}(p, q, w^i) \subseteq \mathbf{B}(\tilde{p}, \tilde{q}, \tilde{w}^i)$. That $\mathbf{B}(\tilde{p}, \tilde{q}, \tilde{w}^i) \subseteq \mathbf{B}(p, q, w^i)$ follows by a similar argument, so $\mathbf{B}(p, q, w^i) = \mathbf{B}(\tilde{p}, \tilde{q}, \tilde{w}^i))$ for all individuals, which implies that $(p, q, w) \in \mathbf{D_E}$. □

Lemma 7. *For any candidate profile of functions* $(\phi^1, \dots, \phi^I)$*,*

1. *if it is admissible (in the no-arbitrage setting), then so is the profile* $(\varphi(\cdot; \phi^1), \dots, \varphi(\cdot; \phi^I))$ *(in the financial markets setting);*
2. *if it is consistent with* $\mathbf{F}$ *over* $\mathbf{D}$*, then the profile* $(\varphi(\cdot; \phi^1), \dots, \varphi(\cdot; \phi^I))$ *is consistent with* F *over*

$$\{(p, w) | ((\frac{1}{p_{s,1}} p_s)_{s=0}^{S}, \sum_{s=1}^{S} V_s(p_{s,1}), w) \in \mathbf{D}\}.$$

Proof. For the first part, Slutsky symmetry is by definition, while the other properties are as the first part of lemma 4, so details are omitted. For the second part, notice that, over D,

$$\begin{aligned}
\sum_i \varphi(p, w^i; \phi^i) &= \sum_i \phi^i((\frac{1}{p_{s,1}} p_s)_{s=0}^{S}, \sum_{s=1}^{S} V_s(p_{s,1}), w^i) \\
&= \mathbf{F}((\frac{1}{p_{s,1}} p_s)_{s=0}^{S}, \sum_{s=1}^{S} V_s(p_{s,1}), w) \\
&= F(p, w).
\end{aligned}$$

□

Notes

[1]This is important since, when there exist uninsurable risks, competitive equilibrium is typically inefficient in a strong sense: a planner could use the existing insurance possibilities to make every individual better off (see Geanakoplos and Polemarchakis, 1986). The question immediately arises of how much information a planner needs to have in order to figure out an improving policy intervention: the transfer paradox, first pointed out by Leontief, illustrates how ambiguous the welfare effects of a policy can be when the fundamentals of the economy are unknown (see, for instance, Geanakoplos and Heal, 1983, or Donsimoni and Polemarchakis, 1994).

[2]Also obtained in independent work by Sergio Turner.

[3]This condition excludes additively separable preferences of the form $u^i(x) = \sum_s u^i_s(x_s)$. In this case, our analysis still holds under the following assumption: for every s and every sequence $(x_n)_{n=1}^{\infty}$ defined in $\mathbb{R}^L_{++}$, if it converges to some x in $\partial\mathbb{R}^L_+$, then it is true that $||Du^i_s(x_n)||^{-1}Du^i_s(x_n)\cdot x_n \to 0$ and $||Du^i_s(x_n)||^{-1} \to \infty$.

[4]So our question is one of identification and not one of testability or refutability, which has been dealt with elsewhere: for the standard Arrow-Debreu model, see Brown and Matzkin (1996); for the case of uncertainty, see Kübler (2003). For a survey of this literature, see Carvajal et al. (2004).

[5]The argument of the function and its derivatives is omitted in the expression; the condition is to hold at every $(p, w) \in \{p \in \mathbb{R}^N_{++}|p_{0,1} = 1\} \times \mathbb{R}^N_{++}$.

[6]Functions $f^i : \mathcal{S}^{N-1}\times\mathbb{R}^N_{++} \to \mathbb{R}^N_{++}$ and $F : \mathcal{S}^{N-1}\times\mathbb{R}^{NI}_{++} \to \mathbb{R}^N_{++}$ are well defined, by assumption 1 guarantees that the range of f^i is contained in $\mathbb{R}^N_{++}$.

[7]Carvajal and Riascos (2005) give an example of a demand system of rank 3 (in the sense of Blanks et al. (1997) and Lewbel (2003)) that satisfies the regularity conditions in the case of complete markets; this example can be extended to incomplete markets.

[8]In fact, for the case of complete markets the argument here strengthens the theorem of Carvajal and Riascos (2005), since closedness, rather than compactness, suffices for the result when markets are complete.

[9]Formally, we define δ only on the domain $\mathcal{S}^{N-1} \times \mathbb{R}^N I_{++}$. Since E is closed, this function is well defined.

[10]To see this, notice that if $\Delta(p, w) = 0$, then there exists a sequence $(\tilde{p}_n, \tilde{w}_n)_{n=1}^{\infty}$ such that $B(\tilde{p}_n, \tilde{w}^i_n) = B(p, w^i)$ and $\delta(\tilde{p}_n, \tilde{w}_n) \to 0$. By definition of δ, there exists a sequence $(\hat{p}_n, \hat{w}_n)_{n=1}^{\infty}$ in E, such that $\delta(\tilde{p}_n, \tilde{w}_n) = ||(\hat{p}_n, \hat{w}_n) - (\tilde{p}_n, \tilde{w}_n)||$. Since E is compact, $(\hat{p}_n, \hat{w}_n)$ has a convergent subsequence $(\hat{p}_{n(k)}, \hat{w}_{n(k)})_{k=1}^{\infty} \to (\hat{p}, \hat{w}) \in E$, and, since $\delta(\tilde{p}_n, \tilde{w}_n) \to 0$, it follows that $(\tilde{p}_{n(k)}, \tilde{w}_{n(k)}) \to (\hat{p}, \hat{w}) \in E$. By lemma 1, it follows that $B(p, w^i) = B(\hat{p}, \hat{w}^i)$ for all i, and hence that $(p, w) \in D_E$

[11]For budget determinacy, observe that if $B(p, w^i) = B(\tilde{p}, \tilde{w}^i)$ for all i, then $p_{0,2} = \tilde{p}_{0,2}$. To see this, fix some i and define $x^*_{0,1} = \max_{x\in B(p,w^i)} x_{0,1}$, $x^*_{0,2} = \max_{x\in B(p,w^i)} x_{0,2}$, $\tilde{x}^*_{0,1} = \max_{x\in B(\tilde{p},\tilde{w}^i)} x_{0,1}$ and $\hat{x}^*_{0,2} = \max_{x\in B(\tilde{p},\tilde{w}^i)} x_{0,2}$. Then, $p_{0,2} = \frac{x^*_{0,1}}{x^*_{0,2}}$ and $\tilde{p}_{0,2} = \frac{\tilde{x}^*_{0,1}}{\tilde{x}^*_{0,2}}$, and, since $B(p, w^i) = B(\tilde{p}, \tilde{w}^i)$, $x^*_{0,1} = \tilde{x}^*_{0,1}$ and $x^*_{0,2} = \tilde{x}^*_{0,2}$.

[12]Again, the condition is to hold at every $(p, w) \in \mathcal{S}^{N-1} \times \mathbb{R}^N_{++}$.

[13]For example, let $v : \mathbb{R}^N_{++} \to \mathbb{R}$ be continuous, $\mathbf{C}^1(\mathbb{R}^N_{++})$, monotone, strongly quasi-concave, differentiable strictly monotone and differentiable strictly concave, and such that for all $x \in \mathbb{R}^N_{++}$, $\{x' \in \mathbb{R}^N_+|v(x') \geq v(x)\} \subseteq \mathbb{R}^N_{++}$; let $\phi : \mathcal{Q} \to \mathbb{R}^N_{++}$ be defined by $\phi(q) = \arg\max_{x\in\mathbf{B}(\bar{p},q,\bar{w})} v(x)$, for given $(\bar{p}, \bar{w})$; and let

$$\pi(q) = ((\frac{\partial v}{\partial x_{0,1}}(\phi(q)))^{-1}\frac{\partial v}{\partial x_{s,1}}(\phi(q)))^S_{s=0}.$$

[14]Obviously, $\tilde{f}(p, p \cdot w; w_{\mathbf{1}}) = f(p, w)$.

[15]We only need E to be closed, to have closure in $\mathcal{S}^{N-1} \times \mathbb{R}_{+}^{NI}$ contained in $\mathcal{S}^{N-1} \times \mathbb{R}_{++}^{NI}$, and to have a bounded projection into the space of p.

[16]Regardless of compactness of E, if E is closed, the closure of E in $\mathcal{S}^{N-1} \times \mathbb{R}_{+}^{NI}$ is contained in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{+}^{NI}$, and the projection of E into the space of p is bounded, the conclusion follows: since (p_n, w_n) is bounded, (p_n) is bounded away from zero, and $\tilde{w}_n^i \in B(p_n, w_n^i)$, it follows that $(\tilde{w}_n)$ is itself bounded; if $(\tilde{p}_n)$ is also bounded, then there is subsequence that converges to $(\tilde{p}, \tilde{w})$. Also, the closure of E in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{+}^{NI}$ is contained in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$, so $(\tilde{p}, \tilde{w}) \in \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$, and, since E is closed (in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$), $(\tilde{p}, \tilde{w}) \in E$.

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