Some particular self-interacting diffusions: Ergodic behaviour and almost sure convergence

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SOME PARTICULAR SELF-INTERACTING DIFFUSIONS: ERGODIC BEHAVIOUR AND ALMOST SURE CONVERGENCE

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ABSTRACT. This paper deals with some self-interacting diffusions $(X_t, t \geq 0)$ living on $\mathbb{R}^d$. These diffusions are solutions to stochastic differential equations:

$$\mathrm{d}X_t = \mathrm{d}B_t - g(t)\nabla V(X_t - \overline{\mu}_t)\mathrm{d}t,$$

where $\overline{\mu}_t$ is the empirical mean of the process X, V is an asymptotically strictly convex potential and g is a given function. We study the ergodic behaviour of X and prove that it is strongly related to g. Actually, we show that X is ergodic (in the limit-quotient sense) if and only if $\overline{\mu}_t$ converges a.s. We also give some conditions (on g and V) for the almost sure convergence of X.

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1. Introduction

Processes with path-interaction have been an intensive research area since the seminal work of Norris, Rogers and Williams [13]. More precisely, self-interacting diffusions have been first introduced by Durrett and Rogers [7] under the name of Brownian polymers. They proposed a model for the shape of a growing polymer. Denoting by X_t the location of the end of the polymer at time t, X satisfies a SDE with a drift term depending on its own occupation measure (in dimension 1, we define it through the local time of X). One is then interested in rescaling X (see [5, 6, 9, 12, 15]). Later, another model of polymers has been proposed by Benaïm, Ledoux and Raimond [2]. They have studied a class of self-interacting diffusions depending on the empirical measure. When the process is living on a compact Riemannian manifold, they have proved that the asymptotic behaviour of the empirical measure can be related to the analysis of some deterministic dynamical flow defined on the space of the Borel probability measures. Benaïm and Raimond [3] went further in this study and in particular, they gave sufficient conditions for the a.s. convergence of the empirical measure. Very recently, Raimond [16] has generalized the previous work: he has studied the asymptotic properties of a process X, living on a Riemannian compact manifold M, solution to the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t - g(t)\nabla V * \mu_t(X_t)\mathrm{d}t, \tag{1.1}$$

with $V * \mu_t(x) = \frac{1}{t}\int_0^t V(x, X_s)\mathrm{d}s,\ \mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s,\ |g(t)| \leq a\log(t)$ and $g'(t) = O(t^{-\gamma})$ with $0 < \gamma \leq 1$. He has proved that, unless g is constant, the approximation of μ_t by a deterministic flow is no longer valid. He has more particularly investigated the example $M = \mathbb{S}^n$ and $V(x,y) = -\cos d(x,y)$ (where d is the geodesic distance on $\mathbb{S}^n$) and proved that a.s. μ_t converges weakly towards a Dirac measure. For an overview on reinforced processes, we refer the reader to Pemantle's survey [14].

In the present paper, we are concerned with some self-interacting processes living on $\mathbb{R}^d$. Consider a smooth potential $V : \mathbb{R}^d \to \mathbb{R}_+$ and an application $g : \mathbb{R}_+ \to \mathbb{R}_+^*$. Our goal is to study the ergodic behaviour of the self-interacting diffusion X solution to

$$\mathrm{d}X_t = \mathrm{d}B_t - g(t)\nabla V(X_t - \overline{\mu}_t)\mathrm{d}t,\ X_0 = x. \tag{1.2}$$

where B is a standard Brownian motion and $\overline{\mu}_t$ denotes the empirical mean of X:

$$\overline{\mu}_t = \frac{1}{r+t}\left(r\bar{\mu} + \int_0^t X_s\mathrm{d}s\right),\ \ \overline{\mu}_0 = \overline{\mu}. \tag{1.3}$$

Here μ is an initial (given) probability measure on $\mathbb{R}^d$, $\bar{\mu}$ denotes the mean of μ and $r > 0$ is an initial weight (it permits to consider any initial probability measure).

First, note that for a quadratic interaction potential V, the process satisfying (1.2) is exactly of the form of (1.1) and, in both cases, the occupation measure is penalized by $g(t)$. Afterwards, a natural generalization of this process is the class of self-interacting diffusions discussed here. The interesting point is that we manage to study precisely the asymptotic behaviour of X and prove a convergence criterion. Moreover, this model could be used to represent the behaviour of social insects, as the ants trails. Indeed, ants mark their paths with the trails' pheromones. Certain ants lay down an initial trail of pheromones as they return to the nest with food. This trail attracts other ants and serves as a guide. As long as the food source remains, the pheromone trail will be continually renewed. Despite the quick evaporation, the path is reinforced and so, the ants manage to gradually find the best route. In this (simplified) model, the function g reflects the speed of evaporation and X denotes the trail.

In order to study the solution to (1.2), it is natural to introduce the process Y, defined by

$$Y_t = X_t - \overline{\mu}_t. \tag{1.4}$$

It is easily seen that $(Y_t, t \geq 0)$ is the solution to the SDE

$$\mathrm{d}Y_t = \mathrm{d}B_t - g(t)\nabla V(Y_t)\mathrm{d}t - Y_t\frac{\mathrm{d}t}{r+t},\ Y_0 = x - \overline{\mu}; \tag{1.5}$$

and $\mathrm{d}\overline{\mu}_t = Y_t\frac{\mathrm{d}t}{r+t}$. As Y is a (non-homogeneous) Markov process, it is easier to study Y than X. Indeed, we will prove that Y converges a.s. and satisfies the pointwise ergodic theorem. Due to that, the behaviour of X could seem a bit easy at first glance. But it really shows unexpected behaviours and, in particular, it does not satisfy the pointwise ergodic theorem in general (because $\overline{\mu}_t$ does not converge, except for functions g going fast to infinity). This explains how difficult is the study of more general self-interacting diffusions in non-compact spaces (see Kurtzmann [10]), driven by the generic equation $\mathrm{d}X_t = \mathrm{d}B_t - \int_{\mathbb{R}^d} \nabla V(X_t, x)\mathrm{d}\mu_t(x)\mathrm{d}t$.

The remainder of the paper is organized as follows. First, we enumerate the hypotheses and state the main results in Section 2. We motivate our study, in Section 3, by the basic case when V is quadratic, for which we have an explicit expression of X (in terms of Brownian martingale). Section 4 deals with the description of the behaviour of Y near the local extrema of V. Finally, Section 5 is devoted to the proof of the main results.

2. Technical assumptions and main results

In the sequel, $(\cdot,\cdot)$ stands for the Euclidian scalar product. We also denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$.

Consider the potential $V : \mathbb{R}^d \to \mathbb{R}_+$. Let $Max = \{M_1, \dots, M_p\}$ be the (finite) set of saddle points and local maxima of V and denote by $Min = \{m_1, \dots, m_n\}$ the (finite) set of the local minima of V. So $Min \cup Max$ is the set of critical points of V. We assume that V is either quadratic (Section 3) or:

1) (*regularity and positivity*) $V \in \mathcal{C}^2(\mathbb{R}^d)$ and $V > 0$;
2) (*convexity*) $V = W + \chi$ where χ is a compactly supported function such that $\nabla\chi$ is $\tilde{C}$-Lipschitz (with $\tilde{C} > 0$) and there exists $c > 0$ such that $\nabla^2 W \geq cId$;
3) (*growth*) there exists $a > 0$ such that for all $x \in \mathbb{R}^d$, we have
$$\Delta V(x) \leq aV(x) \text{ and } \lim_{|x|\to\infty} \frac{|\nabla V(x)|^2}{V(x)} = \infty; \tag{2.1}$$
4) (*critical points*) $\forall m_i$, $\forall \xi \in \mathbb{R}^d$, $(\nabla^2 V(m_i)\xi, \xi) > 0$ and for all M_i, $\nabla^2 V(M_i)$ admits a negative eigenvalue.

Remark 2.1. *By the growth condition* (2.1)*, $|\nabla V|^2 - \Delta V$ is bounded by below.*

Suppose also that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and $g \in \mathcal{C}^1(\mathbb{R}_+)$. We denote by $g(\infty)$ the limit of $g(t)$ and we exclude the trivial case where g is identically zero, so that $g(\infty) > 0$. Let $G(t) := \int_0^t g(s)\mathrm{d}s$ and G^{-1} be its generalized inverse: $G^{-1}(t) := \inf\{u \geq 0; G(u) \geq t\}$.

Remark 2.2. *If $g(\infty) = \infty$, then for all $T > 0$, we have that $G^{-1}(t+T) - G^{-1}(t) \underset{t\to\infty}{\longrightarrow} 0$.*

The following easy result will be very useful in the sequel.

Lemma 2.3. *Suppose that $g'(t)/g^2(t)$ converges to 0. Then the following hold:*

i) for any $c > 0$, $\int_0^t s^2 e^{2cG(s)} \mathrm{d}s = O(t^2 e^{2cG(t)}/g(t))$;
ii) if $g(\infty) = \infty$, *then we have* $\int_0^t \frac{g'(s)}{g(s)^2} G(s) \mathrm{d}s = O(t)$;
iii) for $H(t) := \int_0^t \frac{e^{-cG(u)}}{(r+u)^2} \mathrm{d}u$, *the following expansion holds:*

$$H(t) = H(\infty) - \frac{1}{cg(t)(r+t)^2} e^{-cG(t)} + o\left(\frac{e^{-cG(t)}}{t^2 g(t)}\right).$$

Proof. We deduce all these estimates from an integration by parts:

$$\int_0^t s^2 e^{2cG(s)} \mathrm{d}s = \frac{t^2 e^{2cG(t)}}{2cg(t)} - \int_0^t \left(\frac{s}{g(s)} - \frac{s^2 g'(s)}{2g(s)^2}\right) \frac{e^{2cG(s)}}{c} \mathrm{d}s = O(t^2 e^{2cG(t)}/g(t)),$$

and we obtain $H(t) - H(s) = \frac{e^{-cG(s)}}{r+s} - \frac{e^{-cG(t)}}{r+t} - c\int_s^t g(u) e^{-cG(u)} \frac{\mathrm{d}u}{r+u}$. Similarly for t large enough and u such that $g(u) > 0$, we find $\int_u^t \frac{g'(s)}{g(s)^2} G(s) \mathrm{d}s = -\frac{G(t)}{g(t)} + \frac{G(u)}{g(u)} + t - u = O(t)$. □

2.1. **Existence.** We begin by proving that the SDE studied admits a unique global strong solution.

Proposition 2.4. *For any* $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}(\mathbb{R}^d)$ *and* $r > 0$, *there exists a unique global strong solution* $(X_t, t \geq 0)$ *of* (1.2).

Proof. The local existence and uniqueness of the solution to (1.2) is standard. We only need to prove here that Y, hence X (since $X_t := Y_t + \int_0^t Y_s \frac{\mathrm{d}s}{r+s}$), does not explode in a finite time. To this aim, apply Itô's formula to the function $x \mapsto V(x)$:

$$\mathrm{d}V(Y_t) = (\nabla V(Y_t), \mathrm{d}B_t) + \left(\frac{1}{2}\Delta V(Y_t) - g(t)|\nabla V(Y_t)|^2 - \frac{1}{r+t}(\nabla V(Y_t), Y_t)\right) \mathrm{d}t,$$

and introduce the sequence of stopping times $\tau_0 = 0$ and

$$\tau_n = \inf\{t \geq 0; V(Y_t) + \int_0^t g(s)|\nabla V(Y_s)|^2 \mathrm{d}s > n\}.$$

By the convexity condition, we have $(\nabla V(y), y) \underset{|y|\to+\infty}{\longrightarrow} +\infty$ and so the growth condition (2.1) implies the existence of C such that $\mathbb{E}V(Y_{t\wedge\tau_n}) \leq \mathbb{E}V(Y_0) + e^{Ct}$. □

2.2. **Results.** We give now a description of the asymptotic behaviour of both μ_t and X_t.

Definition 2.5. *The process* X *satisfies the pointwise ergodic theorem if there exists a measure* μ_∞ *such that a.s.* $\mu_t := \frac{1}{r+t}\left(r\mu + \int_0^t \delta_{X_s} \mathrm{d}s\right) \to \mu_\infty$ *for the weak convergence of measures: for all continuous bounded function* f, $\frac{1}{t}\int_0^t f(X_s) \mathrm{d}s \overset{a.s.}{\longrightarrow} \int f \mathrm{d}\mu_\infty$.

Theorem 2.6. (1) *The process Y satisfies the pointwise ergodic theorem: almost surely, the empirical measure of Y converges weakly to a measure, which is a convex combination of Dirac measures taken in the local minima of V.*
(2) *The process X satisfies the pointwise ergodic theorem if and only if the mean-process $\overline{\mu}_t$ converges almost surely.*

A necessary condition for the convergence of $\overline{\mu}_t$ is that 0 is the unique minimum of V. We will prove this result in §5.1. Indeed, what we need here is not only the convergence of Y_t to zero, but the convergence of the integral $\int_0^t Y_s \frac{\mathrm{d}s}{r+s}$, which depends on the speed of convergence of Y_t. The main result of this paper is the following description of the asymptotic behaviour of X, shown in §5.3:

Theorem 2.7. *Suppose that $\sqrt{g(t)^{-1}\log G(t)} = O(h(t)^{-1})$, where G is a primitive of g and $\int_0^\infty \frac{\mathrm{d}s}{(1+s)\,h(s)} < \infty$.*

(1) *The process Y converges almost surely to Y_∞, where Y_∞ belongs to the set of the local minima of V. For each local minimum m of V, one has $\mathbb{P}(Y_\infty = m) > 0$.*
(2) *On the set $\{Y_\infty = 0\}$, both X_t and $\overline{\mu}_t$ converge almost surely to $\overline{\mu}_\infty := \overline{\mu} + \int_0^\infty Y_s \frac{\mathrm{d}s}{r+s}$, whereas on the set $\{Y_\infty \neq 0\}$, we have that $\lim_{t\to\infty} \frac{X_t}{\log t} = Y_\infty$.*

3. A motivating example

We consider $V(x) = \frac{1}{2}(x, cx)$, where c is a symmetric positive definite matrix. Let X be the solution of the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t - g(t)\left(cX_t - \frac{r}{r+t}c\overline{\mu} - \frac{1}{r+t}\int_0^t cX_s \mathrm{d}s\right)\mathrm{d}t,\ X_0 = x. \quad (3.1)$$

Without any loss of generality, we suppose that $d = 1$. When $d \geq 1$, it suffices to diagonalize the matrix c and to remark that, for an orthogonal matrix U, the process $(U \cdot B_s, s \geq 0)$ is also a Brownian motion.

3.1. Explicit expression of X.

Lemma 3.1. *If X is the solution to (3.1), then we have*

$$Y_t := X_t - \bar{\mu}_t = \frac{e^{-cG(t)}}{r+t}\left(\int_0^t (r+s)e^{cG(s)}\mathrm{d}B_s + r(x-\overline{\mu})\right).$$

Proof. The process Y satisfies

$$\mathrm{d}Y_t = \mathrm{d}B_t - \left(cg(t) + \frac{1}{r+t}\right)Y_t\mathrm{d}t,\ \ Y_0 = x - \overline{\mu}. \quad (3.2)$$

To express Y in terms of a Brownian martingale, we consider the function of Y defined by $U_t := (r+t)e^{cG(t)}Y_t$. Then Itô's formula implies

$$\mathrm{d}U_t = (r+t)e^{cG(t)}\mathrm{d}B_t,\ U_0 = r(x-\overline{\mu}).$$

□

Corollary 3.2. *Let $F(t) = \int_0^t e^{-cG(s)} \frac{g(s)}{r+s} \mathrm{d}s$. The solution to the SDE* (3.1) *is given by*

$$X_t = x + rc(\overline{\mu} - x)F(t) + \int_0^t \left[1 - (r+s)ce^{cG(s)}(F(t) - F(s))\right] \mathrm{d}B_s.$$

Proof. Remark that $\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mu}_t = \frac{Y_t}{r+t}$. So, by Fubini's theorem for stochastic integrals (see [17] p.175), we have

$$\overline{\mu}_t = \int_0^t (r+s)e^{cG(s)}(H(t) - H(s))\mathrm{d}B_s + r(x - \bar{\mu})H(t) + \bar{\mu}$$

with $H(t) = \int_0^t \frac{e^{-cG(u)}}{(r+u)^2}\mathrm{d}u = \frac{1}{r} - cF(t) - \frac{e^{-cG(t)}}{r+t}$. As $X_t = Y_t + \overline{\mu}_t$, the latter result implies the desired expression. □

3.2. **Ergodic result.** We begin to prove the pointwise ergodic theorem for the following non-homogeneous (Gauss-)Markov process.

Proposition 3.3. *Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function, $A(t) := \int_0^t a(s)\mathrm{d}s$ and $K(t) := e^{-2A(t)} \int_0^t e^{2A(s)}\mathrm{d}s$. Suppose that $a(\infty) = \lim_{t\to\infty} a(t)$ exists and is nonzero, so that $K(\infty) = \frac{1}{2a(\infty)} < \infty$. Consider the process*

$$\mathrm{d}Z_t = -a(t)Z_t\mathrm{d}t + \mathrm{d}B_t, \;\; Z_0 = z.$$

Then, denoting by γ the centered Gaussian measure with variance $K(\infty)$ (with the convention $\gamma = \delta_0$ for $K(\infty) = 0$), we have for all continuous bounded function φ

$$\frac{1}{t}\int_0^t \varphi(Z_s)\mathrm{d}s \xrightarrow[t\to\infty]{a.s.} \int \varphi(z)\gamma(\mathrm{d}z).$$

Proof. We prove the result for the Fourier transform. First, note that

$$Z_t = e^{-A(t)} \left(\int_0^t e^{A(s)}\mathrm{d}B_s + z \right).$$

Let $\mathcal{F}_s := \sigma(B_u, 0 \le u \le s)$. Knowing $\mathcal{F}_s$, Z_t has the Gaussian distribution with mean $m(s,t) := e^{-(A(t)-A(s))}Z_s$ and variance $K(s,t) := e^{-2A(t)} \int_s^t e^{2A(u)}\mathrm{d}u$. Fix $t \in \mathbb{R}_+, u \in \mathbb{R}$ and define the martingale $M_s^{t,u} := \mathbb{E}\left(e^{iuZ_t}|\mathcal{F}_s\right) = \exp\left\{ium(s,t) - \frac{u^2}{2}K(s,t)\right\}$. Applying Itô's formula to $s \mapsto M_s^{t,u}$, we find that $\mathrm{d}M_s^{t,u} = iue^{-(A(t)-A(s))}M_s^{t,u}\mathrm{d}B_s$, and so

$$e^{iuZ_t} = \mathbb{E}e^{iuZ_t} + \int_0^t iue^{-(A(t)-A(s))}M_s^{t,u}\mathrm{d}B_s.$$

Then, by Fubini's theorem for stochastic integrals, we easily obtain

$$\int_0^t e^{iuZ_s}\mathrm{d}s = \int_0^t \mathbb{E}e^{iuZ_s}\mathrm{d}s + \int_0^t \mathrm{d}B_s \int_s^t iue^{-(A(r)-A(s))}M_s^{r,u}\mathrm{d}r. \qquad (3.3)$$

As Z_t is Gaussian with variance $K(0,t)$, it converges in distribution to a Gaussian variable of law $\gamma = \mathcal{N}(0, K(\infty))$. Because of Cesàro's result, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}e^{iuZ_s}\mathrm{d}s = e^{-\frac{u^2}{2}K(\infty)}.$$

We wish to find an asymptotic equivalent to the stochastic factor of (3.3). To this aim, consider $N^u_{s,t}(v) := \int_s^t iue^{A(v)-A(r)}M_v^{r,u}\,\mathrm{d}r$. First, on the set $\{\int_0^\infty < N^u_{\cdot,t}(s) >_s \mathrm{d}s < \infty\}$, the local martingale $\int_0^t N^u_{s,t}(s)\mathrm{d}B_s$ converges a.s. to a finite variable and so, is of the order of $o(t)$. Actually, we decompose it as

$$\int_0^t N^u_{s,\infty}(s)\mathrm{d}B_s - \int_0^t N^u_{t,\infty}(s)\mathrm{d}B_s. \tag{3.4}$$

On the set $\{\int_0^\infty < N^u_{\cdot,t}(s) >_s \mathrm{d}s = \infty\}$, the LLN for martingales implies a.s.

$$\int_0^t \mathrm{d}B_s \int_s^\infty iue^{-(A(r)-A(s))}M_s^{r,u}\mathrm{d}r = o\left(\int_0^t \left|\int_s^\infty iue^{-(A(r)-A(s))}M_s^{r,u}\mathrm{d}r\right|^2 \mathrm{d}s\right).$$

Indeed, we obtain the following upper bound by using the initial definition of $M_s^{r,u}$:

$$|N^u_{s,t}(s)| \le |u|\int_s^t e^{A(s)-A(r)}\mathrm{d}r = |u|e^{A(s)}(I_t - I_s)$$

where $I_t := \int_0^t e^{-A(r)}\mathrm{d}r = I_\infty - \frac{e^{-A(t)}}{a(t)} + o\left(\frac{e^{-A(t)}}{a(t)}\right)$, we find by the triangle inequality that $\int_0^t e^{2A(s)}(I_t - I_s)^2\mathrm{d}s = O(t)$. So we have

$$\mathbb{E}\left(\int_0^t N^u_{t,\infty}(s)\mathrm{d}B_s\right)^2 = \int_0^t \mathbb{E}(N^u_{t,\infty}(s)^2)\mathrm{d}s \le |u|^2\int_0^t e^{2A(s)}\mathrm{d}s(I_\infty - I_t)^2 = O(1).$$

Borel-Cantelli's lemma permits to conclude that $\frac{1}{t}\int_0^t N^u_{t,\infty}(s)\mathrm{d}B_s$ converges a.s. to 0. □

Theorem 3.4. *Suppose that $g'(t)/g^2(t)$ converges to 0. Then, with probability 1, the empirical measure $\mu_t = \frac{r}{r+t}\mu + \frac{1}{r+t}\int_0^t \delta_{X_s}\mathrm{d}s$ converges weakly to μ_∞. Moreover, the mean $\overline{\mu}_t = \frac{1}{r+t}\int_0^t X_s\mathrm{d}s + \frac{r}{r+t}\overline{\mu}$ also converges almost surely.*

Proof. We remind that reader that $g'(t)/g^2(t)$ converges to 0. We start by proving that $\bar{\mu}_t$ converges a.s. Decompose the process $\overline{\mu}_t = \overline{\mu}_t^1 + \overline{\mu}_t^2 + \overline{\mu}_t^3$ where

$$\begin{aligned}
\overline{\mu}_t^1 &= \overline{\mu} + r(x-\overline{\mu})H(t),\\
\overline{\mu}_t^2 &= (H(t) - H(\infty))\int_0^t (r+s)e^{cG(s)}\mathrm{d}B_s,\\
\overline{\mu}_t^3 &= \int_0^t (r+s)e^{cG(s)}(H(\infty) - H(s))\,\mathrm{d}B_s.
\end{aligned}$$

The convergence of $H(t)$ obviously implies the convergence of $\overline{\mu}_t^1$. The deterministic factor of $\overline{\mu}_t^2$ is equivalent to $\frac{1}{cg(t)t^2}e^{-cG(t)}$ and, due to Lemma 2.3, the quadratic variation of the stochastic factor in $\overline{\mu}_t^2$ is of the order of $\frac{t^2e^{2cG(t)}}{g(t)}$. By Lemma 2.3 and the law of the iterated logarithm ([11] Theorem 3), we finally have $\overline{\mu}_t^2 \xrightarrow[t\to\infty]{a.s.} 0$. Finally, $\overline{\mu}_t^3$ is a L^2-bounded martingale and thus converges a.s. Putting all the pieces together, we conclude that $\overline{\mu}_t \xrightarrow[t\to\infty]{a.s.} \overline{\mu} + H(\infty)r(x-\overline{\mu}) + \int_0^\infty (r+s)e^{cG(s)}(H(\infty)-H(s))\,\mathrm{d}B_s$.

To show that μ_t converges, we first point out that the deterministic factor of X_t converges. Decompose the process X into three parts: $X_t = \overline{\mu}_\infty + \phi(t)U_t + o(1)$ where

$$\begin{aligned}
\overline{\mu}_\infty &:= x + cr(\overline{\mu}-x)F(\infty) + \int_0^\infty \Big[1-(r+s)ce^{cG(s)}(F(\infty)-F(s))\Big]\,\mathrm{d}B_s,\\
U_t &:= \frac{e^{-cG(t)}}{r+t}\int_0^t (r+s)e^{cG(s)}\mathrm{d}B_s,\\
\phi(t) &:= c(r+t)\left(F(\infty)-F(t)\right)e^{cG(t)}.
\end{aligned}$$

Again, we prove the result for the Fourier transform. We have the following:

$$\frac{1}{t}\int_0^t e^{iuX_s}\mathrm{d}s = \frac{e^{iu(\overline{\mu}_\infty+o(1))}}{t}\int_0^t e^{iu\phi(s)U_s}\mathrm{d}s.$$

By Proposition 3.3, $\phi(t)U_t$ is ergodic. So $\frac{1}{t}\int_0^t e^{iu\phi(s)U_s}\mathrm{d}s$ converges a.s. □

Corollary 3.5. *Suppose that $g'(t)/g^2(t)$ converges to 0 and $g(\infty)<\infty$. Then the limit $\lim \mu_t = \mu_\infty$ is the Gaussian measure $\mu_\infty = \mathcal{N}\left(\overline{\mu}_\infty, \frac{1}{2g(\infty)c}\right)$.*

3.3. **Asymptotic behaviour of X.** We prove here that, depending on g, X exhibits three different behaviours: X converges either almost surely, or in probability (and not a.s.), or it diverges. First, we describe roughly the asymptotic behaviour of X.

Proposition 3.6. *Suppose that $g(\infty)<\infty$. Then we have*

$$\mathbb{P}\left(\limsup_{t\to\infty} X_t = +\infty\right) = \mathbb{P}\left(\liminf_{t\to\infty} X_t = -\infty\right) = 1.$$

Proof. Let A be a non negligible subset of $\mathbb{R}$. We have the asymptotic equivalence

$$\int_0^t \delta_{X_s}(A)\mathrm{d}s \underset{t\to\infty}{\sim} tl$$

where l is a positive constant depending on A. So, $\int_0^\infty \delta_{X_s}(A)\mathrm{d}s = \infty$ a.s. and μ_∞ is diffusive. It then implies that for all $K>0$, $\int_0^\infty \delta_{X_s}([K,\infty[)\mathrm{d}s = \infty$ a.s. and so

$$\mathbb{P}\left(\bigcap_{K\geq 1}\left\{\int_0^\infty \mathbb{1}_{\{X_s\geq K\}}\,\mathrm{d}s = \infty\right\}\right) = 1.$$

We conclude that $\mathbb{P}(\limsup_{t\to\infty} X_t = +\infty) = 1$. The proof is the same for $\liminf_{t\to\infty} X_t$. □

Proposition 3.7. *Suppose that $g'(t)/g^2(t)$ converges to 0 and $g(\infty) = \infty$. Then X_t converges in probability to a random variable X_∞ and a.s. μ_t converges weakly to δ_{X_∞}.*

Proof. As Y is Gaussian and $\mathbb{E}(Y_t^2) = O(g(t)^{-1})$, it converges in L^2 and so in probability to 0. Decomposing X as $X_t = Y_t + \int_0^t Y_s \frac{\mathrm{d}s}{r+s}$, it remains to show that the previous integral converges in probability. Using the explicit form of Y, Fubini's theorem for stochastic integrals ensures

$$\int_0^t Y_s \frac{\mathrm{d}s}{r+s} = r(x-\bar{\mu}) \int_0^t e^{-cG(s)} \frac{\mathrm{d}s}{(r+s)^2} + \int_0^t (r+u) e^{cG(u)} \left(\int_u^t e^{-cG(s)} \frac{\mathrm{d}s}{(r+s)^2} \right) \mathrm{d}B_u.$$

The quadratic variation of the Brownian integral converges by Lemma 2.3 and thus X converges to X_∞ in L^2. Remark that the law of the iterated logarithm does not imply here that X converges a.s. since we do not know whether $\log G(t)/g(t)$ converges to 0 or not. We then easily have that μ_t converges toward δ_{X_∞} in probability. By Theorem 3.4, a.s. μ_t converges (weakly) and so we conclude. □

Proposition 3.8. *Suppose that $g'(t)/g^2(t)$ converges to 0 and $g(t)^{-1} \log G(t)$ is bounded for t large enough. Then there exists $C > 0$ such that*

$$\mathbb{P}(\limsup_{t\to\infty} |Y_t| \leq C) = 1.$$

Proof. We write Y as a Brownian (local) martingale: $Y_t = \frac{e^{-cG(t)}}{r+t} \left(Y_0 + \int_0^t (r+s) e^{cG(s)} \mathrm{d}B_s \right)$. To estimate the quadratic variation of Y, we use Lemma 2.3 and thus, by the law of the iterated logarithm, there exists C such that a.s. $\limsup_{t\to\infty} |Y_t| \leq C$. □

Corollary 3.9. *Suppose that $g(t)^{-1} \log G(t)$ is lower bounded away from 0 and upper bounded for t large enough. Then X_t is bounded a.s., converges in probability (but not a.s.) to $X_\infty = \overline{\mu}_\infty$ and a.s. μ_t converges weakly to δ_{X_∞}.*

Proof. Y is a.s. bounded and $\overline{\mu}_t$ converges a.s., so $X_t = Y_t + \overline{\mu}_t$ is also a.s. bounded. As Y_t is Gaussian, it converges (in law) to a centered Gaussian variable. The latter being bounded, Y_t converges in probability to 0. By the law of the iterated logarithm, Y_t does not converge a.s. to 0 (since $\log G(t)/g(t) > 0$ for large t). So, X_t converges in probability to $\overline{\mu}_\infty$. We conclude by uniqueness of the limit that a.s. μ_t converges weakly to $\delta_{\overline{\mu}_\infty}$. □

Proposition 3.10. *Suppose that $g'(t)/g^2(t)$ converges to 0 and $\lim_{t\to\infty} g(t)^{-1} \log G(t) = 0$. Then the process $Y_t := X_t - \overline{\mu}_t$ converges to 0 a.s. Moreover, both X_t and $\overline{\mu}_t$ converge to $\overline{\mu}_\infty$ a.s. and a.s. μ_t converges weakly to $\delta_{\overline{\mu}_\infty}$.*

Proof. We only need to prove that $Y_t := X_t - \overline{\mu}_t$ converges a.s. to 0. We have already seen that $Y_t = \frac{e^{-cG(t)}}{r+t}\int_0^t (r+s)e^{cG(s)}\mathrm{d}B_s + r(x-\overline{\mu})\frac{e^{-cG(t)}}{r+t} =: U_t + v_t$. The deterministic term v_t converges obviously to 0 and the law of the iterated logarithm implies that U_t converges a.s. to 0. By uniqueness of the limit of μ_t, we conclude that $\mu_\infty = \delta_{\overline{\mu}_\infty}$. □

4. Study of Y with respect to the critical points of V

From now on, we assume that $g'(t)/g^2(t)$ converges to 0 (this hypothesis is only needed to study the behaviour of Y near a local minimum of V). We study the process $Y_t = X_t - \bar{\mu}_t$, which is the solution to

$$\mathrm{d}Y_t = \mathrm{d}B_t - \left(g(t)\nabla V(Y_t) + \frac{Y_t}{r+t}\right)\mathrm{d}t;\ \ Y_0 = x - \overline{\mu}. \tag{4.1}$$

More precisely, we study the behaviour of Y near the critical points of V. We show in particular, for each local minimum of V, that Y stays close to it with positive probability; whereas this probability is zero for any unstable critical point.

4.1. Behaviour near the critical points of V.

Proposition 4.1. *Almost surely,* $\forall\varepsilon > 0$, $\forall t > 0$,

$$T_t^\varepsilon := \inf\{s \geq t; d(Y_s, Min \cup Max) < \varepsilon\} < \infty.$$

Proof. Let $\varepsilon > 0$. Applying Itô's formula to $x \mapsto V(x)$, we obtain

$$\mathrm{d}V(Y_t) = (\nabla V(Y_t), \mathrm{d}B_t) - \left(g(t)|\nabla V(Y_t)|^2 + \frac{1}{r+t}(Y_t, \nabla V(Y_t)) - \frac{1}{2}\Delta V(Y_t)\right)\mathrm{d}t.$$

It then follows from the growth condition (2.1) that on the set $\{z; d(z, Min \cup Max) > \varepsilon\}$ and for $t \geq 0$, the function $y \mapsto \frac{1}{r+t}(y, \nabla V(y)) + g(t)|\nabla V(y)|^2 - \frac{1}{2}\Delta V(y)$ is bounded from below. So, there exists $C > 0$ such that, $\forall y \in \{z; d(z, Min \cup Max) > \varepsilon\}$, we have

$$g(t)|\nabla V(y)|^2 + \frac{1}{r+t}(y, \nabla V(y)) - \frac{1}{2}\Delta V(y) \geq \left(g(t) - \frac{g(\infty)}{2}\right)|\nabla V(y)|^2 - C. \tag{4.2}$$

Let us introduce the stopping time $T_t^\varepsilon = \inf\{s \geq t; d(Y_s, Min \cup Max) < \varepsilon\}$ and prove that $\mathbb{P}(T_t^\varepsilon < +\infty) = 1$. It follows from (4.2) that there exists t_0 such that, for $t > t_0$, $\left(V(Y_{s\wedge T_t^\varepsilon}) + C(s \wedge T_t^\varepsilon), s \geq t\right)$ and

$$\left(V(Y_{s\wedge T_t^\varepsilon}) + C(s\wedge T_t^\varepsilon) + \int_0^{s\wedge T_t^\varepsilon}\left(g(u) - \frac{1}{2}g(\infty)\right)|\nabla V(Y_{u\wedge T_t^\varepsilon})|^2 \mathrm{d}u, s \geq t\right)$$

are two super-martingales. As they are positive, they converge a.s. (as $s \to \infty$). So, the process $\left(\int_0^{s\wedge T_t^\varepsilon} g(u)|\nabla V(Y_{u\wedge T_t^\varepsilon})|^2\mathrm{d}u, s \geq t\right)$ also converges a.s. On the set $\{T_t^\varepsilon = +\infty\}$, we have

$$|\nabla V(Y_{s\wedge T_t^\varepsilon})|^2 \xrightarrow[s\to\infty]{a.s.} 0.$$

Thus $Y_{s\wedge T_t^\varepsilon}$ gets close to $Min \cup Max$ and there is a contradiction. Finally, $\mathbb{P}(T_t^\varepsilon < +\infty) = 1$ for all $t > t_0$. For $t \le t_0$, we conclude since $t \mapsto T_t^\varepsilon$ is increasing. □

Corollary 4.2. *Almost surely, the sequence of stopping times* $T_n := \inf\{s > n; d(Y_s, Min \cup Max) < \varepsilon\}$ *satisfies:* $T_n \to \infty$*, and* $\forall n \ge 1$, $\mathbb{P}(T_n < +\infty) = 1$ *and* $d(Y_{T_n}, Min \cup Max) < \varepsilon$.

4.2. **Case of a stable critical point: local minimum.** We will prove that if Y_0 is near a local minimum m, then the set $\{Y_s; s \ge 0\}$ stays near m with a positive probability. Indeed, a second-order Taylor expansion permits to compare $(y - m, \nabla V(y))$ with $|y - m|^2$ and we use a comparison theorem. Let m be a local minimum of V such that $\nabla^2 V(m) > 0$. By Taylor's formula, there exist $a > 0$ and $\varepsilon_0 > 0$ such that for all $|y - m| \le \varepsilon_0$ we have $(y - m, \nabla V(y)) \ge a|y - m|^2$. Without any loss of generality, we suppose $m = 0$ in the proofs.

Proposition 4.3. *Suppose that* $g(t)^{-1} \log G(t)$ *is bounded on* $\mathbb{R}_+$*. Let* $\varepsilon_0 > \varepsilon > 0$*. Then, there exists a positive stopping time* T_0 *such that for all* $T > T_0$*, we have, on the event* $\{|Y_T - m| < \varepsilon\}$*, that* $\mathbb{P}(\forall s \ge 0; |Y_{s+T} - m| < \varepsilon) > 0$*. Moreover, for any* $T > T_0$*, we have on the event* $\{\forall s \ge T; |Y_s - m| < \varepsilon\}$*:*

$$|Y_{t+T} - m| = O\left(\sqrt{g(t+T)^{-1} \log G(t+T)}\right) \text{ a.s.}$$

Proof. Consider the time-shifted process $\widetilde{Y}_t := Y_{t+T}$. Let $\varepsilon > 0$. We will construct a one-dimensional process U such that for all $t \ge 0$, we have a.s. $|\widetilde{Y}_t| \le U_t$.

1) Suppose that $d = 1$. As $V''(m) > 0$, there exists $a > 0$ such that for all $|y| \le \varepsilon$, $yV'(y) \ge ay^2$. Introduce the nonnegative process U, unique solution to the SDE

$$\mathrm{d}U_t = \mathrm{sign}\,(\widetilde{Y}_t)\,\mathrm{d}B_t^T - ag(t+T)U_t\mathrm{d}t + \mathrm{d}L_t,\ U_0 = |\widetilde{Y_0}|, \tag{4.3}$$

where L is the local time of U in 0. Let $\alpha(t)$ be the function such that $\int_0^{\alpha(t)} e^{2aG(s+T)}\mathrm{d}s = t$ and $\alpha(0) = 0$. Then, the process $A_t := \int_0^{\alpha(t)} e^{aG(s+T)}\mathrm{d}L_s$ is the local time in zero of $W_t = \int_0^{\alpha(t)} e^{aG(s+T)}\,\mathrm{sign}\,(\widetilde{Y}_s)\,\mathrm{d}B_s^T$. Denote by W^+ the reflected Brownian motion associated to W. Skorokhod's lemma (see [8]) then entails that $e^{aG(\alpha(t)+T)}U_{\alpha(t)} = W_t^+$. So, the (strong) solution to (4.3) is

$$U_t = U_0 + e^{-aG(t+T)}W^+_{\alpha^{-1}(t)}. \tag{4.4}$$

By a martingale comparison theorem, we prove that $|\widetilde{Y}_t| \le U_t$ a.s. (Indeed, let l be a function of class $\mathcal{C}^2$ such that $\forall x > 0 :\ l(x) > 0$ and $l'(x) > 0$, and $\forall x \le 0 :\ l(x) = 0$. We apply Itô's formula to $l(|\widetilde{Y}_t| - U_t)$ to show that, on the event $\{|\widetilde{Y}_s| > U_s\}$, we have $l(|\widetilde{Y}_t| - U_t) \le 0$ a.s.). Finally, as $\alpha^{-1}(t) = \int_0^t e^{2aG(s+T)}\mathrm{d}s$, we conclude by the law of the iterated logarithm (LIL), that a.s. $U_t = O\left(\sqrt{g(t+T)^{-1}\log G(t+T)}\right)$.

2) Suppose that $d \geq 2$. Define $\tau := \inf\{t > 0; \widetilde{Y}_t = 0\}$. Itô's formula implies

$$d|\widetilde{Y}_{t\wedge\tau}| = dW_{t\wedge\tau} - g(t\wedge\tau + T)\left(\frac{\widetilde{Y}_{t\wedge\tau}}{|\widetilde{Y}_{t\wedge\tau}|}, \nabla V(\widetilde{Y}_{t\wedge\tau})\right)dt - \frac{|\widetilde{Y}_{t\wedge\tau}|}{r + t\wedge\tau + T}dt + \frac{d-1}{2|\widetilde{Y}_{t\wedge\tau}|}dt$$

where $W_t = \int_0^t \left(\frac{\widetilde{Y}_s}{|\widetilde{Y}_s|}, dB_s^T\right)$ is a standard Brownian motion. The condition $\nabla^2 V(0) > 0$ implies that there exists $a > 0$ such that

$$\forall |y| \leq \varepsilon,\ (y, \nabla V(y)) \geq a|y|^2. \tag{4.5}$$

Let us introduce the $(d-1)$-dimensional Bessel process R. Consider the time-shifted process $U_t := e^{-aG(t+T)} R_{\int_0^t e^{2aG(s+T)}ds}$, which is the nonnegative (strong) solution to

$$dU_t = d\beta_t^T - ag(t+T)U_t dt + \frac{d-1}{2U_t}dt, \tag{4.6}$$

where β is a Brownian motion. On the event $\{\forall s \geq T; |Y_s| < \varepsilon\}$, we apply the previous comparison theorem to obtain a.s. $|\widetilde{Y}_t| \leq U_t$. On the other hand, R is the radial part of a d-dimensional Brownian motion. So, the LIL implies a.s. $R_t = O(\sqrt{(t+T)\log\log(t+T)})$ and $U_t = O\left(\sqrt{g(t+T)^{-1}\log G(t+T)}\right)$.

Now, we prove that $\mathbb{P}(\forall s \geq T; |Y_s - m| < \varepsilon) > 0$. Let $\tau_T := \inf\{s > T; |Y_s - m| > \varepsilon\}$. For all $T < t < \tau_T$, we have a.s. $|Y_t - m| \leq U_t$. By the LIL applied to U, we conclude that, for T large enough, $\mathbb{P}\left(\sup_{s\geq T} U_s < \varepsilon\right) > 0$ and finally $\mathbb{P}(\tau_T = \infty) > 0$. □

Corollary 4.4. *Suppose that $g(t)^{-1}\log G(t)$ converges to 0 when t tends to infinity. Then, there exists $T_0 > 0$ such that for all $T > T_0$, the process Y_t converges almost surely to m on the event $\{\forall s \geq T; |Y_s - m| < \varepsilon\}$.*

4.3. Case of an unstable critical point: local maximum or saddle point.

If M is a local maximum of V, then as $\Delta V(M) < 0$, $\varepsilon_1 := \sup\{\varepsilon; \forall |y| < \varepsilon, \Delta V(M+y) < 0\}$ exists and is finite.
If M is a saddle point of V, then as $\nabla^2 V$ admits a negative eigenvalue in M, there exists an unstable direction e associated to M. Let $P_e : \mathbb{R}^d \mapsto \mathbb{R}e$ be the projection on $\mathbb{R}e$. The amount $\varepsilon_2 := \sup\{\varepsilon;\ \forall |y| < \varepsilon, \partial^2_{ee}V(M+y) < 0 \text{ and } (\partial_e V(P_e(y)), \partial_e V(y)) > 0\}$ exists and is finite.

Proposition 4.5. *Let M be an unstable critical point of V. If M is a local maximum, suppose that $0 < \varepsilon < \varepsilon_1$. If M is a saddle point, suppose that $0 < \varepsilon < \varepsilon_2$.*

Let T be a positive stopping time such that $|Y_T - M| < \varepsilon$. Then

$$\mathbb{P}(\forall s \geq T;\ |Y_s - M| < \varepsilon) = 0.$$

Proof. Note that $T < \infty$ a.s. by Proposition 4.1. Suppose that M is a local maximum and $M = 0$, because the method of the proof is the same for $M \neq 0$ (in that case, we have an additional term $M \log(t+T)$). Let $D(t, Y_t)$ be the drift term of $V(Y_t)$. On the event $A := \{\forall s \geq T;\ |Y_s| < \varepsilon\}$, we obtain

$$D(t+T, Y_{t+T}) = g(t+T)|\nabla V(Y_{t+T})|^2 + \frac{(Y_{t+T}, \nabla V(Y_{t+T}))}{r+t+T} - \frac{1}{2}\Delta V(Y_{t+T}) \geq \frac{C_1}{t+T} + C_2$$

where $C_1 = \frac{1}{2}\inf\{(y, \nabla V(y)); |y| < \varepsilon\}$ and $C_2 = -\frac{1}{2}\sup\{\Delta V(y); |y| < \varepsilon\} > 0$. We thus find for t large enough that $D(t+T, Y_{t+T}) \geq C > 0$ and so

$$\mathbb{E}\left(V(Y_{t+T})\mathbb{1}_A\right) \leq \mathbb{E}\left(V(Y_T)\mathbb{1}_A\right) - Ct\mathbb{P}(A) + o(t). \tag{4.7}$$

Finally, this last inequality is impossible (since V is positive) unless $\mathbb{P}(A) = 0$.

Suppose now that M is a saddle point. We apply Itô's formula to $x \mapsto V(P_e(x))$ and follow the previous computation with $C_1 = \frac{1}{2}\inf\{(P_e(y), \partial_e V(P_e(y))); |y| < \varepsilon\}$ and $C_2 = -\frac{1}{2}\sup\{\partial^2_{ee} V(P_e(y)); |y| < \varepsilon\} > 0$. □

5. Asymptotics

Through this section, we always suppose that $g(\infty) = +\infty$ and $g'(t)/g^2(t)$ converges to 0, even if we do not remind the reader in the statements of the results. In particular, it implies that for all $T > 0$, $G^{-1}(t+T) - G^{-1}(t)$ goes to 0 when t tends to infinity.

5.1. **Ergodicity.**

Lemma 5.1. *The process Y is bounded in L^2.*

Proof. We show a stronger result: $\mathbb{E}V(Y_t)$ is bounded. For all $n \in \mathbb{N}$, define the stopping time $\tau_n = \inf\{t \geq 0; |Y_t| > n\}$. Then, there exists $C > 0$ such that we have by localization:

$$\mathbb{E}V(Y_{t\wedge\tau_n}) \leq \mathbb{E}V(Y_0) + e^{Ct} < \infty.$$

Let n go to infinity and use Fatou's lemma to find, for all $t \geq 0$, that $V(Y_t) \in L^1$. For t large enough, we have that $-g(t)V(x) + aV(x) \leq -\frac{1}{2}g(t)V(x)$. So, as W is $c-$strictly convex and by the growth hypothesis (2.1), the following holds for t large enough

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}V(Y_t) \leq -\frac{1}{2}g(t)\mathbb{E}V(Y_t).$$

Now, solving $\dot{u} = -\frac{1}{2}g(t)u$ leads to $\mathbb{E}V(Y_t) = O(1)$. □

In order to obtain the ergodic result for Y, we introduce a dynamical system ϕ, whose asymptotics are close to Y (see [1] for more details):

Definition 5.2. *The process Y is an asymptotic pseudotrajectory for the flow ϕ if*

$$\forall T > 0,\ \forall \alpha > 0, \quad \lim_{t\to+\infty} \mathbb{P}\left(\sup_{0\leq h\leq T} |Y_{t+h} - \phi_h(Y_t)| \geq \alpha\right) = 0. \tag{5.1}$$

Proposition 5.3. *Let $\phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be the flow generated by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) = -\nabla V(\phi_t(x)); \quad \phi_0(x) = x. \tag{5.2}$$

Then $(Y_{G^{-1}(t)}, t \geq 0)$ is an asymptotic pseudotrajectory for ϕ.

Proof. Let $\widetilde{Y}_t = Y_{G^{-1}(t)}$ and $\widetilde{B}_t = B_{G^{-1}(t)}$. We will use Markov's inequality and then prove that $\lim_{t\to\infty} \mathbb{E}\left(\sup_{0\leq h\leq T} |\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t)|\right) = 0$.

Define $\kappa(t) := (r + G^{-1}(t))g(G^{-1}(t))$. A simple computation yields to

$$\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t) = \widetilde{B}_{t+h} - \widetilde{B}_t + \int_0^h \left(\nabla V(\phi_s(\widetilde{Y}_t)) - \nabla V(\widetilde{Y}_{t+s})\right) \mathrm{d}s - \int_0^h \widetilde{Y}_{t+s} \frac{\mathrm{d}s}{\kappa(t+s)}.$$

Applying Itô's formula to $h \mapsto e^{-2\tilde{C}h}|\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t)|^2$, we have:

$$\frac{1}{2}e^{2\tilde{C}h}\mathrm{d}(e^{-2\tilde{C}h}|\widetilde{Y}_{t+h}-\phi_h(\widetilde{Y}_t)|^2) = \left(\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t), \mathrm{d}\widetilde{B}_{t+h}\right) + \frac{\left(\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t), \widetilde{Y}_{t+h}\right)}{\kappa(t+h)}\mathrm{d}h -$$

$$-\tilde{C}|\widetilde{Y}_{t+h}-\phi_h(\widetilde{Y}_t)|^2\mathrm{d}h + \left(\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t), \nabla V(\phi_h(\widetilde{Y}_t)) - \nabla V(\widetilde{Y}_{t+h})\right)\mathrm{d}h + \frac{e^{2\tilde{C}h}}{2g(G^{-1}(t+h))}\mathrm{d}h.$$

First, we notice that $\left(G^{-1}(t)\right)' = 1/g(G^{-1}(t))$. By the convexity assumption on V, we also remark that

$$-\tilde{C}|\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t)|^2 + \left(\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t), \nabla V(\phi_h(\widetilde{Y}_t)) - \nabla V(\widetilde{Y}_{t+h})\right) \leq 0,$$

and so we deduce the following upper bound for all $0 \leq h \leq T$:

$$\frac{1}{2}|\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t)|^2 \leq e^{2\tilde{C}h}\int_0^h e^{-2\tilde{C}s}(\widetilde{Y}_{t+s} - \phi_s(\widetilde{Y}_t), \mathrm{d}\widetilde{B}_{t+s}) + \frac{e^{2\tilde{C}T}}{2}\left(G^{-1}(t+T) - G^{-1}(t)\right)$$
$$+ e^{2\tilde{C}h}\int_0^h \left(\widetilde{Y}_{t+s} - \phi_s(\widetilde{Y}_t), \widetilde{Y}_{t+s}\right)\frac{\mathrm{d}s}{\kappa(t+s)}. \tag{5.3}$$

To conclude, we will now find an upper bound for each right-hand term of (5.3). By BDG's inequality for the local martingale $\int_0^h e^{-2\tilde{C}s}(\widetilde{Y}_{t+s} - \phi_s(\widetilde{Y}_t), \mathrm{d}\widetilde{B}_{t+s})$ and a rough upper bound for its quadratic variation, there exists $\alpha > 0$ such that:

$$\mathbb{E}\left(\sup_{0\leq h\leq T}\int_0^h e^{-2\tilde{C}s}(\widetilde{Y}_{t+s} - \phi_s(\widetilde{Y}_t), \mathrm{d}\widetilde{B}_{t+s})\right) \leq \alpha(G^{-1}(t+T) - G^{-1}(t))\mathbb{E}\left(\sup_{0\leq h\leq T}|\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t)|^2\right)^{\frac{1}{2}}.$$

We now estimate the remaining term of (5.3) by the triangle inequality. As κ is nondecreasing, we have the following bound by Lemma 5.1:

$$\mathbb{E}\int_0^T \left(\frac{|\widetilde{Y}_{t+s}|^2}{\kappa(t+s)} + \frac{|\widetilde{Y}_{t+s} - \phi_s(\widetilde{Y}_t)|^2}{\kappa(t+s)}\right)\mathrm{d}s \leq \frac{MT}{\kappa(t)} + \frac{T}{\kappa(t)}\mathbb{E}\left(\sup_{0\leq h\leq T}|\widetilde{Y}_{t+h} - \phi_h(\widetilde{Y}_t)|^2\right).$$

So, we obtain for t large enough:

$$\mathbb{E}\left(\sup_{0\le h\le T}|\widetilde{Y}_{t+h}-\phi_h(\widetilde{Y}_t)|^2\right)\le 4e^{4\tilde{C}T}(G^{-1}(t+T)-G^{-1}(t))+4Me^{2\tilde{C}T}\frac{T}{\kappa(t)},$$

and the result follows since $G^{-1}(t+T)-G^{-1}(t)$ and $1/\kappa(t)$ converge to 0. ☐

Lemma 5.4. *Suppose that for all $T>0$, $G^{-1}(t+T)-G^{-1}(t)$ vanishes when t tends to infinity. Then, $\left(\mu_t^{G^{-1}}:=\frac{1}{t}\int_0^t\delta_{Y_{G^{-1}(s)}}\mathrm{d}s, t\ge 0\right)$ is a tight family of measures.*

Proof. It is enough to show that a.s. $\varphi(t):=\int_0^t V(Y_{G^{-1}(s)})\mathrm{d}s=O(t)$. Let $A>0$ and K be a compact set such that $\forall x\in K^c$, $V(x)\ge A$. Then $\mu_t^{G^{-1}}(V)\ge A\mu_t^{G^{-1}}(\mathbb{1}_{K^c})$. From the growth assumption (2.1), there exists $a>0$ and for all $\varepsilon>0$, there exists k_ε such that $\Delta V\le aV$ and $V\le k_\varepsilon+\varepsilon|\nabla V|^2$. It then yields

$$\varphi(t)\le k_\varepsilon t+\varepsilon\int_0^t|\nabla V(Y_{G^{-1}(s)})|^2\mathrm{d}s \text{ and } \int_0^t\Delta V(Y_{G^{-1}(s)})\mathrm{d}s\le a\varphi(t). \quad (5.4)$$

Applying Itô's formula to $V(Y_{G^{-1}(t)})$, we obtain

$$\begin{aligned} V(Y_{G^{-1}(t)})-V(Y_0) &= \int_0^{G^{-1}(t)}(\nabla V(Y_s),\mathrm{d}B_s)-\int_0^t\frac{(Y_{G^{-1}(s)},\nabla V(Y_{G^{-1}(s)}))}{(r+G^{-1}(s))g(G^{-1}(s))}\mathrm{d}s \\ &- \int_0^t|\nabla V(Y_{G^{-1}(s)})|^2\mathrm{d}s+\frac{1}{2}\int_0^t\Delta V(Y_{G^{-1}(s)})\frac{\mathrm{d}s}{g(G^{-1}(s))} \quad (5.5)\end{aligned}$$

Consider the (local-)martingale term of (5.5). On the set $\{\int_0^\infty|\nabla V(Y_s)|^2\mathrm{d}s<\infty\}$, it is bounded in L^2 and thus converges. Whereas on the set $\{\int_0^\infty|\nabla V(Y_s)|^2\mathrm{d}s=\infty\}$, the strong LLN implies that, for t large enough, a.s.

$$\int_0^{G^{-1}(t)}(\nabla V(Y_s),\mathrm{d}B_s)\le\frac{1}{2}\int_0^t|\nabla V(Y_{G^{-1}(s)})|^2\mathrm{d}s.$$

By (5.5), we then find for t large enough

$$\begin{aligned}\int_0^t|\nabla V(Y_{G^{-1}(s)})|^2\mathrm{d}s &\le \int_0^t\Delta V(Y_{G^{-1}(s)})\mathrm{d}s-2V(Y_{G^{-1}(t)})+2V(Y_0) \\ &- 2\int_0^t\frac{(Y_{G^{-1}(s)},\nabla V(Y_{G^{-1}(s)}))}{(r+G^{-1}(s))g(G^{-1}(s))}\mathrm{d}s \\ &\le \frac{a\varphi(t)}{g(G^{-1}(t))}+2V(Y_0)+O\left(\int_0^t\frac{\mathrm{d}s}{G^{-1}(s)g(G^{-1}(s))}\right).\end{aligned}$$

So, we have a.s. $\int_0^t|\nabla V(Y_{G^{-1}(s)})|^2\mathrm{d}s=O(t)+a\varphi(t)$. Putting this result in (5.4) and choosing ε small enough, we conclude that $\varphi(t)=O(t)$ a.s. ☐

Theorem 5.5. *The process Y satisfies the pointwise ergodic theorem. More precisely, there exist some (deterministic) constants $a_i \geq 0$, such that $\sum a_i = 1$ and $\mu_t^Y = \frac{1}{t}\int_0^t \delta_{Y_s}\mathrm{d}s$ converges (for the weak convergence of measures) toward $\sum_{1\leq i\leq n} a_i\delta_{m_i}$.*

Proof. By Benaïm & Schreiber [4], Proposition 5.3 implies that the limit points of the empirical measure of $Y_{G^{-1}(t)}$ are included in the set of all the "invariant measures" for $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) = -\nabla V(\phi_t(x))$ with the initial condition $\phi_0(x) = x$. All these invariant measures are included in $Span\{\delta_{m_1}, \cdots, \delta_{m_n}, \delta_{M_1}, \cdots, \delta_{M_p}\}$. Let $\mu_t^{G^{-1}} = \frac{1}{t}\int_0^t \delta_{Y_{G^{-1}(s)}}\mathrm{d}s$. By Lemma 5.4, the empirical measure $\mu_t^{G^{-1}}$ converges. One also shows that μ_t is a Cauchy sequence in L^1: there exists $C > 0$ such that for any $s > 0$,

$$|\mathbb{E}\overline{\mu}_{t+s} - \mathbb{E}\overline{\mu}_t| \leq \frac{s}{t(t+s)}\int_0^t \mathbb{E}|X_u|\mathrm{d}u + \frac{1}{t+s}\int_t^{t+s}\mathbb{E}|X_u|\mathrm{d}u \leq C\frac{s}{t+s}.$$

So, the limit-measure of $\mu_t^{G^{-1}}$ writes $\sum_{i=1}^n a_i\delta_{m_i} + \sum_{i=1}^p b_i\delta_{M_i}$ (where a_i, b_i are nonnegative constants such that $\sum(a_i + b_i) = 1$). And the same result holds for μ_t. Indeed, for all continuous bounded function ψ and $t > s$, we have (by an integration by parts)

$$\begin{aligned}
\int_s^t \psi(Y_u)\mathrm{d}u &= \frac{G(t)}{g(t)}\mu_{G(t)}^{G^{-1}}\psi - \frac{G(s)}{g(s)}\mu_{G(s)}^{G^{-1}}\psi + \int_s^t \frac{g'(u)G(u)}{g^2(u)}\mu_{G(u)}^{G^{-1}}\psi\mathrm{d}u \\
&= (t-s)\mu_{G(t)}^{G^{-1}}\psi + \frac{G(s)}{g(s)}\left(\mu_{G(t)}^{G^{-1}}\psi - \mu_{G(s)}^{G^{-1}}\psi\right) \\
&+ \int_s^t \frac{g'(u)G(u)}{g^2(u)}(\mu_{G(u)}^{G^{-1}}\psi - \mu_{G(t)}^{G^{-1}}\psi)\mathrm{d}u.
\end{aligned}$$

As $\mu_{G(t)}^{G^{-1}}\psi$ converges a.s., we deduce that

$$\mu_t\psi = o(1) + \mu_{G(t)}^{G^{-1}}\psi + \frac{1}{t}\int_s^t \frac{g'(u)G(u)}{g^2(u)}(\mu_{G(u)}^{G^{-1}}\psi - \mu_{G(t)}^{G^{-1}}\psi)\mathrm{d}u.$$

So, by Lemma 2.3, μ_t converges. We also wish to show that $b_i = 0$ for all i. Proposition 4.5 implies that, for an unstable critical point M, there exists a direction j such that for all $\varepsilon > 0$, $\mathbb{P}(\forall s \geq T, |Y_s^{(j)} - M^{(j)}| \leq \varepsilon) = 0$. Consider a continuous function f, supported by a small ball (of radius $\alpha > 0$) around M: f vanishes in all critical points except M and $f(M) = 1$. Then, we have a.s. $\int_0^t \mathbb{1}_{\{|Y_s^{(j)} - M^{(j)}| \leq \alpha\}}\mathrm{d}s = o(t)$ and $\frac{1}{t}\int_0^t f(Y_s)\mathrm{d}s$ converges almost surely to b. So, we conclude that $b = 0$. □

At this stage, we have proved that Y satisfies the pointwise ergodic theorem. The main question is whether X also satisfies the pointwise ergodic theorem or not. To answer it, we remind that $\overline{\mu}_t$ converges a.s. if and only if $\int_0^t Y_s\frac{\mathrm{d}s}{r+s}$ converges, and if $Y_t \xrightarrow[t\to\infty]{a.s.} 0$ polynomially fast then $\overline{\mu}_t$ converges a.s.

Proposition 5.6. *The measure μ_t converges weakly if and only if $\overline{\mu}_t$ converges a.s.*

Proof. We have shown in Theorem 5.5 that Y is pointwise ergodic. Consider the Fourier transform of μ_t and recall that $X_t = Y_t + \overline{\mu}_t$. We have for all $u \in \mathbb{R}^d$:

$$\frac{1}{t}\int_0^t e^{i(u,X_s)}\mathrm{d}s = \frac{e^{i(u,\overline{\mu}_\infty)}}{t}\int_0^t e^{i(u,Y_s)}\mathrm{d}s + \frac{1}{t}\int_0^t e^{i(u,Y_s)}\left(e^{i(u,\overline{\mu}_s)} - e^{i(u,\overline{\mu}_\infty)}\right)\mathrm{d}s.$$

The first right member converges a.s. to $e^{i(u,\overline{\mu}_\infty)}\sum_{1\le p\le n} e^{i(u,m_p)}$. For the second right member, Cesàro asserts that it converges a.s. to 0 if and only if $\overline{\mu}_t$ converges a.s. □

5.2. **Almost sure convergence toward the local minima of V.** Let $0 < \varepsilon < \varepsilon_0$ and $T > T_0$ be as in Section 4. Let m be a local minimum of V such that $|Y_T - m| < \varepsilon$.

Lemma 5.7. *If $\lim_{t\to\infty} g(t)^{-1}\log G(t) = 0$, then for all $\alpha > 0$, we have $\int_0^\infty e^{-\alpha g(t)}\mathrm{d}t < +\infty$.*

Proof. For all $\varepsilon > 0$, there exists t such that, for all $s \ge t$, we have $g(s) \ge \varepsilon^{-1}\log G(s)$. Moreover, there exists $a > 0$ such that for t large enough $g(t) \ge a$ and then $G(t) \ge at$. So, we conclude that $\int_1^\infty e^{-\varepsilon^{-1}\alpha\log(at)}\mathrm{d}t < \infty$ (choose for instance $\varepsilon = \alpha/2$). □

Proposition 5.8. *If $g(t)^{-1}\log G(t)$ converges to 0, then Y_t converges a.s. and for all i, we have $\mathbb{P}\left(\lim_{t\to\infty} Y_t = m_i\right) > 0$ and $\mathbb{P}\left(\lim_{t\to\infty} Y_t = M_i\right) = 0$.*

Proof. Benaïm ([1] Proposition 4.6) asserts that if $-\nabla V(x)$ is a continuous globally integrable vector field, and if for all $\alpha > 0$, we have $\int_0^\infty e^{-\alpha g\circ G^{-1}(t)}\mathrm{d}t < +\infty$ and $\mathbb{P}(\sup_t |Y_t| < \infty) = 1$, then Y is a.s. an asymptotic pseudotrajectory for the flow induced by $-\nabla V$. Actually, the first and last conditions are fulfilled here. Moreover, as G^{-1} is nondecreasing, the (finite) integral $\int_0^\infty e^{-\alpha g(t)}\mathrm{d}t$ is a upper bound for the preceding one. Thus, Y is a.s. an asymptotic pseudotrajectory for the flow ϕ defined by (5.2) and ϕ restricted to the limit points of Y does not admit any other attractor than the set of limit points. Finally, Y converges a.s. and its limit points are included into the set $\{x; \nabla V(x) = 0\}$.

If Y converges to Y_∞, then due to Proposition 4.5, the limit Y_∞ is not a local maximum. On the event $\{\forall s \ge T; |Y_s - m_i| < \varepsilon\}$, occuring with a positive probability by Proposition 4.3, we have a.s. $|Y_{t+T} - m_i| \le U_t$. As $\overline{\lim_{t\to\infty}} U_t\sqrt{\frac{g(t)}{\log G(t)}} = 1$ a.s., we conclude that $U_t \xrightarrow[t\to\infty]{a.s.} 0$. □

Corollary 5.9. *Suppose that $\lim g(t)^{-1}\log G(t) = 0$.*

1) If V is a strictly uniformly convex function (m its unique minimum), then $Y_t \xrightarrow[t\to\infty]{a.s.} m$.

2) A necessary condition for the almost sure convergence of Y *to 0 is that the potential* V *admits a unique minimum at 0 (e.g.* V *is symmetric and strictly convex).*

5.3. **Back to** X.

Theorem 5.10. *Assume that* $g(t)^{-1}\log G(t) = O(h(t)^{-2})$*, with* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\int_0^\infty \frac{\mathrm{d}s}{(1+s)h(s)} < +\infty$*. Then, on the set* $\{Y_\infty \neq 0\}$*,* $\frac{X_t}{\log t}$ *converges to* Y_∞*. Moreover:*

(1) *If 0 is the unique local minimum of* V*, then*

$$\mathbb{P}\left(\lim_{t\to\infty} X_t = \overline{\mu} + \int_0^\infty Y_s \frac{\mathrm{d}s}{r+s}\right) = \mathbb{P}\left(\lim_{t\to\infty} \overline{\mu}_t = \overline{\mu} + \int_0^\infty Y_s \frac{\mathrm{d}s}{r+s}\right) = 1;$$

(2) *If* V *admits 0 as a (non-unique) local minimum, then* X_t *converges on the event* $\{Y_t \xrightarrow{a.s.} 0\}$ *and diverges elsewhere. More precisely, one has*

$$\mathbb{P}\left(\lim_{t\to\infty} X_t = \overline{\mu} + \int_0^\infty Y_s \frac{\mathrm{d}s}{r+s}\right) + \mathbb{P}(\lim_{t\to\infty} |X_t| = \infty) = 1$$

and $$1 > \mathbb{P}\left(\lim_{t\to\infty} X_t = \overline{\mu} + \int_0^\infty Y_s \frac{\mathrm{d}s}{r+s}\right) = \mathbb{P}\left(\lim_{t\to\infty} \overline{\mu}_t = \overline{\mu} + \int_0^\infty Y_s \frac{\mathrm{d}s}{r+s}\right) > 0;$$

(3) *If 0 is not a local minimum of* V*, then*

$$\mathbb{P}\left(\lim_{t\to\infty} |X_t| = \infty\right) = \mathbb{P}\left(\lim_{t\to\infty} |\overline{\mu}_t| = \infty\right) = 1.$$

Proof. Denote $m = (m^{(1)}, \cdots, m^{(d)})$. First, suppose that $m = 0$. By Proposition 5.8, Y_t converges toward 0 with a positive probability. On this event, Proposition 4.3 implies that the integral $\int_0^t \frac{Y_s}{r+s}\mathrm{d}s$ converges. So, $\overline{\mu}_t$ converges toward this (limit) integral and the result follows for X. On the other hand, if $m \neq 0$, then $\mathbb{P}(Y_t \to m) > 0$ and so the j^{th}-coordinate of $\overline{\mu}_t$ converges to $sgn(m^{(j)})\infty$. So, the direction j is unstable and X_t does not converge a.s. Moreover, on the set $\{Y_\infty \neq 0\}$, we have

$$\left|\frac{\overline{\mu}_t}{\log t} - Y_\infty\right| \leq \frac{1}{\log t}\int_0^t |Y_s - Y_\infty| \frac{\mathrm{d}s}{r+s} \leq \frac{1}{\log t}\int_0^t \sqrt{\frac{\log G(s)}{g(s)}}\frac{\mathrm{d}s}{r+s}.$$

The latter upper bound tends to 0 by the law of the iterated logarithm (Proposition 4.3). As $\frac{X_t}{\log t} = \frac{\overline{\mu}_t}{\log t} + \frac{Y_t}{\log t}$, the result follows. □

Remark 5.11. *Any polynomial* h *satisfies the required condition. In particular, one can choose* $g(t) = t^\alpha(\log(1+t))^\beta$ *with* $\alpha > 0$*, or* $\alpha = 0$ *and* $\beta > 2$*.*

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